Maximal Hyperplane Sections of Schubert Varieties over Finite Fields

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Grassmann Varieties : A Quick Introduction

V : vector space of dimension m over a field $\mathbb{F}$

For $1 \le \ell \le m$, we have the Grassmann variety:

$$G_{\ell,m} = G_\ell(V) := \{\ell\text{-dimensional subspaces of } V\}.$$

Plücker embedding: $G_{\ell,m} \hookrightarrow \mathbb{P}^{k-1}$, where $k := \binom{m}{\ell}$.

Explicitly, $\mathbb{P}^{k-1} = \mathbb{P}(\wedge^\ell V)$ and

$$W = \langle w_1, \dots, w_\ell \rangle \longleftrightarrow [w_1 \wedge \cdots \wedge w_\ell] \in \mathbb{P}(\wedge^\ell V).$$

For example, $G_{1,m} = \mathbb{P}^{m-1}$. In terms of coordinates,

$$W = \langle w_1, \dots, w_\ell \rangle \in G_\ell(V) \longleftrightarrow p(W) = \left(p_\alpha(A_W)\right)_{\alpha \in I(\ell,m)},$$

where $A_W = (a_{ij})$ is a $\ell \times m$ matrix whose rows are (the coordinates of) a basis of W and $p_\alpha(A_W)$ is the α^{th} minor of A_W, viz., $\det\left(a_{i\alpha_j}\right)_{1 \le i,j \le \ell}$.

Introduction to Grassmann Varieties Contd.

Notation: $I(\ell, m) := \left\{\alpha = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{Z}^\ell : 1 \le \alpha_1 < \cdots < \alpha_\ell \le m\right\}$.

Facts:

- $G_{\ell,m}$ is a projective algebraic variety given by the common zeros of certain quadratic homogeneous polynomials in k variables. As a projective algebraic variety $G_{\ell,m}$ is nondegenerate, irreducible, nonsingular, and rational.
- There is a natural transitive action of GL_m on $G_{\ell,m}$ and if P_ℓ denotes the stabilizer of a fixed $W_0 \in G_{\ell,m}$, then P_ℓ is a maximal parabolic subgroup of GL_m and $G_{\ell,m} \simeq \mathrm{GL}_m/P_\ell$.
- If $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, then $G_{\ell,m}$ is a (real or complex) manifold, and its cohomology spaces and Betti numbers are explicitly known. In fact, $b_\nu = \dim H^{2\nu}(G_{\ell,m}; \mathbb{C})$ is precisely the number of partitions of ν into at most ℓ parts, each part $\le m - \ell$,

Grassmannian Over Finite Fields

Suppose $\mathbb{F} = \mathbb{F}_q$ is the finite field with q elements. Then $G_{\ell,m} = G_{\ell,m}(\mathbb{F}_q)$ is a finite set and its cardinality is given by the Gaussian binomial coefficient:

$$\begin{bmatrix} m \\ \ell \end{bmatrix}_q := \frac{(q^m - 1)(q^m - q)\cdots(q^m - q^{\ell-1})}{(q^\ell - 1)(q^\ell - q)\cdots(q^\ell - q^{\ell-1})}.$$

This is a polynomial in q of degree $\delta := \ell(m - \ell)$ and in fact,

$$|G_{\ell,m}(\mathbb{F}_q)| = \begin{bmatrix} m \\ \ell \end{bmatrix}_q = \sum_{\nu=0}^{\delta} b_\nu q^\nu = q^\delta + q^{\delta-1} + 2q^{\delta-2} + \cdots + 1,$$

where the coefficients b_ν are nonnegative integers that have combinatorial and topological interpretation mentioned earlier. Note that

$$\lim_{q\to 1} \begin{bmatrix} m \\ \ell \end{bmatrix}_q = \binom{m}{\ell}.$$

Schubert Varieties in Grassmannians

Fix a basis $\{e_1, \ldots, e_m\}$ of V and any $\alpha \in I(\ell, m)$, that is,

$$\alpha = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{Z}^\ell, \; 1 \le \alpha_1 < \cdots < \alpha_\ell \le m.$$

The corresponding Schubert variety is defined by

$$\Omega_\alpha := \{W \in G_{\ell,m} : \dim(W \cap A_i) \ge i \; \forall \, i = 1, \ldots, \ell\},$$

where $A_i = \langle e_1, \ldots, e_{\alpha_i} \rangle$ for $1 \le i \le \ell$. Alternatively,

$$\Omega_\alpha := \{[v_1 \wedge \ldots \wedge v_\ell] : v_1, \ldots, v_\ell \in V \text{ linearly independent and } v_i \in A_i \; \forall \, i\}.$$

