Linear sets in the projective line over the endomorphism ring of a finite field

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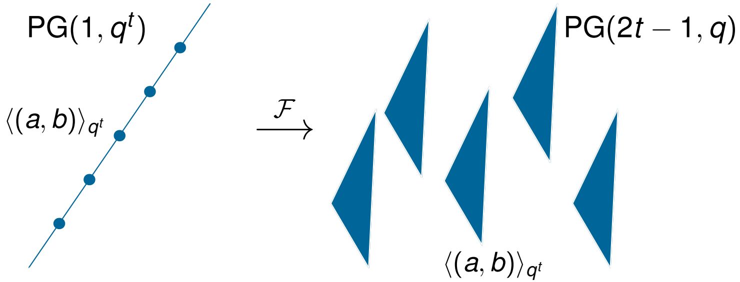
- $\mathcal{G}$ denotes the Grassmannian of $(t-1)$-dimensional subspaces of $\mathrm{PG}(2t-1, q)$.

Field reduction map $\mathcal{F}$: $\mathrm{PG}(1, q^t) \to \mathcal{G}$



The *field reduction map* $\mathcal{F}$ assigns to each point $\langle (a, b) \rangle_{q^t}$ that element of the Grassmannian $\mathcal{G}$ which is given by $\langle (a, b) \rangle_{q^t}$ (considered as subspace of the vector space $\mathbb{F}_{q^t}^2$ over $\mathbb{F}_q$).

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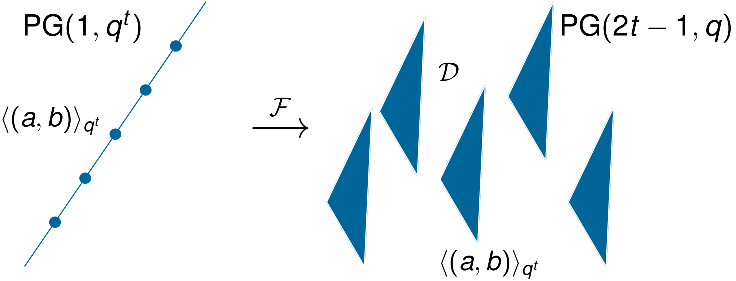

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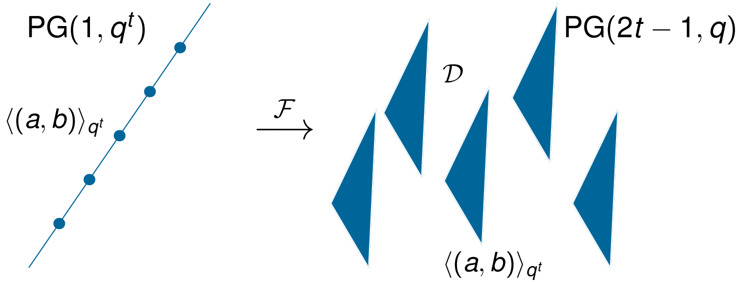

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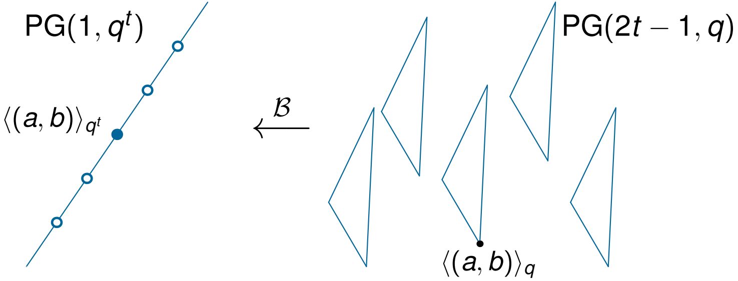
The map $\mathcal{F}$ is injective.

Blow up map $\mathcal{B}$: PG$(2t-1, q) \to$ PG$(1, q^t)$



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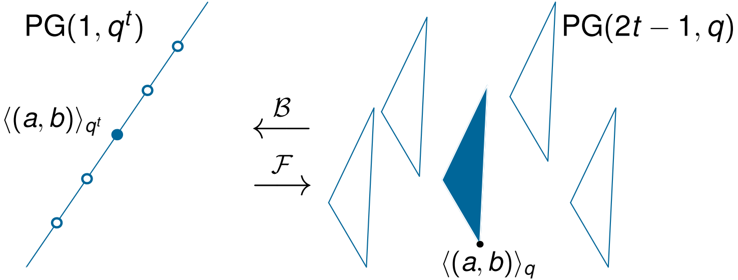

