Strongly Regular Graphs Related to Polar Spaces

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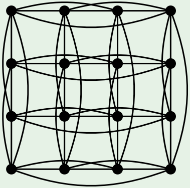
Strongly Regular Graphs

Definition

A **strongly regular graph** (SRG) with parameters (v, k, λ, μ) is a k-regular graph on v vertices s.t. two adjacent vertices have λ common neighbours and two non-adjacent vertices have μ common neighbours.

Example ($K_4 \times K_4$)



$v = 16,\ k = 6,\ \lambda = 2,\ \mu = 2.$

The Spectrum

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Lemma

The adjacency matrix of a strongly regular graph has **three different eigenvalues** k, e^+, e^-, *where* $k > e^+ > 0 > e^-$.

Given (v, k, λ) one can calculate (k, e^+, e^-) and vice versa.

Godsil-McKay Switching

Question

Is a strongly regular graph **uniquely determined by its parameters/spectrum***?*

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Lemma (Godsil-McKay Switching (simplified))

Let G be a graph. Let $\{X, Y\}$ be a **partition of the vertex set** *of G such that*

- **each $z \in Y$ is adjacent to 0, $|X|/2$ or $|X|$ vertices in X,**
- **each $z \in X$ has the same number of neighbours in X.**

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Change the adjacencies of $z \in Y$ with $|X|/2$ neighbours in X:

- *Old neighbourhood:* $N(z)$.
- *New neighbourhood:* $N(z) \triangle X$.

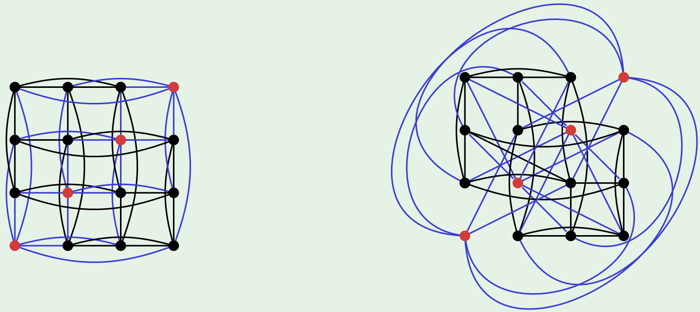
The new graph has **same spectrum** *as G. If G is a SRG, then the new graph is an SRG with the same parameters.*

Godsil-McKay Switching Example

Example

- The graph $G = K_4 \times K_4$ has $v = 16$, $k = 6$, $\lambda = 2$, $\mu = 2$.
- Let X be a coclique of size 4. Then G' is the (strongly regular) Shrikhande graph with $v = 16$, $k = 6$, $\lambda = 2$, $\mu = 2$.

Example (From $K_4 \times K_4$ to the Shrikhande graph)



The Symplectic Polar Space

- Vector space: $\mathbb{F}_q^6$.
- Symplectic form: $\sigma(x, y) = x_1 y_2 - x_2 y_1 + x_3 y_4 - x_4 y_3 + x_5 y_6 - x_6 y_5$.

Define $\mathrm{Sp}(6, q)$ as follows:

- The **vertices** are the 1-dimensional subspaces of $\mathbb{F}_q^6$.
- Two vertices $\langle x \rangle$ and $\langle y \rangle$ are **adjacent** if $\sigma(x, y) = 0$.
- Parameters for $q = 2$: $v = 63$, $k = 30$, $\lambda = 13$, $\mu = 15$.

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Theorem (Abiad & Haemers (2015))

The SRG $\mathrm{Sp}(2d, 2)$, $d > 2$, *is not determined by* (v, k, λ, μ).

Proof idea for $d = 3$.

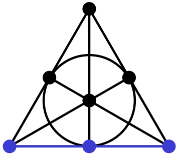
A switching set of size 4 yields a non-isomorphic graph. □

To understand this, let's look at $Sp(6, 2)$ ...

One Possible Switching Set?

The following is based on Barwick, Jackson, Penttila (2016).

