On Tight Sets of Hyperbolic Quadrics

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Fifth Irsee Conference
Finite Geometry
2017

Motivation for tight sets

- Introduced and first studied by Payne in GQ (1987)
- Extended to Polar Spaces by Drudge (1998)
- Bamberg, Kelly, Law, Penttila (2006, 2007):
  - Unification: tight sets and m-ovoids as Intriguing Sets
  - New constructions
  - Connections to: m-systems, covers, eggs, minihypers,..
  - .. and projective two-character sets.
- Many more recent results
- Tight sets of hyperbolic quadrics generalise (and inherit a very special property of) Cameron-Liebler line classes in $PG(3, q)$.

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Point sets in Polar spaces

- Let $\mathcal{P}$ be a finite polar space of rank r.
- A **generator** of $\mathcal{P}$ is a maximal t.i./t.s. subspace.
- A generator $\cong \mathrm{PG}(r-1, q)$, so every two of its points are collinear.

Let T be a subset of points of $\mathcal{P}$.

Question (Payne, 1987, for the case $r = 2$)

How many pairs of collinear points may T contain?
(What could be the maximum average number of points of T collinear to a point of T?)

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The average number κ of points of T collinear with a point of T satisfies:

$$\kappa \leq |T| \frac{q^{r-1} - 1}{q^r - 1} + q^{r-1}.$$

If equality holds, then $|T| = x\frac{q^r - 1}{q - 1}$ for some integer x, and for every point $p \in \mathcal{P}$,

$$|T \cap \{p\}^\perp| = \begin{cases} x\frac{q^{r-1}-1}{q-1} + q^{r-1}, & \text{if } p \in T, \\ x\frac{q^{r-1}-1}{q-1}, & \text{if } p \notin T. \end{cases}$$

In case of equality, we call such a set T:

tight / x-**tight** / tight with parameter x.

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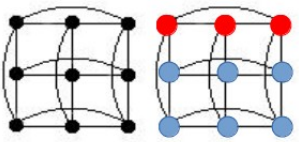
Tight sets of Polar spaces: equitable partitions

T is an x-tight set $\Rightarrow \mathcal{P} \setminus T$ is $(q^{r-1}+1-x)$-tight



1-tight, 2-tight sets in $\mathsf{Q}^+(3, \mathbb{F}_2)$

Question

How many pairs of collinear points may T contain?

An extremal set T will be x-tight:

$$T \approx \text{the union of } x \text{ disjoint generators}$$

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Are there any other tight sets? (For given rank, type, $\mathbb{F}_q$)

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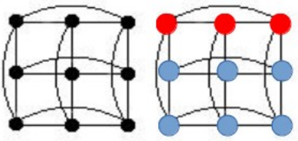

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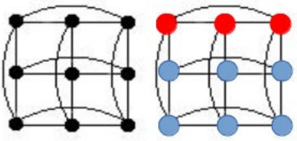

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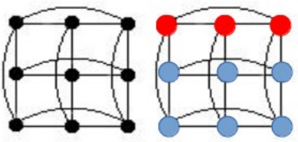

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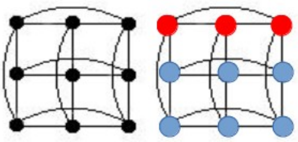

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Hyperbolic polar spaces

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The water tower by Vladimir Shukhov (Russia, 1896)

Tight sets of hyperbolic polar spaces

Disjoint generators in $\mathsf{Q}^+(2r-1,q)$:

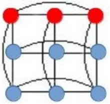
(r even) $\swarrow$ $\searrow$ (r odd)

large sets of them | no more than 2

Any other tight sets? | Any tight sets with $x > 2$?

Small ranks:

$r = 2$



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Klein $\updownarrow$ *correspondence*

C.-L. line classes in $PG(3,q)$
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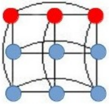

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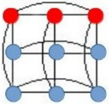

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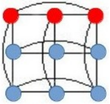

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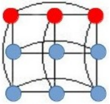

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Irreducible: Bounds on x

An x-tight set of $\mathsf{Q}^+(2r-1, q)$ must be reducible if:

- $x \lesssim \sqrt{q}$,

  K. Drudge, 1998

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- $x \leq q$ and $r \in \{2, 3, 4\}$,
- $x \leq q - 1$ and $r \geq 5$ and and $q \geq 71$,

  L. Beukemann, K. Metsch, 2013

- $x \leq Cq^{4/3}$ and $r = 3$,

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Recall, however, that x may take values $0, 1, \ldots, \dfrac{q^{r-1}+1}{2}$.

