A note on the weight distribution of the Schubert code $C_\alpha(2, m)$

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Definition (Grassmannian)

The Grassmannian of all ℓ–subspaces of $\mathbb{F}_q^m$ is given by

$$G_{\ell,m} := \{W \subseteq \mathbb{F}_q^m : W \text{ is a subspace and } \dim W = \ell\}.$$

- For each $W \in G$, we pick an $\ell \times m$ matrix, M_W whose rowspace is W.

Definition

$$I(\ell, m) := \{(\alpha_1 < \alpha_2 < \cdots < \alpha_\ell) \mid 1 \leq \alpha_i \leq m\}$$

Definition

For a generic $\ell \times m$ matrix $\mathbf{X}$, and $I \in I(\ell, m)$ denote by $det_I(\mathbf{X})$ the $\ell \times \ell$ minor given by the columns indexed by I.

Definition

We denote by

$$\Delta(\ell, m) := \{ \sum_{I \in I(\ell,m)} f_I det_I(\mathbf{X}) \mid f_I \in \mathbb{F}_q \}.$$

Definition (Evaluation Map)

$$\begin{array}{rccc} ev_{G_{\ell,m}}: & \Delta(\ell,m) & \to & \mathbb{F}_q \\ & f & \mapsto & (f(\mathbf{M}_W))_{W\in G_{\ell,m}} \end{array}$$

Definition (Grassmann code)

$$C_\alpha(\ell,m) := \{ev_{G_{\ell,m}}(f) \mid W \in G_{\ell,m}, f \in \Delta(\ell,m)\} \subseteq \mathbb{F}_q^{G_{\ell,m}}$$

- $C_\alpha(\ell,m)$ is a linear code obtained from the Plücker embedding of $G_{\ell,m}$ onto a linear space.

Definition (Bruhat Order)

For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_\ell)$ we say that $\alpha \leq \beta$ if and only if $\alpha_i \leq \beta_i$ for all i.

Definition (Schubert variety)

Let $\alpha \in I(\ell, m)$. The Schubert variety may be defined as:

$$\Omega_\alpha(\ell, m) := \{W \in G_{\ell,m} \mid det_I(M_W) = 0,\ \forall I \not\leq \alpha\}$$

Definition (Evaluation Map (for Schubert varieties)

$$\begin{array}{rccl} ev_{\Omega_\alpha(\ell,m)} : & \Delta(\ell, m) & \to & \mathbb{F}_q \\ & f & \mapsto & (f(\mathbf{M}_W))_{W \in \Omega_\alpha(\ell,m)} \end{array}$$

Definition (Schubert codes)

$$C_\alpha(\ell, m) := \{ev_{\Omega_\alpha(\ell,m)}(f) \mid f \in \Delta(\ell, m)\} \subseteq \mathbb{F}_q^{\Omega_\alpha(\ell,m)}$$

- Ryan introduced Grassmann codes for $q = 2$ in 1988.
- Nogin generalized Grassmann codes to arbitrary q in 1996.
- Grassmann codes are $\left[\begin{bmatrix} m \\ \ell \end{bmatrix}_q, \binom{m}{\ell}, q^{\ell(m-\ell)}\right]$ codes.

- Nogin determined the weight distribution for $\ell = 2$.
- He observed that for $M_W = \begin{pmatrix} \cdots & \mathbf{x}_i & \ldots \mathbf{x}_j & \cdots \\ \cdots & \mathbf{y}_i & \ldots \mathbf{y}_j & \cdots \end{pmatrix}$, the function $det_{\{i,j\}}(M_w) = \mathbf{xDy}^T$ where most of the entries of $\mathbf{D}$ are zero except for $\mathbf{D}_{i,j} = -\mathbf{D}_{j,i} = 1$.

Definition

Let

$$f = \sum_{i=1}^{i} \sum_{j=i+1}^{m} f_{i,j} det_{\{i,j\}}(\mathbf{X}) \in \Delta(2, m).$$

Define **F** to be the skew–symmetric matrix corresponding to f, that is $\mathbf{F}_{a_1,a_2} = -\mathbf{F}_{a_2,a_1} = f_{\{(a_1,a_2)\}}$.

