Graph decompositions in projective geometries

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Joint work with Marco Buratti and Alfred Wassermann

Irsee, September 2017

Definition (Steiner system)

A t-$(v,k,1)$ Steiner system is a pair $(\mathcal{P},\mathcal{B})$ such that

- $\mathcal{P}$ is the set of v points
- $\mathcal{B}$ is a set of k-subsets of $\mathcal{P}$
- each t-subset of $\mathcal{P}$ in 1 block.

- P. Cameron. *Locally symmetric designs*. Geom. Dedicata 3, 56-76, 1974.
- P. Delsarte. *Association schemes and t-designs in regular semilattices*. J. Combin. Theory Ser. A 20(2), 230-243, 1976.

Definition (Steiner system)

A t-$(v,k,1)$ Steiner system over $\mathbb{F}_q$ is a pair $(\mathcal{P},\mathcal{B})$ such that

- $\mathcal{P}$ is the set of points of PG$(\mathbb{F}_q^v)$
- $\mathcal{B}$ is a set of $(k-1)$-dimensional subspaces $\mathrm{PG}(\mathbb{F}_q^v)$
- each $(t-1)$-dimensional subspace is contained in 1 block.

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- $\begin{bmatrix} v \\ t \end{bmatrix}_q / \begin{bmatrix} k \\ t \end{bmatrix}_q = \dfrac{(q^v-1)(q^{v-1}-1)\cdots(q^{v-t+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q^{k-t+1}-1)}$ blocks

- $t=1$: $(k-1)$-spread:
- each point is contained in one block

Theorem

A 1-$(v,k,1)$ design over $\mathbb{F}_q$ exists if ans only if $k|v$.

Definition (Steiner system)

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- $\begin{bmatrix} v \\ t \end{bmatrix}_q / \begin{bmatrix} k \\ t \end{bmatrix}_q = \dfrac{(q^v-1)(q^{v-1}-1)\cdots(q^{v-t+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q^{k-t+1}-1)}$ blocks

- $t = 2$: $(1, k-1)$-spread:
- each line is contained in one block

Theorem (Braun, Etzion, Ostergaard, Vardy, Wassermann, 2017)

A 2-$(13, 3, 1)$ Steiner system over $\mathbb{F}_2$ exists.

Definition (Graph decomposition)

- A decomposition $\mathcal{D}$ of a graph G is a collection of subgraphs of G (*blocks*) whose edges partition $E(G)$.
- One says that $\mathcal{D}$ is a (G, Γ)-design if $B \simeq \Gamma$, $\forall B \in \mathcal{D}$.

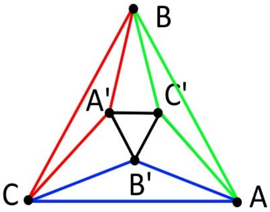


Figure: $(\mathbb{K}_{2,2,2}, \mathbb{K}_3)$-design.

Remark

2-$(v,k,1)$ Steiner system

$\Leftrightarrow$

decomposition of $\mathbb{K}_v$ into cliques of size k

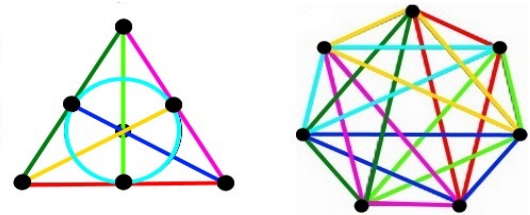


Figure: 2-(7,3,1) design and $(\mathbb{K}_7, \mathbb{K}_3)$-design.

Definition (Steiner system)

A 2-$(v,k,1)$ Steiner system over $\mathbb{F}_q$ is a pair $(\mathcal{P},\mathcal{B})$ such that

- $\mathcal{P}$ is the set of points of PG$(\mathbb{F}_q^v)$
- $\mathcal{B}$ is a set of $(k-1)$-dimensional subspaces $\mathrm{PG}(\mathbb{F}_q^v)$
- each line is contained in 1 block.

