A STEP TOWARDS THE WEAK CYLINDER CONJECTURE

Joint work with J. De Beule (VUB), J. Demeyer (UGent), P. Sziklai (ELTE Budapest)

Sam Mattheus
September 14, 2017
The cylinder conjecture(s)

Let \( p \) be a prime.

Definition

A cylinder in \( AG(\mathbb{F}_p, \mathbb{F}_p) \) is the union of \( p \) parallel lines.

Example

A plane in \( AG(\mathbb{F}_p, \mathbb{F}_p) \) is a trivial example of a cylinder.
The cylinder conjecture(s)

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A plane in $AG(3, p)$ is a trivial example of a cylinder.
The cylinder conjecture(s) (Ball, 2006)

Weak Cylinder Conjecture
Let $S$ be a set of $p$ points in $\mathbb{A}G(3,p)$, not determining at least $p$ directions, then $S$ is a cylinder.

Strong Cylinder Conjecture
Let $S$ be a set of $p$ points in $\mathbb{A}G(3,p)$ such that every plane intersects it in zero ($\mod p$) points, then $S$ is a cylinder.

Theorem
Let $S$ be a set of $p$ points in $\mathbb{A}G(3,p)$ not determining at least $p$ directions, then every plane intersects it in zero ($\mod p$) points.
Weak Cylinder Conjecture

Let $S$ be a set of $p^2$ points in $AG(3, p)$, not determining at least $p$ directions, then $S$ is a cylinder.
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Theorem (Ball, zero zero zero 6)
If the set of non-determined directions contains a conic, then $S$ is a plane.
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Weaker Cylinder Conjecture

Let $S$ be a set of $p^2$ points in $\mathbb{A}G(3, p)$, not determining at least $p + 1$ directions, then $S$ is a cylinder.
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Facts about $S$

Embed $\mathbb{A}G(3, p)$ in $\mathbb{P}G(3, p)$ with plane at infinity $W = 0$. The affine point $(x, y, z)$ then has projective coordinates $(x, y, z, 1)$.

Let $S = \{(a_i, b_i, c_i, 1) \mid i = 1, \ldots, p^2\}$, and suppose it is not a cylinder.
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iii. every plane contains $0 \pmod{p}$ points;

iv. $\sum_{i=1}^{p^2} a_i^k b_i^l = 0$ for all $k + l \leq p$. 
Reduction by projection

\[ \text{PG}(3, p) \]

\[ W = 0 \]

\[ \text{AG}(3, p) \]
Reduction by projection

\[ X = 0 \]

\[ Y = 0 \]

\[ (0, 0, 1, 0) \]

\[ W = 0 \]
Reduction by projection

\[(0, 0, 1, 0)\]

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4. the weight distribution is given by a function $w(X, Y) = XYg(X, Y)$, where $g$ is of degree at most $p - 5$. 
Example

\[ p = 2 \]
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Example

\( p = 3 \)

\[
\begin{array}{ccc}
0 & & \\
0 & & \\
0 & & 0
\end{array}
\]
Example

$p = 3$

\[
\begin{array}{ccc}
0 & - & 0 \\
0 & - & 0 \\
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\]
Example

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0 1 2

0 0 0

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Results for small $p$

Theorem (De Beule, Demeyer, M., Sziklai)

There exists no such function for all primes $p \leq 13$. 
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Corollary

Weaker Cylinder Conjecture is true for all primes $p \leq 13$. 
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Corollary

Weaker Cylinder Conjecture is true for all primes $p \leq 13$.

Problem

Does there exist such function for any prime $p$?
Why three planes?
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Recall

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Why three planes?

Shearer’s lemma

Suppose \( n \) points in \( \mathbb{F}^3 \) have \( n_1, n_2 \) and \( n_3 \) points of projection on the XY-, YZ- and XZ-plane respectively, then \( n_1 n_2 n_3 \geq n^2 \).
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Shearer’s lemma

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Corollary

*We can assume that at least* \( p^{4/3} \) *points in* \( \text{AG}(2, p) \) *have non-zero weight.*
What about the WCC/SCC?
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1. Every point has weight at most $p - 1$;
2. on every line the total weight is a multiple of $p$;
3. every point on the X-axis or the Y-axis has weight 0;
4. the weight distribution is given by a function $w(X, Y)$ of degree at most $p - 2$. 
Counterexample

For all $p \geq 5$,

\[ f(t) = 1 - \frac{t^p - t}{t^2 - t}, \]

\[ w(X, Y) = f(X) + f(Y) - f(X + Y). \]
Conclusion

Theorem

*The cylinder conjectures are hard.*
Thank you for your attention!

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