Extension Sets, Affine Designs, and Hamada's Conjecture

Dieter Jungnickel
University of Augsburg

(Joint work with Yue Zhou and Vladimir D. Tonchev)

Finite Geometries, Fifth Irsee Conference
Sept. 12, 2017

Overview

Designs and Gaussian Coefficients

Let V be a set of cardinality v, and let $\mathcal{B}$ be a subset of $\mathcal{P}(V)$. One calls

- the elements of V *points*, and
- the elements of $\mathcal{B}$ *blocks*.

The pair $(V, \mathcal{B})$ is said to be a (v, k, λ)-*design* provided that:

- Each block contains exactly k points.
- Given any two distinct points, there are exactly λ blocks containing both points.

Designs and Gaussian Coefficients

Let V be a set of cardinality v, and let $\mathcal{B}$ be a subset of $\mathcal{P}(V)$. One calls

- the elements of V *points*, and
- the elements of $\mathcal{B}$ *blocks*.

The pair $(V, \mathcal{B})$ is said to be a (v, k, λ)-*design* provided that:

- Each block contains exactly k points.
- Given any two distinct points, there are exactly λ blocks containing both points.

The *Gaussian coefficient* $\begin{bmatrix} n \\ i \end{bmatrix}_q$ is the number of i-dimensional subspaces of an n-dimensional vector space over $GF(q)$:

$$\begin{bmatrix} n \\ i \end{bmatrix}_q = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-i+1} - 1)}{(q^i - 1)(q^{i-1} - 1) \cdots (q - 1)}$$

Geometric Designs

Let $\Sigma = AG(n,q)$ be the n-dimensional affine space over $GF(q)$.

The points and d-spaces of Σ form a resolvable 2-(v,k,λ) design $\mathcal{D} = AG_d(n,q)$ with parameters

$$v = q^n, \quad k = q^d, \quad \lambda = \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q,$$

$$r = \begin{bmatrix} n \\ d \end{bmatrix}_q, \quad b = r \cdot q^{n-d}.$$

These designs – and their projective analogues $PG_d(n,q)$ – are called *classical* or *geometric*.

Designs with Classical Parameters

Theorem. (DJ 1984,2011, DJ & VDT 2009, 2011, DJ & KM 2016)
Let q be any prime power and d an integer in the range $1 \leq d \leq n-1$. If we fix either d or $n-d$, then the number of (resolvable) non-isomorphic designs having the same parameters as $AG_d(n,q)$ or $PG_d(n,q)$ grows exponentially with linear growth of n.

Designs with Classical Parameters

Theorem. (DJ 1984,2011, DJ & VDT 2009, 2011, DJ & KM 2016)
Let q be any prime power and d an integer in the range $1 \leq d \leq n-1$. If we fix either d or $n-d$, then the number of (resolvable) non-isomorphic designs having the same parameters as $AG_d(n,q)$ or $PG_d(n,q)$ grows exponentially with linear growth of n.

Example. $(n = 3,\ q = 4,\ d = 1, 2)$

- There are at least

$$2^{19} \cdot 3^{12} \cdot 5^7 \cdot 7^7 \cdot 143^4 \ > \ 10^{30}$$

  non-isomorphic resolvable $2-(64,4,1)$ designs. (DJ & KM 2016)
- There are at least 21,621,600 non-isomorphic resolvable $2-(64,16,5)$ designs. (Harada, Lam & VDT 2003)

Codes from Designs

Let $\mathcal{D} = (V, \mathcal{B}, I)$ be a (v, k, λ)-design and label the points as $p_1, \ldots, p_v$ and the blocks as $B_1, \ldots, B_b$. The matrix $M = (m_{ij})_{i=1,\ldots,b;\, j=1,\ldots,v}$ defined by

$$m_{ij} := \begin{cases} 1 & \text{if } p_j \in B_i \\ 0 & \text{otherwise} \end{cases}$$

is called an *incidence matrix* for $\mathcal{D}$. The row of M belonging to a block B is called the *incidence vector* of B.

Codes from Designs

Let $\mathcal{D} = (V, \mathcal{B}, I)$ be a (v, k, λ)-design and label the points as $p_1, \ldots, p_v$ and the blocks as $B_1, \ldots, B_b$. The matrix $M = (m_{ij})_{i=1,\ldots,b;\, j=1,\ldots,v}$ defined by

$$m_{ij} := \begin{cases} 1 & \text{if } p_j \in B_i \\ 0 & \text{otherwise} \end{cases}$$

is called an *incidence matrix* for $\mathcal{D}$. The row of M belonging to a block B is called the *incidence vector* of B.

