Classes and equivalence of linear sets in $PG(1, q^n)$

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Joint work with B. Csajbók and O. Polverino

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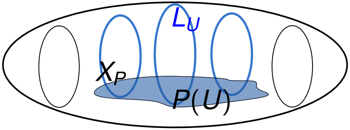

$$L_U = \{P \in \Lambda \colon X_P \cap P(U) \neq \emptyset\}$$

Definition of linear set

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$L \subseteq \Lambda$ is an $\mathbb{F}_q$-linear set if

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$\dim_{\mathbb{F}_q} U = k \quad \Rightarrow L_U$ is an $\mathbb{F}_q$–linear set of Λ of *rank* k

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- Every projective subspace of $\mathrm{PG}(r-1, q^n)$ is an $\mathbb{F}_{q^n}$-linear set.

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U subspace of V over $\mathbb{F}_q$

- Every projective subspace of $\mathrm{PG}(r-1, q^n)$ is an $\mathbb{F}_{q^n}$-linear set.
- Every subgeometry $\mathrm{PG}(s, q)$ of $\mathrm{PG}(r-1, q^n)$ $(s < r$ and $n > 1)$ is an $\mathbb{F}_q$-linear set.

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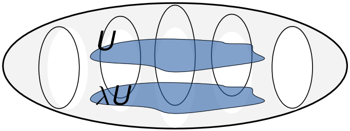
$\forall\lambda \in \mathbb{F}_{q^n} \Rightarrow L_{\lambda U} = L_U$



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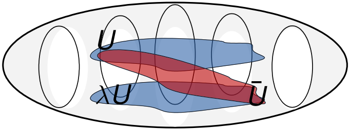

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An $\mathbb{F}_q$-linear set and the vector space defining it must be considered as coming in pair

Linear sets and applications

- Blocking sets in finite projective spaces
- Two intersection sets in finite projective spaces
- Translation spreads of the Cayley Generalized Hexagon
- Translation ovoids of polar spaces
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- Finite semifields and finite semifield planes

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[M. Lavrauw: Scattered spaces in Galois Geometry, *Contemporary Developments in Finite Fields and Applications*, 2016, 195–216.]

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Example

$\mathbb{F}_q$-vector subspaces of $W = V(r, q^n)$ of rank $k \geq rn - n + 1$ determine the whole projective space but there is no semilinear map between two $\mathbb{F}_q$-subspaces with different rank

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Example

$\mathbb{F}_q$-vector subspaces of $W = V(2, q^n)$ of rank $k \geq 2n - n + 1$ determine the whole projective space but there is no semilinear map between two $\mathbb{F}_q$-subspaces with different rank

Equivalence issue linear sets of rank n in $PG(1, q^n)$

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Question

Is it possible to have an $\mathbb{F}_q$-subspace of rank different from n defining L_U?

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Theorem (Ball, Blokhuis, Brouwer, Storme, Szőnyi, 1999 - Ball, 2003)

Let f be a function from $\mathbb{F}_q$ to $\mathbb{F}_q$, $q = p^h$, and let N be the number of directions determined by f. Let $s = p^e$ be maximal such that any line with a direction determined by f that is incident with a point of the graph of f is incident with a multiple of s points of the graph of f. Then one of the following holds.

1. $s = 1$ *and* $(q + 3)/2 \leq N \leq q + 1$,
2. $e|h$, $q/s + 1 \leq N \leq (q - 1)/(s - 1)$,
3. $s = q$ *and* $N = 1$.

Moreover if $s > 2$, then the graph of f is $\mathbb{F}_s$-linear.

