Dimension bounds for constant rank subspaces of bilinear forms

Rod Gow

School of Mathematics
University College Dublin, Ireland
rod.gow@ucd.ie

September 15, 2017

Background and Notation

Let q be a power of a prime p and let $\mathbb{F}_q$ denote the finite field of order q.

Background and Notation

Let q be a power of a prime p and let $\mathbb{F}_q$ denote the finite field of order q.

Let V be a vector space of dimension n over $\mathbb{F}_q$.

Background and Notation

Let q be a power of a prime p and let $\mathbb{F}_q$ denote the finite field of order q.

Let V be a vector space of dimension n over $\mathbb{F}_q$.

Let $\mathrm{Bil}(V)$ denote the $\mathbb{F}_q$-vector space of bilinear forms defined on $V \times V$.

Background and Notation

Let q be a power of a prime p and let $\mathbb{F}_q$ denote the finite field of order q.

Let V be a vector space of dimension n over $\mathbb{F}_q$.

Let $\mathrm{Bil}(V)$ denote the $\mathbb{F}_q$-vector space of bilinear forms defined on $V \times V$.

Let $\mathrm{Alt}(V)$, $\mathrm{Symm}(V)$ denote the subspaces of alternating, symmetric bilinear forms, respectively, in $\mathrm{Bil}(V)$.

Background and Notation

Let q be a power of a prime p and let $\mathbb{F}_q$ denote the finite field of order q.

Let V be a vector space of dimension n over $\mathbb{F}_q$.

Let $\mathrm{Bil}(V)$ denote the $\mathbb{F}_q$-vector space of bilinear forms defined on $V \times V$.

Let $\mathrm{Alt}(V)$, $\mathrm{Symm}(V)$ denote the subspaces of alternating, symmetric bilinear forms, respectively, in $\mathrm{Bil}(V)$.

If preferred, we can identify $\mathrm{Bil}(V)$ with the vector space of $n \times n$ matrices over $\mathbb{F}_q$, and then $\mathrm{Alt}(V)$, $\mathrm{Symm}(V)$ correspond to the subspaces of skew-symmetric, symmetric matrices, respectively.

Background and Notation

Let q be a power of a prime p and let $\mathbb{F}_q$ denote the finite field of order q.

Let V be a vector space of dimension n over $\mathbb{F}_q$.

Let $\mathrm{Bil}(V)$ denote the $\mathbb{F}_q$-vector space of bilinear forms defined on $V \times V$.

Let $\mathrm{Alt}(V)$, $\mathrm{Symm}(V)$ denote the subspaces of alternating, symmetric bilinear forms, respectively, in $\mathrm{Bil}(V)$.

If preferred, we can identify $\mathrm{Bil}(V)$ with the vector space of $n \times n$ matrices over $\mathbb{F}_q$, and then $\mathrm{Alt}(V)$, $\mathrm{Symm}(V)$ correspond to the subspaces of skew-symmetric, symmetric matrices, respectively.

Given a vector space U, we let $U^\times$ denote the subset of non-zero vectors in U.

Constant rank subspaces

Definition

Let $\mathcal{M}$ be a subspace of $\mathrm{Bil}(V)$ and let m be an integer satisfying $1 \leq m \leq n$.

Constant rank subspaces

Definition

Let $\mathcal{M}$ be a subspace of $\mathrm{Bil}(V)$ and let m be an integer satisfying $1 \leq m \leq n$.

We say that $\mathcal{M}$ is a constant rank m subspace if each element of $\mathcal{M}^{\times}$ has rank m.

Constant rank subspaces

Definition

Let $\mathcal{M}$ be a subspace of $\mathrm{Bil}(V)$ and let m be an integer satisfying $1 \le m \le n$.

We say that $\mathcal{M}$ is a constant rank m subspace if each element of $\mathcal{M}^\times$ has rank m.

We are interested in finding an upper bound for $\dim \mathcal{M}$ when $\mathcal{M}$ is a constant rank m subspace and also in trying to describe $\mathcal{M}$ in some way if $\dim \mathcal{M}$ achieves a known upper bound.

Basic isotropy theorem

The following theorem underlies our investigations of constant rank subspaces.

Theorem 1

Let $\mathcal{M}$ be a constant rank m subspace of $\mathrm{Bil}(V)$ and let f be an element of $\mathcal{M}^{\times}$.

Basic isotropy theorem

The following theorem underlies our investigations of constant rank subspaces.

