A STEP TOWARDS THE WEAK CYLINDER CONJECTURE

Joint work with J. De Beule (VUB),
J. Demeyer (UGent), P. Sziklai (ELTE Budapest)

Sam Mattheus

September 14, 2017

VRIJE
UNIVERSITEIT
BRUSSEL

The cylinder conjecture(s)

The cylinder conjecture(s)

Let p be a prime.

The cylinder conjecture(s)

Let p be a prime.

Definition

A **cylinder** in $\mathrm{AG}(3, p)$ is the union of p parallel lines.

The cylinder conjecture(s)

Let p be a prime.

Definition

A **cylinder** in $\mathrm{AG}(3, p)$ is the union of p parallel lines.

Example

A plane in $\mathrm{AG}(3, p)$ is a trivial example of a cylinder.

The cylinder conjecture(s) (Ball, 2006)

The cylinder conjecture(s) (Ball, 2006)

Weak Cylinder Conjecture

Let S be a set of p^2 points in $\mathrm{AG}(3, p)$, not determining at least p directions, then S is a cylinder.

The cylinder conjecture(s) (Ball, 2006)

Weak Cylinder Conjecture

Let S be a set of p^2 points in $\mathrm{AG}(3, p)$, not determining at least p directions, then S is a cylinder.

Strong Cylinder Conjecture

Let S be a set of p^2 points in $\mathrm{AG}(3, p)$ such that every plane intersects it in $0 \pmod{p}$ points, then S is a cylinder.

The cylinder conjecture(s) (Ball, 2006)

Weak Cylinder Conjecture

Let S be a set of p^2 points in $\mathrm{AG}(3,p)$, not determining at least p directions, then S is a cylinder.

Strong Cylinder Conjecture

Let S be a set of p^2 points in $\mathrm{AG}(3,p)$ such that every plane intersects it in $0 \pmod p$ points, then S is a cylinder.

Theorem

Let S be a set of p^2 points in $\mathrm{AG}(3,p)$ not determining at least p directions, then every plane intersects it in $0 \pmod p$ points.

The weaker cylinder conjecture

The weaker cylinder conjecture

Weaker Cylinder Conjecture

Let S be a set of p^2 points in $\mathrm{AG}(3, p)$, not determining at least $p + 1$ directions, then S is a cylinder.

The weaker cylinder conjecture

Weaker Cylinder Conjecture

Let S be a set of p^2 points in $\mathrm{AG}(3, p)$, not determining at least $p + 1$ directions, then S is a cylinder.

Theorem (Ball, 2006)

If the set of non-determined directions contains a conic, then S is a plane.

Facts about S

Embed $\mathrm{AG}(3,p)$ in $\mathrm{PG}(3,p)$ with plane at infinity $W=0$. The affine point (x,y,z) then has projective coordinates $(x,y,z,1)$.

Let $S=\{(a_i,b_i,c_i,1) \mid i=1,\ldots,p^2\}$, and suppose it is not a cylinder.

Facts about S

Embed $\mathrm{AG}(3,p)$ in $\mathrm{PG}(3,p)$ with plane at infinity $W=0$. The affine point (x,y,z) then has projective coordinates $(x,y,z,1)$.

Let $S=\{(a_i,b_i,c_i,1) \mid i=1,\ldots,p^2\}$, and suppose it is not a cylinder.

i. S cannot contain any lines (Blokhuis);

Facts about S

Embed $\mathrm{AG}(3,p)$ in $\mathrm{PG}(3,p)$ with plane at infinity $W=0$. The affine point (x,y,z) then has projective coordinates $(x,y,z,1)$.

Let $S=\{(a_i,b_i,c_i,1) \mid i=1,\ldots,p^2\}$, and suppose it is not a cylinder.

i. S cannot contain any lines (Blokhuis);
ii. there exist three pairwise intersecting planes not containing any point of S;

Facts about S

Embed $\mathrm{AG}(3,p)$ in $\mathrm{PG}(3,p)$ with plane at infinity $W=0$. The affine point (x,y,z) then has projective coordinates $(x,y,z,1)$.

Let $S=\{(a_i,b_i,c_i,1) \mid i=1,\ldots,p^2\}$, and suppose it is not a cylinder.

i. S cannot contain any lines (Blokhuis);
ii. there exist three pairwise intersecting planes not containing any point of S;
iii. every plane contains 0 (mod p) points;

