Approaches to the Erdős-Ko-Rado Theorems

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Set systems

An intersecting 3-set system from $\{1, \ldots, 6\}$:

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135	136	234	235	236

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The second type of set system has many names:

- Trivially intersecting or Dictatorship
- I prefer *canonically intersecting* for the set of all k-sets that contain a fixed element.

Erdős-Ko-Rado Theorem

Theorem

Let $\mathcal{F}$ be a t-intersecting k-set system on an n-set. If $n > f(k,t)$, then

1. $|\mathcal{F}| \leq \binom{n-t}{k-t}$,

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- 1961 - Erdős, Ko and Rado had $f(k,t) \geq t + (k-t)\binom{k}{t}^3$.
- 1978 - Frankl proved $f(k,t) = (t+1)(k-t+1)$ when t is large.
- 1984 - Wilson gave an algebraic proof of the bound for all t.
- 1997 - Ahslwede and Khachatrian detemined the largest system for all values of t, k and n.

We can ask the same question for other objects

Object	Definition of intersection
k-Sets	a common element
Blocks in a design	a common element
Multisets	a common element
Vector spaces over a field	a common 1-D subspace
Lines in a partial geometry	a common point
Integer sequences	same entry in same position
Permutations	both map i to j
Permutations	a common cycle
Set Partitions	a common class
Tilings	a tile in the same place
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What is the size and structure of the largest set of intersecting objects?

General Framework

- Each *object* is made of k *atoms*.

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Objects have the *EKR property* if a canonically intersecting set is the largest intersecting set.

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The objects have the EKR property, if

$$k^2 P(2) < P(1).$$

Simple Counting Bound

- For uniform k-partitions of $\{1, \ldots, k\ell\}$ this is

$$k^2 \frac{1}{(k-2)!} \prod_{i=2}^{k} \binom{n-i\ell}{\ell} < \frac{1}{(k-1)!} \prod_{i=1}^{k} \binom{n-i\ell}{\ell}.$$

  Need $(k-1)k^2 < \binom{n-\ell}{\ell}$ (Works for all $\ell > 2$).

- For blocks in a 2-$(n, m, 1)$ design this bound is

$$m^2 \leq \frac{n-1}{m-1}$$

  So any such design with $m^3 - m^2 + 1 < n$ has the EKR property.

When Counting Fails

1. For permutations this never works since

$$k^2P(2) = n^2(n-2)! > (n-1)! = P(1).$$

2. For triangulations of a convex polygon the counting never works since

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In these examples the number of atoms in an object is not independent from the total number of atoms.

Left-compression

Erdős, Ko and Rado used a *compression operation*.

- In each k-set, replace j with a smaller i, unless the new set is already in the system.
- Sets are smaller in the colexicographic order (system has more structure).
- It doesn't change the size of the system.
- If the original system was intersecting, the new system is too.

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$(5 \to 1)$-Compression

$$\begin{array}{ccc} \{1,2,3\} & \to & \{1,2,3\} \\ \{1,2,4\} & \to & \{1,2,4\} \\ \{3,4,5\} & \to & \{1,3,4\} \\ \{2,5,6\} & \to & \{1,2,6\} \\ \{2,3,5\} & \to & \{2,3,5\} \\ \{1,2,5\} & \to & \{1,2,5\} \end{array}$$

Compression

Compression advantages:

- Ahslwede and Khachatrian's proof uses compression.
- Talbot used a clever compression to show the *seperated sets* have the EKR property.
- Holroyd, Talbot, and Borg use compression to prove that the cocliques in a family of graphs have EKR property.
- Ku and Wong used compression for *cycle*-intersecting permutations and set partitions.

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Compression pitfalls:

- The obvious compressions for permutations and perfect matchings are not 1-1 (Purdy's thesis).
- There are published paper mistakenly claiming to have a compression for both vectors spaces and permutations.

Derangement Graphs

For a set of objects, define the *derangement graph*

- the vertices are the objects,
- two vertices are adjacent if they are *not* intersecting.
- A coclique in the graph is an intersecting set of objects.

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What is the size of the maximum coclique in the derangement graph?

Which cocliques achieve this bound?

Graph Homomorphism

A *graph homomorphism* is a map $f : V(X) \to V(Y)$ such that if x_1, x_2 are adjacent in X, then $f(x_1)$ and $f(x_2)$ are adjacent in Y.

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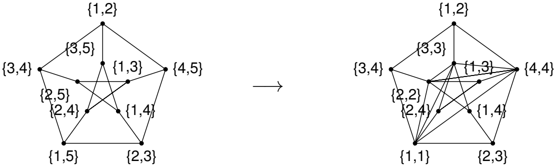

Derangement graph for sets
$K(5,2)$

Derangement graph for multisets
$M(4,2)$

If X is a spanning subgraph of Y, then $\alpha(Y) \leq \alpha(X)$.