Equivalence between $\mathbb{F}_q$-linear sets of $\mathrm{PG}(1,q^n)$ of rank n

$\mathbb{F}_{q^t}$ is the maximum field of linearity of L_U if $t|n$ and L_U is an $\mathbb{F}_{q^t}$-linear set

Theorem (B. Csajbók, G.M., O. Polverino)

Let L_U be an $\mathbb{F}_q$-linear set of $\mathrm{PG}(W,\mathbb{F}_{q^n})=\mathrm{PG}(1,q^n)$ of rank n. The maximum field of linearity of L_U is $\mathbb{F}_{q^d}$, where

$$d=\min\{\dim_q(U\cap\langle\mathbf{u}\rangle_{q^n})\colon \mathbf{u}\in U\setminus\{\mathbf{0}\}\}.$$

If the maximum field of linearity of L_U is $\mathbb{F}_q$, then the rank of L_U as an $\mathbb{F}_q$-linear set is uniquely defined, i.e. for each $\mathbb{F}_q$-subspace V of W if $L_U=L_V$, then $\dim_q(V)=n$.

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The $\Gamma\mathrm{L}$-class of a linear set is a $\mathrm{P\Gamma L}$-invariant

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Simple linear sets have been also studied by Csajboók-Zanella and Van de Voorde

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Example

Subgeometries (trivial).

Simple linear sets

Definition

An $\mathbb{F}_q$-linear set L of $\mathrm{PG}(r-1, q^n) = \mathrm{PG}(W, \mathbb{F}_{q^n})$ of rank k with maximum field of linearity $\mathbb{F}_q$ is called *simple* if all the $\mathbb{F}_q$-subspaces of W of dimension k defining L are in the same orbit of $\Gamma\mathrm{L}(r, q^n)$.

Example

Subgeometries (trivial).

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Let L_U and L_V be two $\mathbb{F}_q$-linear sets of $PG(r-1, q^n)$ of rank k.

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Example (Bonoli-Polverino, 2005)

$\mathbb{F}_q$-linear sets of $\mathrm{PG}(2,q^n)$ of rank $n+1$ with $(q+1)$-secants are simple. This allowed a complete classification of $\mathbb{F}_q$-linear blocking sets in $\mathrm{PG}(2,q^4)$.

Non-simple $\mathbb{F}_q$-linear sets of $\mathrm{PG}(1, q^n)$ of rank n

Example (Csajbók-Zanella, 2016)

Linear sets of pseudoregulus type of $\mathrm{PG}(1, q^n)$

$$L_U = \{\langle (x, x^{q^s}) \rangle \colon x \in \mathbb{F}_{q^n}^*\}, \qquad \gcd(s, n) = 1$$

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It is not hard to find non-simple linear sets!

Dual of a linear set

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Up to projective equivalence such a linear set does not depend on τ

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If τ is symplectic then $L_U = L_U^\tau = L_{U^\perp}$

Dual of a linear set

In practice:

$$L_U, \quad U := U_f = \{(x, f(x)) \colon x \in \mathbb{F}_{q^n}\},$$

for some q-polynomial $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$, $a_i \in \mathbb{F}_{q^n}$

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Hence, usually, the $\Gamma\mathrm{L}$-class of L_U is at least 2, i.e. L_U is non-simple

Non-simple linear sets of rank n in $\mathrm{PG}(1, q^n)$

Example (Csajbók-Zanella, 2016)

$\mathbb{F}_q$-linear sets of $\mathrm{PG}(1, q^n)$ of psudoregulus type

$$L_U = \{\langle (x, x^{q^s}) \rangle : x \in \mathbb{F}_{q^n}^*\}, \qquad \gcd(s, n) = 1$$

The $\Gamma\mathrm{L}$-class of L_U is $\varphi(n)/2$. Hence, for $n \geq 5$ and $n = 6$, L_U is not simple.

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Proposition (Csajbók-G.M.-Polverino)

The $\mathbb{F}_q$-linear sets of $\mathrm{PG}(1, q^n)$ introduced by Lunardon-Polverino (2001)

$$L_U = \{\langle (x, \delta x^q + x^{q^{n-1}}) \rangle : x \in \mathbb{F}_{q^n}^*\}, \qquad n > 3, q \geq 3$$

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Other examples in $\mathrm{PG}(1, q^n)$, $n \in \{6, 8\}$ (Csajbók-G.M.-Polverino-Zanella, Csajbók-G.M.-Zullo)

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Simple $\mathbb{F}_q$-linear sets of $\mathrm{PG}(1, q^n)$ of rank n

Question

Is it possible to find a simple $\mathbb{F}_q$-linear set of rank n in $\mathrm{PG}(1, q^n)$ <u>for each</u> n?