- identify the points of PG$(\mathbb{F}_q^v)$ with the elements of the Singer Group $S_{[v]_q} := \mathbb{F}_{q^v}^*/\mathbb{F}_q^*$
- $[\mathbb{K}_v]_q$ is the complete graph whose vertices are the points of $\mathrm{PG}(\mathbb{F}_q^v)$

Definition (Graph decomposition over finite field)

Let G be a graph with $V(G) = V([\mathbb{K}_v]_q)$.

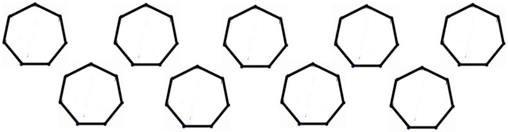
We say that a (G,Γ)-design $\mathcal{D}$ is over $\mathbb{F}_q$ if $V(B)$ is a subspace of $\mathrm{PG}(\mathbb{F}_q^v)$ $\forall B \in \mathcal{D}$.

- A 2-$(v,k,1)$ Steiner system over $\mathbb{F}_q$ is a $([\mathbb{K}_v]_q, [\mathbb{K}_k]_q)$-design over $\mathbb{F}_q$
- 2-(13,3,1) is a $([\mathbb{K}_{13}]_2, \mathbb{K}_7)$-design over $\mathbb{F}_2$
- Reduce the search space: impose automorphism group
- Constructions for CYCLIC graph decompositions over finite fields

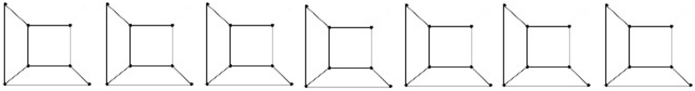
- $([\mathbb{K}_7]_2, \mathbb{K}_7)$-design over $\mathbb{F}_2$ $\rightsquigarrow$ $381 = 3 \cdot 127$ blocks

Braun, Nakic, Kiermaier, 2016: is does not exist

- $([\mathbb{K}_7]_2, C_7)$-design over $\mathbb{F}_2$ $\rightsquigarrow$ $1143 = 9 \cdot 127$ blocks



- $([\mathbb{K}_7]_2, Q_3 - v)$-design over $\mathbb{F}_2$ $\rightsquigarrow$ $889 = 7 \cdot 127$ blocks



- the Singer Group $S_{[v]_q} := \mathbb{F}^*_{q^v}/\mathbb{F}^*_q$
- Let G be a Cayley graph on $S_{[v]_q}$ and let Ω be its connection set:

$$V(G) = S_{[v]_q}; \quad \{x, y\} \in E(G) \Longleftrightarrow xy^{-1} \in \Omega$$

- For any subgraph B of G we set $Q(B) = \{xy^{-1}, yx^{-1} \ : \ \{x, y\} \in E(B)\}$

Definition

A collection $\mathcal{F}$ of subgraphs of G is a (G, Γ)-QF if we have:

- $B \simeq \Gamma$ for every $B \in \mathcal{F}$
- $\displaystyle\bigcup_{B \in \mathcal{F}} Q(B) = \Omega.$

A (G, Γ)-QF is over $\mathbb{F}_q$ if $V(B)$ is a subspace of $\mathrm{PG}(\mathbb{F}_q^v)$, for every $B \in \mathcal{F}$.

Proposition

If $\mathcal{F}$ is a (G, Γ)-QF, then

$$dev\mathcal{F} := \{g \cdot B \mid g \in S_{[v]_q};\ B \in \mathcal{F}\}$$

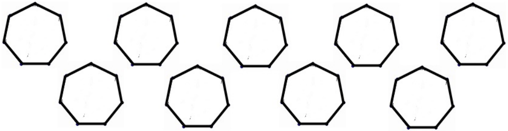
is a CYCLIC (G, Γ)-design.

If $\mathcal{F}$ is over $\mathbb{F}_q$, then $dev\mathcal{F}$ is over $\mathbb{F}_q$ as well.

