An inequality for the line–size sum in a finite linear space.

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Outline

1. Inequalities in finite linear spaces
2. Finite linear spaces and cliques (partitions) of graphs
3. A weigthed inequality on the number of lines in finite linear spaces and clique partitions of complete graphs
4. A generalization

Some inequalities in finite linear spaces

Let $\mathcal{P} \neq \emptyset$ (*points*), $\mathcal{L} \subseteq 2^{\mathcal{P}}$ (*lines*), $(\mathcal{P}, \mathcal{L})$ is a *linear space* if *any pair of two distinct points belongs to exactly one line, every line has size at least two and there are at least two lines*.

$|\mathcal{P}| < \infty$, $v := |\mathcal{P}|$ and $b := |\mathcal{L}|$

- De Bruijn–Erdös (1948): *$b \geq v$. Equality holds iff $(\mathcal{P}, \mathcal{L})$ is a (possibly degenerate) projective plane.*

Some inequalities in finite linear spaces

Let n be the unique positive integer with
$(n-1)^2+(n-1)+1 \leq v \leq n^2+n+1$, $v \geq 4$,

$$B(v) := \begin{cases} n^2+n-1, & \text{if } \; v = n^2-n+2 \text{ and } v \neq 4 \\ n^2+n, & \text{if } \; n^2-n+3 \leq v \leq n^2+1 \text{ or } v=4 \\ n^2+n+1, & \text{if } \; n^2+2 \leq v \end{cases}$$

- Erdös – Mullin – Sós – Stinson (1983): $b \geq B(v)$

For any non–incident point line pair (p, ℓ) $\pi(p, \ell) :=$ *the number of lines through p missing* ℓ

• K. Metsch (1991): $b \geq v + \pi(p, \ell)$ *for every non–incident point–line pair.* (Proof of the Dowling–Wilson conjecture (1985))

$\forall \ell \in \mathcal{L}$, $i_\ell :=$ *the number of lines meeting* ℓ *and different from* ℓ

• Hanani (1954)–Varga (1985): *If* ℓ *is a line of maximal size, then* $i_\ell \geq v - 1$ *and equality holds iff* $(\mathcal{P}, \mathcal{L})$ *is a (possibly degenerate) projective plane.*

A *clique* of a simple graph G is a subset of mutually adjacent vertices

Graphs with large cliques have interesting geometric meanings:

- Geometrisable graphs: R.C. Bose, *Strongly Regular Graphs, Partial Geometries and Partially Balanced Designs* (1963)–Pacific J. Math.).

- Embeddability of finite linear spaces:

A. Beutelspacher–K.Metsch, *Embedding finite linear spaces in projective planes,* Annals Discrete Math. (1986) and *Embedding finite linear spaces in projective planes II,* D.M. (1987)

A *clique partition* $\mathcal{C}$ of G is a family of cliques in G such that the endpoints of every edge of G lie in exactly one member of $\mathcal{C}$.

The minimum size of a clique partition of G is the *clique partition number* of G and is denoted by cp(G).

The *sigma clique partition number* of a graph G, denoted by *scp*(G), is the smallest integer s for which there exists a clique partition of G where the sum of the sizes of its cliques is equal to s.

The *complement* $\overline{G}$ of a graph $G = (V, E)$ is the graph with vertex set V and with edge set $[V]^2 \setminus E$ (that is all the 2–element subset of V not in E (recall: $E \subset [V]^2$)).

For every graph G with v vertices, the union of a clique partition of G and a clique partition of its complement $\overline{G}$, form a linear space on v points.

The *cocktail party graph* T_v is the unique $(v-2)$–regular graph on $v = 2n$ vertices u_i, $1 = 1, 2, \ldots, v$ with u_i nonadjacent to u_{i+n} for each $i = 1, 2, \ldots, n$ and all other pairs of vertices are adjacent.

The connection between linear spaces and clique partitions of graphs has been deployed to estimate cp(G), when G is some special graph such as $K_v - K_w$, the Cocktail party graphs and complement of paths and cycles.

Let $(\mathcal{P}, \mathcal{L})$ be a finite linear space. For every line ℓ let k_ℓ denote its size.

• Akbar Davoodi – Ramin Javadi – Behnaz Omoomi (D.M. 2016): $\sum_{\ell \in \mathcal{L}} k_\ell \geq 3v - 3$ *and equality holds iff* $(\mathcal{P}, \mathcal{L})$ *is a degenerate projective plane.*

In terms of clique partitions:

• Akbar Davoodi – Ramin Javadi – Behnaz Omoomi (D.M. 2016): *Let* $\mathcal{C}$ *be a clique partition of the complete graph* K_n *whose cliques are of size at most* $n-1$*. Then* $\sum_{C \in \mathcal{C}} |C| \geq 3n - 3$*.*

• Akbar Davoodi – Ramin Javadi – Behnaz Omoomi (D.M. 2016): *For every graph* G *on* n *vertices except the empty and complete graph, we have* $scp(G) + scp(\overline{G}) \geq 3n - 3$*.*

Theorem (V.N., 2017 A.J.C.)

Let $(\mathcal{P}, \mathcal{L})$ be a finite linear space on v points. Let $m \geq 2$ denote the minimum point degree. Then

$$\sum_{\ell \in \mathcal{L}} k_\ell \geq (v - m + 1)(m + 1).$$

The equality holds if and only if $m = 2$ and $(\mathcal{P}, \mathcal{L})$ is a degenerate–projective plane.