Now let F be some field. The F-vector space spanned by the rows of M is called the (*block*) *code* of $\mathcal{D}$ over F and will be denoted by $\mathcal{C}_F(\mathcal{D})$.

For most fields, this notion is not interesting:

The p-rank

Proposition. Assume $v > k$. Then M has rank v over any field of characteristic 0 as well as over any field of characteristic p, where p is a prime not dividing any of the numbers r, k and $n := r - \lambda$.

Moreover, If p divides one of r or k, but not n, the rank of M over F is either v or $v - 1$.

The p-rank

Proposition. Assume $v > k$. Then M has rank v over any field of characteristic 0 as well as over any field of characteristic p, where p is a prime not dividing any of the numbers r, k and $n := r - \lambda$.

Moreover, If p divides one of r or k, but not n, the rank of M over F is either v or $v - 1$.

Assume that F has a prime characteristic p dividing n. One calls rank M $=$ dim $\mathcal{C}_F(\mathcal{D})$ the *p-rank* of $\mathcal{D}$.

For designs with classical parameters, the natural choice for p is the characteristic of $GF(q)$.

The p-rank

Proposition. Assume $v > k$. Then M has rank v over any field of characteristic 0 as well as over any field of characteristic p, where p is a prime not dividing any of the numbers r, k and $n := r - \lambda$.

Moreover, If p divides one of r or k, but not n, the rank of M over F is either v or $v - 1$.

Assume that F has a prime characteristic p dividing n. One calls rank M $= \dim \mathcal{C}_F(\mathcal{D})$ the *p-rank* of $\mathcal{D}$.

For designs with classical parameters, the natural choice for p is the characteristic of $GF(q)$.

Explicit summation formulas for the p-rank of the incidence matrix of a geometric design were given by Hamada in 1968.

Hamada's Conjecture

The following conjecture was proposed by Hamada in 1973:

Conjecture. Let $\mathcal{D}$ be a design with the parameters of a geometric design $PG_d(n,q)$ or $AG_d(n,q)$, where q is a power of a prime p.

Then the p-rank of the incidence matrix of $\mathcal{D}$ is greater than or equal to the p-rank of the corresponding geometric design. *(Weak Hamada Conjecture)*

Hamada's Conjecture

The following conjecture was proposed by Hamada in 1973:

Conjecture. Let $\mathcal{D}$ be a design with the parameters of a geometric design $PG_d(n,q)$ or $AG_d(n,q)$, where q is a power of a prime p.

Then the p-rank of the incidence matrix of $\mathcal{D}$ is greater than or equal to the p-rank of the corresponding geometric design. *(Weak Hamada Conjecture)*

Moreover, equality holds if and only if $\mathcal{D}$ is isomorphic to the geometric design. *(Strong Hamada Conjecture)*

The significance of Hamada's Conjecture:

- It indicates that the geometric designs are the best choice for the construction of majority-logic decodable codes.

The significance of Hamada's Conjecture:

- It indicates that the geometric designs are the best choice for the construction of majority-logic decodable codes.
- It provides a computationally simple characterization of geometric designs.

The significance of Hamada's Conjecture:

- It indicates that the geometric designs are the best choice for the construction of majority-logic decodable codes.
- It provides a computationally simple characterization of geometric designs.
- It implies that any finite projective plane of prime order is Desarguesian.

Hamada's Conjecture (in its strong version) is known to hold for the designs corresponding to the following cases:

- hyperplanes in a binary projective or affine space (Hamada & Ohmori 1975);
- lines in a binary projective or ternary affine space (Doyen, Hubaut & Vandensavel 1978);
- planes in a binary affine space (Teirlinck 1980).

Counterexamples to Hamada's Conjecture

- four 2-$(31, 7, 7)$ designs with the same parameters as $PG_2(4, 2)$, all of 2-rank 16 (Tonchev 1986);
- four 3-$(32, 8, 7)$ designs with the same parameters as $AG_3(5, 2)$, all of 2-rank 16 (Tonchev 1986);

Counterexamples to Hamada's Conjecture

- four 2-$(31, 7, 7)$ designs with the same parameters as $PG_2(4, 2)$, all of 2-rank 16 (Tonchev 1986);
- four 3-$(32, 8, 7)$ designs with the same parameters as $AG_3(5, 2)$, all of 2-rank 16 (Tonchev 1986);
- two 2-$(64, 16, 5)$ designs with the same parameters as $AG_2(3, 4)$, all of 2-rank 16 (Harada, Lam and Tonchev 2005).