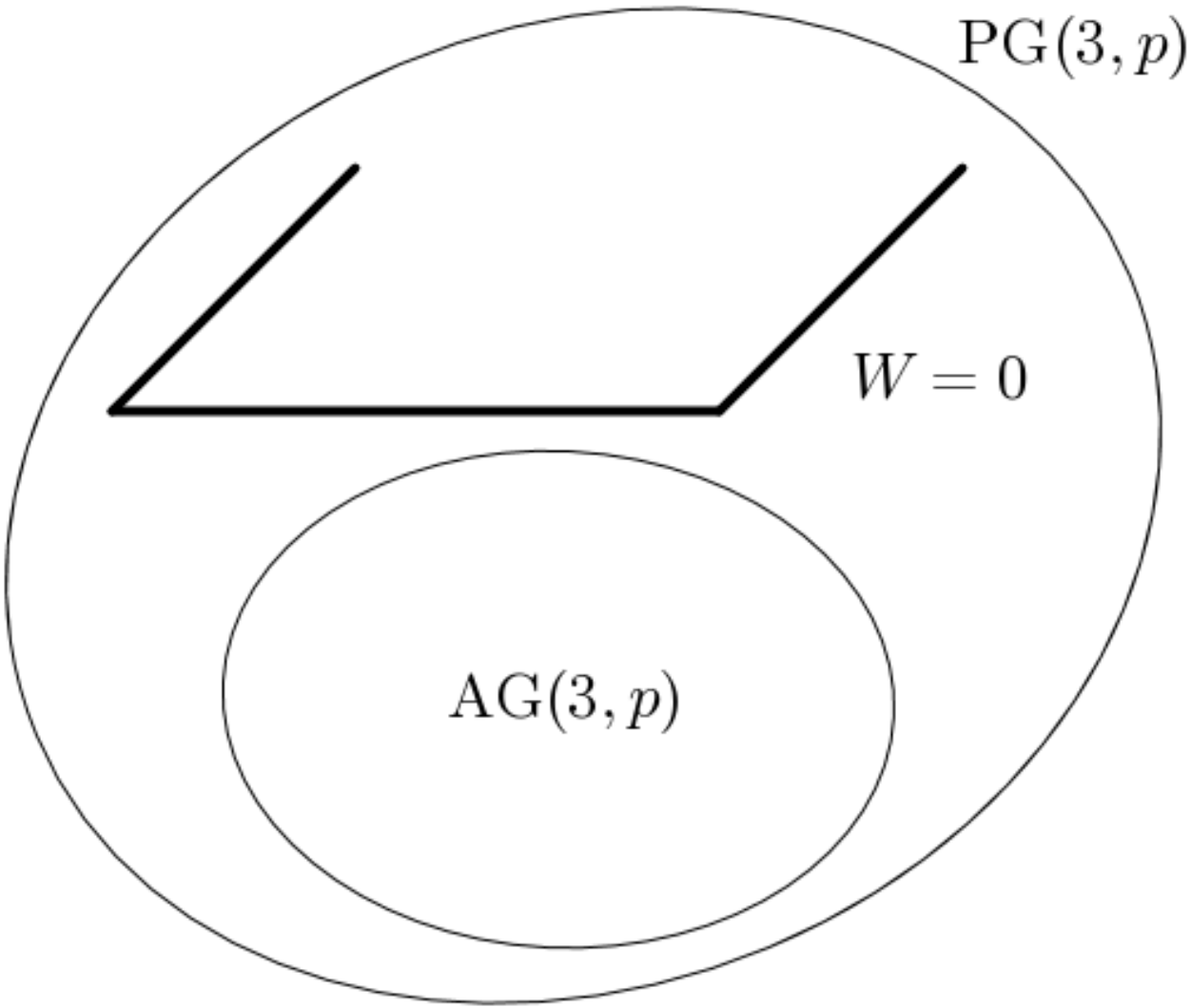
Facts about S

Embed $\mathrm{AG}(3,p)$ in $\mathrm{PG}(3,p)$ with plane at infinity $W=0$. The affine point (x,y,z) then has projective coordinates $(x,y,z,1)$.

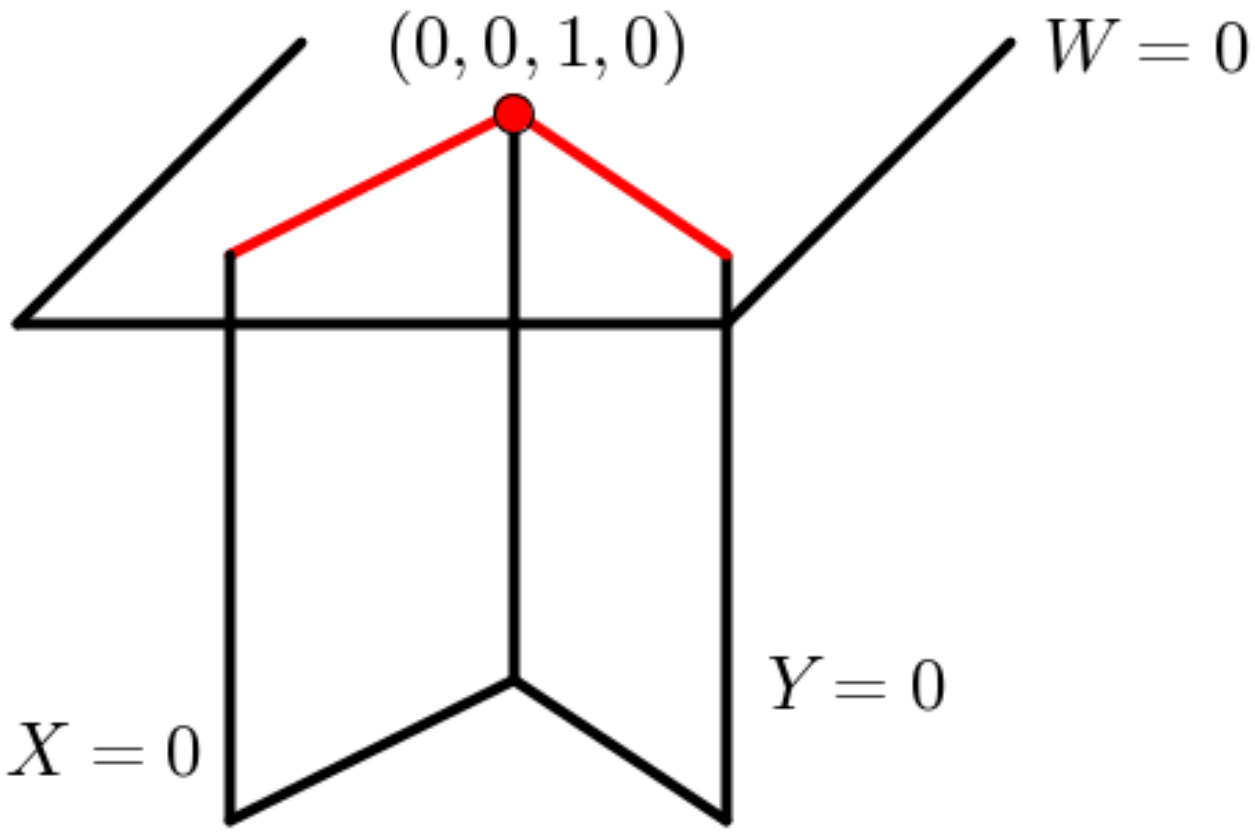
Let $S=\{(a_i,b_i,c_i,1) \mid i=1,\ldots,p^2\}$, and suppose it is not a cylinder.

i. S cannot contain any lines (Blokhuis);
ii. there exist three pairwise intersecting planes not containing any point of S;
iii. every plane contains 0 (mod p) points;
iv. $\sum_{i=1}^{p^2} a_i^k b_i^l = 0$ for all $k+l \leq p$.