The Plücker embedding of $G_{\ell,m}$ induces a nondegenerate embedding

$$\Omega_\alpha(\mathbb{F}_q) \hookrightarrow \mathbb{P}^{k_\alpha - 1} \quad \text{where} \quad k_\alpha = |\{\beta \in I(\ell, m) : \beta \le \alpha\}|,$$

with $\le$ being the componentwise partial order (Bruhat-Chevalley):

$$\beta = (\beta_1, \ldots, \beta_\ell) \le \alpha = (\alpha_1, \ldots, \alpha_\ell) \Longleftrightarrow \beta_i \le \alpha_i \; \forall \, i = 1, \ldots, \ell.$$

Hyperplane Sections of Schubert Varieties

Fix $\alpha \in I(\ell, m)$ and consider $\Omega_\alpha(\mathbb{F}_q) \hookrightarrow \mathbb{P}^{k_\alpha - 1}$. We are interested in

$$e_\alpha(\ell, m) := \max_H |\Omega_\alpha(\mathbb{F}_q) \cap H| \text{ and } M_\alpha(\ell, m) := |\{H : |\Omega_\alpha(\mathbb{F}_q) \cap H| = e_\alpha(\ell, m)\}|,$$

where the maximum is taken over all hyperplanes H in $\mathbb{P}^{k_\alpha - 1}$, or equivalently, all hyperplanes H in $\mathbb{P}(\bigwedge^\ell V)$ such that $\Omega_\alpha \not\subseteq H$. In the special case when $\alpha = (m - \ell + 1, \ldots, m - 1, m)$, that is, when $\Omega_\alpha = G_{\ell,m}$, we shall denote $e_\alpha(\ell, m)$ and $M_\alpha(\ell, m)$ simply by $e(\ell, m)$ and $M(\ell, m)$.

Theorem (Nogin, 1996)

$$e(\ell, m) = \begin{bmatrix} m \\ \ell \end{bmatrix}_q - q^\delta \quad \textit{and} \quad M(\ell, m) = (q-1)\begin{bmatrix} m \\ \ell \end{bmatrix}_q.$$

In fact, the hyperplanes H that attain $e(\ell, m)$ are precisely those that correspond to decomposable elements of $\bigwedge^{m-\ell} V = \left(\bigwedge^\ell V\right)^*$.

Connection with Coding Theory

Fix representatives $P_1, \ldots, P_{n_\alpha}$ in $\bigwedge^\ell V$ of points of the Schubert variety $\Omega_\alpha(\mathbb{F}_q) \subseteq G_{\ell,m}(\mathbb{F}_q) \subseteq \mathbb{P}(\bigwedge^\ell V)$. We have the evaluation map

$$\bigwedge^{m-\ell} V \longrightarrow \mathbb{F}_q^{n_\alpha} \quad \text{given by} \quad f \longmapsto c_f = \left(f \wedge P_1, \ldots, f \wedge P_{n_\alpha}\right).$$

This is clearly linear and the image is denoted by $C_\alpha(\ell, m)$ and called the Schubert code. When $\alpha = (m - \ell + 1, \ldots, m)$, it is called the Grassmann code and denoted by $C(\ell, m)$. In this case, the evaluation map is injective and thus the length n and the dimension k of the Grassmann code $C(\ell, m)$ are given by

$$n = \begin{bmatrix} m \\ \ell \end{bmatrix}_q \quad \text{and} \quad k = \binom{m}{\ell}.$$

The result of Nogin (1996) mentioned earlier says that

$$d\left(C(\ell, m)\right) = q^\delta \ \text{ where } \ \delta := \ell(m - \ell).$$

Further, the number of minimum weight codewords is given by $M(\ell, m)$.

Minimum Distance of Schubert Codes

The problem mentioned eariler corresponds exactly to finding the minimum distance of Schubert codes and the number of minimum weight codewords.