Counterexamples to Hamada's Conjecture

- four 2-$(31, 7, 7)$ designs with the same parameters as $PG_2(4, 2)$, all of 2-rank 16 (Tonchev 1986);
- four 3-$(32, 8, 7)$ designs with the same parameters as $AG_3(5, 2)$, all of 2-rank 16 (Tonchev 1986);
- two 2-$(64, 16, 5)$ designs with the same parameters as $AG_2(3, 4)$, all of 2-rank 16 (Harada, Lam and Tonchev 2005).

Theorem (DJ & VDT 2009). Let $q = p$ be a prime, and let $d \geq 2$. Then there exists a design with the same parameters and the same p-rank as, but not isomorphic to, the geometric design $PG_d(2d, p)$.

Counterexamples to Hamada's Conjecture

- four 2-$(31, 7, 7)$ designs with the same parameters as $PG_2(4, 2)$, all of 2-rank 16 (Tonchev 1986);
- four 3-$(32, 8, 7)$ designs with the same parameters as $AG_3(5, 2)$, all of 2-rank 16 (Tonchev 1986);
- two 2-$(64, 16, 5)$ designs with the same parameters as $AG_2(3, 4)$, all of 2-rank 16 (Harada, Lam and Tonchev 2005).

Theorem (DJ & VDT 2009). Let $q = p$ be a prime, and let $d \geq 2$. Then there exists a design with the same parameters and the same p-rank as, but not isomorphic to, the geometric design $PG_d(2d, p)$.

Theorem (Clark, DJ & VDT 2010). Let $d \geq 2$. Then there exists a design with the same parameters and the same 2-rank as, but not isomorphic to, the geometric design $AG_{d+1}(2d + 1, 2)$.

Good blocks

- $\mathcal{D}$ – an affine design with the parameters of $AG_2(3, q)$
- A block B is called *good* if the incidence structure $\mathcal{D}(B)$ induced on B by the intersections of non-parallel blocks is a q-**fold multiple** of an affine plane $\mathcal{A}$.

Good blocks

- $\mathcal{D}$ – an affine design with the parameters of $AG_2(3,q)$
- A block B is called *good* if the incidence structure $\mathcal{D}(B)$ induced on B by the intersections of non-parallel blocks is a q-**fold multiple** of an affine plane $\mathcal{A}$.

- $\mathcal{D}_B$ – the residual structure induced on the points not in B
- View blocks parallel to B as "groups" in $\mathcal{D}_B$
- $\mathcal{D}_B$ becomes a resolvable GDD $\mathcal{E}$ with parameters

$$m = q-1,\ n = q^2,\ k = q^2 - q,\ \lambda_1 = q,\ \lambda_2 = q+1$$

Good blocks

- $\mathcal{D}$ – an affine design with the parameters of $AG_2(3,q)$
- A block B is called *good* if the incidence structure $\mathcal{D}(B)$ induced on B by the intersections of non-parallel blocks is a q-**fold multiple** of an affine plane $\mathcal{A}$.

- $\mathcal{D}_B$ – the residual structure induced on the points not in B
- View blocks parallel to B as "groups" in $\mathcal{D}_B$
- $\mathcal{D}_B$ becomes a resolvable GDD $\mathcal{E}$ with parameters

$$m = q-1,\ n = q^2,\ k = q^2 - q,\ \lambda_1 = q,\ \lambda_2 = q+1$$

- *bundle* – q blocks of $\mathcal{D}$ intersecting B in a fixed line ℓ of $\mathcal{A}$
- The bundles give a second resolution of $\mathcal{E}$, orthogonal to the natural parallelism.

Extension sets

An *extension set* for $\mathcal{A}$ is a collection $\mathcal{F}$ of q^2 sets of lines of $\mathcal{A}$, called *factors*, such that

(F1) Each factor in $\mathcal{F}$ contains precisely one line from each parallel class of $\mathcal{A}$.

(F2) Any two distinct factors in $\mathcal{F}$ have precisely one line of $\mathcal{A}$ in common.

Extension sets

An *extension set* for $\mathcal{A}$ is a collection $\mathcal{F}$ of q^2 sets of lines of $\mathcal{A}$, called *factors*, such that

(F1) Each factor in $\mathcal{F}$ contains precisely one line from each parallel class of $\mathcal{A}$.

(F2) Any two distinct factors in $\mathcal{F}$ have precisely one line of $\mathcal{A}$ in common.