Theorem 1

Let $\mathcal{M}$ be a constant rank m subspace of $\mathrm{Bil}(V)$ and let f be an element of $\mathcal{M}^\times$.

Let U denote the left radical of f, and W the right radical of f (note that $\dim U = \dim W$ and $\mathrm{rank} f = n - \dim U$, but we need not have $U = W$).

Basic isotropy theorem

The following theorem underlies our investigations of constant rank subspaces.

Theorem 1

Let $\mathcal{M}$ be a constant rank m subspace of $\mathrm{Bil}(V)$ and let f be an element of $\mathcal{M}^{\times}$.

Let U denote the left radical of f, and W the right radical of f (note that $\dim U = \dim W$ and $\mathrm{rank} f = n - \dim U$, but we need not have $U = W$).

Then provided that $q \geq m + 1$, we have

$$g(u, w) = 0$$

for all $u \in U$, $w \in W$, and all $g \in \mathcal{M}$.

Alternating and symmetric cases

When f is an alternating or symmetric bilinear form, its left radical equals its right radical, and in that case, we call these radicals simply *the radical* and write $\mathrm{rad}\, f$ for this subspace.

Alternating and symmetric cases

When f is an alternating or symmetric bilinear form, its left radical equals its right radical, and in that case, we call these radicals simply *the radical* and write $\mathrm{rad}\, f$ for this subspace.

We can restate Theorem 1 in terms of what we call a totally isotropic subspace for $\mathcal{M}$.

Corollary 1

Let $\mathcal{M}$ be a constant rank m subspace of $\mathrm{Alt}(V)$ or $\mathrm{Symm}(V)$ and suppose that $q \geq m+1$.

Alternating and symmetric cases

When f is an alternating or symmetric bilinear form, its left radical equals its right radical, and in that case, we call these radicals simply *the radical* and write $\operatorname{rad} f$ for this subspace.

We can restate Theorem 1 in terms of what we call a totally isotropic subspace for $\mathcal{M}$.

Corollary 1

Let $\mathcal{M}$ be a constant rank m subspace of $\mathrm{Alt}(V)$ or $\mathrm{Symm}(V)$ and suppose that $q \geq m + 1$.

Let f be an element of $\mathcal{M}^{\times}$. Then we have

Alternating and symmetric cases

When f is an alternating or symmetric bilinear form, its left radical equals its right radical, and in that case, we call these radicals simply *the radical* and write $\operatorname{rad} f$ for this subspace.

We can restate Theorem 1 in terms of what we call a totally isotropic subspace for $\mathcal{M}$.

Corollary 1

Let $\mathcal{M}$ be a constant rank m subspace of $\mathrm{Alt}(V)$ or $\mathrm{Symm}(V)$ and suppose that $q \geq m + 1$.

Let f be an element of $\mathcal{M}^{\times}$. Then we have

$$g(u, w) = 0$$

for all u and w in $\operatorname{rad} f$ and all g in $\mathcal{M}$.

Totally isotropic subspaces

In the context of Corollary 1, we say that $\mathrm{rad}\, f$ is totally isotropic for $\mathcal{M}$ (and $\dim \mathrm{rad}\, f = n - m$).

Totally isotropic subspaces

In the context of Corollary 1, we say that $\operatorname{rad} f$ is totally isotropic for $\mathcal{M}$ (and $\dim \operatorname{rad} f = n - m$).

Since this holds for all $f \in \mathcal{M}^{\times}$, we have a supply of subspaces that are totally isotropic for $\mathcal{M}$.

Totally isotropic subspaces

In the context of Corollary 1, we say that $\operatorname{rad} f$ is totally isotropic for $\mathcal{M}$ (and $\dim \operatorname{rad} f = n - m$).

Since this holds for all $f \in \mathcal{M}^{\times}$, we have a supply of subspaces that are totally isotropic for $\mathcal{M}$.

In our study of constant rank subspaces, these totally isotropic subspaces, not surprisingly, play a major role.

Upper bound for $\dim \mathcal{M}$

Using Theorem 1, we have proved the following theorem.

Upper bound for $\dim \mathcal{M}$

Using Theorem 1, we have proved the following theorem.

Theorem 2

Let $\mathcal{M}$ be a constant rank m subspace of $\mathrm{Bil}(V)$ and suppose that $q \geq m+1$.

Upper bound for $\dim \mathcal{M}$

Using Theorem 1, we have proved the following theorem.

Theorem 2

Let $\mathcal{M}$ be a constant rank m subspace of $\mathrm{Bil}(V)$ and suppose that $q \geq m+1$.