Examples

- $([\mathbb{K}_7]_2, C_7)$-design over $\mathbb{F}_2$
- $([\mathbb{K}_7]_2, Q_3 - v)$-design over $\mathbb{F}_2$
- $([\mathbb{K}_4]_3, P_{40})$-design over $\mathbb{F}_3$
- $([\mathbb{K}_4]_3, M_{40})$-design over $\mathbb{F}_3$

- $[\mathbb{K}_7]_2$ is the Cayley graph on $S_{[7]_2}$ with connection set $S_{[7]_2} \setminus \{1\}$
- 1143 blocks
- find a $([\mathbb{K}_7]_2, C_7)$-QF over $\mathbb{F}_2$ with 9 cycles



- select the set Π of all planes of $\mathrm{PG}(\mathbb{F}_2^7)$ passing through the identity
- "translate" everything in the language of $\mathbb{Z}_{127} \simeq S_{[7]_2}$ $\qquad (Log)$
- find a $(\mathbb{Z}_{127}, C_7)$-DF such that
  - every block is an arrangement of a suitable $Log(\pi), \pi \in \Pi$
- a bit of counting: choose 9 blocks out of $|\Pi| \cdot \frac{6!}{2} = 93 \cdot 360 = 33480$

$(0, 1, 3, 7, 15, 31, 63)$ $\quad$ $(0, 7, 1, 71, 74, 79, 92)$ $\quad$ $(0, 29, 19, 8, 95, 65, 56)$

$(0, 18, 42, 14, 2, 114, 53)$ $\quad$ $(0, 14, 47, 70, 91, 2, 22)$ $\quad$ $(0, 37, 20, 89, 63, 3, 46)$

$(0, 55, 3, 111, 63, 13, 96)$ $\quad$ $(0, 80, 2, 75, 41, 14, 102)$ $\quad$ $(0, 51, 10, 72, 108, 40, 85)$

Example: $([\mathbb{K}_7]_2, C_7)$-design over $\mathbb{F}_2$

- here are the nine cycles of the required $([\mathbb{K}_7]_2, C_7)$-QF:

$$C_1 = (1, g, g^3, 1+g, g+g^3, 1+g+g^3, 1+g^3)$$

$$C_2 = (1, 1+g, g, g+g^2+g^4+g^5, 1+g^2+g^4+g^5, 1+g+g^2+g^4+g^5, g^2+g^4+g^5),$$

$$C_3 = (1, g+g^5, 1+g+g^5, g+g^2, 1+g^2+g^5, g^2+g^5, 1+g+g^2),$$

$$C_4 = (1, g^4+g^6, 1+g^2+g^4+g^6, 1+g^2, g^2, g^2+g^4+g^6, 1+g^4+g^6),$$

$$C_5 = (1, 1+g^2, 1+g+g^2+g^3+g^4, 1+g+g^3+g^4, g+g^3+g^4, g^2, g+g^2+g^3+g^4)$$

$$C_6 = (1, 1+g+g^2+g^3+g^6, g+g^2+g^6, 1+g+g^2+g^6, 1+g^3, g^3, g+g^2+g^3+g^6),$$

$$C_7 = (1, g+g^6, g^3, 1+g+g^3+g^6, 1+g^3, 1+g+g^6, g+g^3+g^6),$$

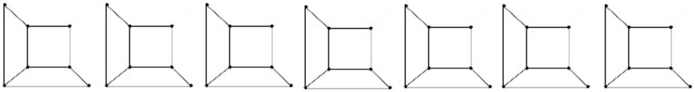
$$C_8 = (1, g+g^2+g^3+g^5+g^6, g^2, g+g^3+g^5+g^6, 1+g+g^3+g^5+g^6, 1+g^2, 1+g+g^2+g^3+g^5+g^6),$$

$$C_9 = (1, 1+g^2+g^4+g^5+g^6, g^3+g^4, g^2+g^3+g^5+g^6, 1+g^3+g^4, g^2+g^4+g^5+g^6, 1+g^2+g^3+g^5+g^6)$$