Examples.

- *pencil type*: Associate with each point p of $\mathcal{A}$ the factor F_p consisting of the $q+1$ lines through p.
- *translation type*: Choose a first factor F by selecting an arbitrary line from each parallel class of $\mathcal{A}$, and then apply the translation group T of $\mathcal{A}$ to obtain q^2 factors.

Lemma.

Define a new incidence structure $\mathcal{A}'$ as follows:

- The points are the factors in $\mathcal{F}$.
- The lines are the lines of $\mathcal{A}$.
- A point ($=$ factor F) is incident with a line ℓ if and only if $\ell \in F$.

Lemma.

Define a new incidence structure $\mathcal{A}'$ as follows:

- The points are the factors in $\mathcal{F}$.
- The lines are the lines of $\mathcal{A}$.
- A point ($=$ factor F) is incident with a line ℓ if and only if $\ell \in F$.

Then $\mathcal{A}'$ is an affine plane of order q with the same line set.

Two distinct lines are parallel in $\mathcal{A}'$ if and only if they are parallel in $\mathcal{A}$.

The basic construction

Modify $\mathcal{D}$ as follows:

- Consider the groups of $\mathcal{E}$ again as blocks.
- Adjoin the factors in $\mathcal{F}$ to $\mathcal{E}$ as new points, forming a new block B'.
- Let a new point F be incident with the $q(q+1)$ blocks of $\mathcal{E}$ in the $q+1$ bundles determined by the lines in F.

The basic construction

Modify $\mathcal{D}$ as follows:

- Consider the groups of $\mathcal{E}$ again as blocks.
- Adjoin the factors in $\mathcal{F}$ to $\mathcal{E}$ as new points, forming a new block B'.
- Let a new point F be incident with the $q(q+1)$ blocks of $\mathcal{E}$ in the $q+1$ bundles determined by the lines in F.

Every block of $\mathcal{D}_B$ is a point set of the form $C \setminus \ell$ for a unique block C of $\mathcal{D}$, where the intersection $\ell = C \cap B$ of the good block B with C is a line of the affine plane $\mathcal{A}$ induced on B.

Then $C \setminus \ell$ is extended to a block C' of the new incidence structure $\mathcal{D}'$ by adjoining the q points of the line ℓ considered as a line of $\mathcal{A}'$, that is, by adjoining the q factors in $\mathcal{F}$ containing ℓ.

The basic construction

Theorem.

The new incidence structure $\mathcal{D}' = \mathcal{D}'(\mathcal{F})$ is an affine design with the same parameters as $\mathcal{D}$.

B' is a good block of $\mathcal{D}'$ such that

- $\mathcal{D}'(B')$ is the q-fold multiple of $\mathcal{A}'$;
- $\mathcal{D}'_{B'} = \mathcal{D}_B$.

The basic construction

Theorem.

The new incidence structure $\mathcal{D}' = \mathcal{D}'(\mathcal{F})$ is an affine design with the same parameters as $\mathcal{D}$.

B' is a good block of $\mathcal{D}'$ such that

- $\mathcal{D}'(B')$ is the q-fold multiple of $\mathcal{A}'$;
- $\mathcal{D}'_{B'} = \mathcal{D}_B$.

If $\mathcal{D} = AG_2(3, q)$, where $q \geq 3$, and if $\mathcal{F}$ is not of pencil type, then $\mathcal{D}'$ is not isomorphic to $\mathcal{D}$.

Assume that $\mathcal{F}$ is of translation type. Then the lines in $\mathcal{D}'$ are as follows:

- A line ℓ of $\mathcal{D}$ which is not parallel to B loses the point $B \cap \ell$ in $\mathcal{D}'$ and becomes a line of size $q-1$.
- The lines in B' have size q in $\mathcal{D}'$.
- The lines of $\mathcal{D}$ parallel to, but not in, B remain unchanged in $\mathcal{D}'$.
- A point in B' and a point of $\mathcal{D}_B$ always determine a line of size 2 in $\mathcal{D}'$.