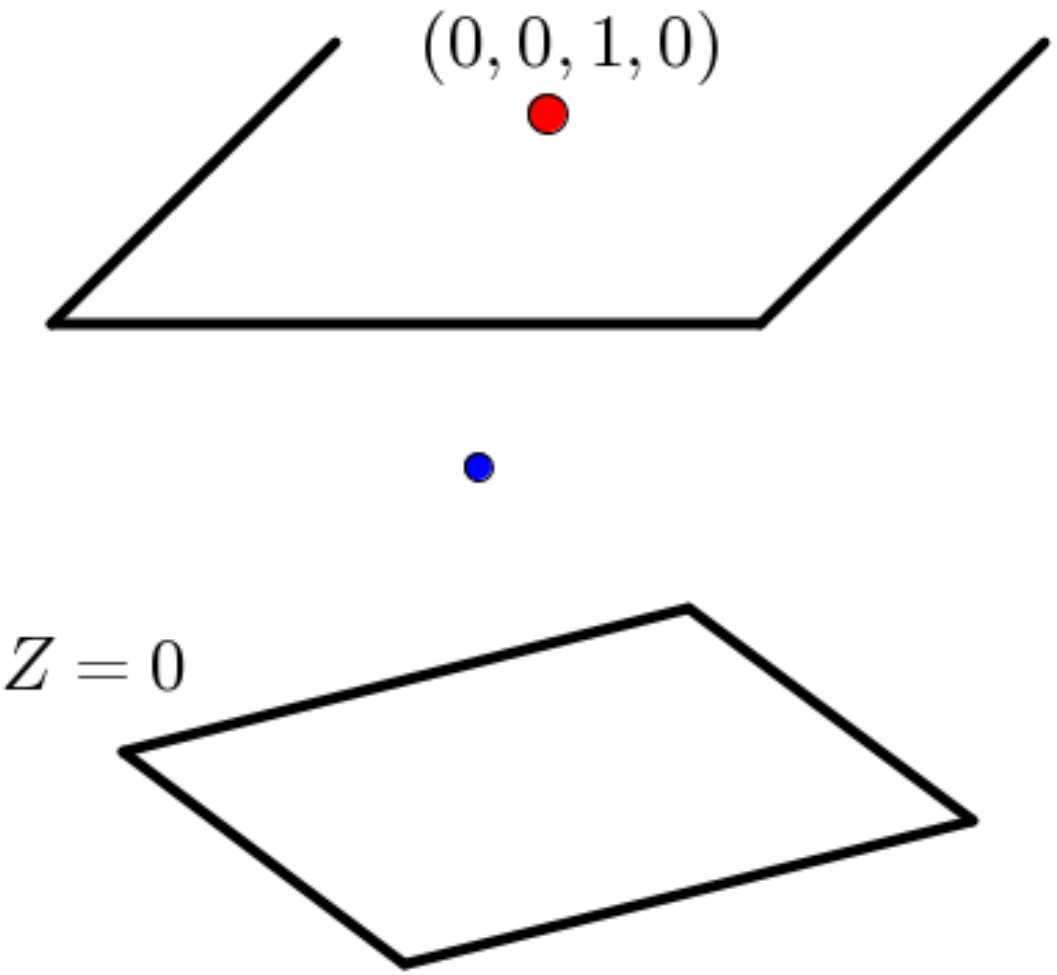
Reduction by projection



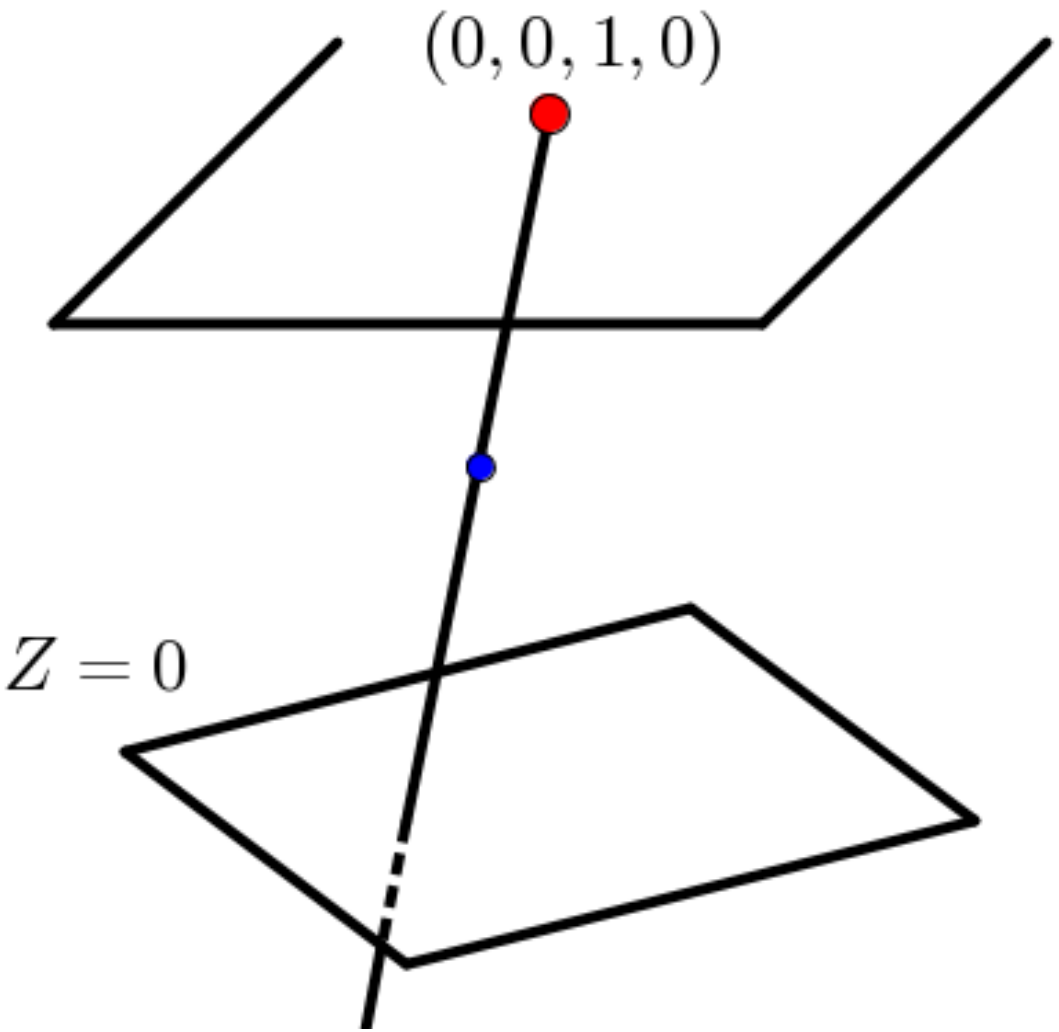
Reduction by projection



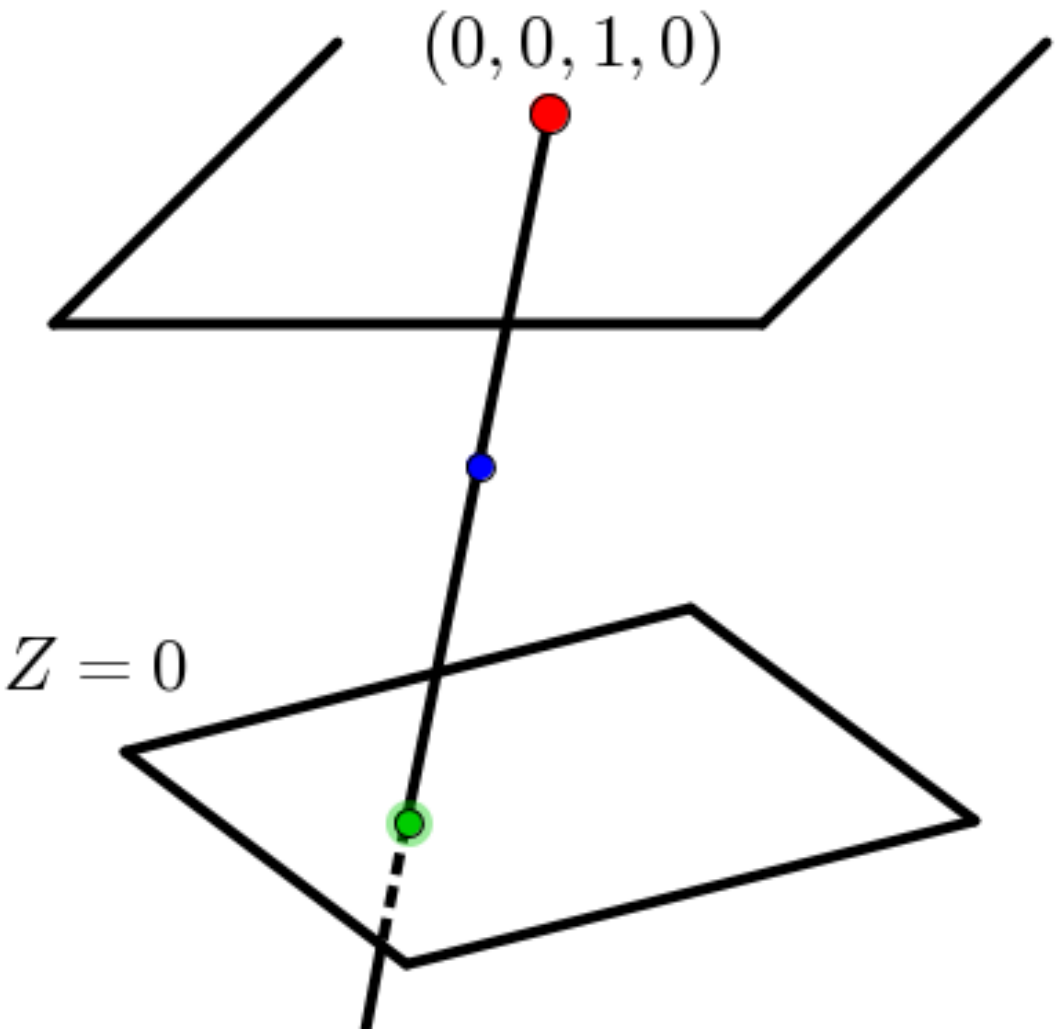
Reduction by projection



Reduction by projection



Reduction by projection



Reduction by projection

i. S cannot contain any lines (Blokhuis);
ii. every plane contains 0 (mod p) points;
iii. the planes $X = 0$ and $Y = 0$ do not contain any point of S;
iv. $\sum_{i=1}^{p^2} a_i^k b_i^l = 0$ for all $k + l \leq p$.

Reduction by projection

i. S cannot contain any lines (Blokhuis);
ii. every plane contains 0 (mod p) points;
iii. the planes $X = 0$ and $Y = 0$ do not contain any point of S;
iv. $\sum_{i=1}^{p^2} a_i^k b_i^l = 0$ for all $k + l \leq p$.

1. Every point has weight at most $p - 1$;

Reduction by projection

i. S cannot contain any lines (Blokhuis);
ii. every plane contains 0 (mod p) points;
iii. the planes $X = 0$ and $Y = 0$ do not contain any point of S;
iv. $\sum_{i=1}^{p^2} a_i^k b_i^l = 0$ for all $k + l \leq p$.

1. Every point has weight at most $p - 1$;
2. on every line the total weight is a multiple of p;

Reduction by projection