- by the action of the Singer group $S_{[7]_2}$ we obtain $9 \cdot 127 = 1143$ blocks of $([\mathbb{K}_7]_2, C_7)$-design over $\mathbb{F}_2$

Example: $([\mathbb{K}_7]_2, Q_3 - v)$-design over $\mathbb{F}_2$

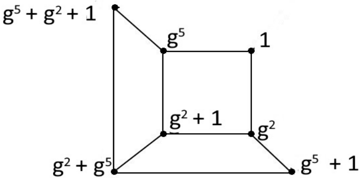
- $[\mathbb{K}_7]_2$ is the Cayley graph on $S_{[7]_2}$ with connection set $S_{[7]_2} \setminus \{1\}$
- 889 blocks
- find a $([\mathbb{K}_7]_2, Q_3 - v)$-QF over $\mathbb{F}_2$ with 7 $Q_3 - v$



- select the set Π of all planes of $\mathrm{PG}(\mathbb{F}_2^7)$ passing through the identity
- "translate" everything in the language of $\mathbb{Z}_{127} \simeq S_{[7]_2}$ $\quad (Log)$
- find a $(\mathbb{Z}_{127}, Q_3 - v)$-DF such that
  - every block is an arrangement of a suitable $Log(\pi), \pi \in \Pi$
  - the image of $Frob(S_{[7]_2})$ under Log
- a bit of counting: choose 1 block out of $|\Pi| \cdot 7 \cdot \frac{6!}{6} = 93 \cdot 840 = 78120$

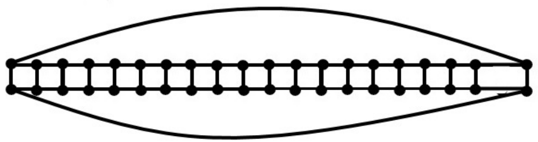
Example:$([\mathbb{K}_7]_2, Q_3 - v)$-design over $\mathbb{F}_2$

- here is $Q_3 - v$ required $([\mathbb{K}_7]_2, Q_3 - v)$-QF:



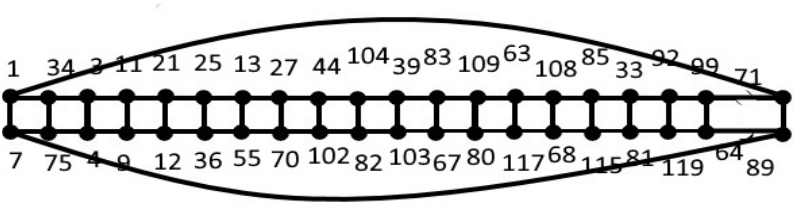
- by the action of the Singer group $S_{[7]_2}$ and $Frob(S_{[7]_2})$ we obtain $7 \cdot 127 = 889$ blocks of $([\mathbb{K}_7]_2, Q_3 - v)$-design over $\mathbb{F}_2$

- $[\mathbb{K}_5]_3$ is the Cayley graph on $S_{[5]_3}$ with connection set $S_{[5]_3} \setminus \{1\}$
- 121 blocks
- find a $([\mathbb{K}_5]_3, P_{40})$-QG over $\mathbb{F}_3$



- take any hyperplane of $\mathrm{PG}(\mathbb{F}_3^5)$
- "translate" everything in the language of $\mathbb{Z}_{121} \simeq S_{[5]_3}$ (Log)
- $B = \{0, 1, 2, 3, 5, 6, 7, 10, 11, 15, 17, 18, 22, 28, 30, 36, 39, 46, 47, 49, 51, 61, 69, 70, 71, 74, 75, 77, 79, 86, 88, 89, 93, 95, 101, 102, 106, 109, 112, 115\}$
- find a $(\mathbb{Z}_{121}, P_{40})$-DG with vertex set B
- $> 10^{30}$ possible rearrangements of the points

