m-ovoids of regular near polygons

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joint work with John Bamberg *and* Melissa Lee

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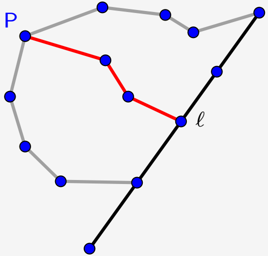
near polygons

A *near polygon* or $2d$-gon is an incidence geometry which satisfies:

- Any two points on at most one line.
- Diameter of collinearity graph is d.
- Given a line ℓ and non-incident point P, there is a unique nearest path from P to a point on ℓ (wrt to the collinearity graph).



Regular near polygons

If a near polygon has parameters $(s, t_2, \ldots, t_{d-1}, t)$ such that:

- each point is incident with $t+1$ lines,
- each line is incident with $s+1$ points,
- if x and y are points such that $d(x, y) = i$, then there are $t_i + 1$ lines on y with a point at distance $i-1$ from x.

Equivalently, they have distance regular collinearity graphs, with intersection numbers:

$$a_i = (s-1)(t_i+1), \quad b_i = s(t - t_i), \quad c_i = t_i + 1$$

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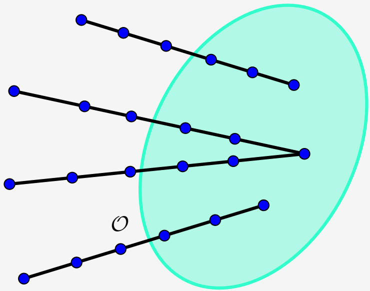
Example

Finite dual polar spaces are regular near $2d$-gons.

In particular $DW(2d-1, q)$, $DQ(2d, q)$ and $DH(2d-1, q^2)$ are regular near polygons.

m-ovoids

An *m-ovoid* of a near $2d$-gon is a set of points $\mathcal{O}$ such that every line meets exactly m elements of $\mathcal{O}$.



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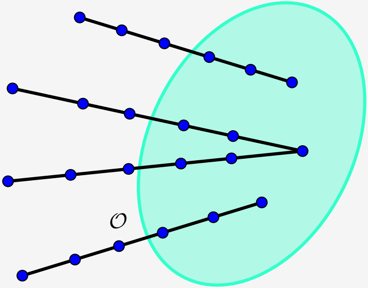

If m is half the number of points on a line $(s+1/2)$, then the m-ovoid is called a *hemisystem*.

Some m-ovoid results of dual polar spaces

- No 1-ovoids of $DQ(4, q)$, q odd (Thas)
- No 1-ovoids of $DQ^-(5, q)$ (Thas)
- No 1-ovoids of $DQ(3, q)$, q odd (Thas)
- No 1-ovoids of $DH(4, 2^2)$ (Brouwer)
- 1-ovoids of $DQ^-(7, q)$ not known
- 1-ovoids of $DH(6, q^2)$ not known
- No 1-ovoids $DW(5, q)$, q even (Payne, Thas), q odd (Thomas)
- No 1-ovoids of generalised hexagons of order (s, s^2) (De Bruyn, Vanhove)
- m-ovoids of $DH(3, q^2)$ are hemisystems, q odd (Segre)
- m-ovoids of generalised quadrangles of order (q, q^2) are hemisystems, q odd (Cameron, Goethals, Seidel)
- m-ovoids of regular near $2d$-gons with $t_i + 1 = \frac{s^{2i}-1}{s^2-1}$ are hemisystems (Vanhove)

Prior theorems

Theorem (De Bruyn, Vanhove)

A regular near 2d-gon satisfies

$$\frac{(s^i - 1)(t_{i-1} + 1 - s^{i-2})}{s^{i-2} - 1} \leqslant t_i + 1 \leqslant \frac{(s^i + 1)(t_{i-1} + 1 + s^{i-2})}{s^{i-2} + 1}$$

for $s, d \geqslant 2$ *and* $i \in \{3, \ldots, d\}$.