i. S cannot contain any lines (Blokhuis);
ii. every plane contains 0 (mod p) points;
iii. the planes $X = 0$ and $Y = 0$ do not contain any point of S;
iv. $\sum_{i=1}^{p^2} a_i^k b_i^l = 0$ for all $k + l \leq p$.

1. Every point has weight at most $p - 1$;
2. on every line the total weight is a multiple of p;
3. every point on the X-axis or the Y-axis has weight 0;

Reduction by projection

i. S cannot contain any lines (Blokhuis);
ii. every plane contains 0 (mod p) points;
iii. the planes $X = 0$ and $Y = 0$ do not contain any point of S;
iv. $\sum_{i=1}^{p^2} a_i^k b_i^l = 0$ for all $k + l \leq p$.

1. Every point has weight at most $p - 1$;
2. on every line the total weight is a multiple of p;
3. every point on the X-axis or the Y-axis has weight 0;
4. the weight distribution is given by a function $w(X, Y)$ of degree at most $p - 3$.

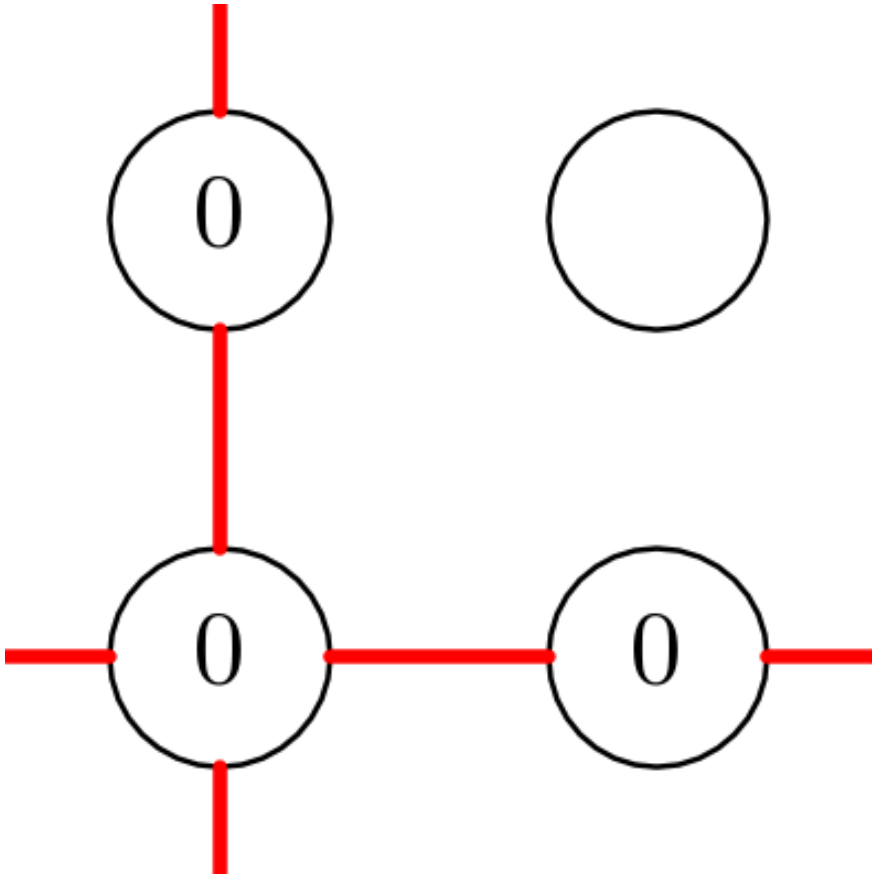
Reduction by projection

i. S cannot contain any lines (Blokhuis);
ii. every plane contains 0 (mod p) points;
iii. the planes $X = 0$ and $Y = 0$ do not contain any point of S;
iv. $\sum_{i=1}^{p^2} a_i^k b_i^l = 0$ for all $k + l \leq p$.