Proposition (G - Lachaud(2000))

For any $\alpha \in I(\ell, m)$,

$$d\left(C_\alpha(\ell, m)\right) \leq q^{\delta_\alpha} \quad \textit{where} \quad \delta_\alpha := \sum_{i=1}^{\ell} (\alpha_i - i).$$

When $\alpha = (m - \ell + 1, \ldots, m - 1, m)$, the inequality is an equality, thanks to Nogin. The following conjecture was made in the same paper:

Minimum Distance Conjecture (MDC)

For any $\alpha \in I(\ell, m)$,

$$\boxed{d\left(C_\alpha(\ell, m)\right) = q^{\delta_\alpha}.}$$

Length of Schubert Codes

- If $\ell = 2$ and $\alpha = (m - h - 1, m)$, then

$$n_\alpha = \frac{(q^m - 1)(q^{m-1} - 1)}{(q^2 - 1)(q - 1)} - \sum_{j=1}^{h} \sum_{i=1}^{j} q^{2m-j-2-i}$$

and

$$k_\alpha = \frac{m(m-1)}{2} - \frac{h(h+1)}{2}.$$

[Hao Chen (2000)]

- In general,

$$n_\alpha = \sum \prod_{i=0}^{\ell-1} \begin{bmatrix} \alpha_{i+1} - \alpha_i \\ k_{i+1} - k_i \end{bmatrix}_q q^{(\alpha_i - k_i)(k_{i+1} - k_i)}$$

where the sum is over $(k_1, \dots, k_{\ell-1}) \in \mathbb{Z}^\ell$ satisfying $i \le k_i \le \alpha_i$ and $k_i \le k_{i+1}$ for $1 \le i \le \ell - 1$; by convention, $\alpha_0 = 0 = k_0$ and $k_\ell = \ell$.

[Vincenti (2001)]

Length of Schubert Codes (Contd.)

- $n_\alpha = \sum_{\beta \le \alpha} q^{\delta_\beta}$, [Ehresmann (1934); G - Tsfasman (2005)]
- Suppose α has $u+1$ consecutive blocks:

  $\alpha = (\alpha_1, \dots, \alpha_{p_1}, \ \dots, \ \alpha_{p_u+1}, \dots, \alpha_{p_{u+1}})$. Then

$$n_\alpha = \sum_{s_1=p_1}^{\alpha_{p_1}} \cdots \sum_{s_u=p_u}^{\alpha_{p_u}} \prod_{i=0}^{u} \lambda(\alpha_{p_i}, \alpha_{p_{i+1}}; s_i, s_{i+1})$$

  where, $s_0 = p_0 = 0$; $s_{u+1} = p_{u+1} = \ell$, and

  $\lambda(a, b; s, t) := \sum_{r=s}^{t} (-1)^{r-s} q^{\binom{r-s}{2}} \begin{bmatrix} a-s \\ r-s \end{bmatrix}_q \begin{bmatrix} b-r \\ t-r \end{bmatrix}_q$. [G - Tsfasman (2005)]

- $n_\alpha = \det \left(q^{(j-i)(j-i-1)/2} \begin{bmatrix} \alpha_j - j + 1 \\ i - j + 1 \end{bmatrix}_q \right)_{1 \le i,j \le \ell}$. [G - Krattenthaler]

Dimension of Schubert Codes [G-Tsfasman (2005)]

- Let $\alpha = (\alpha_1, \ldots, \alpha_\ell) \in I(\ell, m)$. The dimension of $C_\alpha(\ell, m)$ is the $\ell \times \ell$ determinant:

$$k_\alpha = \det_{1\le i,j\le \ell}\left(\binom{\alpha_j - j + 1}{i - j + 1}\right) = \begin{vmatrix} \binom{\alpha_1}{1} & 1 & 0 & \ldots & 0 \\ \binom{\alpha_1}{2} & \binom{\alpha_2 - 1}{1} & 1 & \ldots & 0 \\ \vdots & & & & \vdots \\ \binom{\alpha_1}{\ell} & \binom{\alpha_2 - 1}{\ell - 1} & \binom{\alpha_3 - 2}{\ell - 2} & \ldots & \binom{\alpha_\ell - \ell + 1}{1} \end{vmatrix}.$$

- If $\alpha_1, \ldots, \alpha_\ell$ are in arithmetic progression, i.e., $\alpha_i = c(i-1) + d \ \forall i$ for some $c, d \in \mathbb{Z}$, then

$$k_\alpha = \frac{\alpha_1}{\ell!}\prod_{i=1}^{\ell-1}(\alpha_{\ell+1} - i) = \frac{\alpha_1}{\alpha_{\ell+1}}\binom{\alpha_{\ell+1}}{\ell},$$

where $\alpha_{\ell+1} = c\ell + d = \ell\alpha_2 + (1-\ell)\alpha_1$.