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for $s, d \geqslant 2$ *and* $i \in \{3, \ldots, d\}$.

Theorem (De Bruyn, Vanhove)

A regular 2d-gon ($d \geqslant 3$, $s \geqslant 2$ *) attaining the lower bound for* $i = 3$ *is isomorphic to* $DQ(2d, q)$, $DW(2d-1, q)$ *or* $DH(2d-1, q^2)$.

New result

Theorem (Bamberg, JL, Lee)

Given a 2d-gon satisfying

$$t_i + 1 = \frac{(s^i + (-1)^i)(t_{i-1} + 1 + (-1)^i s^{i-2})}{s^{i-2} + (-1)^i}$$

a non-trivial m-ovoid is a hemisystem.

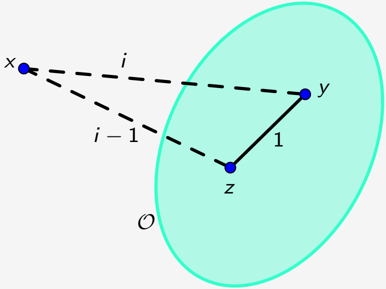
Sketch of proof

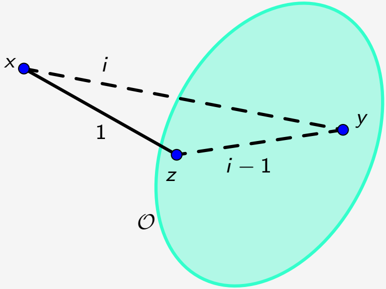
Fix $x \notin \mathcal{O}$. Count $y, z \in \mathcal{O}$ such that $d(x, y) = i$ and

- $d(y, z) = i - 1$
  $d(x, z) = 1$

or

- $d(y, z) = 1$
  $d(x, z) = i - 1$





Sketch of proof

Counting y then z:

For x and y at distance i define
$v_{x,y} := s(c_{i-1} - s)(\chi_x + \chi_y) + \chi_{\Gamma_1(x)\cap\Gamma_{i-1}(y)} + \chi_{\Gamma_{i-1}(x)\cap\Gamma_1(y)}$

Theorem

$v_{x,y} \cdot \chi_{\mathcal{O}} = \frac{2(s(c_{i-1} - s^{i-2}) + c_i)m}{s+1}$ *(design-orthogonal)*

Sketch of proof

Counting first y and then z, the number of pairs is

$$\sum_{y\in\mathcal{O}\cap\Gamma_i(x)} \left(|\Gamma_1(x)\cap\Gamma_{i-1}(y)\cap\mathcal{O}| + |\Gamma_{i-1}(x)\cap\Gamma_1(y)\cap\mathcal{O}|\right)$$

$$= \sum_{y\in\mathcal{O}\cap\Gamma_i(x)} (v_{x,y} - s(c_{i-1} + (-1)^i s^{i-2})(\chi_x + \chi_y)) \cdot \chi_{\mathcal{O}}$$

$$= \dots$$

$$= \frac{mk_{i-1}(t - t_{i-1})}{s+1}\left(1 - \left(-\frac{1}{s}\right)^i\right)\frac{2c_i ms - (s+1-2m)\left(c_{i-1}s^2 + (-1)^i s^i\right)}{c_i(s+1)}.$$