- Take a 2-space ℓ of $\mathrm{Sp}(6, 2)$ (3 **vertices**).
- There is a 3-space S of $\mathrm{Sp}(6, 2)$ containing ℓ.



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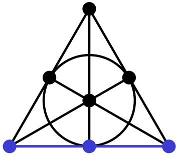


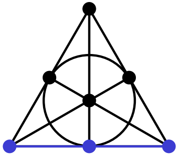
- The switching set X consists of the 4 **vertices** of $S \setminus \ell$.
- A vertex x not in S is adjacent to a 2-space ℓ' of S:
  - If $\ell' = \ell$, then x has 0 neighbours in X.
  - If $\ell' \neq \ell$, then x has $2 = |X|/2$ **neighbours** in X.

Godsil-McKay switching applicable! **:-)**

What about $Sp(6, q)$, $q > 2$?

The following is based on Barwick, Jackson, Penttila (2016).

- Take a 2-space ℓ of $\mathrm{Sp}(6, q)$ ($q + 1$ **vertices**).
- There is a 3-space S of $\mathrm{Sp}(6, q)$ containing ℓ.



- The switching set X consists of the q^2 **vertices** of $S \setminus \ell$.
- A vertex x not in S is adjacent to a 2-space ℓ' of S:
  - If $\ell' = \ell$, then x has 0 neighbours in X.
  - If $\ell' \neq \ell$, then x has $q \neq q^2/2 = |X|/2$ **neighbours** in X.

Godsil-McKay switching not applicable! **:-(**

More Polar Spaces

Finite classical polar spaces are geometries embedded in $\mathbb{F}_q^n$: 1-spaces (points), 2-spaces (lines), 3-spaces (planes), ..., d-spaces.

- $\Omega^-(2d+2,q)$: Elliptic quadric.
- $\Omega(2d+1,q)$: Parabolic quadric.
- $\Omega^+(2d,q)$: Hyperbolic quadric.
- $Sp(2d,q)$: Symplectic polar space.
- $U(2d,q^2)$: Hermitian polar space.
- $U(2d+1,q^2)$: Hermitian polar space.

(In this talk I usually identify a polar space with its collinearity graph.)

More Results for Polar Spaces

The following results were obtained by Godsil-McKay switching:

Theorem (Kubota (2016))

More non-isomorphic graphs with the same parameters as $\mathrm{Sp}(2d, 2)$.

Theorem (Barwick, Jackson, Penttila (2016))

Non-isomorphic graphs with the same parameters as $\Omega^{-}(2d+2, 2)$, $\Omega(2d+1, 2)$, $\Omega^{+}(2d, 2)$.

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For all polar spaces and for all q (no switching):

Theorem (Kantor (1982))

Constructs a possibly new SRG with the same parameters as the collinearity graph if there is a partition into d*-spaces (spread).*

Problem: existence of partitions and non-isomorphy.

A Geometric Construction

Define a SRG as follows:

- The **vertices** are the 2-dimensional subspaces of $\mathbb{F}_q^4$.
- Two vertices x and y are **adjacent** if $\dim(x \cap y) = 1$.

Theorem (Jungnickel (1984))

There are many SRGs with the same parameters.

1

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Idea (ad libitum): Permute the 2-spaces of an affine space while preserving parallel classes.

Vaguely similar ideas: Wallis (1971), Fon-Der-Flaass (2002), Muzychuk (2006), Jungnickel–Tonchev (2009), and surely many more.

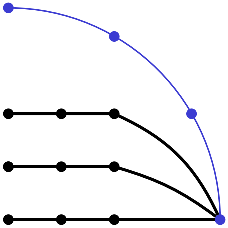
Pointed out to me by: Klaus Metsch for a different project (on the MMS conjecture[1] with Karen Meagher).

[1] Which I am interested in thanks to Simeon Ball.

Solution for $q = 3$

S: 3-space.
ℓ: 2-space in S.

Blue: ℓ. Black: $S \setminus \ell$.