There is a homomorphism from $K(n,k) \to M(n-k+1,k)$, which implies that the multisets have the (strict) EKR property.

Fractional Chromatic Number

If X is a vertex-transitive graph, then the fractional chromatic number is

$$\chi_f(X) = \frac{|V(X)|}{\alpha(X)}.$$

If $X \to Y$, then $\chi_f(X) \leq \chi_f(Y)$.

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Theorem

If $X \to Y$, and X and Y are vertex transitive then

$$\alpha(Y) \leq \frac{|V(Y)|\,\alpha(X)}{|V(X)|}.$$

Circulant Graphs

Define $C(n,k)$ to be graph with vertices cyclic k-intervals from $\{1,\ldots,n\}$ and two intervals are adjacent if they are disjoint.

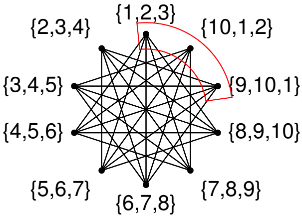

Figure: The graph $C(10,3)$.

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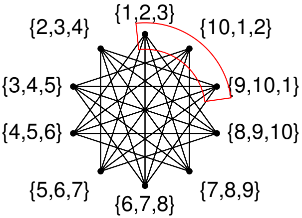

Figure: The graph $C(10,3)$.

- $C(n,k)$ is a subgraph of $K(n,k)$.
- $C(n,k)$ is vertex transitive.
- $\alpha(C(n,k)) = k$.
- $\chi_f(C(n,k)) = \frac{n}{k}$.

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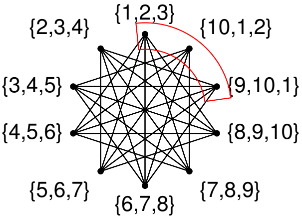

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There is a homomorphism $C(n,k) \to K(n,k)$:

$$\chi_f(K(n,k)) \geq \frac{n}{k} \quad \to \quad \alpha(K(n,k)) \leq \frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$$

Clique-Coclique Bound

If X is vertex-transitive, then $K_{\omega(X)} \to X$ and

$$\omega(X) = \chi_f(K_{\omega(X)}) \le \chi_f(X) = \frac{|V(X)|}{\alpha(X)}$$

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Theorem

If X is a vertex-transitive graph then

$$\alpha(X)\omega(X) \leq |V(X)|.$$

If equality holds, then every maximum coclique and every maximum clique intersect.

Clique-Coclique Bound Examples

1. For length-n integer sequences with entries from $\mathbb{Z}_q$

$$\alpha(X) = \frac{q^n}{q} = q^{n-1}.$$

2. For perfect matchings (1-factorization is a clique),

$$\alpha(M(2k)) = \frac{(2k-1)!!}{2k-1} = (2k-3)!!$$

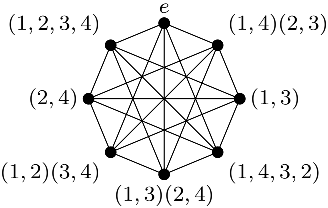
3. For permutations (sharply 1-transitive set is a clique)

$$\alpha(\Gamma_n) = \frac{n!}{n} = (n-1)!$$

Permutation Groups

Let $G \leq \mathrm{Sym}(n)$,

1. Γ_G denotes the derangement graph for a group G.
2. Vertices $\sigma, \pi \in G$ are adjacent if and only if $\pi\sigma^{-1}$ is a derangment.



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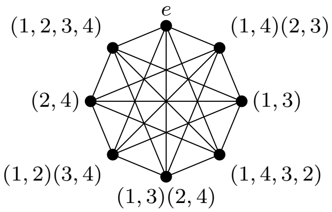

The graph $\Gamma_{D(4)}$.

Properties of Γ_G:

- Γ_G is vertex transitive.
- If G has a sharply 1-transitive set, then G has the EKR property.
- Γ_G is a normal Cayley graph.
- Γ_G is a graph in the conjugacy class scheme.

Conjugacy Class Scheme

Define a family of graphs X_i:

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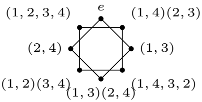
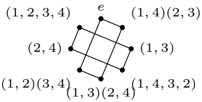
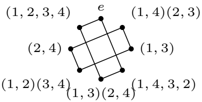
The conjugacy class scheme for D_4.



A_1 ($\pi\sigma^{-1}$ is conjugate to $(1,3)$.)