Then we have $\dim \mathcal{M} \leq n = \dim V$, and this bound is optimal.

Upper bound for $\dim \mathcal{M}$

Using Theorem 1, we have proved the following theorem.

Theorem 2

Let $\mathcal{M}$ be a constant rank m subspace of $\mathrm{Bil}(V)$ and suppose that $q \geq m+1$.

Then we have $\dim \mathcal{M} \leq n = \dim V$, and this bound is optimal.

We remark that it is generally supposed that Theorem 2 holds for all infinite fields as well, but as far as we know, this has only been established in special cases.

The dimension n case

We can obtain additional information in Theorem 2 when the upper bound is attained.

The dimension n case

We can obtain additional information in Theorem 2 when the upper bound is attained.

Theorem 3

Let $\mathcal{M}$ be a constant rank m subspace of $\mathrm{Bil}(V)$ of dimension n and suppose that $q \geq m+1$.

The dimension n case

We can obtain additional information in Theorem 2 when the upper bound is attained.

Theorem 3

Let $\mathcal{M}$ be a constant rank m subspace of $\mathrm{Bil}(V)$ of dimension n and suppose that $q \geq m + 1$.

Suppose also that $n \geq 2m + 1$.

The dimension n case

We can obtain additional information in Theorem 2 when the upper bound is attained.

Theorem 3

Let $\mathcal{M}$ be a constant rank m subspace of $\mathrm{Bil}(V)$ of dimension n and suppose that $q \geq m + 1$.

Suppose also that $n \geq 2m + 1$.

Then the elements of $\mathcal{M}^{\times}$ all have the same left radical, or they have the same right radical.

The dimension n case

We can obtain additional information in Theorem 2 when the upper bound is attained.

Theorem 3

Let $\mathcal{M}$ be a constant rank m subspace of $\mathrm{Bil}(V)$ of dimension n and suppose that $q \geq m+1$.

Suppose also that $n \geq 2m+1$.

Then the elements of $\mathcal{M}^{\times}$ all have the same left radical, or they have the same right radical.

We remark that we really require $\dim \mathcal{M} = n$ in this theorem, and also that some restriction on the size of m is required.

Improvements in alternating and symmetric cases

Theorem 2 is optimal in general, but there is reason to suppose that improvements can often be made for subspaces of $\mathrm{Alt}(V)$ and $\mathrm{Symm}(V)$.

Improvements in alternating and symmetric cases

Theorem 2 is optimal in general, but there is reason to suppose that improvements can often be made for subspaces of $\mathrm{Alt}(V)$ and $\mathrm{Symm}(V)$.

We devote the rest of this talk to examining some sharpening of Theorem 2 that we have achieved.

Improvements in alternating and symmetric cases

Theorem 2 is optimal in general, but there is reason to suppose that improvements can often be made for subspaces of $\mathrm{Alt}(V)$ and $\mathrm{Symm}(V)$.

We devote the rest of this talk to examining some sharpening of Theorem 2 that we have achieved.

Theorem 4

Let $\mathcal{M}$ be a constant rank m subspace of $\mathrm{Symm}(V)$ and suppose that m is odd.

Improvements in alternating and symmetric cases

Theorem 2 is optimal in general, but there is reason to suppose that improvements can often be made for subspaces of $\mathrm{Alt}(V)$ and $\mathrm{Symm}(V)$.

We devote the rest of this talk to examining some sharpening of Theorem 2 that we have achieved.

Theorem 4

Let $\mathcal{M}$ be a constant rank m subspace of $\mathrm{Symm}(V)$ and suppose that m is odd.

Then we have $\dim \mathcal{M} \leq m$ (this holds for all q).

Improvements in alternating and symmetric cases

Theorem 2 is optimal in general, but there is reason to suppose that improvements can often be made for subspaces of $\mathrm{Alt}(V)$ and $\mathrm{Symm}(V)$.

We devote the rest of this talk to examining some sharpening of Theorem 2 that we have achieved.

Theorem 4

Let $\mathcal{M}$ be a constant rank m subspace of $\mathrm{Symm}(V)$ and suppose that m is odd.

Then we have $\dim \mathcal{M} \leq m$ (this holds for all q).

Moreover, if $\dim \mathcal{M} = m$ and $q \geq m+1$, then all elements of $\mathcal{M}^\times$ have the same radical.