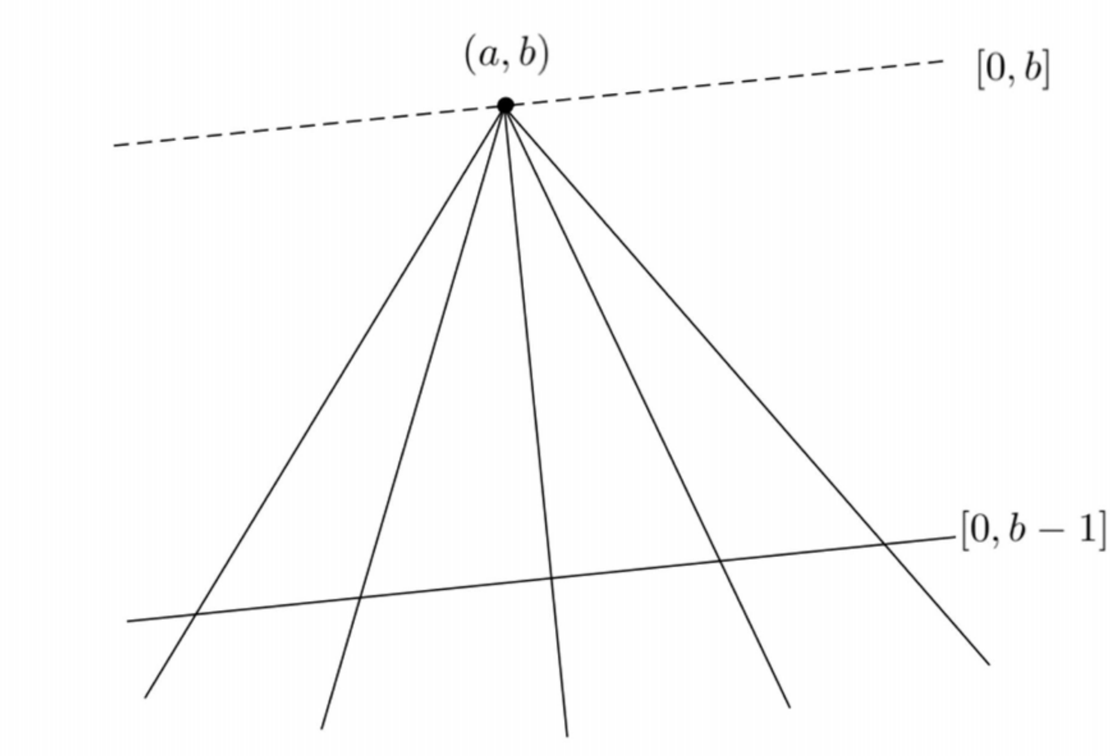
Assume that $\mathcal{F}$ is of translation type. Then the lines in $\mathcal{D}'$ are as follows:

- A line ℓ of $\mathcal{D}$ which is not parallel to B loses the point $B \cap \ell$ in $\mathcal{D}'$ and becomes a line of size $q-1$.
- The lines in B' have size q in $\mathcal{D}'$.
- The lines of $\mathcal{D}$ parallel to, but not in, B remain unchanged in $\mathcal{D}'$.
- A point in B' and a point of $\mathcal{D}_B$ always determine a line of size 2 in $\mathcal{D}'$.

Hence the good blocks of $\mathcal{D}'$ are B' and its parallel blocks.

Nearpencils

Replacing the line $[0, b]$ of the pencil through the point (a, b) by $[0, b-1]$ gives the *nearpencil* through (a, b):



The q^2 nearpencils form an extension set of translation type.

Nearpencils

Theorem. Let $\mathcal{D}$ be a classical affine design $AG_2(3,q)$, where $q = p^n \geq 3$ and p is a prime. Let B be an arbitrary block of $\mathcal{D}$, and let $\mathcal{F}$ be the extension set of nearpencil type.

Then the affine design $\mathcal{D}' = \mathcal{D}'(\mathcal{F})$ has the same parameters as $\mathcal{D}$, and its p-rank exceeds that of $\mathcal{D}$ by an integer d satisfying

$$1 \leq d \leq q - p^{n-1} - 1.$$

Nearpencils

Theorem. Let $\mathcal{D}$ be a classical affine design $AG_2(3, q)$, where $q = p^n \geq 3$ and p is a prime. Let B be an arbitrary block of $\mathcal{D}$, and let $\mathcal{F}$ be the extension set of nearpencil type.

Then the affine design $\mathcal{D}' = \mathcal{D}'(\mathcal{F})$ has the same parameters as $\mathcal{D}$, and its p-rank exceeds that of $\mathcal{D}$ by an integer d satisfying

$$1 \leq d \leq q - p^{n-1} - 1.$$

- The bounds coincide for $q \in \{3, 4\}$.

Nearpencils

Theorem. Let $\mathcal{D}$ be a classical affine design $AG_2(3, q)$, where $q = p^n \geq 3$ and p is a prime. Let B be an arbitrary block of $\mathcal{D}$, and let $\mathcal{F}$ be the extension set of nearpencil type.

