Pre-sympletic Semifields

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Joint work with Guglielmo Lunardon

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The **kernel** of $\mathcal{U}(\mathcal{S})$ is the field of all $\mathbb{F}_q$-endomorphisms of V fixing all the element of $\mathcal{S}$.

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A spread $\mathcal{S}$ of $PG(2n-1, q)$ is a **sympletic spread** with respect to a non singular sympletic polarity $\perp$ of $PG(2n-1, q)$ when all subspaces of $\mathcal{S}$ are totally isotropic with respect to $\perp$.

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An algebra satisfying all the axioms of semifield except (4) is called a **presemifield**.

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$$\lambda(a \star b) = (\lambda a) \star b = a \star (\lambda b)\ \forall a, b \in \mathbb{S}, \forall \lambda \in \mathbb{F}_s$$

If L_b is the map of $\mathbb{S}$ in itself defined by $L_b : a \to a \star b$, let $\mathbb{F}_s$ be the maximal subfield of $\mathbb{S}$ s. t. all the L_b are $\mathbb{F}_s$-linear maps.

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- $\mathbb{S}$ and $\mathbb{S}'$ are **isotopic** $\iff$ $\mathcal{U}(\mathbb{S}) \cong \mathcal{U}(\mathbb{S}')$

Adjoint map and dual semifield

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The **dual** of $\mathbb{S}$ is $\mathbb{S}^d = (\mathbb{F}_{q^n}, +, \circ)$ defined by

$$x \circ y = y \star x$$

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$$\Sigma = \{(x, x^q, \ldots, x^{q^{n-1}}) \in PG(n-1, q^n) \,|\, x \in \mathbb{F}_{q^n}\}$$

Geometric spread set (with respect to Σ) and sympletic cone

Let $\perp$ be the polarity of $\mathbb{P} = PG(n-1, q^n)$ defined by the bilinear form

$$b(\mathbf{x}, \mathbf{y}) = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{F}_{q^n}^n$$

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A **geometric spread set** Γ of $\mathbb{P}$ (with respect to Σ) is a $\mathbb{F}_s$-linear set of rank nh, $q = s^h$, s. t.

$$\Gamma^\star = \{(y_1, y_2, \ldots, y_n) \in \mathbb{P} \mid Tr_s(b(\mathbf{x}, \mathbf{y})) = 0 \ \forall \mathbf{x} \in \Gamma\}$$

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The $\mathbb{F}_q$-linear set of rank $\frac{n(n+1)}{2}$ of $\mathbb{P}$

$$\Omega = \{(\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{P} \mid i = 2, 3, \ldots, n : \beta_i^{q^{n-i+1}} = \beta_{n-i+2}\}$$

is called **sympletic cone** .

Geometric construction

Let $\Gamma = \{(x, f(x)^q, f(x)) : x \in \mathbb{F}_{q^3}\}$ be a geometric spread set of $\mathbb{P} = PG(2, q^3)$ (with respect to Σ).

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where $\tilde{f}(y)$ is the adjoint map of f with respect to the quadratic form $q(\alpha) = Tr_s(\alpha^2)$.

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Let $\Delta := \langle \Sigma, \Omega^\star \rangle$ be, Let $\Delta := \langle \Sigma, \Omega^\star \rangle$ be,

$$P \in (\Delta \setminus \Sigma) \cap r \iff P(x\alpha, 0, x\alpha^{q^2} + x^{q^2}\alpha), \; x \in \mathbb{F}_{q^3}, \, \alpha \in \mathbb{F}^*_{q^3}$$

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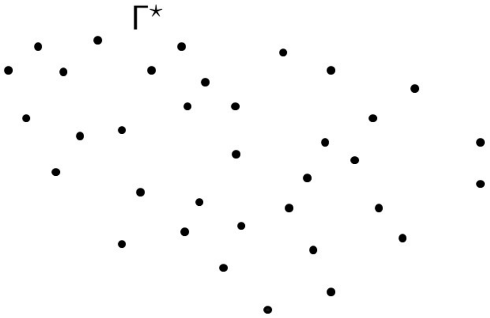
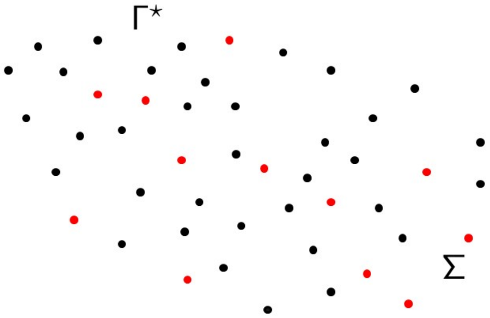
Then

$$\begin{vmatrix} x\alpha & x\alpha^{q^2} + x^{q^2}\alpha \\ -\tilde{f}(y) & y \end{vmatrix} \neq 0$$

for all $\alpha, x, y \in \mathbb{F}^*_{q^3}$.