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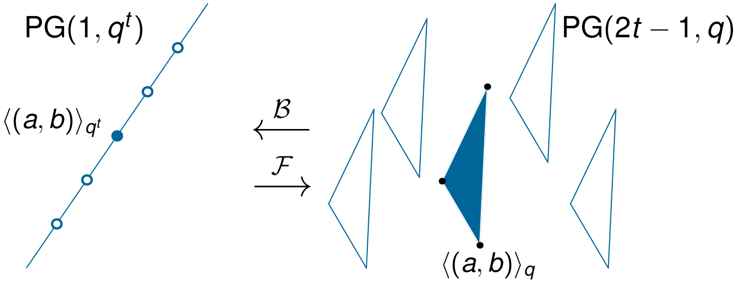

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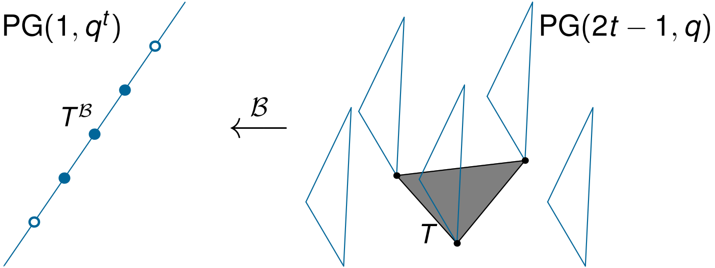
The map $\mathcal{B}$ is not injective (due to $t \geq 2$).

Linear sets



By blowing up all points of an element $T \in \mathcal{G}$ we obtain a subset $T^{\mathcal{B}}$ of PG$(1, q^t)$, which is called an $\mathbb{F}_q$-linear set of rank t.

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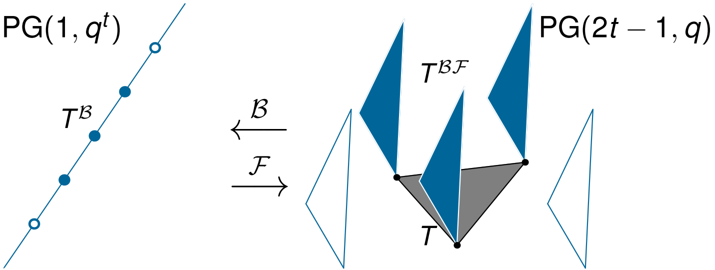

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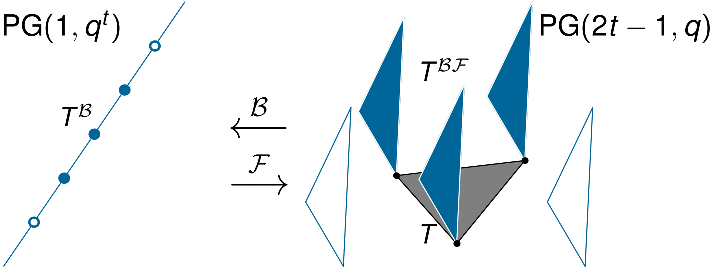

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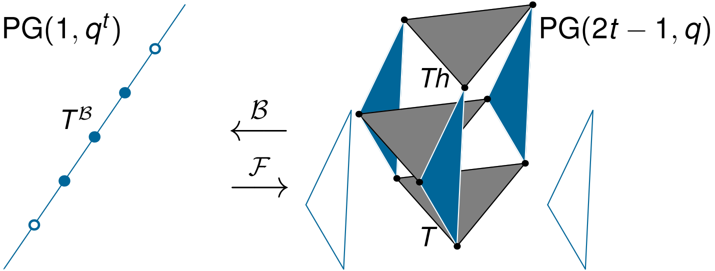
An element $T \in \mathcal{G}$ and its corresponding linear set $T^{\mathcal{B}}$ are said to be scattered if the restriction of $\mathcal{B}$ to T is injective.

Scattered linear sets – Two families



Let T be scattered and write $Th := \{\langle(ah, bh)\rangle_q \mid \langle(a, b)\rangle_q \in T\}$, where $h \in \mathbb{F}_{q^t} \setminus \{0\} =: \mathbb{F}^*_{q^t}$.