Simple $\mathbb{F}_q$-linear sets of $\mathrm{PG}(1, q^n)$ of rank n

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Lemma

Let $f(x)=\sum_{i=0}^{n-1} a_i x^{q^i}$ *and* $g(x)=\sum_{i=0}^{n-1} b_i x^{q^i}$ *be two q-polynomials over* $\mathbb{F}_{q^n}$*, such that* $\boxed{L_f = L_g}$*, i.e.*

$$\left\{\frac{f(x)}{x} : x \in \mathbb{F}_{q^n}^*\right\} = \left\{\frac{g(x)}{x} : x \in \mathbb{F}_{q^n}^*\right\}.$$

Then

$$a_0 = b_0, \tag{1}$$

and for $k=1,2,\ldots,n-1$ *it holds that*

$$a_k a_{n-k}^{q^k} = b_k b_{n-k}^{q^k}, \tag{2}$$

for $k=2,3,\ldots,n-1$ *it holds that*

$$a_1 a_{k-1}^{q} a_{n-k}^{q^k} + a_k a_{n-1}^{q} a_{n-k+1}^{q^k} = b_1 b_{k-1}^{q} b_{n-k}^{q^k} + b_k b_{n-1}^{q} b_{n-k+1}^{q^k}. \tag{3}$$

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Theorem

Let $T = \{(x, Tr_{q^n/q}(x)) : x \in \mathbb{F}_{q^n}\} \subset \mathrm{PG}(1, q^n) = \mathrm{PG}(W, \mathbb{F}_{q^n})$. For each $\mathbb{F}_q$-subspace U of W it turns out $L_U = L_T$ only if $T = \lambda U$ for some $\lambda \in \mathbb{F}_{q^n}^*$.

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Summing up:

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Question

What happens for $n = 4$?

$\mathbb{F}_q$-linear sets of $\mathrm{PG}(1, q^4)$ of rank 4

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Linear sets of rank 4 of $\mathrm{PG}(1, q^4)$*, with maximum field of linearity* $\mathbb{F}_q$*, are simple.*

$\mathbb{F}_q$-linear sets of $\mathrm{PG}(1, q^4)$ of rank 4

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Sketch of Proof.

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Sketch of Proof.

1. Simplicity is $\mathrm{P\Gamma L}$-invariant, so we can consider linear sets of type $L_f = L_{U_f}$, $U_f = \{(x, f(x)) : \ x \in \mathbb{F}_{q^4}\}$, with $f(x) = \sum_{i=0}^{4} a_i x^{q^i}$

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2. Let $g(x) = \sum_{i=0}^{4} b_i x^{q^i}$ such that $L_f = L_g$. By technical lemma we have

$$a_0 = b_0,\ a_1 a_3^q = b_1 b_3^q,\ a_2^{q^2+1} = b_2^{q^2+1},\ a_1^{q+1} a_2^{q^2} + a_2 a_3^{q+q^2} = b_1^{q+1} b_2^{q^2} + b_2 b_3^{q+q^2}$$

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$$a_0 = b_0,\; a_1 a_3^q = b_1 b_3^q,\; a_2^{q^2+1} = b_2^{q^2+1},\; a_1^{q+1} a_2^{q^2} + a_2 a_3^{q+q^2} = b_1^{q+1} b_2^{q^2} + b_2 b_3^{q+q^2}$$

3. Also, for $n = 4$, we have

$$N_{q^n/q}(a_1) + N_{q^n/q}(a_2) + N_{q^n/q}(a_3) + a_1^{1+q^2} a_3^{q+q^3} + a_1^{q+q^3} a_3^{1+q^2} + Tr_{q^4/q}\left(a_1 a_2^{q+q^2} a_3^{q^3}\right) =$$

$$N_{q^n/q}(b_1) + N_{q^n/q}(b_2) + N_{q^n/q}(b_3) + b_1^{1+q^2} b_3^{q+q^3} + b_1^{q+q^3} b_3^{1+q^2} + Tr_{q^4/q}\left(b_1 b_2^{q+q^2} b_3^{q^3}\right)$$

$\mathbb{F}_q$-linear sets of $\mathrm{PG}(1, q^4)$ of rank 4

Theorem

Linear sets of rank 4 of $\mathrm{PG}(1, q^4)$, with maximum field of linearity $\mathbb{F}_q$, are simple.