Then the affine design $\mathcal{D}' = \mathcal{D}'(\mathcal{F})$ has the same parameters as $\mathcal{D}$, and its p-rank exceeds that of $\mathcal{D}$ by an integer d satisfying

$$1 \leq d \leq q - p^{n-1} - 1.$$

- The bounds coincide for $q \in \{3, 4\}$.
- The p-rank of $\mathcal{D}'$ attains the upper bound for all $q \leq 19$.

Line ovals

Let $\mathcal{F}$ be an extension set in $\mathcal{A}$ for which every point is on at most two lines from any given factor: the factors are *line ovals*.

Line ovals

Let $\mathcal{F}$ be an extension set in $\mathcal{A}$ for which every point is on at most two lines from any given factor: the factors are *line ovals*.

- Any line oval in $\mathcal{A}$ determines a maximal arc of degree $q/2$, namely the $q(q-1)/2$ points which are not contained in any of its lines (and conversely).
- Let $\mathcal{A} = AG(2,q)$, and choose the line oval F as the affine part of a regular dual hyperoval in $PG(2,q)$. Then F is a *line conic*.

Line ovals

Theorem (Kantor 1975).

- An extension set of translation type determined by a line oval F is the set of blocks of a symmetric design $\mathcal{S}$ with parameters

$$v = q^2, \quad k = \tfrac{1}{2}q(q+1) \quad \text{and} \quad \lambda = \tfrac{1}{4}q(q+2).$$

- Assume that $\mathcal{A} = AG(2,q)$ and that F is a line conic. Then $\mathcal{S}$ admits a 2-transitive automorphism group G.

The case $q = 4$

Theorem.

- Let B be a block of the affine design $\mathcal{D} = AG_2(3, 4)$, and let $\mathcal{F}$ be an extension set determined by a line conic in the affine plane $\mathcal{A}$ of order 4 induced on B.

  Then the affine design $\mathcal{D}' = \mathcal{D}'(\mathcal{F})$ is not isomorphic to $\mathcal{D}$, but has the same parameters and the same 2-rank as $\mathcal{D}$.

The case $q = 4$

Theorem.

- Let B be a block of the affine design $\mathcal{D} = AG_2(3,4)$, and let $\mathcal{F}$ be an extension set determined by a line conic in the affine plane $\mathcal{A}$ of order 4 induced on B.

  Then the affine design $\mathcal{D}' = \mathcal{D}'(\mathcal{F})$ is not isomorphic to $\mathcal{D}$, but has the same parameters and the same 2-rank as $\mathcal{D}$.

- Let $\widetilde{B}$ be one of the three good blocks of $\mathcal{D}'$ distinct from B', and let $\widetilde{\mathcal{F}}$ be an extension set determined by a line conic in the affine plane $\widetilde{\mathcal{A}}$ of order 4 induced on $\widetilde{B}$.

  Then the affine design $\mathcal{D}'' = \mathcal{D}''(\widetilde{\mathcal{F}})$ is not isomorphic to either $\mathcal{D}$ or $\mathcal{D}'$, but has the same parameters and the same 2-rank.

The case $q = 4$

Theorem.

- Let B be a block of the affine design $\mathcal{D} = AG_2(3,4)$, and let $\mathcal{F}$ be an extension set determined by a line conic in the affine plane $\mathcal{A}$ of order 4 induced on B.

  Then the affine design $\mathcal{D}' = \mathcal{D}'(\mathcal{F})$ is not isomorphic to $\mathcal{D}$, but has the same parameters and the same 2-rank as $\mathcal{D}$.

- Let $\widetilde{B}$ be one of the three good blocks of $\mathcal{D}'$ distinct from B', and let $\widetilde{\mathcal{F}}$ be an extension set determined by a line conic in the affine plane $\widetilde{\mathcal{A}}$ of order 4 induced on $\widetilde{B}$.

  Then the affine design $\mathcal{D}'' = \mathcal{D}''(\widetilde{\mathcal{F}})$ is not isomorphic to either $\mathcal{D}$ or $\mathcal{D}'$, but has the same parameters and the same 2-rank.

- Iterating the construction two more times gives an affine design $\mathcal{D}'''$ isomorphic to $\mathcal{D}'$, and then an affine design isomorphic to $\mathcal{D}$.

The case $q = 4$

Essential steps:

- $\mathcal{D}$ has 2-rank 16.
- $\mathcal{D}_B$ is *linearly embeddable* in $\mathcal{D}$: it has 2-rank $15 = 16 - 1$.