- This means that if the rows of M_W are **x** and **y** then $f(M_W) = \mathbf{x}\mathbf{F}\mathbf{y}^T$

Theorem (Nogin)

Let $f \in \Delta(\ell, m)$. Let **F** *be the matrix corresponding to f. Suppose the rank of* **F** *is r. Then*

$$wt(ev_{G_{\ell,m}}(f)) = q^{2(m-r+1)}\frac{q^{2r}-1}{q^2-1}.$$

- Counting the number of skew–symmetric matrices of a particular rank will give the weight distribution for the Grassmann code in the case $\ell = 2$.

Let $A_1 = \{e_1, e_2, \ldots, e_{a_1}\}$. For the case $\ell = 2$ we give an alternative definition of $\Omega_\alpha(\ell, m)$, where $\alpha = (a_1, m)$.

Definition (Schubert variety)

$$\Omega_\alpha(\ell, m) := \{W \in G_{\ell,m} \mid \dim(W \cap A_i) \geq 1\}$$

Definition (Schubert variety)

$$\Omega_\alpha(\ell, m) := \{W \in G_{\ell,m} \mid det_I(M_W) = 0,\ \forall I = (b_1, b_2), b_1 > a_1\}$$

We know the parameters of $C_\alpha(2,m)$ where $\alpha = (a_1, m)$

- The length is $\dfrac{(q^m-1)(q^{m-1}-1)}{(q^2-1)(q-1)} - \sum_{j=1}^{m-a_1+1} \sum_{i=1}^{j} q^{2m-j-2-i}$.
- The dimension is $\dfrac{m(m-1)}{2} - \dfrac{(m-a_1)(m-a_1-1)}{2}$.
- (H. Chen (2000) and Guerra–Vincenti (2002)) The minimum distance is $q^{m+\alpha_1-3}$.
- (Guerra – Vincenti 2004) determined the weight spectrum of $C_\alpha(2,m)$.

For the Schubert code $C_\alpha(2, m)$ we may consider the codewords corresponding to block matrices of the form:

$$\mathbf{F} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{B}^T & \mathbf{0}_{m-a_1 \times m-a_1} \end{pmatrix}$$

where **A** is a skew–symmetric $a_1 \times a_1$ matrix with zeros on the diagonal and **B** is a $a_1 \times m - a_1$ matrix.

We investigate the weight of $\mathbf{xFy}^T$ where $x \in A_1$. We may represent $\mathbf{xFy}^T$ as

$$\begin{pmatrix} \mathbf{x}_1 & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{B}^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{y}_1^T \\ \mathbf{y}_2^T \end{pmatrix} = \mathbf{x}_1 A \mathbf{y}_1^T + \mathbf{x}_1 B \mathbf{y}_2^T$$

where $\mathbf{y}_2 \neq \mathbf{0}$.

- Suppose that **A** has corank r_1, **B** has corank r_2
- and their left kernels intersect in a space of dimension r.
- For how many vectors $\mathbf{x} \in A_1$ and $\mathbf{y} \in \mathbb{F}_q^m \setminus \mathbf{A_1}$ is $\mathbf{x}_1 A\mathbf{y}_1^T + \mathbf{x}_1 B\mathbf{y}_2^T \neq 0$?

	$\#\mathbf{x}_1$	$\#\mathbf{y}_1 \times \#\mathbf{y}_2$
$\mathbf{x}_1 A = 0, \mathbf{x}_1 B = 0$	q^r	0
$\mathbf{x}_1 A \neq 0, \ \mathbf{x}_1 B = 0$	$q^{r_2} - q^r$	$(q^{a_1} - q^{a_1-1})(q^{m-a_1} - 1)$
$\mathbf{x}_1 A = 0, \ \mathbf{x}_1 B \neq 0$	$q^{r_1} - q^r$	$q^m - q^{m-1}$
$\mathbf{x}_1 A \neq 0, \mathbf{x}_1 B \neq 0$	$q^m - q^{r_1} - q^{r_2} + q^r$	$(q^{a_1} - q^{a_1-1})(q^{m-a_1} - 1)$

In total we have

$$(q^m - q^{r_1})(q^{a_1} - q^{a_1-1})(q^{m-a_1} - 1) + (q^{r_1} - q^r)(q^m - q^{m-1})$$

pairs of vectors.