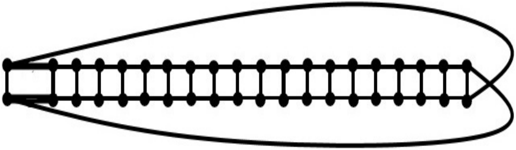
- $(\mathbb{Z}_{121}, P_{40})$-DG with vertex set B exists



- by the action of the Singer group $S_{[5]_3}$ we obtain 121 blocks of $([\mathbb{K}_5]_3, P_{40})$-design over $\mathbb{F}_3$

Example: $([\mathbb{K}_5]_3, M_{40})$-design over $\mathbb{F}_3$

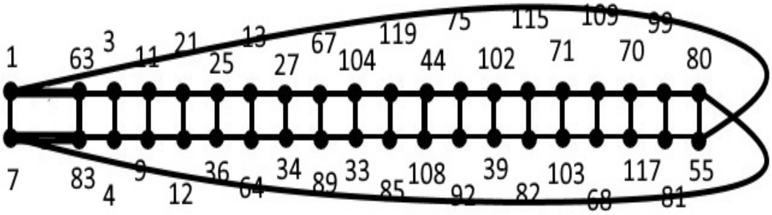
- $[\mathbb{K}_5]_3$ is the Cayley graph on $S_{[5]_3}$ with connection set $S_{[5]_3} \setminus \{1\}$
- find a $([\mathbb{K}_5]_3, M_{40})$-QG over $\mathbb{F}_3$



- take any hyperplane of $\mathrm{PG}(\mathbb{F}_3^5)$
- "translate" everything in the language of $\mathbb{Z}_{121} \simeq S_{[5]_3}$ $\quad (Log)$
- $B = \{0, 1, 2, 3, 5, 6, 7, 10, 11, 15, 17, 18, 22, 28, 30, 36, 39, 46, 47, 49, 51, 61, 69, 70, 71, 74, 75, 77, 79, 86, 88, 89, 93, 95, 101, 102, 106, 109, 112, 115\}$
- find a $(\mathbb{Z}_{121}, M_{40})$-DG with vertex set B
- $> 10^{30}$ possible rearrangements of the points

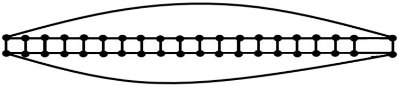
Example:$([\mathbb{K}_5]_3, M_{40})$-design over $\mathbb{F}_3$

- $(\mathbb{Z}_{121}, M_{40})$-DG with vertex set B exists



- by the action of the Singer group $S_{[5]_3}$ we obtain 121 blocks of $([\mathbb{K}_5]_3, M_{40})$-design over $\mathbb{F}_3$

- $([\mathbb{K}_4]_3, P_{40})$-design



- $B = \{0, 1, 2, 3, 5, 6, 7, 10, 11, 15, 17, 18, 22, 28, 30, 36, 39, 46, 47, 49, 51, 61, 69, 70, 71, 74, 75, 77, 79, 86, 88, 89, 93, 95, 101, 102, 106, 109, 112, 115\}$
- $\Delta B = \mathbb{Z}_{121} \setminus \{0\}$
- Base block is a $(121, 40, 13)$ Singer difference set in $S_{[5]_3}$

Remark

- Let D be a $(\frac{q^v-1}{q-1}, \frac{q^{v-1}-1}{q-1}, \frac{q^{v-2}-1}{q-1})$ Singer difference set
- Let Γ be a graph of order $\frac{q^{v-1}-1}{q-1}$
- If there exists $B \simeq \Gamma$ such that

$$V(B) = D \qquad \text{and} \qquad \Delta B = \mathbb{Z}_{(q^v-1)/(q-1)} \setminus \{0\}$$

- then $devB$ is a $([\mathbb{K}_v]_q, \Gamma)$-design over $\mathbb{F}_q$

Thank you!