1. Every point has weight at most $p - 1$;
2. on every line the total weight is a multiple of p;
3. every point on the X-axis or the Y-axis has weight 0;
4. the weight distribution is given by a function $w(X, Y) = XYg(X, Y)$, where g is of degree at most $p - 5$.

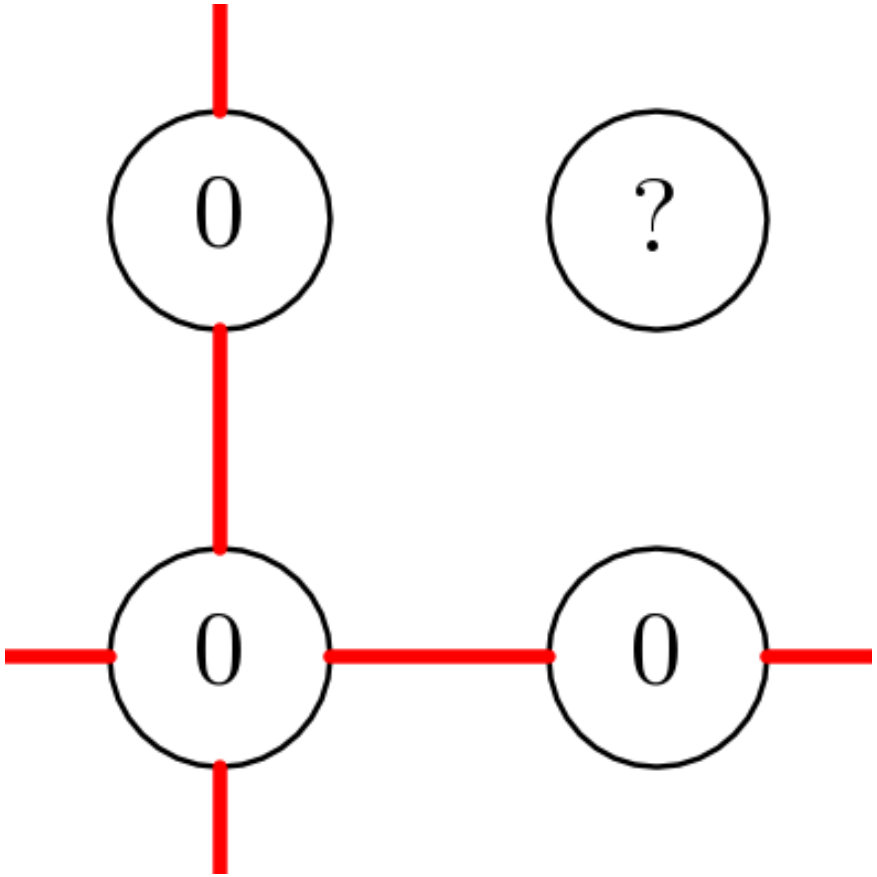
Example

$p = 2$



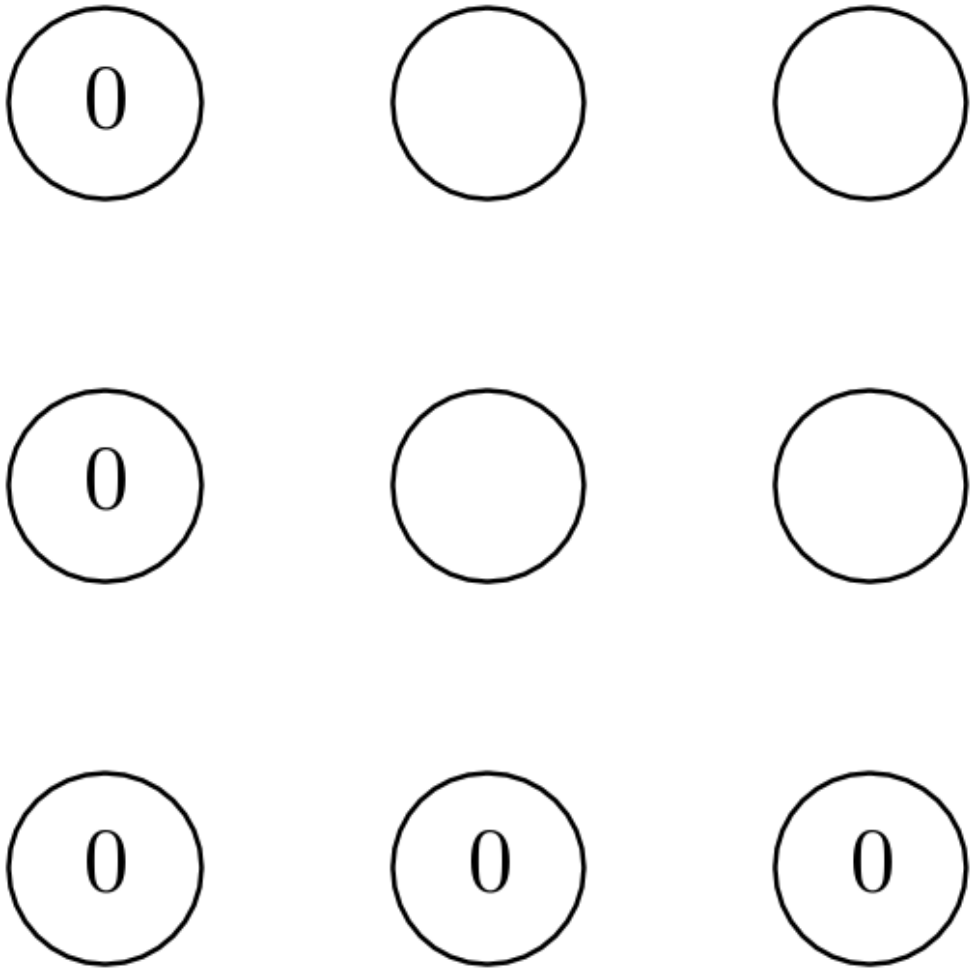
Example

$p = 2$



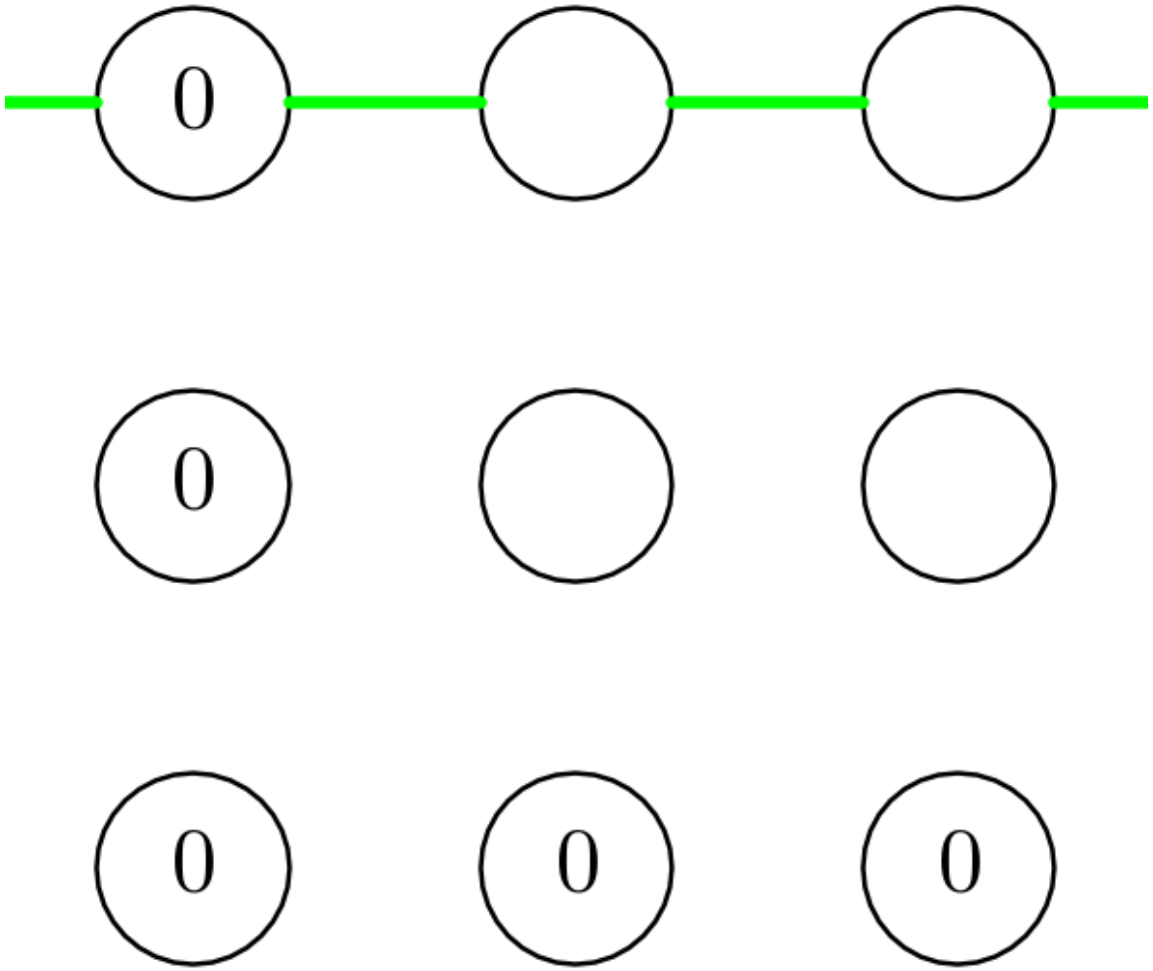
Example

$p = 3$



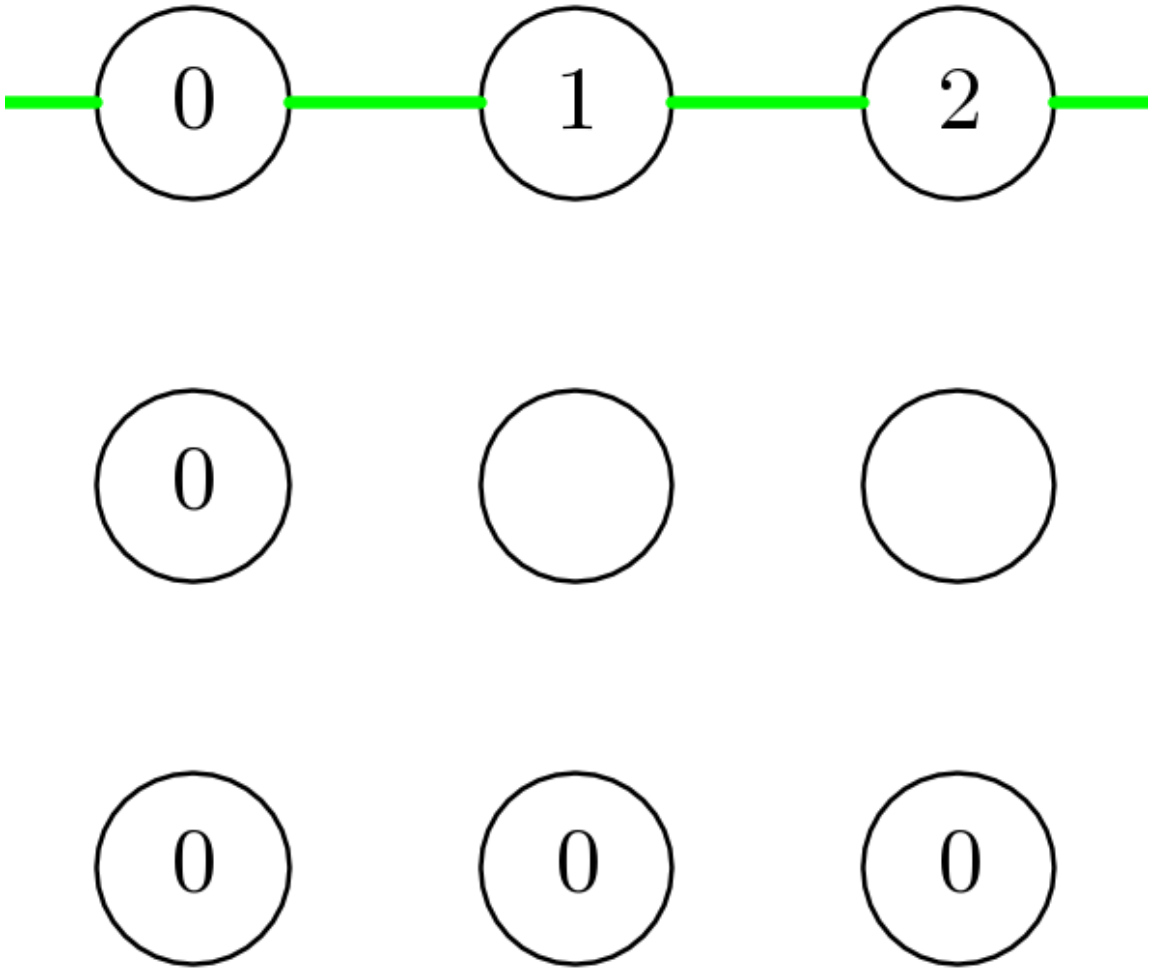
Example

$p = 3$



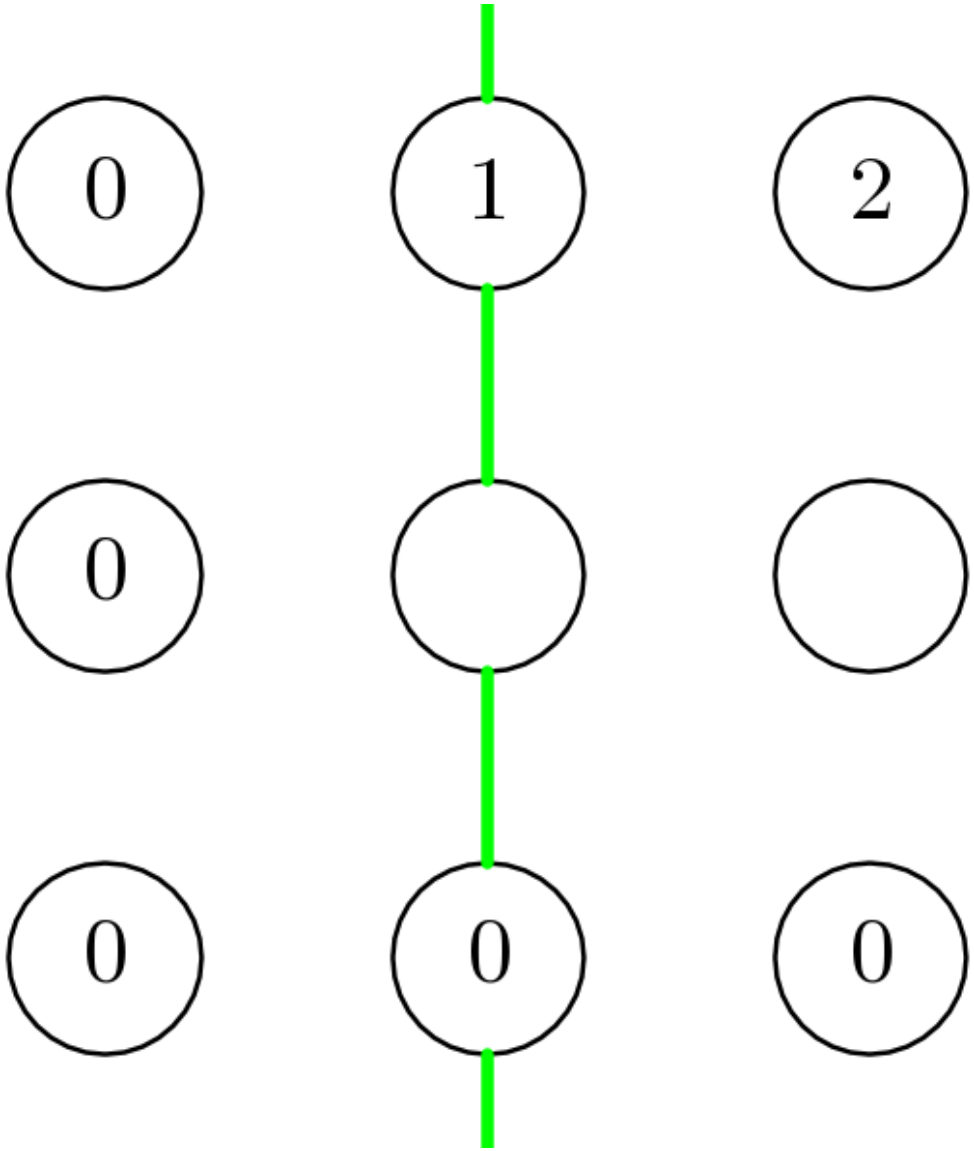
Example

$p = 3$



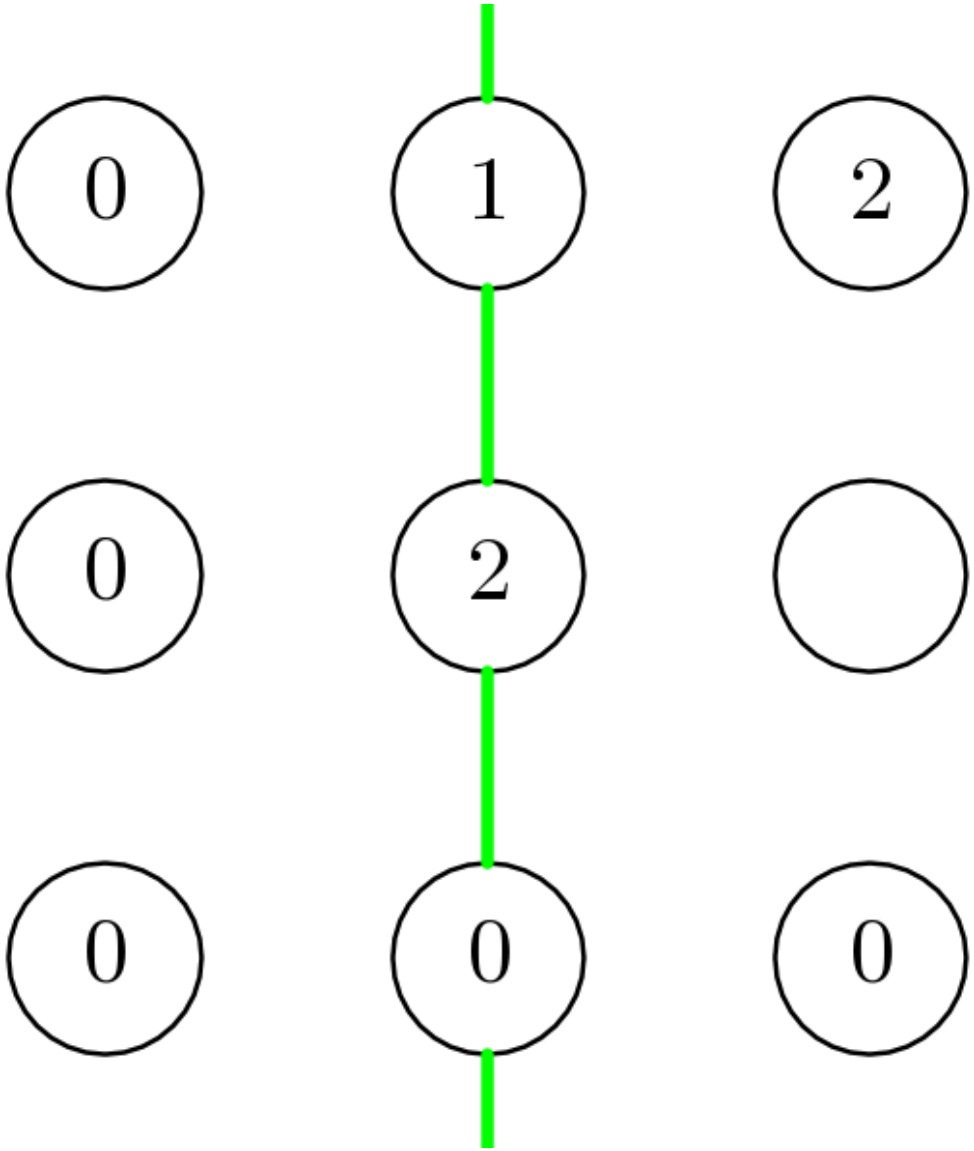
Example

$p = 3$



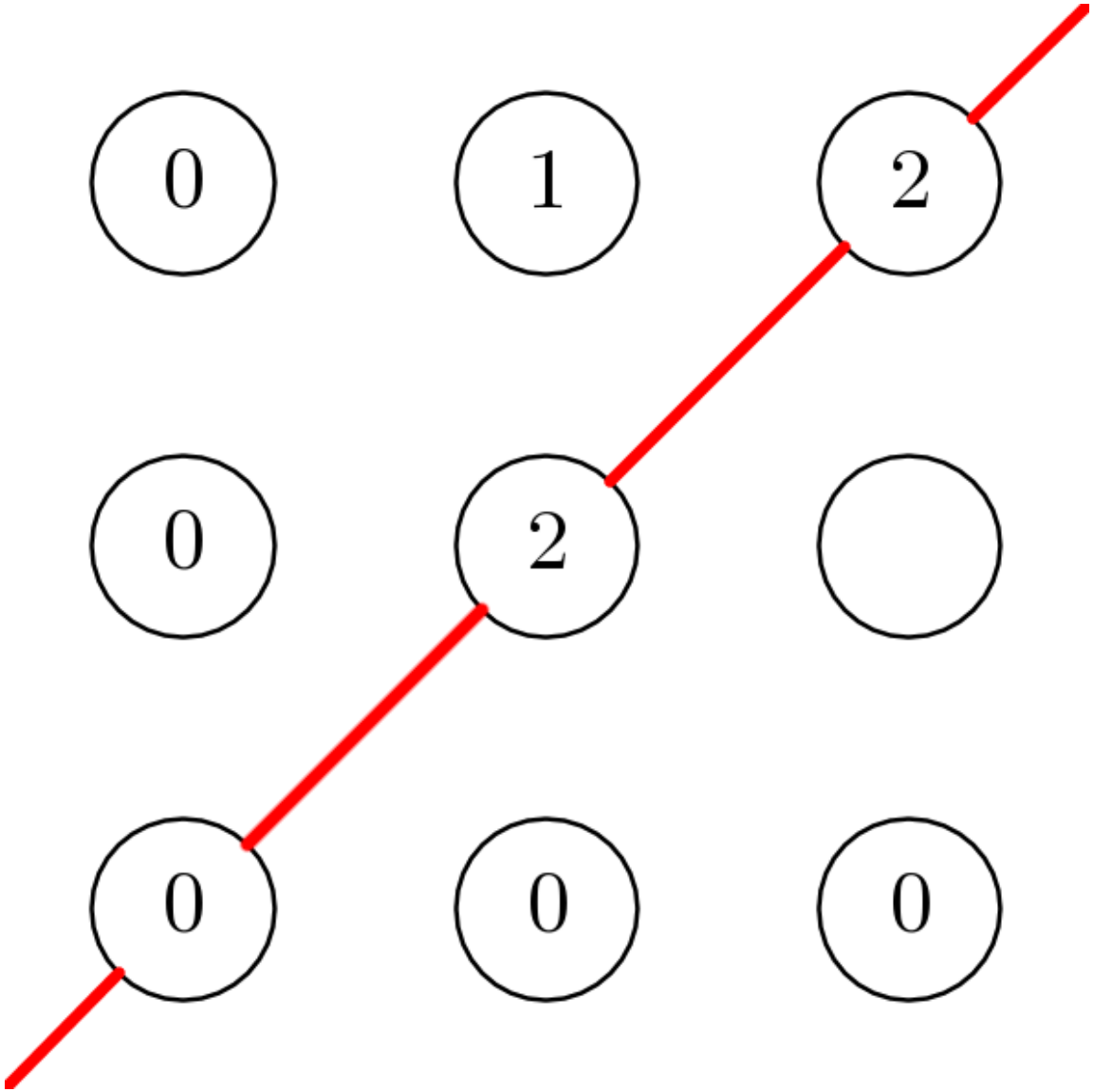
Example

$p = 3$



Example

$p = 3$



Results for small p

Theorem (De Beule, Demeyer, M., Sziklai)