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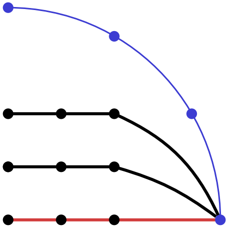

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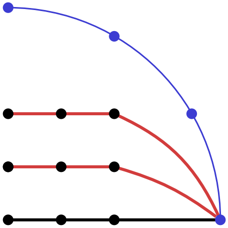

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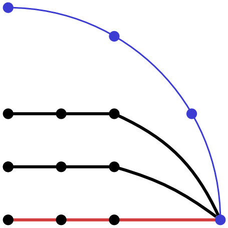

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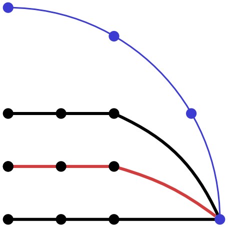

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- x is adjacent to a 2-space ℓ' with 3 vertices in S.
- The complement of ℓ' has too many vertices.
- Better: take a 2-space parallel to ℓ' instead of the complement.

Solution: permute parallel classes of subspaces to change adjacency.

New Strongly Regular Graphs

The sketched construction works for ...

- All finite classical polar spaces: Ω^-, Ω, Ω^+, Sp, U.
- All finite fields $\mathbb{F}_q$.
- All ranks $d > 2$ (the dimension of the maximal subspaces, in the examples usually 3).

Theorem (I. (2017))

No collinearity graph of a finite classical polar space with rank at least 3 is **determined** *by its parameters* (v, k, λ, μ).

What about non-isomorphy?

Are These New Graphs?

How to distinguish graphs?

- *p*-ranks (Abiad, Haemers)
- automorphism groups (Barwick, Jackson, Penttila)
- common neighbours of triples ($K_4 \times K_4$ vs Shrikhande graph)

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Lemma

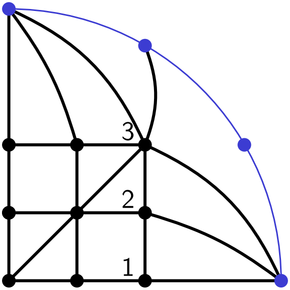
A clique of size 3 of $\mathrm{Sp}(6, q)$ *has* $q^3 + q^2 + q - 2$ *or* $q^2 + q - 2$ *neighbours.*

Lemma

If the permutation switches exactly two 2-spaces, then the modified graph has a clique of size 3 with $q^3 + q^2 + q - 3$ *common neighbours.*

What changes?

Take three points such that their (common) neighbours in the affine subspace are as follows:



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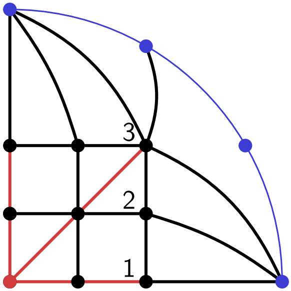

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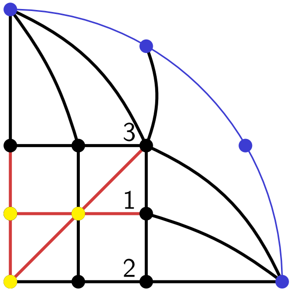

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Open Problems

There other strongly regular graphs from finite geometries, e.g. one can use polarities of subgeometries to obtain cospectral graphs:

Theorem (Cossidente, Pavese (2016))

The strongly regular point graph of a $GQ(s,t)$ with $t \in \{s, s\sqrt{s}\}$, s an even power of a prime, is not determined by its spectrum.

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Related finite geometry problems that are open (as far as I know):

1. Other generalized quadrangles.
2. There are (slightly exceptional) families of strongly regular graphs defined on defined for $\Omega^-(2d-1,3)$, $\Omega^+(2d+1,3)$, $\Omega^-(2d-1,5)$, and $\Omega^+(2d+1,5)$.
3. Strongly regular graphs from E_6.
4. Distance-regular graphs from E_7.
5. Distance-regular graphs from maximals of polar spaces.

Thank you for your attention!