Transparent embeddings of point–line geometries

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Joint work with Ilaria Cardinali and Antonio Pasini

Projective embeddings

- $\Gamma = (\mathcal{P}, \mathcal{L})$: point-line geometry
- $\varepsilon : \mathcal{P} \to \Sigma := \mathrm{PG}(V)$,
- $\mathcal{P}^\varepsilon := \{\varepsilon(p) : p \in \mathcal{P}\}$,
- $\mathcal{L}^\varepsilon := \{\ell^\varepsilon := \{\varepsilon(p) : p \in \ell\}, \ell \in \mathcal{L}\}$.
- $\Gamma^\varepsilon := (\mathcal{P}^\varepsilon, \mathcal{L}^\varepsilon)$.

Definition

$\varepsilon : \mathcal{P} \to \Sigma$ is a *full projective embedding* if

(E0) ε injective;

(E1) $\langle \mathcal{P}^\varepsilon \rangle = \Sigma$;

(E2) $\forall \ell \in \mathcal{L}$, ℓ^ε is a line of Σ.

Transparent embeddings

Problem

Suppose $r \subseteq \mathcal{P}^{\varepsilon}$ to be a line of Σ; when is $r \in \mathcal{L}^{\varepsilon}$?

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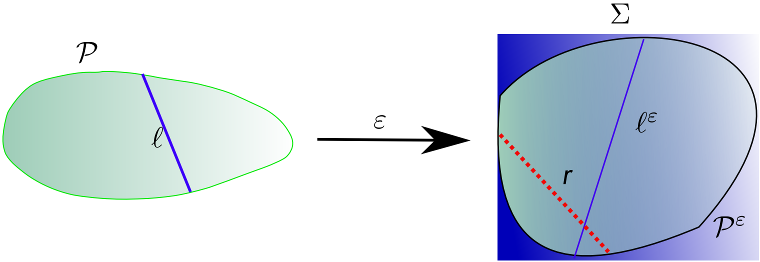
Definition

ε is transparent if

$$\forall r \subseteq \Sigma : r \subseteq \mathcal{P}^{\varepsilon} \Rightarrow r \in \mathcal{L}^{\varepsilon}.$$

Transparent embeddings



Motivation

- If $\varepsilon : \mathcal{P} \to \Sigma$ is transparent, given $\mathcal{P}^\varepsilon \subseteq \Sigma$,

$$\mathcal{L}^\varepsilon = \{r \subseteq \Sigma : r \subseteq \mathcal{P}^\varepsilon\}.$$

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- $\mathrm{Stab}(\mathcal{P}^\varepsilon)$: stabilizer of $\mathcal{P}^\varepsilon$ in $\mathrm{Aut}(\Sigma) := \mathrm{P\Gamma L}(V)$

Then,

- Any collineation in $\mathrm{Stab}(\mathcal{P}^\varepsilon)$ lifts to an automorphism of the geometry Γ^ε.

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- Useful to determine $\mathrm{Stab}(\mathcal{P}^\varepsilon)$.
- Applications to projective systems, varieties, codes, etc.

Motivation/Automorphism groups

- $\mathrm{Aut}(\Gamma)$: automorphism group of the geometry Γ
- $\mathrm{Stab}(\mathcal{P}^{\varepsilon})$: stabilizer of $\mathcal{P}^{\varepsilon}$ in $\mathrm{Aut}(\Sigma) := \mathrm{P\Gamma L}(V)$
- $\mathrm{Aut}(\varepsilon)$: set-wise stabilizer of $\mathcal{L}^{\varepsilon}$ in $\mathrm{Stab}(\mathcal{P}^{\varepsilon})$
- $\mathrm{Aut}(\Gamma)_{\varepsilon}$: elements of $\mathrm{Aut}(\Gamma)$ lifting to Γ^{ε}, i.e.

$$g \in \mathrm{Aut}(\Gamma)_{\varepsilon} \Leftrightarrow \exists g^{\varepsilon} \in \mathrm{Aut}(\varepsilon) : \varepsilon g = g^{\varepsilon}\varepsilon$$