The case $q = 4$

Essential steps:

- $\mathcal{D}$ has 2-rank 16.
- $\mathcal{D}_B$ is *linearly embeddable* in $\mathcal{D}$: it has 2-rank $15 = 16 - 1$.

  Note: The residual designs of $AG_{n-1}(n, q)$ and of $PG_{n-1}(n, q)$ are always linearly embeddable. (Tonchev 2016)

The case $q = 4$

Essential steps:

- $\mathcal{D}$ has 2-rank 16.
- $\mathcal{D}_B$ is *linearly embeddable* in $\mathcal{D}$: it has 2-rank $15 = 16 - 1$.

  Note: The residual designs of $AG_{n-1}(n, q)$ and of $PG_{n-1}(n, q)$ are always linearly embeddable. (Tonchev 2016)
- An incidence matrix M for $\mathcal{D}$:

$$
\begin{array}{cccccc}
80 & & & \widetilde{B} & B & \\
\left(\begin{array}{c|cccc}
M(\mathcal{D}(B)) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{j} \\
\hline
 & \mathbf{0} & \mathbf{0} & \mathbf{j} & \mathbf{0} \\
M(\mathcal{E}) & \mathbf{0} & \mathbf{j} & \mathbf{0} & \mathbf{0} \\
 & \mathbf{j} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{array}\right)
\end{array}
\begin{array}{l} 16 \\ \\ 48 \end{array}
$$

The case $q = 4$

- Let $\mathcal{C}$ be the code spanned by the last 48 rows of M. Then the sum $\mathbf{r}_p + \mathbf{r}_{p'}$ of any two of the first 16 row vectors belongs to $\mathcal{C}$.
- It suffices to prove the analogous statement for any two of the first 16 row vectors of a corresponding incidence matrix M' for $\mathcal{D}'$.
- Check that the sum $\mathbf{s} = \mathbf{r}_F + \mathbf{r}_{F'}$ of any two "oval vectors" (with $F, F' \in \mathcal{F}$) can also be written as the sum $\mathbf{r}_p + \mathbf{r}_{p'}$ of two "pencil vectors".

The case $q = 4$

- Let $\mathcal{C}$ be the code spanned by the last 48 rows of M. Then the sum $\mathbf{r}_p + \mathbf{r}_{p'}$ of any two of the first 16 row vectors belongs to $\mathcal{C}$.
- It suffices to prove the analogous statement for any two of the first 16 row vectors of a corresponding incidence matrix M' for $\mathcal{D}'$.
- Check that the sum $\mathbf{s} = \mathbf{r}_F + \mathbf{r}_{F'}$ of any two "oval vectors" (with $F, F' \in \mathcal{F}$) can also be written as the sum $\mathbf{r}_p + \mathbf{r}_{p'}$ of two "pencil vectors".
- Show that the residual structure $\mathcal{D}_{\widetilde{B}}$ is linearly embeddable in $\mathcal{D}'$.
- Then an analogous proof works for $\mathcal{D}''$.

Linear extension sets

An extension set in $\mathcal{A} = AG(2, q)$ is called *linear* if it yields a non-classical linear embedding of the residual structure $RAG_2(3, q)$ of $AG_2(3, q)$.

Example: The extension set defined by a line conic in $AG(2, 4)$ is linear.

Linear extension sets

An extension set in $\mathcal{A} = AG(2,q)$ is called *linear* if it yields a non-classical linear embedding of the residual structure $RAG_2(3,q)$ of $AG_2(3,q)$.

Example: The extension set defined by a line conic in $AG(2,4)$ is linear.

Example: The extension sets defined by a line conic in $AG(2,8)$ and by a line conic or the Lunelli-Sce hyperoval in $AG(2,16)$ are *not* linear:

- $\mathcal{D} = AG_2(3,8)$ has 2-rank 64, but $\mathcal{D}'$ has 2-rank 70;
- $\mathcal{D} = AG_2(3,16)$ has 2-rank 256, but $\mathcal{D}'$ has 2-rank 280 resp. 288.

Linear extension sets

Theorem.

Let $q \geq 3$ be a prime power, and let $\mathcal{F}$ be a linear extension set for the classical affine plane $\mathcal{A} = \mathrm{AG}(2,q)$ which is not of pencil type.

Then necessarily $q = 4$, and $\mathcal{F}$ belongs to a line conic in $AG(2,4)$.

Linear extension sets

Theorem.