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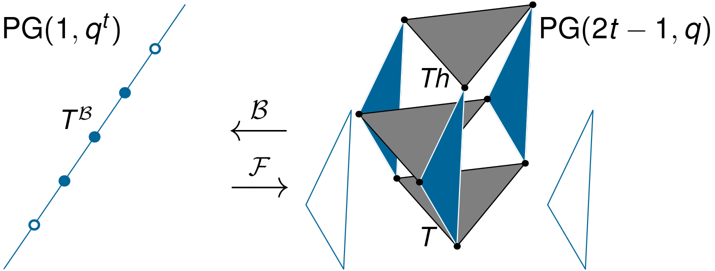

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$$\mathcal{U}(T) := T^{\mathcal{B}\mathcal{F}} \text{ and } \mathcal{U}'(T) := \{Th \mid h \in \mathbb{F}^*_{q^t}\},$$

constitute two partitions (by elements of $\mathcal{G}$) of the same hypersurface of degree t in PG$(2t - 1, q)$.

See M. Lavrauw, J. Sheekey, C. Zanella [15, Prop. 2].

The projective line over E

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- An element $(\alpha, \beta) \in E^2$ is called *admissible* if it can be extended to a basis of the left E-module E^2.
- The *projective line over E* is the set $\mathrm{PG}(1, E)$ of all cyclic submodules $E(\alpha, \beta)$ of E^2, where $(\alpha, \beta) \in E^2$ is admissible. The elements of $\mathrm{PG}(1, E)$ are called *points*.

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- The map

$$\Psi : \mathrm{PG}(1, E) \to \mathcal{G} : E(\alpha, \beta) \mapsto \left\{ \langle (u^\alpha, u^\beta) \rangle_q \mid u \in \mathbb{F}^*_{q^t} \right\}$$

is a bijection (X. Hubaut [11], Z.-X. Wan [24], and others).

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- P and Q are called *distant*, in symbols $P \bigtriangleup Q$, if $((\alpha, \beta), (\gamma, \delta))$ is a basis of E^2.
- $P \bigtriangleup Q$ if, and only if, the subspaces P^Ψ and Q^Ψ are skew (see, among others, A. Blunck [1, Thm. 2.4]).

Embedding of $PG(1, q^t)$ in $PG(1, E)$

- The mapping

$$\mathbb{F}_{q^t} \to E\colon a \mapsto (\rho_a\colon x \mapsto xa)$$

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- This allows us to define an embedding

$$\iota : \mathrm{PG}(1, q^t) \to \mathrm{PG}(1, E) : \langle (a, b) \rangle_{q^t} \mapsto E(\rho_a, \rho_b).$$

Projectivities

- Given a matrix
$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathsf{GL}_2(E)$$
we obtain a projectivity of $\mathsf{PG}(1, E)$ by letting
$$E(\xi, \eta) \mapsto E\left((\xi, \eta) \cdot \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right)$$
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- All projectivities of $\mathrm{PG}(1, E)$ and $\mathrm{PG}(2t-1, q)$ can be obtained in this way (S. Lang [13, 642–643]).
- The actions of $\mathrm{GL}_2(E)$ on $\mathrm{PG}(1, E)$ and $\mathcal{G}$ are isomorphic.

Dictionary

$\mathrm{PG}(1, E)$	Grassmannian $\mathcal{G}$
point T	subspace $T^{\Psi} \in \mathcal{G}$
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$L_T := \left\{ X \in \mathrm{PG}(1, q^t)^{\iota} \mid X \not\triangle T \right\}$	$\mathcal{U}(T^{\Psi}) = (T^{\Psi})^{\mathcal{BF}}$
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The sets L_T, with T varying in $\mathrm{PG}(1, E)$, are precisely the images under ι of the $\mathbb{F}_q$-linear sets of rank t in $\mathrm{PG}(1, q^t)$.

Linear sets of pseudoregulus type

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Cf. B. Czajbók, C. Zanella [4],
G. Donati, N. Durante [6],
M. Lavrauw, J. Sheekey, C. Zanella [15],
G. Lunardon, G. Marino, O. Polverino, R. Trombetti [20].

Main result

Theorem (H. H., C. Zanella [9])

A scattered linear set of $\mathrm{PG}(1, q^t)$, $t \geq 3$, *arising from* $T \in \mathrm{PG}(1, E)$ *is of pseudoregulus type if, and only if, there exists a projectivity* φ *of* $\mathrm{PG}(1, E)$ *such that* $L_T^{\varphi} = L'_T$.

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"$\Leftarrow$" See M. Lavrauw, J. Sheekey, C. Zanella [15, Cor. 18] or [9].

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Essence: We establish the existence of a cyclic group of projectivities of $\mathrm{PG}(1, E)$ acting regularly on L_T and fixing L'_T pointwise.

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The list of references contains also relevant work that went uncited on the previous slides.

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