The structure of the minimum size supertail of a subspace partition

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- $V = V(n, q)$ the vector space of dimension n over $\mathrm{GF}(q)$.
- A subspace partition or partition $\mathcal{P}$ of V, is a collection of subspaces $\{W_1, \ldots, W_k\}$ s.t.
  - $V = W_1 \cup \cdots \cup W_k$
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  - $W_i \cap W_j = \{\mathbf{0}\}$ for $i \neq j$.
- size of a subspace partition $\mathcal{P}$ is the number of subspaces in $\mathcal{P}$.

Applications

- translation planes
- error-correcting codes
- orthogonal arrays
- designs

Let $\mathcal{P}$ be any partition of V.

- $\mathcal{P}$ has type $[d_1^{n_1}, \ldots, d_m^{n_m}]$, if for each i, there are $n_i > 0$ subspaces of dim d_i in $\mathcal{P}$, and $d_1 < d_2 < \cdots < d_m$.

Problem

- What are the necessary and sufficient conditions for the existence of a partition of V of a given type?

Every partition $\mathcal{P}$ of V satisfies:

- packing condition

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- Heden and Lehmann (2012) derived some other necessary conditions

Let $\mathcal{P}$ be a partition of V of type $[d_1^{n_1}, \ldots, d_m^{n_m}]$.

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- $\sigma_q(n, s) =$ the min size of any partition of $V(n, q)$ in which the largest subspace has dim s.

- For any s such that $d_1 < s \leq d_m$, the set ST of subspaces in $\mathcal{P}$ of dim less than s and with greatest subspace dim t is called the st-supertail of $\mathcal{P}$.

Theorem (Heden, Lehmann, N., and Sissokho, 2011-2). Let n, m, s, and r_s be integers such that $1 \leq r_s < s$, $m \geq 1$, and $n = ms + r_s$. Then

$$\sigma_q(n, s) = \begin{cases} q^s + 1 \ \text{ for } 3 \leq n < 2s, \\ \\ q^{s+r_s} \sum_{i=0}^{m-2} q^{is} + q^{\lceil \frac{s+r_s}{2} \rceil} + 1 \text{ for } n \geq 2s. \end{cases}$$

Theorem (Heden, Lehmann, N., and Sissokho, 2013). Let $\mathcal{P}$ be a partition of V of type $[d_1^{n_1}, \ldots, d_m^{n_m}]$. If ST is an *st*-supertail of $\mathcal{P}$, then

$$|ST| \geq \sigma_q(s, t) .$$

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$$|ST| \geq \sigma_q(s,t) \ .$$

Corollary. If $s \geq 2t$ and $|ST| = \sigma_q(s,t)$, then the union of the subspaces in ST forms a subspace of dim s.

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Theorem [Heden, 2009]. Let $\mathcal{P}$ be a partition of V of type

$$[d_1^{n_1}, \ldots, d_m^{n_m}].$$

If ST is the tail of $\mathcal{P}$, i.e., all subspaces in ST have the same dim $d_1 = t$, s.t.

$$|ST| = q^t + 1 \text{ and } d_2 = s < 2t,$$

then the subspaces of ST form a subspace of dim $2t$.

Theorem [N. and Sissokho, 2017] Let $\mathcal{P}$ be a partition of V of type $[d_1^{n_1}, \ldots, d_m^{n_m}]$, and let ST be an *st*-supertail of $\mathcal{P}$ s.t. $|ST| = \sigma_q(s,t)$ and $t < s < 2t$. If one of the following conditions holds

(i) ST contains subspaces of at most 2 different dimensions

(ii) $s = 2t - 1$

(iii) All the subspaces in $\mathcal{P} \setminus ST$ have the same dimension s,

then the union of the subspaces in ST forms a subspace W, and either

(a) $d_1 = t$, $n_1 = q^t + 1$, and $\dim W = 2t$, or

(b) $d_1 = a$ and $d_2 = t$, with $n_1 = q^t$ and $n_2 = 1$, and $\dim W = a + t$.

Lemma 1 (Beutelspacher, Bu, Schönheim). Let d be an integer such that $1 \leq d \leq n/2$. Then $V(n, q)$ admits a partition with

- 1 subspace of dim $n - d$, and
- q^{n-d} subspaces of dim d.

Lemma 2. Let $\mathcal{P}$ be a partition of V. If $d_1 \leq t < s < 2t$, $s = t + r_t$ and ST is an st-supertail of $\mathcal{P}$ of size $\sigma_q(s,t)$, then $d_1 \geq r_t$.