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Our result

Theorem (G., Metsch, 2014)

Let T be an x-tight set of $\mathsf{Q}^+(5,q)$ (the Klein quadric). Then for every plane π of $\mathsf{Q}^+(5,q)$, the number $n := |\pi \cap T|$ satisfies

$$\binom{x}{2} + n(n-x) \equiv 0 \bmod (q+1).$$

Theorem

Let T be an x-tight set of $\mathsf{Q}^+(2r-1,q)$. Then, for every generator γ, the number $n := |\gamma \cap T|$ satisfies

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If an x-tight set of $\mathsf{Q}^+(2r-1,q)$ exists, then there must exist at least one integer n satisfying the equation above.

Example: $q=3$, $r=5 \Rightarrow x \in \{0,1,2,\dots,\frac{3^4+1}{2}=41\}$.

For $x \in \{3,4,6,7,11,12,\dots\}$ (20 values of $\{0,1,2,\dots,41\}$), there are no integer numbers n satisfying the equation with given x and q. $\Rightarrow$ There are no x-tight sets with given x.

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For $x \in \{3, 4, 6, 7, 11, 12, \ldots\}$ (20 values of $\{0, 1, 2, \ldots, 41\}$), there are no integer numbers n satisfying the equation with given x and q. $\Rightarrow$ There are no x-tight sets with given x.

Sketch of the proof: basic lemma

Let T be an x-tight set of a hyperbolic quadric $\mathcal{P}$ of rank r.

Lemma (L. Beukemann, K. Metsch, 2013)

For every subspace S of dimension $d \leq r-1$ of $\mathrm{PG}(2r-1,q)$

$$|T \cap S^{\perp}| = x\frac{q^{r-d-1}-1}{q-1} + q^{r-d-1}|T \cap S|.$$

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Theorem (K. Drudge, 1998)

For every pair $\{p_1, p_2\}$ of non-collinear points of $\mathcal{P}$:

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Sketch of the proof: Double counting, 1

Fix a point $p_1 \in T$.

We count the number E of pairs of incident points in

$$(T \cap \{p_1\}^\perp) \quad \times \quad (T \cap (\mathcal{P} \setminus \{p_1\}^\perp))$$

$$\begin{array}{ccc} ? & \Longleftarrow & \circlearrowleft \; p_2 \\ \| & & \end{array}$$

$$|T \cap \{p_1, p_2\}^\perp| = x\frac{q^{r-2}-1}{q-1} + q^{r-2}|T \cap \{p_1, p_2\}|.$$

This gives E in terms of x, q, r:

$$\sum_{p_2 \in T \cap (\mathcal{P} \setminus \{p_1\}^\perp)} |T \cap \{p_1, p_2\}^\perp|$$

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This enables to express E via $\sum_{\ell \ on \ p_1} |T \cap \ell|^2$.

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Applying this lemma to $\pi = \langle \ell_1, \ell_2 \rangle$, where ℓ_1, ℓ_2 on p_1, $\ell_1 \not\perp \ell_2$:

$$\sum_{\ell' \text{ on } p_1:\ \ell' \perp \ell_1,\ \ell' \perp \ell_2} |T \cap \ell'| = x\frac{q^{r-3}-1}{q-1} + q^{r-3}(|T \cap \ell_1| + |T \cap \ell_2|).$$

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The result

Theorem

Let T be an x-tight set of $\mathsf{Q}^+(2r-1,q)$. Then, for every generator γ, the number $n := |\gamma \cap T|$ satisfies

$$\binom{x}{2} + n(n-x) \equiv 0 \bmod (q+1), \text{ (if the rank } r \text{ is odd)},$$

or

$$n(n-x) \equiv 0 \bmod (q+1), \text{ (if the rank } r \text{ is even)}.$$

Concluding remarks

- $r = 2$: only reducible (i.e., trivial) examples
- $r = 3$: many examples of C.-L. line classes
- $r = 4$: the field reduction

$$\mathrm{PG}(3, q^2) \rightsquigarrow PG(2 \cdot (3+1) - 1, q)$$

a tight set of $\mathsf{Q}^+(3, q) \mapsto$ a tight set of $\mathsf{Q}^+(7, q)$

(so it gives only trivial tight sets)

- $r = 5$: $\mathsf{Q}^+(9, q)$, completely open, but our result eliminates about a half of possible values for x.

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