Improvements in alternating and symmetric cases

Theorem 2 is optimal in general, but there is reason to suppose that improvements can often be made for subspaces of $\mathrm{Alt}(V)$ and $\mathrm{Symm}(V)$.

We devote the rest of this talk to examining some sharpening of Theorem 2 that we have achieved.

Theorem 4

Let $\mathcal{M}$ be a constant rank m subspace of $\mathrm{Symm}(V)$ and suppose that m is odd.

Then we have $\dim \mathcal{M} \leq m$ (this holds for all q).

Moreover, if $\dim \mathcal{M} = m$ and $q \geq m+1$, then all elements of $\mathcal{M}^{\times}$ have the same radical.

It is not entirely clear to us why such a theorem holds for odd m and we ask if there is any analogue of the theorem for arbitrary fields.

Improved symmetric dimension bound

There is of course no analogue of Theorem 4 for even m, but we can nevertheless make some small improvements to Theorem 2 in the symmetric case.

Improved symmetric dimension bound

There is of course no analogue of Theorem 4 for even m, but we can nevertheless make some small improvements to Theorem 2 in the symmetric case.

Theorem 5

Let $\mathcal{M}$ be a constant rank m subspace of $\mathrm{Symm}(V)$, where m is even.

Improved symmetric dimension bound

There is of course no analogue of Theorem 4 for even m, but we can nevertheless make some small improvements to Theorem 2 in the symmetric case.

Theorem 5

Let $\mathcal{M}$ be a constant rank m subspace of $\mathrm{Symm}(V)$, where m is even.

Suppose that q is odd and at least $m+1$.

Improved symmetric dimension bound

There is of course no analogue of Theorem 4 for even m, but we can nevertheless make some small improvements to Theorem 2 in the symmetric case.

Theorem 5

Let $\mathcal{M}$ be a constant rank m subspace of $\mathrm{Symm}(V)$, where m is even.

Suppose that q is odd and at least $m+1$.

Then provided that $m \le 2n/3$, we have $\dim \mathcal{M} \le n-1$.

Improved symmetric dimension bound

There is of course no analogue of Theorem 4 for even m, but we can nevertheless make some small improvements to Theorem 2 in the symmetric case.

Theorem 5

Let $\mathcal{M}$ be a constant rank m subspace of $\mathrm{Symm}(V)$, where m is even.

Suppose that q is odd and at least $m+1$.

Then provided that $m \leq 2n/3$, we have $\dim \mathcal{M} \leq n-1$.

This bound also holds if $m = n-1$ and n is odd.

Improved alternating dimension bound

What can we say about constant rank m subspaces of $\mathrm{Alt}(V)$? Recall that m must be even here.

Improved alternating dimension bound

What can we say about constant rank m subspaces of $\mathrm{Alt}(V)$?
Recall that m must be even here.

Theorem 6

Let $\mathcal{M}$ be a constant rank m subspace of $\mathrm{Alt}(V)$.

Improved alternating dimension bound

What can we say about constant rank m subspaces of $\mathrm{Alt}(V)$?
Recall that m must be even here.

Theorem 6

Let $\mathcal{M}$ be a constant rank m subspace of $\mathrm{Alt}(V)$.

Suppose that $q \geq m+1$ and $m \leq n/2$.

Improved alternating dimension bound

What can we say about constant rank m subspaces of $\mathrm{Alt}(V)$?
Recall that m must be even here.

Theorem 6

Let $\mathcal{M}$ be a constant rank m subspace of $\mathrm{Alt}(V)$.

Suppose that $q \geq m + 1$ and $m \leq n/2$.

Then we have $\dim \mathcal{M} \leq n - 1$.

Improved alternating dimension bound

What can we say about constant rank m subspaces of $\mathrm{Alt}(V)$?
Recall that m must be even here.

Theorem 6

Let $\mathcal{M}$ be a constant rank m subspace of $\mathrm{Alt}(V)$.

Suppose that $q \geq m+1$ and $m \leq n/2$.

Then we have $\dim \mathcal{M} \leq n-1$.

Furthermore, if $m \geq 4$, we have $\dim \mathcal{M} \leq n-2$

Dimension n constant rank

We conclude the talk by discussing properties of constant rank m subspaces of $\mathrm{Alt}(V)$ of dimension n.

Dimension n constant rank

We conclude the talk by discussing properties of constant rank m subspaces of $\mathrm{Alt}(V)$ of dimension n.

Theorem 7

Let $\mathcal{M}$ be a constant rank m subspace of $\mathrm{Alt}(V)$ of dimension n, and suppose that $q \geq m+1$.