Lemma 2. Let $\mathcal{P}$ be a partition of V. If $d_1 \leq t < s < 2t$, $s = t + r_t$ and ST is an st-supertail of $\mathcal{P}$ of size $\sigma_q(s,t)$, then $d_1 \geq r_t$.

Lemma 3. Let $\mathcal{P}$ be a partition of V. If $d_1 \leq t < s < 2t$ and ST is an st-supertail of $\mathcal{P}$ of size $\sigma_q(s,t)$, then there exists an integer k s.t.

$$\sum_{i=1}^{t} n_i(q^i - 1) = kq^s - 1.$$

Lemma 4. Let $\mathcal{P}$ be a partition of V with an *st*-supertail ST of type $[t^1, a^{q^t}]$. Then the union of the subspaces in ST forms a subspace of dim $t + a$.

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Lemma 5. Let $n = ms + r_s$ and $s = t + r_t$, with $1 \le r_s < s$ and $1 \le r_t < t$. Let $\ell_q(n,s) = \dfrac{q^n - q^{s+r_s}}{q^s - 1}$ and let $\mathcal{P}$ be a partition of $V(n,q)$ with an *st*-supertail ST of size $\sigma_q(s,t)$.

If s the largest subspace dim of $\mathcal{P}$ and $n_s = \ell_q(n,s)$, then the union of the subspaces in ST is a subspace W of dimension $s + r_s$.

Theorem Let $\mathcal{P}$ be a partition of V of type $[d_1^{n_1}, \ldots, d_m^{n_m}]$, and let ST be an st-supertail of $\mathcal{P}$ with $t < s < 2t$ and size $\sigma_q(s,t)$.

(i) If ST has subspaces of at most 2 different dimensions then the union of subspaces in ST forms a subspace W.

Proof.

- If ST has only subspaces of dim $t \Rightarrow$ result holds by Heden's Theorem, and $\dim W = 2t$.

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- If ST has only subspaces of dim $t \Rightarrow$ result holds by Heden's Theorem, and dim $W = 2t$.
- Suppose ST has subspaces of dimensions a and t. Since $s < 2t$, we have $\sigma_q(s,t) = q^t + 1$ by Theorem 1. Thus,

$$|ST| = n_t + n_a = q^t + 1,$$

and from Lemma 3,

$$n_t(q^t - 1) + n_a(q^a - 1) = kq^s - 1.$$

Since $s = t + r_t$,

$$n_t = \frac{q^t(kq^{r_t - a} - 1)}{q^{t-a} - 1} + 1.$$

Proof cont. Since $\gcd(q^t, q^{t-a}-1) = 1$,

$$n_t = \frac{q^t(kq^{r_t - a} - 1)}{q^{t-a} - 1} + 1 \Rightarrow (q^{t-a} - 1) \mid (kq^{r_t - a} - 1).$$

Proof cont. Since $\gcd(q^t, q^{t-a} - 1) = 1$,

$$n_t = \frac{q^t(kq^{r_t - a} - 1)}{q^{t-a} - 1} + 1 \Rightarrow (q^{t-a} - 1) \mid (kq^{r_t - a} - 1).$$

Hence $n_t = q^t \cdot c + 1$, where c is either 0 or 1 since $n_t \leq q^t + 1$.

Proof cont. Since $\gcd(q^t, q^{t-a} - 1) = 1$,

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Hence $n_t = q^t \cdot c + 1$, where c is either 0 or 1 since $n_t \leq q^t + 1$.

- If $c = 0$, then $n_t = 1$ and $n_a = q^t$. Hence by Lemma 4 the union of the subspaces in ST is a subspace W of dim $t + a$.

Proof cont. Since $\gcd(q^t, q^{t-a} - 1) = 1$,

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Hence $n_t = q^t \cdot c + 1$, where c is either 0 or 1 since $n_t \leq q^t + 1$.

- If $c = 0$, then $n_t = 1$ and $n_a = q^t$. Hence by Lemma 4 the union of the subspaces in ST is a subspace W of dim $t + a$.
- If $c = 1$, then $n_t = q^t + 1$ and $n_a = 0$ and hence, by Heden's Theorem, the union of the subspaces in ST form a subspace W of dim $2t$. □

Theorem. Let $\mathcal{P}$ be a partition of V of type $[d_1^{n_1}, \ldots, d_m^{n_m}]$, and let ST be an st-supertail of $\mathcal{P}$ with $t < s < 2t$ and of size $\sigma_q(s,t)$.

