Finite flag-transitive affine planes with a solvable automorphism group

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Definition

A finite incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ consists of

1. two finite nonempty sets $\mathcal{P}$ (points) and $\mathcal{L}$ (lines/blocks),
2. an incidence relation $\mathcal{I} \subset \mathcal{P} \times \mathcal{L}$.

A flag is an incident point-line pair.

The classification of finite incidence structures in terms of a group theoretical hypothesis is now commonplace.

Example (Ostrom-Wagner Theorem)

A finite projective plane having a 2-transitive collineation group must be Desarguesian.

The general study of flag-transitive planes was initiated by Higman and McLaughlin, and they posed the problem of classifying the finite flag-transitive projective planes.

Theorem (Kantor)

A finite flag-transitive projective plane is desarguesian with the possible exception where the collineation group G is a Frobenius group of prime degree.

Remark

"...the Frobenius case remains elusive, but presumably occurs only for $\mathrm{PG}(2,2)$ *and* $\mathrm{PG}(2,8)$*" (Kantor)*

The affine case

Theorem (Wagner)

A finite flag-transitive affine plane must be a translation plane.

Unlike the projective case,

1. there are many examples of such planes;
2. the classification and construction are more of a combinatorial flavor rather than group theoretical.

The translation group T is elementary abelian, and acts regularly on points. The collineation group$=T\rtimes$ translation complement.

Spread

A spread of $V = \mathbb{F}_q^{2n}$ is a set of n-dimensional subspaces $W_0, W_1, \cdots, W_{q^n}$ that partitions the nonzero vectors of V.

Example (regular spread)

Take $V = \mathbb{F}_{q^n} \times \mathbb{F}_{q^n}$, and define $L_a = \{(x, ax) : x \in \mathbb{F}_{q^n}\}$ for $a \in \mathbb{F}_{q^n}$, $L_\infty = \{(0, y) : y \in \mathbb{F}_{q^n}\}$. They form a spread of V.

Translation plane

We can define an affine plane from a spread.

1. points: vectors of $V = \mathbb{F}_q^{2n}$;
2. lines: $W_i + v$, $0 \leq i \leq q^n$, $v \in V$.
3. incidence: inclusion.

The translation group T consists of the translation τ_u's defined by

$$\tau_u(v) = u + v, \quad \tau_u(W_i + v) = W_i + u + v.$$

The regular spread defines $\mathrm{AG}(2, q)$ in this way.

Solvability of collineation group

Theorem (Foulser)

With a finite number of exceptions, a solvable flag transitive group of a finite affine plane has its translation complement contained in the group consisting of $x \mapsto ax^\sigma$ with $a \in \mathbb{F}^*_{q^{2n}}$ and $\sigma \in \mathrm{Gal}(\mathbb{F}_{q^{2n}})$.

Theorem (Kantor)

The only odd order flag-transitive planes with nonsolvable automorphism groups are the nearfield planes of order 9 and Hering's plane of order 27.

$\mathcal{C}$-planes and $\mathcal{H}$-planes

Assuming that the plane is not Hering plane of order 27, Ebert showed that the translation complement must contain a Singer subgroup $H = \langle \gamma^2 \rangle$ of order $\frac{q^n+1}{2}$ under the restriction

$$\gcd\left(\frac{1}{2}(q^n+1), ne\right) = 1, \quad q \text{ odd},$$
$$\gcd(q^n+1, ne) = 1, \quad q \text{ even}.$$

If the translation complement is isomorphic to $\langle \gamma \rangle$, then we say that the plane is type $\mathcal{C}$. If the translation complement contains an isomorphic copy of $\langle \gamma^2 \rangle$ but not $\langle \gamma \rangle$, then we call the plane type $\mathcal{H}$.

Examples

There are two general constructions

1. Odd order: Kantor-Suetake family[1]
2. Even order case: Kantor-Williams family[2]

The dimensions of these planes over their kernels are odd.

Remark

It remains open whether there is a flag-transitive affine plane of even order and even dimension.

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[1] The dimension two case is also due to Baker and Ebert.

[2] prolific, arising from symplectic spread

Classifications

Prince has completed the determination of all the flag-transitive affine planes of order at most 125.

Ebert and collaborators classified the (odd order, dim 2/3) case.

1. approach: geometric
2. Baer subgeometry partition

The starting point: coordinatization

Let $\mathcal{S}$ be a spread of type $\mathcal{H}$ or type $\mathcal{C}$. Let W be a component of $\mathcal{S}$, so that $\mathcal{S} = \{g(W) : g \in \mathrm{Aut}(\mathcal{S})\}$. Since the regular spread of $\mathbb{F}_{q^{2n}}$ has $q^n + 1$ components, there exists $\delta \in \mathbb{F}_{q^{2n}} \setminus \mathbb{F}_{q^n}$ such that $W \cap \mathbb{F}_{q^n} \cdot \delta = \{0\}$. From $\mathbb{F}_{q^{2n}} = \mathbb{F}_{q^n} \oplus \mathbb{F}_{q^n} \cdot \delta$, we can write the $\mathbb{F}_q$-subspace W as follows:

$$W = \{x + \delta \cdot L(x) : x \in \mathbb{F}_{q^n}\}, \qquad (1)$$

where $L(X) \in \mathbb{F}_{q^n}[X]$ is a reduced q-polynomial. We also define

$$Q(X) := (X + \delta L(X)) \cdot (X + \delta^{q^n} L(X)), \qquad (2)$$

which is a DO polynomial over $\mathbb{F}_{q^n}$.