Let $q \geq 3$ be a prime power, and let $\mathcal{F}$ be a linear extension set for the classical affine plane $\mathcal{A} = \mathrm{AG}(2, q)$ which is not of pencil type.

Then necessarily $q = 4$, and $\mathcal{F}$ belongs to a line conic in $AG(2, 4)$.

Corollary.

The class of affine designs with the parameters of some $\mathrm{AG}_2(3, q)$, where $q \geq 5$, which arise from $RAG_2(3, q)$ and some extension set for $AG(2, q)$ satisfies Hamada's conjecture.

Linear extension sets

The essential tool:

Theorem. (Polverino and Zullo 2016)

Let $\mathcal{C}$ be the block code of some classical symmetric design $PG_{d-1}(d,q)$.

Then the minimum weight of $\mathcal{C}$ is $q^{d-1}+\cdots+q+1$, and the second smallest weight is $2q^{d-1}$.

Linear extension sets

The essential tool:

Theorem. (Polverino and Zullo 2016)

Let $\mathcal{C}$ be the block code of some classical symmetric design $PG_{d-1}(d,q)$.

Then the minimum weight of $\mathcal{C}$ is $q^{d-1} + \cdots + q + 1$, and the second smallest weight is $2q^{d-1}$.

Moreover, all codewords of minimum weight belong to incidence vectors of hyperplanes,

Linear extension sets

The essential tool:

Theorem. (Polverino and Zullo 2016)

Let $\mathcal{C}$ be the block code of some classical symmetric design $PG_{d-1}(d,q)$.

Then the minimum weight of $\mathcal{C}$ is $q^{d-1} + \cdots + q + 1$, and the second smallest weight is $2q^{d-1}$.

Moreover, all codewords of minimum weight belong to incidence vectors of hyperplanes,
and all codewords of weight $2q^{d-1}$ arise from the difference of the incidence vectors of two distinct hyperplanes (up to scalar multiples).

Linear extension sets

Corollary.

Let $\mathcal{C}$ be the point code of some classical affine design $AG_{d-1}(d, q)$.

Then the minimum weight of $\mathcal{C}$ is $q^{d-1} + \cdots + q + 1$, and the second smallest weight is $2q^{d-1}$.

Moreover, all codewords of minimum weight belong to incidence vectors of points, and all codewords of weight $2q^{d-1}$ arise from the difference of the incidence vectors of two distinct points (up to scalar multiples).

Some Problems

- The weak version of Hamada's conjecture is still entirely open.

Some Problems

- The weak version of Hamada's conjecture is still entirely open.
- Find infinite families of counterexamples to the strong version of the conjecture when q is not a prime.
- Can one find an infinite series of counterexamples for the cases $AG_{d-1}(d, 4)$?

Some Problems

- The weak version of Hamada's conjecture is still entirely open.
- Find infinite families of counterexamples to the strong version of the conjecture when q is not a prime.
- Can one find an infinite series of counterexamples for the cases $AG_{d-1}(d, 4)$?
- Settle the status for $n = 2$ (projective and affine planes) and, more generally, for $d = n - 1$ (symmetric and affine designs).

Some Problems

- The weak version of Hamada's conjecture is still entirely open.
- Find infinite families of counterexamples to the strong version of the conjecture when q is not a prime.
- Can one find an infinite series of counterexamples for the cases $AG_{d-1}(d, 4)$?
- Settle the status for $n = 2$ (projective and affine planes) and, more generally, for $d = n - 1$ (symmetric and affine designs).
- Show that the extension set of nearpencil type for $AG(2, q)$, $q = p^n \geq 3$, always transforms $AG_2(3, q)$ into an affine design with p-rank raised by $q - p^{n-1} - 1$.

Some Problems

- The weak version of Hamada's conjecture is still entirely open.
- Find infinite families of counterexamples to the strong version of the conjecture when q is not a prime.
- Can one find an infinite series of counterexamples for the cases $AG_{d-1}(d, 4)$?
- Settle the status for $n = 2$ (projective and affine planes) and, more generally, for $d = n - 1$ (symmetric and affine designs).
- Show that the extension set of nearpencil type for $AG(2, q)$, $q = p^n \geq 3$, always transforms $AG_2(3, q)$ into an affine design with p-rank raised by $q - p^{n-1} - 1$.
- Find (and prove) a formula for the 2-rank of affine designs with the parameters of $AG_2(3, q)$, q even, constructed from the extension set given by a line conic in $AG(2, q)$.

Thanks for your attention.