- Suppose that **A** has corank r_1, **B** has corank r_2
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- For how many vectors $\mathbf{x} \in A_1$ and $\mathbf{y} \in \mathbb{F}_q^m \setminus \mathbf{A_1}$ is $\mathbf{x}_1 A \mathbf{y}_1^T + \mathbf{x}_1 B \mathbf{y}_2^T \neq 0$?

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$\mathbf{x}_1 A = 0, \ \mathbf{x}_1 B \neq 0$	$q^{r_1} - q^r$	$q^m - q^{m-1}$
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In total we have

$$(q^m - q^{r_1})(q^{a_1} - q^{a_1-1})(q^{m-a_1} - 1) + (q^{r_1} - q^r)(q^m - q^{m-1})$$

pairs of vectors.

Theorem

Let $f \in \Delta(2, m)$ such that its associated matrix, **F** *is of the form*

$$\mathbf{F} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{B}^T & \mathbf{0}_{m-a_1 \times m-a_1} \end{pmatrix}$$

where **F** *is as before,*
then the weight of the codeword of $C_\alpha(2, m)$ corresponding to f is

$$wt(c_f) = \frac{(q^{a_1} - q^{r_1})(q^{a_1} - q^{a_1-1})(q^{m-a_1} - 1)}{(q-1)(q^2-q)} \quad (1)$$

$$+\frac{(q^{r_1} - q^{r})(q^{m} - q^{m-1})}{(q-1)(q(q-1))} \quad (2)$$

$$+q^{2(r_1-1)}\frac{q^{a_1-r_1} - 1}{q^2-1} \quad (3)$$

Definition

Let $f \in \Delta(2, m)$ such that its associated matrix, $\mathbf{F}$ is of the form

$$\mathbf{F} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{B}^T & \mathbf{0}_{m-a_1 \times m-a_1} \end{pmatrix}$$

where

- $\mathbf{A}$ is a skew–symmetric $a_1 \times a_1$ matrix
- $\mathbf{B}$ is a $a_1 \times m - a_1$ matrix,
- $\mathbf{A}$ has corank r_1, $\mathbf{B}$ has corank r_2
- and their left kernels intersect in a space of dimension r.

We denote by

$$d(r_1, r) := wt(ev(f)).$$

- $w(c_f)$ depends only on r_1 and r.
- **and** $(s, s_1) \neq (r, r_1)$ implies $d(s, s_1) \neq d(r, r_1)$.

Main Result

For $\alpha = (a_1, m)$, the weight distribution of $C_\alpha(2, m)$ is given by

$$\sum_{r_1=0}^{a_1} \sum_{r=0}^{r_1} N_1(\alpha_1, \alpha_1 - r_1) \begin{bmatrix} r_1 \\ r \end{bmatrix}_q \left(\sum_{r_2=r}^{\alpha_1} \Lambda(\alpha_1, \alpha_1 - r_2) \Delta(\alpha_1, r_1, r_2, r) \right) X^{d(r_1, r)}$$

So in order to find the weight distribution we sum the product of

- The number of skew–symmetric $a_1 \times a_1$ matrices **A** of rank $a_1 - r_1$

$$N_1(a_1, \alpha_1 - r_1) := q^{\left(\frac{a_1-r_1}{2}\right)\left(\frac{a_1-r_1}{2}-1\right)} \frac{\prod_{i=0}^{a_1-r_1-1}(q^{a_1-i}-1)}{\prod_{i=0}^{\frac{a_1-r_1}{2}-1}(q^{a_1-r_1-2i}-1)}$$

times

- The number of r dimensional subspaces of the left kernel

$$\begin{bmatrix} r_1 \\ r \end{bmatrix}_q$$

times

- The number of r_2–dimensional spaces intersecting the left kernel of **A** in exactly an r–dimensional subspace

$$\Delta(\alpha_1, r_1, r_2, r) = \sum_{j=r}^{r_2} (-1)^{j-r} q^{\binom{j-r}{2}} \begin{bmatrix} r_1 - r \\ j - r \end{bmatrix}_q \begin{bmatrix} \alpha_1 - j \\ r_2 - j \end{bmatrix}_q$$

times

- The number of $a_1 \times m - a_1$ matrices whose left kernel is exactly a fixed r_2–dimensional subspace.

$$\Lambda(\alpha_1, r_2, m) = \prod_{i=0}^{r_2-1} (q^{m-a_1} - q^i).$$

Thank you for your attention!