The key lemma

Additional notation:

1. $\Theta(u)$: the map $x \mapsto ux$, $x \in \mathbb{F}_{q^{2n}}$;
2. β: an element of order $(q^n+1)(q-1)$;
3. $\mathcal{S}_H := \{W^g : g \in \langle \Theta(\beta^2) \rangle\}$;
4. $\mathcal{S}_C := \{W^g : g \in \langle \Theta(\beta) \rangle\}$.

Lemma

1. *If q is odd, then $\mathcal{S}_H$ is a partial spread iff $Q(x)$ is a planar function, and $\mathcal{S}_C$ is a spread iff $x \mapsto Q(x)$ permutes $\mathbb{F}_{q^n}^*/\mathbb{F}_q^*$.*
2. *If q is even, then $\mathcal{S}_C$ is a spread iff $x \mapsto Q(x)$ permutes $\mathbb{F}_{q^n}$.*

Idea of the proof

A function $f : \mathbb{F}_q \mapsto \mathbb{F}_q$ is *planar* if $x \mapsto f(x+a) - f(x) - f(a)$ is a permutation of $\mathbb{F}_q$ for any $a \neq 0$. It is known that there are no planar functions in even characteristic.

Lemma (Weng, Zeng, 2012)

Let $f : \mathbb{F}_q \mapsto \mathbb{F}_q$ be a DO polynomial. Then f is planar if and only if f is 2-to-1, namely, every nonzero element has 0 or 2 preimages.

Immediate consequences

Corollary

In the case q and n are both odd, if $\mathcal{S}_H$ forms a partial spread, then $\mathcal{S}_C$ forms a spread.

Theorem

There is no type $\mathcal{C}$ spread with ambient space $(\mathbb{F}_{q^{2n}}, +)$ and kernel $\mathbb{F}_q$ when n is even and q is odd.

Characterization of Kantor-Suetake family

Menichetti (1977, 1996): Let S be a finite semifield of prime dimension n over the nucleus $\mathbb{F}_q$. Then there is an integer $\nu(n)$ such that if $q \geq \nu(n)$ then S is isotopic to a finite field or a generalized twisted field. Moreover, we have $\nu(3) = 0$.

Theorem (F., 2017)

Let n be an odd prime, $\nu(n)$ be as above, and $q \geq \nu(n)$. A type $\mathcal{C}$ spread $\mathcal{S}$ of $(\mathbb{F}_{q^{2n}}, +)$ with kernel $\mathbb{F}_q$ is isomorphic to the orbit of $W = \{x + \delta \cdot x^{q^i} : x \in \mathbb{F}_{q^n}\}$ under $\langle \Theta(\beta) \rangle$ for some δ and i such that $\delta^{q^n - 1} = -1$, $\gcd(i, n) = 1$.

Idea of the proof

By Prop 11.31 of the Handbook, which is essentially due to Albert, a generalized twisted field that has a commutative isotope must be isotopic to the commutative presemifield defined by a planar function x^{1+p^α} over $\mathbb{F}_{p^e}$, where $e/\gcd(e,\alpha)$ is odd.

Lemma (Coulter, Henderson, 2008)

Let p be an odd prime and $q = p^e$. Let f be a planar function of DO type over $\mathbb{F}_q$ and $S_f = (\mathbb{F}_q, +, *)$ be the associated presemifield with $x * y = f(x+y) - f(x) - f(y)$. There exist linearized permutation polynomials M_1 and M_2 such that

1. *if S_f is isotopic to a finite field, then $f(M_2(x)) = M_1(x^2)$;*
2. *if S_f is isotopic to a commutative twisted field, then $f(M_2(x)) = M_1(x^{p^\alpha+1})$, where α is as above.*

The case q even

The following lemma describes how to study the permutation behavior of a DO polynomial via quadratic forms.

Lemma

Let $Q(X) = \sum_{i,j} a_{ij} X^{q^i+q^j} \in \mathbb{F}_{q^n}[X]$ with q even. Then $Q(X)$ is a PP iff $Q_y(x) = Tr_{\mathbb{F}_{q^n}/\mathbb{F}_q}(yQ(x))$ has odd rank for $y \neq 0$.

We are able to characterize type $\mathcal{C}$ planes up to dimension four. This is the first characterization result in the even order case.

Thanks for your attention!