There exists no such function for all primes $p \leq 13$.

Results for small p

Theorem (De Beule, Demeyer, M., Sziklai)

There exists no such function for all primes $p \leq 13$.

Corollary

Weaker Cylinder Conjecture is true for all primes $p \leq 13$.

Results for small p

Theorem (De Beule, Demeyer, M., Sziklai)

There exists no such function for all primes $p \leq 13$.

Corollary

Weaker Cylinder Conjecture is true for all primes $p \leq 13$.

Problem

Does there exist such function for any prime p?

Why three planes?

Why three planes?

Recall

i. S cannot contain any lines (Blokhuis);
ii. there exist three pairwise intersecting planes not containing any point of S;
iii. every plane contains 0 (mod p) points;
iv. $\sum_{i=1}^{p^2} a_i^k b_i^l = 0$ for all $k + l \leq p$.

Why three planes?

Recall

i. S cannot contain any lines (Blokhuis);
ii. there exist three pairwise intersecting planes not containing any point of S;
iii. every plane contains 0 (mod p) points;
iv. $\sum_{i=1}^{p^2} a_i^k b_i^l = 0$ for all $k + l \leq p$.

Why three planes?

Recall

i. S cannot contain any lines (Blokhuis);
ii. there exist three pairwise intersecting planes not containing any point of S;
iii. every plane contains 0 (mod p) points;
iv. $\sum_{i=1}^{p^2} a_i^k b_i^l = 0$ for all $k + l \leq p$.

Why three planes?

Shearer's lemma

Suppose n points in $\mathbb{F}^3$ have n_1, n_2 and n_3 points of projection on the XY-,YZ- and XZ-plane respectively, then $n_1 n_2 n_3 \geq n^2$.

Why three planes?

Shearer's lemma

Suppose n points in $\mathbb{F}^3$ have n_1, n_2 and n_3 points of projection on the XY-,YZ- and XZ-plane respectively, then $n_1 n_2 n_3 \geq n^2$.

Corollary

We can assume that at least $p^{4/3}$ points in $\mathrm{AG}(2, p)$ have non-zero weight.

What about the WCC/SCC?

What about the WCC/SCC?

1. Every point has weight at most $p - 1$;
2. on every line the total weight is a multiple of p;
3. every point on the X-axis or the Y-axis has weight 0;
4. the weight distribution is given by a function $w(X, Y)$ of degree at most $p - 2$.

What about the WCC/SCC?

Counterexample

For all $p \geq 5$,

$$f(t) = 1 - \frac{t^p - t}{t^2 - t},$$

$$w(X, Y) = f(X) + f(Y) - f(X + Y).$$

Conclusion

Theorem

The cylinder conjectures are hard.

Thank you for your attention!

sammattheus.wordpress.com