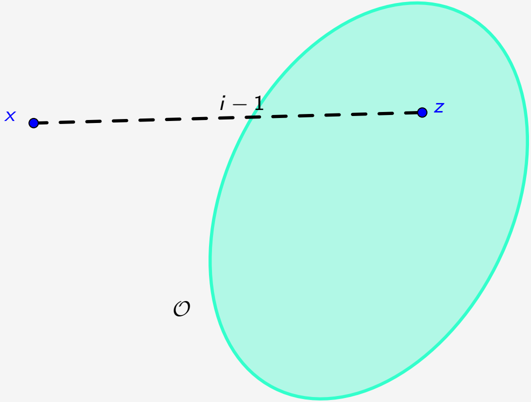
Sketch of proof

Counting z then y:

$$\sum_{z \in \mathcal{O} \cap \Gamma_1(x)} |\Gamma_{i-1}(z) \cap \Gamma_i(x) \cap \mathcal{O}| + \sum_{z \in \mathcal{O} \cap \Gamma_{i-1}(x)} |\Gamma_1(z) \cap \Gamma_i(x) \cap \mathcal{O}|.$$

Sketch of proof

$d(x,z) = i-1, \quad |\Gamma_1(z) \cap \Gamma_i(x) \cap \mathcal{O}| = ?$



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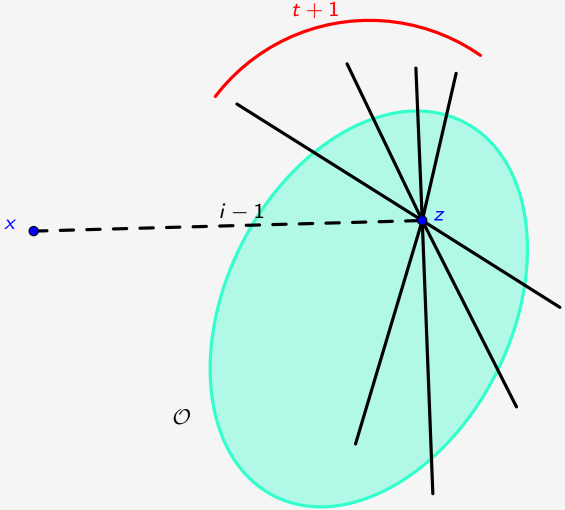

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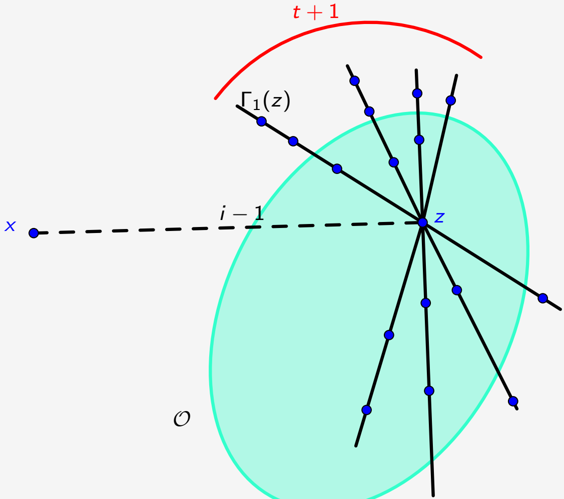

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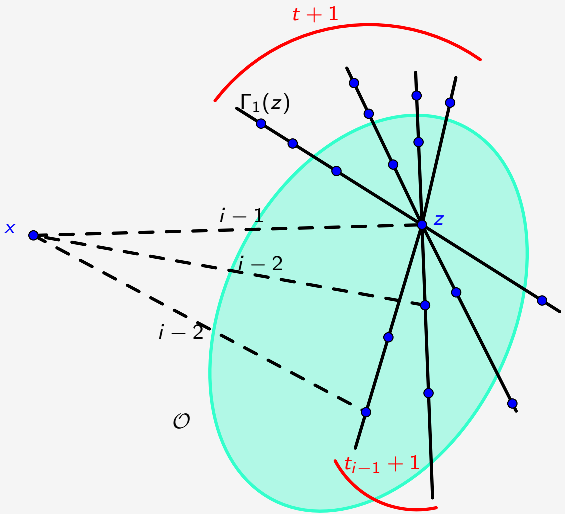