- $k_\alpha = \sum_{s_1=p_1}^{\alpha_{p_1}} \sum_{s_2=p_2}^{\alpha_{p_2}} \cdots \sum_{s_u=p_u}^{\alpha_{p_u}} \prod_{i=0}^{u} \binom{\alpha_{p_{i+1}} - \alpha_{p_i}}{s_{i+1} - s_i}.$

What do we know about the MDC?

Recall that the MDC states that

$$d\left(C_\alpha(\ell,m)\right) = q^{\delta_\alpha}, \quad \text{where} \quad \delta_\alpha := (\alpha_1 - 1) + \cdots + (\alpha_\ell - \ell).$$

The MDC is:

- True if $\alpha = (m-\ell+1, \ldots, m-1, m)$. [Nogin (1996)]
- True if $\ell = 2$. [Hao Chen (2000)]; independently [Guerra-Vincenti (2002)]. In general, one has a lower bound for $d\left(C_\alpha(\ell,m)\right)$ [G-V (2002)]:

$$\frac{q^{\alpha_1}(q^{\alpha_2} - q^{\alpha_1}) \cdots (q^{\alpha_\ell} - q^{\alpha_{\ell-1}})}{q^{1+2+\cdots+\ell}} \geq q^{\delta_\alpha - \ell}.$$

- True for $C_{(2,4)}(2,4)$. [Vincenti (2001)]
- True for all Schubert divisors in $G_{\ell,m}$. [G - Tsfasman (2005)]

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- True for $C_{(2,4)}(2, 4)$. [Vincenti (2001)]
- True for all Schubert divisors in $G_{\ell,m}$. [G - Tsfasman (2005)]
- True, in general! [Xu Xiang (2008)], [G - Singh (2016)]

Minimum Weight Codewords of Schubert Codes

The first natural question is the following.

Question: Does every decomposable elements of $\bigwedge^{m-\ell} V$ correspond to a minimum weight codeword of $C_\alpha(\ell, m)$?

The answer is No, in general. For example, consider $\alpha = (\alpha_1, \alpha_2) \in I(2, m)$ with $\alpha_1 \geq 2$. As before, let $A_1 = \langle e_1, \ldots, e_{\alpha_1} \rangle$ and $A_2 = \langle e_1, \ldots, e_{\alpha_2} \rangle$, where $\{e_1, \ldots, e_m\}$ is a fixed basis of V. Let

$$f = e_3 \wedge \cdots \wedge e_m \in \bigwedge^{m-2} V \text{ and } c_f \text{ the corresponding codeword in } C_\alpha(2, m).$$

Then it can be shown that $\mathrm{wt}(c_f) = q^{\alpha_1+\alpha_2-3} + q^{\alpha_1+\alpha_2-4} - q^{2\alpha_1-3}$, and so

$$\mathrm{wt}(c_f) = q^{\delta(\alpha)} \iff \alpha_2 = \alpha_1 + 1, \text{ i.e., } C_\alpha(2, m) = C(2, \alpha_2).$$

On the other hand, $h = e_1 \wedge e_3 \wedge e_5 \wedge \cdots \wedge e_m \in \bigwedge^{m-2} V$ is decomposable and it can be seen that $\mathrm{wt}(c_h) = q^{\alpha_1+\alpha_2-3}$.

Schubert Decomposability

It turns out that we need a notion more subtle than decomposability. Let us

- write α uniquely as

$$\alpha = (\alpha_1, \ldots, \alpha_{p_1},\ \alpha_{p_1+1}, \ldots, \alpha_{p_2},\ \ldots,\ \alpha_{p_{u-1}+1}, \ldots, \alpha_{p_u},\ \alpha_{p_u+1}, \ldots, \alpha_\ell)$$

so that $1 \le p_1 < \cdots < p_u < \ell$ and $\alpha_{p_i+1}, \ldots, \alpha_{p_{i+1}}$ are consecutive for $0 \le i \le u$. By convention, $p_0 = 0$ and $p_{u+1} = \ell$.

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- α is called completely nonconsecutive if $\alpha_i - \alpha_{i-1} \geq 2$ for all $2 \leq i \leq \ell$

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Definition

A decomposable element $f = f_1 \wedge \ldots \wedge f_{m-\ell} \in \bigwedge^{m-\ell} V$ is said to be Schubert decomposable if $\dim (V_f \cap A_{p_i}) = \alpha_{p_i} - p_i$ for all $i = 1, \ldots, u$, where V_f denotes the *annihilator* of f, i.e., $V_f := \{v \in V : v \wedge f = 0\} = \langle f_1, \ldots, f_{m-\ell} \rangle$.