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$\varepsilon : \mathcal{P} \to \Sigma$

$$\mathrm{Aut}(\varepsilon) \cong \mathrm{Aut}(\Gamma)_{\varepsilon} \leq \mathrm{Aut}(\Gamma)$$

$$\mathrm{Aut}(\varepsilon) \leq \mathrm{Stab}(\mathcal{P}^{\varepsilon})$$

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$\varepsilon : \mathcal{P} \to \Sigma$ transparent and homogeneous

$$\mathrm{Stab}(\mathcal{P}^{\varepsilon}) = \mathrm{Aut}(\varepsilon) \cong \mathrm{Aut}(\Gamma)_{\varepsilon} = \mathrm{Aut}(\Gamma)$$

Degree of opacity/notation

- $\Gamma = (\mathcal{P}, \mathcal{L})$: point line geometry
- $\widehat{\Gamma} := (\mathcal{P}, \mathcal{E})$ collinearity graph of Γ:

$$(p, q) \in \mathcal{E} \Leftrightarrow \exists \ell \in \mathcal{L} \colon p, q \in \ell$$

- $\delta := \mathrm{Diam}(\widehat{\Gamma})$
- $\forall p, q \in \mathcal{P}$, $d(p, q)$: distance between p and q in $\widehat{\Gamma}$

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Upper and lower degree

- How "far" is ε from being transparent?

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Definition

- Lower degree of opacity:

$$\chi_\varepsilon^{\uparrow}+1 := \max\{1 \leq h \leq \delta : \langle \varepsilon(x), \varepsilon(y)\rangle \subseteq \mathcal{P}^\varepsilon, \\ \forall x, y \in \mathcal{P}, d(x,y) \leq h\}$$

- Upper degree of opacity:

$$\chi_\varepsilon^{\downarrow}+1 := \min\{1 \leq h \leq \delta : \langle \varepsilon(x), \varepsilon(y)\rangle \not\subseteq \mathcal{P}^\varepsilon, \\ \forall x, y \in \mathcal{P}, d(x,y) > h\}$$

Upper and lower degree

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Definition

- Lower degree of opacity:

$$\chi_\varepsilon^{\uparrow}+1:=\max\{1\leq h\leq\delta : \langle\varepsilon(x),\varepsilon(y)\rangle\subseteq\mathcal{P}^{\varepsilon},\\ \forall x,y\in\mathcal{P}, d(x,y)\leq h\}$$

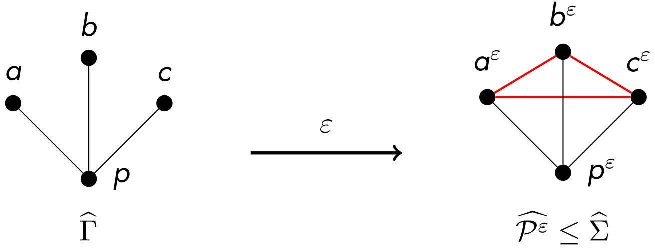
- Upper degree of opacity:

$$\chi_\varepsilon^{\downarrow}+1:=\min\{1\leq h\leq\delta : \langle\varepsilon(x),\varepsilon(y)\rangle\not\subseteq\mathcal{P}^{\varepsilon},\\ \forall x,y\in\mathcal{P}, d(x,y)> h\}$$

- $0\leq\chi_\varepsilon^{\uparrow}\leq\chi_\varepsilon^{\downarrow}\leq\delta-1$
- ε transparent if and only if $\chi_\varepsilon=\chi_\varepsilon^{\uparrow}=\chi_\varepsilon^{\downarrow}=0.$

Lower degree

$$\widehat{\Gamma} := (\mathcal{P}, \mathcal{E}) \text{ where } (x, y) \in \mathcal{E}' \Leftrightarrow \exists \ell \in \mathcal{L} : x, y \in \ell$$
$$\widehat{\mathcal{P}^\varepsilon} := (\mathcal{P}^\varepsilon, \mathcal{E}') \text{ where } (x, y) \in \mathcal{E}' \Leftrightarrow \langle x, y\rangle \subseteq \mathcal{P}^\varepsilon$$