Dimension n constant rank

We conclude the talk by discussing properties of constant rank m subspaces of $\mathrm{Alt}(V)$ of dimension n.

Theorem 7

Let $\mathcal{M}$ be a constant rank m subspace of $\mathrm{Alt}(V)$ of dimension n, and suppose that $q \geq m + 1$.

Then the different subspaces $\mathrm{rad}\, f$, as f runs over $\mathcal{M}^{\times}$, form a spread of V consisting of subspaces of dimension $n - m$.

Dimension n constant rank

We conclude the talk by discussing properties of constant rank m subspaces of $\mathrm{Alt}(V)$ of dimension n.

Theorem 7

Let $\mathcal{M}$ be a constant rank m subspace of $\mathrm{Alt}(V)$ of dimension n, and suppose that $q \geq m + 1$.

Then the different subspaces $\mathrm{rad}\, f$, as f runs over $\mathcal{M}^{\times}$, form a spread of V consisting of subspaces of dimension $n - m$.

Thus $n - m$ divides n.

Dimension n constant rank

We conclude the talk by discussing properties of constant rank m subspaces of $\mathrm{Alt}(V)$ of dimension n.

Theorem 7

Let $\mathcal{M}$ be a constant rank m subspace of $\mathrm{Alt}(V)$ of dimension n, and suppose that $q \geq m+1$.

Then the different subspaces $\mathrm{rad}\, f$, as f runs over $\mathcal{M}^{\times}$, form a spread of V consisting of subspaces of dimension $n-m$.

Thus $n-m$ divides n.

There is a corresponding spread decomposition of $\mathcal{M}$ into subspaces of dimension $n-m$, in which all non-zero elements in the given spread subspace have the same radical.

Partial converse

There is a partial converse to Theorem 7, which, for the sake of simplicity, we will restrict to the case where n is odd.

Partial converse

There is a partial converse to Theorem 7, which, for the sake of simplicity, we will restrict to the case where n is odd.

Theorem 8

Suppose that n is odd and m is a positive integer such that $n - m$ divides n.

Partial converse

There is a partial converse to Theorem 7, which, for the sake of simplicity, we will restrict to the case where n is odd.

Theorem 8

Suppose that n is odd and m is a positive integer such that $n - m$ divides n.

Then $\mathrm{Alt}(V)$ contains a constant rank m subspace of dimension n.

Partial converse

There is a partial converse to Theorem 7, which, for the sake of simplicity, we will restrict to the case where n is odd.

Theorem 8

Suppose that n is odd and m is a positive integer such that $n - m$ divides n.

Then $\mathrm{Alt}(V)$ contains a constant rank m subspace of dimension n.

Maximal constant rank subspaces of $\mathrm{Alt}(V)$

We have emphasized constant rank subspaces of maximal dimension but we can also investigate constant rank m subspaces that are maximal with respect to containment in constant rank m subspaces.

Maximal constant rank subspaces of $\mathrm{Alt}(V)$

We have emphasized constant rank subspaces of maximal dimension but we can also investigate constant rank m subspaces that are maximal with respect to containment in constant rank m subspaces.

Maximal constant rank m subspaces of $\mathrm{Alt}(V)$ or $\mathrm{Symm}(V)$ can have very small dimension compared with n. Here is a representative example.

Maximal constant rank subspaces of $\mathrm{Alt}(V)$

We have emphasized constant rank subspaces of maximal dimension but we can also investigate constant rank m subspaces that are maximal with respect to containment in constant rank m subspaces.

Maximal constant rank m subspaces of $\mathrm{Alt}(V)$ or $\mathrm{Symm}(V)$ can have very small dimension compared with n. Here is a representative example.

Theorem 9

Let $k \geq 3$ be an odd integer such that $k \leq n$ and suppose that $q \geq k$.

Maximal constant rank subspaces of $\mathrm{Alt}(V)$

We have emphasized constant rank subspaces of maximal dimension but we can also investigate constant rank m subspaces that are maximal with respect to containment in constant rank m subspaces.

Maximal constant rank m subspaces of $\mathrm{Alt}(V)$ or $\mathrm{Symm}(V)$ can have very small dimension compared with n. Here is a representative example.

Theorem 9

Let $k \geq 3$ be an odd integer such that $k \leq n$ and suppose that $q \geq k$.

Then $\mathrm{Alt}(V)$ contains a maximal constant rank $k-1$ subspace of dimension k.