A_2 ($\pi\sigma^{-1}$ is conjugate to $(1,2)(3,4)$.)



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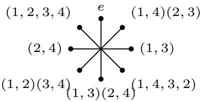

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$\rightarrow$

Clique-Coclique bound in an Association Scheme

Set up:

1. $\mathcal{A} = \{A_0, A_1, \ldots, A_d\}$ an association scheme,
2. T is a subset of $\{1, \ldots, d\}$, and
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- $M = \sum_{i=0}^{d} \frac{x^T A_i x}{v v_i} A_i, \qquad N = \sum_{i=0}^{d} \frac{y^T A_i y}{v v_i} A_i$.

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- So $M \circ N = \alpha I$ (scalar matrix). So

$$tr(MN) = sum(M \circ N) = \alpha v, \quad tr(M)\, tr(N) = \alpha v^2$$

- Since M and N are p.s.d., we can show that $tr(MN) \geq \frac{sum(M)\, sum(N)}{v^2}$

Clique-Coclique bound

Putting these together we get:

$$\frac{\mathrm{sum}(M)}{\mathrm{tr}(M)}\ \frac{\mathrm{sum}(N)}{\mathrm{tr}(N)} \leq v.$$

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We have that

$$|C| = \frac{\mathrm{sum}(M)}{\mathrm{tr}(M)}, \qquad |S| = \frac{\mathrm{sum}(N)}{\mathrm{tr}(N)},$$

so the inequality becomes:

$$\alpha(X)\,\omega(X) \le v.$$

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Theorem

If $\mathcal{A}$ is an association scheme and S a coclique in $\sum_{i\in T} X_i$, then

$$|S| = \frac{\operatorname{sum}(N)}{\operatorname{tr}(N)} \le \min_{M} v\,\frac{\operatorname{tr}(M)}{\operatorname{sum}(M)}$$

where M is positive semi-definite matrix in $\mathbb{C}[\mathcal{A}]$ with $M \circ N$ is a constant matrix.

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where M is positive semi-definite matrix in $\mathbb{C}[\mathcal{A}]$ with $M \circ N$ is a constant matrix.

We don't even need the clique, just the matrix!
Set $M = A - \tau I$, with τ least eigenvalue we get

$$|S| \leq \frac{v}{1 - \frac{d}{\tau}}.$$

Delsarte-Hoffman Bound for cocliques

$\leftarrow$

Theorem

If X is a union of graphs in an association scheme, then

$$\alpha(X) \leq \frac{|V(X)|}{1 - \frac{d}{\tau}}$$

where d the largest eigenvalue and τ is the least eigenvalue.

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If equality holds in the ratio bound and v_S is a characteristic vector for a maximum coclique S, then

$$v_S - \frac{\alpha(X)}{|V(X)|}\mathbf{1}$$

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If X is single graph in an association scheme, then

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Is there a maximum set of intersecting blocks that is not canonical?

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- Equality in the ratio bound means that if S is maximum intersecting set then $\underbrace{v_S}_{\text{characteristic vector}} - \underbrace{\frac{|S|}{n}\mathbf{1}}_{\text{balanced}}$ is an eigenvector.

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- Columns of H are in span of the τ-eigenspace and $\mathbf{1}$.
- H has full rank (check $H^T H$) so the columns span this space.
- If S is a maximum intersecting set, then v_S is a linear combination of the columns of H.

Strict EKR property

$$Hx = \begin{array}{r|c:c|}
 & 1,\dots,6 & 7,\dots,22 \\ \hline
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- Every clique is canonical.

Ratio Bound

The ratio bound holds with equality for

1. k-sets of an n-set (Johnson scheme),
2. k-dimensional subspaces (Grassmann scheme),
3. integer sequences (Hamming scheme),
4. perfect matchings (Perfect matching scheme),
5. permutations (Conjugacy class scheme).

2-Transitive Groups

This approach also works for many 2-transitive groups.

- The eigenvalues of Γ_G are

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- For many 2-transitive groups this gives the least eigenvalue: $Sym(n)$, $Alt(n)$, $PSL(2,q)$, $PGL(2,q)$, $PGL(3,q)$, the Mathieu groups.
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Using this approach we can prove all 2-transitive groups have the EKR property.

Related Problems

There are lots and lots of related problems:

1. Find EKR theorems in more general settings.
2. What is the size of the largest t-intersecting set of objects?
3. What is the largest set of intersecting objects in which not all contain a common element? (Hilton-Milner-type results)

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5. What is the size of the largest triangle-free subgraph in the derangement graph.
6. Are there interesting examples of object that do not have EKR propery?