EKR-Sets in Finite Buildings

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The EKR-problem

Let M be a finite set and $R \subseteq M \times M$ a reflexive and symmetric relation.

We put

- $\mathbf{Y} := \{Y \subseteq X \mid (A, B) \in R \text{ for all } A, B \in Y\}$;
- $e(R) := \max\{|Y| \mid Y \in \mathbf{Y}\}$;
- $\mathcal{E}(R) := \{Y \in \mathbf{Y} \mid |Y| = e(R)\}$.

An element of $\mathcal{E}(R)$ is called an *EKR-set for R*.

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Problem: Determine $e(R)$ and describe the EKR-sets for R.

The standard example

Let

- X be a finite set, $n := |X|$ and $k < \frac{n}{2}$;
- $M := \{Y \subseteq X \mid |Y| = k\}$;
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Theorem (Erdös-Ko-Rado):

(i) $e(R) := \binom{n-1}{k-1}$;

(ii) $\mathcal{E}(R) := \{M_p \mid p \in X\}$.

The EKR-problem for projective Spaces and polar Spaces

Let X be a finite projective or a finite non-degenerate polar space of rank n.

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Remarks:

1. Contributions from several authors;
2. X is a building of type A_n (resp. C_n, D_n).

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$$\mathrm{Vert}(X) := \{U \leq_K X \mid \{0\} \neq U \neq X\}$$

and we have a *type function*

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  $:\Leftrightarrow \exists U'$ incident with U and opposite to W;
- close $:\Leftrightarrow U$ is not in general position to W.

The simplicial building associated with X

A *flag of* X is a clique of the incidence graph on $\mathrm{Vert}(X)$ and $\mathrm{Flag}(X)$ denotes the set of flags of X.

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On the set $\mathrm{Flag}(X)$ we have

- an incidence relation;
- an opposition relation;
- a relation of being in general position;
- a relation of being close.

Spherical buildings

Let $\Delta := (\mathcal{F}, \subseteq)$ be a spherical building over the type set S. Then we have:

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The EKR-problem for finite buildings
Let $\Delta = (\mathcal{F}, \subseteq)$ be a finite (and hence spherical) building with type set S and let $\emptyset \neq J \subseteq S$.
We put

$$M := \{F \in \mathcal{F} \mid \text{typ}(F) = J\}$$

and

$$R := \{(F, G) \in M \times M \mid F \text{ and } G \text{ are close } \}.$$

Known results

Finite buildings of type A_n are projective spaces over some F_q. The EKR-problem is solved for the cases

- $|J| = 1$ (essentially due to Hsieh 1975);
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Finite buildings of type C_n or D_n are polar spaces associated with a non-degenerate sesquilinear or quadratic form on some vector space over F_q. The EKR-problem is solved for the cases

- $J = \{1\}$ easy;
- $J = \{n\}$ (Pepe, Storme, Vanhove 2011, almost all cases);
- $J = \{2\}$ (Metsch 2016, $q \geq 2(n-1)$);
- $J = \{n-1, n\}$ for D_n-buildings with n even (Ihringer, Metsch, M. 2016, partial for $q = 2$).

Buildings of exceptional type

Finite buildings of type F_4 are metasymplectic spaces over F_q (i.e. related to the groups $F_4(q)$ or ${}^2E_6(q)$).

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Observation: In almost cases, for which one knows the solution of the EKR-problem, the EKR-sets are contained in a ball of radius $\frac{\pi}{2}$.

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$$A^{\text{op}} := \{B \in \mathcal{F} \mid B \text{ is opposite to } A\}.$$

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The ball of radius $\frac{\pi}{2}$ around A is defined as

$$B_{\frac{\pi}{2}}(A) := \{B \in \mathcal{F} \mid \mathrm{dist}(B, A) \leq \mathrm{dist}(B, A^{\mathrm{op}})\}.$$

A strong version of Tits' center conjecture for spherical buildings

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Conjecture: Let $\mathcal{K} \subseteq \mathcal{F}$ be convex. Then one of the following holds:

(A) $\mathcal{K}$ is a subbuilding of Δ;

(B) there exists $C \in \mathcal{F}$ such that $\mathcal{K} \subseteq B_{\frac{\pi}{2}}(C)$.

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This conjecture is known to be true

- if $|S| \leq 3$ (Balser, Lytchak 2005);
- if Δ is of type A_n (Ramos-Cuevas 2015).

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Earlier contributions: Serre (2002); M., Tits (2006); Leeb, Ramos-Cuevas (2011); Parker, Tent (2012).

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Earlier contributions: Serre (2002); M., Tits (2006); Leeb, Ramos-Cuevas (2011); Parker, Tent (2012).

3. The proof for the weaker version of the center conjecture is not particularly illuminating for attacking its stronger version.

The proofs for the EKR-results obtained so far are constructive and might provide useful hints for attacking the conjecture above.

A question about EKR-sets in finite buildings

Does the following hold?

Let $\Delta = (\mathcal{F}, \subseteq)$ be a finite building with type set S.
Let $\mathcal{E} \subseteq \mathcal{F}$ be an EKR-set of J-flags (for some $\emptyset \neq J \subseteq S$).

Then we have one of the following:

(A) $\mathcal{E}$ corresponds to a subbuilding;

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Remark: Proving this is probably easier than proving the strong version of the center conjecture, because the arguments used so far always provide the center of the ball of radius $\frac{\pi}{2}$.