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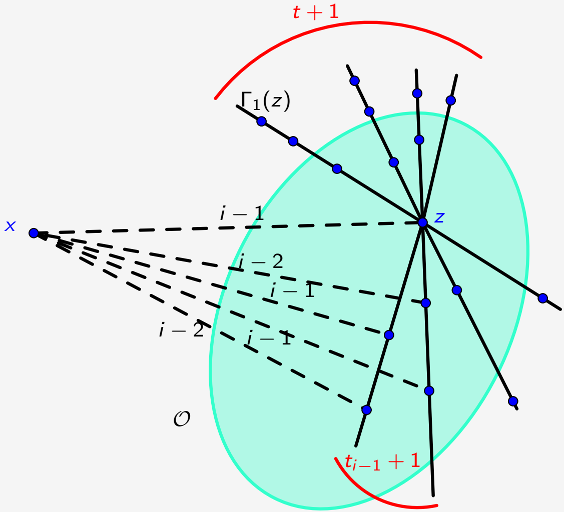

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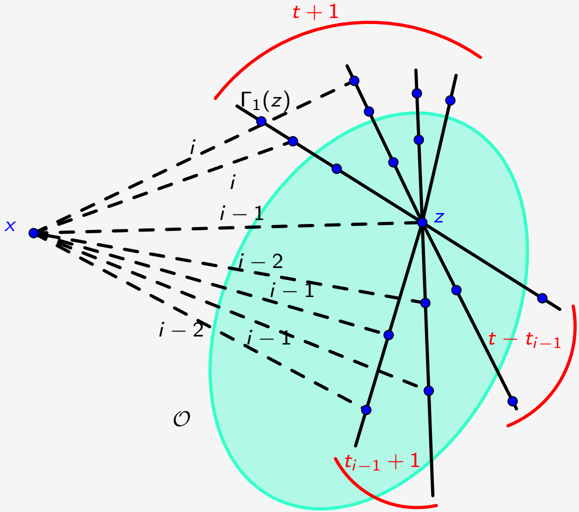

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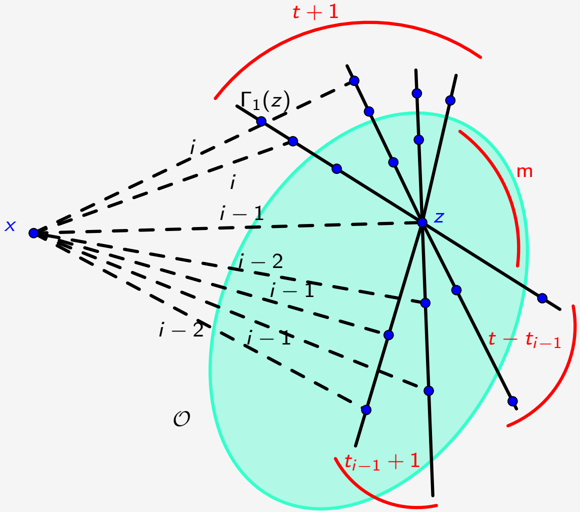
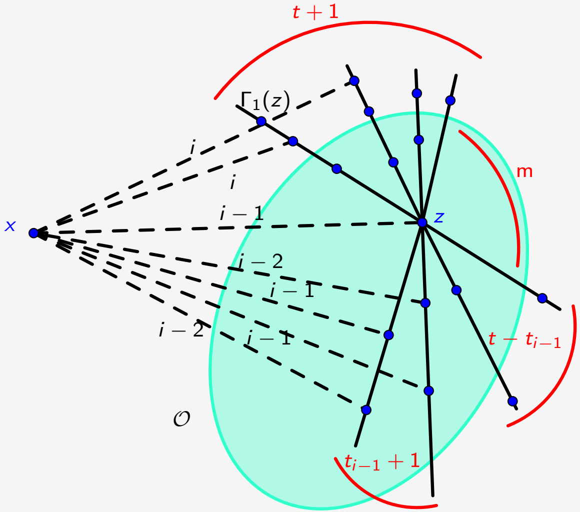


Sketch of proof

$$d(x, z) = i - 1, \quad |\Gamma_1(z) \cap \Gamma_i(x) \cap \mathcal{O}| = (t - t_{i-1})(m - 1)$$



Sketch of proof

$$d(x,z) = 1, \quad |\Gamma_{i-1}(z) \cap \Gamma_i(x) \cap \mathcal{O}| = ?$$

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Harder - can't just take points on lines through z.