Sketch of Proof.

1. Simplicity is $\mathrm{P\Gamma L}$-invariant, so we can consider linear sets of type $L_f = L_{U_f}$, $U_f = \{(x, f(x)) : x \in \mathbb{F}_{q^4}\}$, with $f(x) = \sum_{i=0}^4 a_i x^{q^i}$
4. Let $g(x) = \sum_{i=0}^4 b_i x^{q^i}$ such that $L_f = L_g$. Then there exists $\lambda \in \mathbb{F}_{q^4}^*$ such that

$$U_g = \lambda U_f \quad \text{or} \quad U_g = \lambda U_{\hat{f}}.$$

$\mathbb{F}_q$-linear sets of $\mathrm{PG}(1, q^4)$ of rank 4

Theorem

Linear sets of rank 4 of $\mathrm{PG}(1, q^4)$*, with maximum field of linearity* $\mathbb{F}_q$*, are simple.*

Sketch of Proof.

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Hence the $\Gamma\mathrm{L}$-class of L_f is at most 2.

$\mathbb{F}_q$-linear sets of $\mathrm{PG}(1, q^4)$ of rank 4

Theorem

Linear sets of rank 4 of $\mathrm{PG}(1, q^4)$, with maximum field of linearity $\mathbb{F}_q$, are simple.

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5. Prove that U_f and $U_{\hat{f}}$ are in the same $\Gamma\mathrm{L}(2, q^4)$-orbit.

$\mathbb{F}_q$-linear sets of $\mathrm{PG}(1, q^4)$ of rank 4

6. U_f and $U_{\hat{f}}$ are in the same $\Gamma\mathrm{L}(2, q^4)$-orbit iff there exist $A, B, C, D \in \mathbb{F}_{q^4}$, $AD - BC \neq 0$, and $\sigma = p^k$,

$\mathbb{F}_q$-linear sets of $\mathrm{PG}(1, q^4)$ of rank 4

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$$C + Da_0^\sigma - a_0 A = Ba_0 a_0^\sigma + (Ba_1 a_1^\sigma)^{q^3} + (Ba_2 a_2^\sigma)^{q^2} + (Ba_3 a_3^\sigma)^q,$$

$$Da_1^\sigma - (a_3 A)^q = Ba_0 a_1^\sigma + (Ba_1 a_2^\sigma)^{q^3} + (Ba_2 a_3^\sigma)^{q^2} + (Ba_3 a_0^\sigma)^q,$$

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$\mathbb{F}_q$-linear sets of $\mathrm{PG}(1,q^4)$ of rank 4

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Determine $A,B,C,D \in \mathbb{F}_{q^4}$ and $\sigma = p^k$ is not hard.

$\mathbb{F}_q$-linear sets of $\mathrm{PG}(1, q^4)$ of rank 4

6. U_f and $U_{\hat{f}}$ are in the same $\Gamma\mathrm{L}(2, q^4)$-orbit iff there exist $A, B, C, D \in \mathbb{F}_{q^4}$, $AD - BC \neq 0$, and $\sigma = p^k$, satisfying the following system of four equations

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Determine $A, B, C, D \in \mathbb{F}_{q^4}$ and $\sigma = p^k$ is not hard. The delicate part is to show that