- $\left.\begin{array}{l} d(a,b) = 2 \text{ and } \langle a^\varepsilon, b^\varepsilon\rangle \subseteq \mathcal{P}^\varepsilon \\ d(a,c) = 2 \text{ and } \langle a^\varepsilon, c^\varepsilon\rangle \subseteq \mathcal{P}^\varepsilon \\ d(b,c) = 2 \text{ and } \langle b^\varepsilon, c^\varepsilon\rangle \subseteq \mathcal{P}^\varepsilon \end{array}\right\} \Rightarrow \chi_\varepsilon^\uparrow + 1 \geq 2$

Upper degree

$$\widehat{\Gamma} := (\mathcal{P}, \mathcal{E}) \text{ where } (x, y) \in \mathcal{E}' \Leftrightarrow \exists \ell \in \mathcal{L} : x, y \in \ell$$

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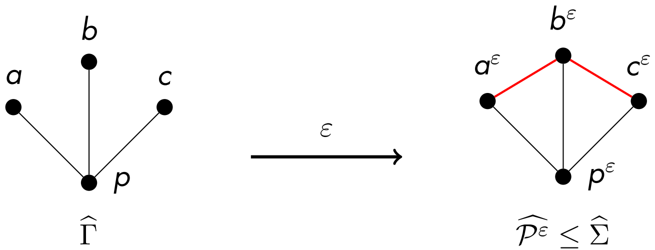

- $d(a, c) = 2$ and $\langle a^\varepsilon, c^\varepsilon\rangle \not\subseteq \mathcal{P}^\varepsilon$, so $\chi^\uparrow_\varepsilon + 1 < 2$.
- $d(a, b) = 2$ and $\langle a^\varepsilon, b^\varepsilon\rangle \subseteq \mathcal{P}^\varepsilon$, so $\chi^\downarrow_\varepsilon + 1 \geq 2$.

Lower and upper degree

$$\widehat{\Gamma} := (\mathcal{P}, \mathcal{E}) \text{ where } (x, y) \in \mathcal{E}' \Leftrightarrow \exists \ell \in \mathcal{L} : x, y \in \ell$$

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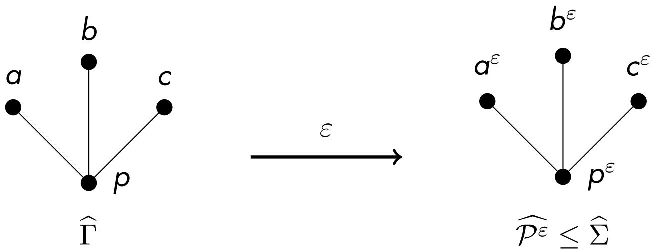

- $\forall x, y : d(x, y) \geq 2 \Rightarrow \langle x^\varepsilon, y^\varepsilon \rangle \not\subseteq \mathcal{P}^\varepsilon$ so $\chi_\varepsilon^\downarrow + 1 < 2$.
- $\chi_\varepsilon^\uparrow = \chi_\varepsilon^\downarrow = 0$; $\widehat{\Gamma} \cong \widehat{\mathcal{P}^\varepsilon}$

Embeddings

- Projective geometries and k-Grassmannians (type A_n)

$$\varepsilon : \mathcal{P} \to \Sigma \qquad \text{Graßmann embedding}$$

- Polar spaces and polar k-Grassmannians (type C_n)

$$\varepsilon : \mathcal{P} \to \Sigma \qquad \begin{cases} \text{Graßmann embedding} \\ \text{Spin embedding for} \begin{cases} DQ(2n, \mathbb{K}) \\ DQ^-(2n+1, \mathbb{K}) \end{cases} \end{cases}$$

- Oriflamme and half-spin geometries (type D_n)

$$\varepsilon : \mathcal{P} \to \Sigma \qquad \begin{cases} \text{Graßmann embedding} \\ \text{Spin embedding for } HS(2n+1, \mathbb{K}) \end{cases}$$

Projective Grassmannians: type A_n

- $V := V(n, \mathbb{K})$ vector space
- $\mathcal{G}_k(V) = (\mathcal{P}, \mathcal{L})$: k-Grassmannian of V
- $\mathcal{P}$: k-dimensional vector subspaces of V
- $\mathcal{L}$: sets of the form

$$\ell_{X,Y} := \{Z \in \mathcal{P} : X < Z < Y\},$$

  with $\dim X = k - 1, \dim Y = k + 1$
- Incidence: containment.
- Graßmann (Plücker) embedding:

$$\varepsilon_k : \begin{cases} \mathcal{G}_k(V) \to \mathrm{PG}(\bigwedge^k V) \\ \langle v_1, \dots, v_k \rangle \to \langle v_1 \wedge \cdots \wedge v_k \rangle \end{cases}$$

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- ε_k is full, projective, homogeneous, transparent.