(ii) If $s = 2t - 1$, then the union of subspaces in ST forms a subspace W.

Proof. If $s = 2t - 1$, then it follows from Lemma 2 that the smallest subspace dim in ST is $a \geq t - 1$. Thus, ST contains subspaces of at most two different dimensions, namely t and $t - 1$. Now the theorem follows from part (i).

□

Theorem Let $\mathcal{P}$ be a partition of V of type $[d_1^{n_1}, \ldots, d_m^{n_m}]$, and let ST be an s-supertail of $\mathcal{P}$ with $t < s < 2t$ and size $\sigma_q(s, t)$.

(iii) If all the subspaces in $\mathcal{P} \setminus ST$ have dim $s < 2t$ then the union of the subspaces in ST form a subspace W and either

(a) $d_1 = t$, $n_1 = q^t + 1$, and $\dim W = 2t$, or

(b) $d_1 = a$ and $d_2 = t$, with $n_1 = q^t$ and $n_t = 1$, and $\dim W = a + t$.

Proof Sketch. Let $n = ms + r_s$ and $s = t + r_t < 2t$.

- Let $\ell_q(n,s) = \dfrac{q^n - q^{s+r_s}}{q^s - 1}$. Then,

$$n_s < \ell_q(n,s) + \frac{q^{r_s} + q^{r_s - 1}}{2} + 1,$$

  by the Drake and Freeman bound (1979), and the result of Theorem 1

$$|\mathcal{P}| \geq \sigma_q(n,s) = \ell_q(n,s) + q^{\lceil \frac{s+r_s}{2} \rceil} + 1,$$

  we derive that

$$|\mathcal{P} \setminus ST| = n_s = \ell_q(n,s) \text{ and } r_t + r_s \leq t.$$

- By Lemma 5, $W = \bigcup_{X \in ST} X$ is a subspace of dim

$$s + r_s = t + r_t + r_s \leq 2t.$$

Proof Sketch, cont.

- If $r_t + r_s = t$, then $\dim W = 2t$ and ST is a partition of W into subspaces of dim t only.

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- If $r_t + r_s = t$, then $\dim W = 2t$ and ST is a partition of W into subspaces of dim t only.
- If $r_t + r_s < t$, then $\dim W = t + r_t + r_s < 2t$ and ST is a partition of W with 1 subspace of dim t and q^t subspaces of dim $a = r_t + r_s$.

□

Remarks.

- If $r_t + r_s = t$, then $s + r_s = 2t$, and $\mathcal{P}$ is of type $[s^{\ell_q(n,s)}, t^{q^t+1}]$, i.e. $|\mathcal{P}| = \sigma_q(n, s)$.

Remarks.

- If $r_t + r_s = t$, then $s + r_s = 2t$, and $\mathcal{P}$ is of type $[s^{\ell_q(n,s)}, t^{q^t+1}]$, i.e. $|\mathcal{P}| = \sigma_q(n,s)$.
- If $r_t + r_s = t - 1$, then $\mathcal{P}$ is of type $[s^{\ell_q(n,s)}, t^1, (t-1)^{q^t}]$, i.e. $|\mathcal{P}| = \sigma_q(n,s)$.

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- If $r_t + r_s = t - 1$, then $\mathcal{P}$ is of type $[s^{\ell_q(n,s)}, t^1, (t-1)^{q^t}]$, i.e. $|\mathcal{P}| = \sigma_q(n,s)$.
- Partitions of type $[s^{\ell_q(n,s)}, t^{q^t+1}]$ and $[s^{\ell_q(n,s)}, t^1, (t-1)^{q^t}]$ exist.

- If $r_t + r_s < t - 1$, then the resulting partitions are not necessarily of min size.

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  E.g. If $n = 34$, $m = 3$, $s = 11$, $r_s = 1$, $t = 7$, and $r_t = 4$, we can apply (several times) Lemma 1 to construct a partition $\mathcal{P}$ of $V(34, q)$ of type

$$[11^{q^{23}+q^{12}}, 7^1, 5^{q^7}]$$

  which has size

$$|\mathcal{P}| = q^{23} + q^{12} + q^7 + 1 > \sigma_q(34, 11) = q^{23} + q^{12} + q^6 + 1.$$

Conjecture

Let $\mathcal{P}$ be a partition of V of type $[d_1^{n_1}, \ldots, d_m^{n_m}]$, and let ST be an st-supertail of $\mathcal{P}$ with $t < s < 2t$ of size $\sigma_q(s, t)$. Then ST has only subspaces of at most 2 different dimensions.

Thank you!