$\Gamma^\star$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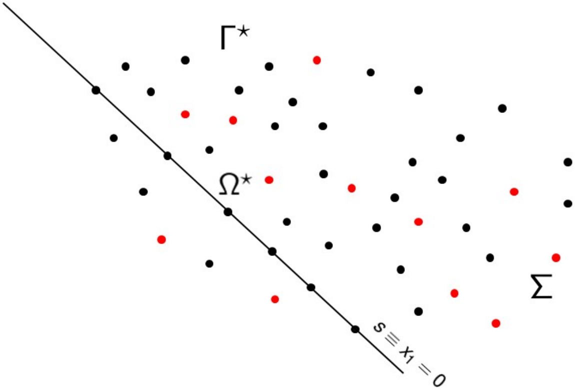
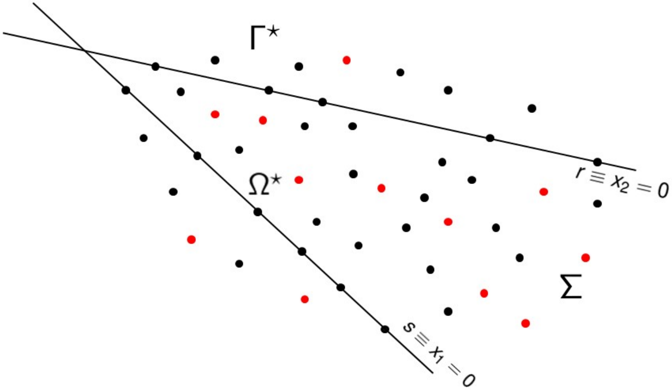
Σ


Γ*
Ω*
Σ
s ≡ x1 = 0



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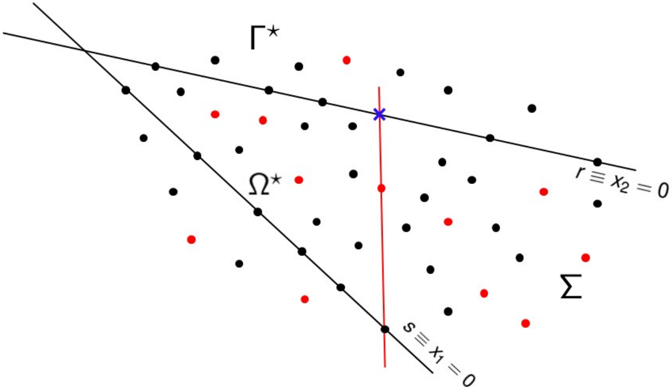


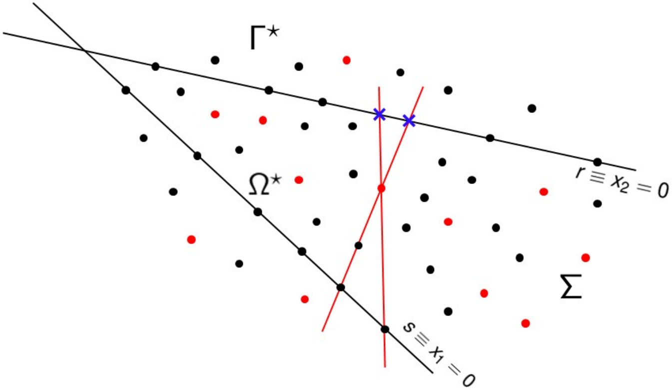
Pre-sympletic Semifields

G. Longobardi, G. Lunardon

Preliminaries

Geometric construction

Results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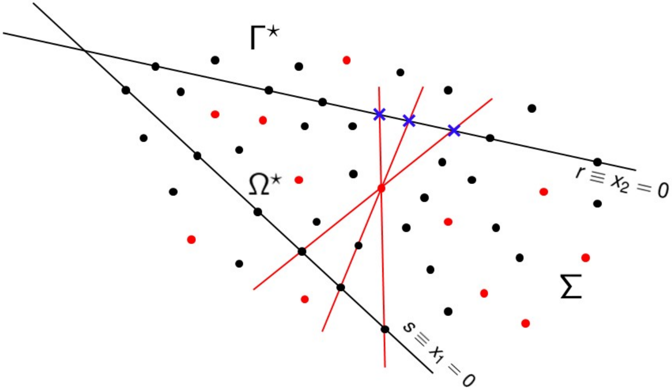
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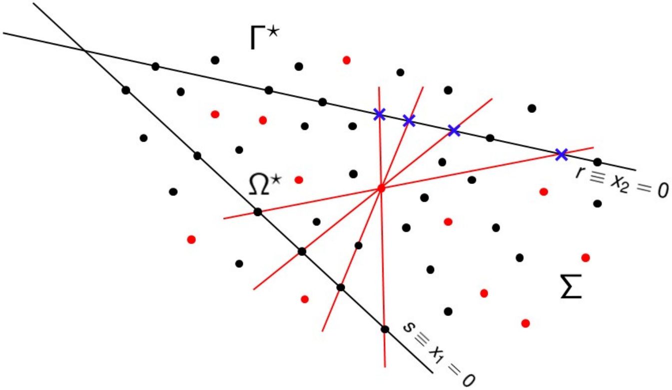
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Theorem [L., Lunardon]

Γ is a geometric spread set if and only if, $\mathbb{S}_\alpha = (\mathbb{F}_{q^3}, +, \star_\alpha)$ with multiplication defined by

$$x \star_\alpha y = (\tilde{f}(y)\alpha^{q^2-1} + y)x + \tilde{f}(y)x^{q^2}$$

is, for all $\alpha \in \mathbb{F}^*_{q^3}$, a presemifield central over $\mathbb{F}_s$ and the kernel of the plane $\mathcal{U}_\alpha$ contains $\mathbb{F}_q$.

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Corollary [L., Lunardon]

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$$x \cdot y = xy + f(y)^q x^q + f(y)x^{q^2}$$

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Remark: $|\{\mathbb{S}_\alpha\}_{\alpha \in \mathbb{F}^*_{q^3}}| = q^2 + q + 1$.

Even characteristic Results

$\Gamma_f = \{(x, f(x)^q, f(x)) : x \in \mathbb{F}_{q^3}\}$, with $q = 2^h$,

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Known semifields of order q^3, $q = 2^h$, with geometric spread sets on a line are:

- **Fields**;
- **Generalized Twisted fields** with center of order q.

Vielen Dank für Ihre Aufmerksamkeit!