Consequences/Chow's theorem

- $\mathbb{G}_k := \mathcal{P}^\varepsilon$: Graßmann variety in $\mathrm{PG}(\bigwedge^k V)$
- for each $r \subseteq \mathbb{G}_k$ (r line of $\mathrm{PG}(\bigwedge^k V)$),

$$\exists X, Y \leq V : \dim X = k-1, \dim Y = k+1$$

and

$$\langle z \rangle \in r \Leftrightarrow \langle z \rangle = \varepsilon_k(Z) \text{ with } X < Z < Y.$$

- $\mathrm{Aut}(\mathbb{G}_k) = \mathrm{Stab}(\mathbb{G}_k) \cong \mathrm{Aut}(\mathcal{G}_k)_\varepsilon = \mathrm{Aut}(\mathcal{G}_k) \cong \mathrm{P\Gamma L}(V)$

Graßmann Embedding

Theorem (I. Cardinali, LG, A. Pasini, 2017)

The Graßmann embedding of

1. *A projective Grassmannian $\mathcal{G}_k$ is transparent.*
2. *A polar Grassmannian $\mathcal{S}_k$ with $k < n$ of either orthogonal or Hermitian type is transparent.*
3. *The symplectic dual polar space $DW(2n-1, \mathbb{K})$ is transparent.*
4. *The Hermitian dual polar space $DH(2n-1, \mathbb{K})$ is transparent.*
5. *$\mathcal{S}_1 = W(2n-1, \mathbb{K})$ is $(1,1)$-opaque.*
6. *$\mathcal{S}_k$ with $1 < k < n$ of symplectic type is $(0,1)$-opaque.*

Spin Embedding

Theorem (I. Cardinali, LG, A. Pasini, 2017)

The spin embedding of

1. *A half-spin geometry is transparent.*
2. *The dual polar space associated to the orthogonal group* $\mathrm{O}(2n+1,\mathbb{K})$ *is* $(1,1)$*-opaque.*
3. *The dual polar space associated to the orthogonal group* $\mathrm{O}^-(2n+2,\mathbb{K})$ *is transparent.*

Automorphism groups/Graßmann embedding

Theorem (I. Cardinali, LG, A. Pasini, 2017)

Let $\mathbb{K}$ be a field. If Γ is one of the following point-line geometries:

1. *The projective Grassmannian $\mathcal{G}_k$ of a vector space over $\mathbb{K}$*
2. *The polar Grassmannian $\mathcal{S}_k$ of a polar space of rank $n > k$ of either orthogonal or Hermitian type over a finite dimensional vector space over $\mathbb{K}$;*
3. *$DH(2n-1, \mathbb{K})$ or $DW(2n-1, \mathbb{K})$*

and $\varepsilon : \Gamma \to \Sigma$ is Graßmann embedding, then

$$\mathrm{Aut}(\mathcal{P}^{\varepsilon}) = \mathrm{Aut}(\varepsilon) \cong \mathrm{Aut}(\Gamma)_{\varepsilon} = \mathrm{Aut}(\Gamma).$$

Automorphism groups/Spin embedding

Theorem (I. Cardinali, LG, A. Pasini, 2017)

Let $\mathbb{K}$ be a field. If Γ is one of the following point-line geometries:

1. *The dual polar space associated to the orthogonal group $O^-(2n+2,\mathbb{K})$.*
2. *A half-spin geometry $HS(2n-1,\mathbb{K})$*

and $\varepsilon : \Gamma \to \Sigma$ is the spin embedding, then

$$\mathrm{Aut}(\mathcal{P}^\varepsilon) = \mathrm{Aut}(\varepsilon) \cong \mathrm{Aut}(\Gamma)_\varepsilon = \mathrm{Aut}(\Gamma).$$

Open problems

Transparecy/degrees of opacity for embeddings of

- Grassmannians of exceptional geometries
- Flag varieties of geometries or buildings (in general)