Sketch of proof

$$d(x,z)=1, \quad |\Gamma_{i-1}(z)\cap\Gamma_i(x)\cap\mathcal{O}|=?$$

Harder - can't just take points on lines through z.

Obtain an iterative formula,

$$|\Gamma_{i-1}(z)\cap\Gamma_i(x)\cap\mathcal{O}|=p^1_{i-1,i-2}\frac{t-t_{i-1}}{t_{i-1}+1}m-\frac{t-t_{i-1}}{t_{i-1}+1}|\Gamma_{i-2}(z)\cap\Gamma_{i-1}(x)\cap\mathcal{O}|$$

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And after manipulation:

$$\begin{aligned}
|\Gamma_{i-1}(z)\cap\Gamma_i(x)\cap\mathcal{O}| &= p^1_{i,i-1}\left(\frac{m-s\left(-\frac{1}{s}\right)^i(-m+s+1)}{s+1}\right)\\
&= \frac{k_{i-1}(t-t_{i-1})}{t+1}\left(\frac{m-s\left(-\frac{1}{s}\right)^i(-m+s+1)}{s+1}\right)\\
&= \frac{k_{i-1}(t-t_{i-1})}{s^{i-1}(t+1)}\left(\frac{m}{s+1}\left(s^{i-1}+(-1)^{i-2}\right)+(-1)^{i-1}\right).
\end{aligned}$$

Sketch of proof

Summing the terms:

$$\sum_{z \in \mathcal{O} \cap \Gamma_1(x)} |\Gamma_{i-1}(z) \cap \Gamma_i(x) \cap \mathcal{O}| + \sum_{z \in \mathcal{O} \cap \Gamma_{i-1}(x)} |\Gamma_1(z) \cap \Gamma_i(x) \cap \mathcal{O}|$$

$$= \frac{mk_{i-1}(t - t_{i-1})}{s+1} \left(2m - 1 + \left(\frac{-1}{s} \right)^{i-1} (s - 2m + 2) \right).$$

Sketch of proof

Equating the two counts "y then z" and "z then y":

$$\left(1-\left(-\frac{1}{s}\right)^i\right)\frac{2(t_i+1)ms-(s+1-2m)\left((t_{i-1}+1)s^2+(-1)^i s^i\right)}{(t_i+1)(s+1)}$$

$$=2m-1+\left(\frac{-1}{s}\right)^{i-1}(s-2m+2).$$

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$$=2m-1+\left(\frac{-1}{s}\right)^{i-1}(s-2m+2).$$

Making use of the assumption on t_i+1, solving for m yields:

$$m=(s+1)/2.$$

Additional results

Theorem (Bamberg, JL, Lee)

The only nontrivial m-ovoids of $DQ(2d,q)$, $DW(2d-1,q)$ and $DH(2d-1,q^2)$, for $d \geqslant 3$, are hemisystems.

Computational results:

- $DW(5,3)$ no hemisystems (De Bruyn, Vanhove)
  - $\implies DH(5,9)$ no hemisystems
- $DW(5,5)$ no hemisystems (Bamberg, JL, Lee)
  - $\implies DH(5,25)$ no hemisystems
- $DQ(6,3)$ unique hemisystem (De Bruyn, Vanhove)
- $DQ(6,5)$ multiple non-isomorphic hemisystems (Bamberg, JL, Lee)

Conjectures

Conjecture

There are no hemisystems of $DW(5, q)$, for all prime powers q.

Conjecture

There exists a hemisystems of $DQ(6, q)$, for all prime powers q.

Danke!