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = AD - BC \neq 0$$

$\mathbb{F}_q$-linear sets of $\mathrm{PG}(1, q^4)$ of rank 4

$\mathbb{F}_q$-linear sets of $\mathrm{PG}(1, q^4)$ of rank 4

$AD - BC \neq 0$ iff

$\mathbb{F}_q$-linear sets of $\mathrm{PG}(1, q^4)$ of rank 4

$AD - BC \neq 0$ iff a given projective subspace H of dimension at least 1 of

$$\Sigma := \{\langle (x, x^q, x^{q^2}, x^{q^3}) \rangle_{q^4} : x \in \mathbb{F}_{q^4}\} = \mathrm{Fix}\,\xi \simeq \mathrm{PG}(3, q) \subset \Sigma^* = \mathrm{PG}(3, q^4)$$

$\mathbb{F}_q$-linear sets of $\mathrm{PG}(1, q^4)$ of rank 4

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is not contained in the quadric of Σ^*

$$Q: \left(\sum_{i=0}^{3} c_i X_i\right)^2 + X_0(X_1 a_3^{2q} + X_2 + X_3 a_1^{2q^3})(N(a_1) - N(a_3))^2 = 0,$$

where

$$c_0 = a_1^{1+q^2+q^3} a_3^q - a_1^{q^3} a_3^{1+q+q^2},$$

$$c_1 = a_3^{2q+q^2+q^3} - a_1^{q+q^3} a_3^{q+q^2},$$

$$c_2 = a_3^{q+q^2+q^3} a_1^{q^2} - a_1^{q+q^2+q^3} a_3^{q^2},$$

$$c_3 = a_1^{q^2+q^3} a_3^{q+q^3} - a_1^{q+q^2+2q^3}.$$

$\mathbb{F}_q$-linear sets of $\mathrm{PG}(1, q^4)$ of rank 4

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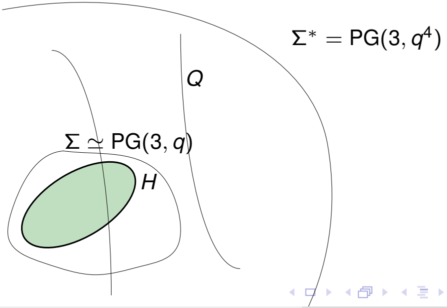
AIM: $H \not\subset Q$



$\mathbb{F}_q$-linear sets of $\mathrm{PG}(1, q^4)$ of rank 4

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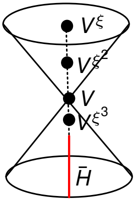
Q has rank 3 or 2.

$\mathbb{F}_q$-linear sets of $\mathrm{PG}(1, q^4)$ of rank 4

If Q has rank 3, then the vertex $V \notin H$. Also if $H \subset Q \Rightarrow H$ is a subline $\Rightarrow V \in \bar{H} \Rightarrow$ $V, V^{\xi}, V^{\xi^2}, V^{\xi^3} \in \bar{H}$, a contradiction.



$H \subset \Sigma = \mathrm{Fix}\,\xi \simeq \mathrm{PG}(3, q) \subset \Sigma^*$

$\mathbb{F}_q$-linear sets of $\mathrm{PG}(1, q^4)$ of rank 4

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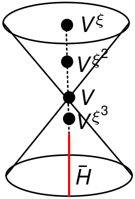
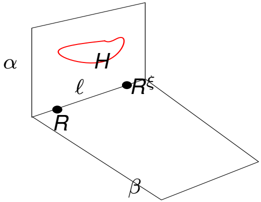


$H \subset \Sigma = \mathrm{Fix}\,\xi \simeq \mathrm{PG}(3, q) \subset \Sigma^*$

If Q has rank 2



There exists a point $R \in \ell \setminus H$, with $\langle R, R^{\xi}, R^{\xi^2}, R^{\xi^3} \rangle = \Sigma^*$. Also $R^{\xi} \in \ell \setminus H$.
If $H \subset Q \Rightarrow H \subset \alpha$ or $H \subset \beta$. Suppose $H \subset \alpha \Rightarrow \alpha = \langle H, R \rangle = \alpha^{\xi}$, a contradiction.

THANK YOU FOR YOUR ATTENTION!