Main Results: Schubert Decomposability and Min Weight Codewords

Note that in the Grassmann case, i.e., when $\alpha = (m-\ell+1, \ldots, m-1, m)$, or more generally, when $\alpha_1, \ldots, \alpha_\ell$ are consecutive, we have $u = 0$ and in this case, the notions of decomposability and Schubert decomposability coincide.

Conjecture

Minimum weight codewords of the Schubert code $C_\alpha(\ell, m)$ are precisely the codewords corresponding to Schubert decomposable elements of $\bigwedge^{m-\ell} V$.

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Theorem (G – Singh)

If $f \in \bigwedge^{m-\ell} V$ is Schubert decomposable, then c_f is a minimum weight codeword of the Schubert code $C_\alpha(\ell, m)$.

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Theorem (G – Singh)

Assume that $f \in \bigwedge^{m-\ell} V$ is decomposable. If c_f is a minimum weight codeword of $C_\alpha(\ell, m)$, then f is Schubert decomposable.

Theorem (G – Singh)

Assume that α is completely non-consecutive. If c is a minimum weight codeword of $C_\alpha(\ell, m)$, then $c = c_h$ for some decomposable $h \in \bigwedge^{m-\ell} V$.

Theorem (G – Singh)

Assume that α is completely non-consecutive. If c is a minimum weight codeword of $C_\alpha(\ell, m)$, then $c = c_h$ for some decomposable $h \in \bigwedge^{m-\ell} V$.

Theorem (G – Singh)

The number of codewords of $C_\alpha(\ell, m)$ corresponding to Schubert decomposable elements of $\bigwedge^{m-\ell} V$ is equal to

$$N_\alpha(\ell, m) := (q-1)q^{\mathsf{P}} \prod_{j=0}^{u} \begin{bmatrix} \alpha_{p_{j+1}} - \alpha_{p_j} \\ p_{j+1} - p_j \end{bmatrix}_q,$$

where

$$\mathsf{P} = \sum_{j=1}^{u} p_j \left(\alpha_{p_{j+1}} - \alpha_{p_j} - p_{j+1} + p_j\right).$$

Thank you!

Idea of Proof

- Let $\alpha' = (\alpha_1, \ldots, \alpha_{\ell-1})$ and $C_{\alpha'}(\ell - 1, m)$ be the corresponding Schubert code

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- $Z(\alpha, f) = \{(L', x) \in \Omega_{\alpha'}(\ell-1, m) \times A_\ell : f \wedge x(L) \neq 0\}$

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Lemma

If $\operatorname{codim}_{A_\ell} E \leq t$, then $A_{\ell-t} \subset E$

Fiber Lemma

Lemma

For a given $L \in W(f)$ the following holds

1. If $L \nsubseteq A_{\ell-1}$, then $|\phi^{-1}(L)| = q^{\ell-1}(q-1)$
2. If $L \subseteq A_{\ell-1}$ and $t := \operatorname{codim}_{A_\ell} E$, then $|\phi^{-1}(L)| \leq q^{\ell-1}(q^t - 1)$
3. If f is Schubert decomposable, then $|\phi^{-1}(L)| = q^{\ell-1}(q^t - 1)$

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For a given $L \in W(f)$ the following holds

1. If $L \not\subseteq A_{\ell-1}$, then $|\phi^{-1}(L)| = q^{\ell-1}(q-1)$
2. If $L \subseteq A_{\ell-1}$ and $t := \operatorname{codim}_{A_\ell} E$, then $|\phi^{-1}(L)| \le q^{\ell-1}(q^t - 1)$
3. If f is Schubert decomposable, then $|\phi^{-1}(L)| = q^{\ell-1}(q^t - 1)$

Lemma

For any $f \in \bigwedge^{m-\ell} V$ the weight of the codeword c_f satisfies

$$\mathrm{wt}(c_f) \ge \frac{1}{q^{\ell-1}(q-1)} \sum_{x \in F \cap A_{\ell-1}} \mathrm{wt}(c_{f \wedge x}) + \frac{1}{q^{\ell-1}(q^t-1)} \sum_{x \in F \setminus A_{\ell-1}} \mathrm{wt}(c_{f \wedge x})$$