

Some recent developments in the theory of linear MRD-codes

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Outline

- 1 **Linear MRD-codes**
- 2 **Generalized Gabidulin Codes and linearized polynomials**
- 3 **Generalized Twisted Gabidulin Codes**
- 4 **MRD-codes-Maximum Scattered Spaces-Segre Variety**

RD-codes

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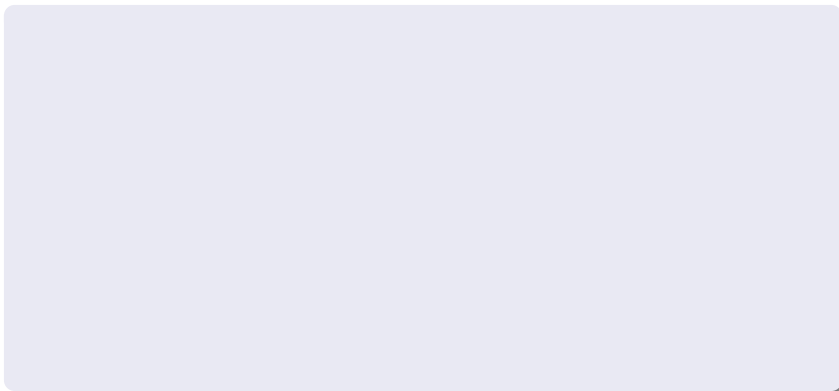
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$Aut(\mathcal{C})$ Automorphism group of $\mathcal{C}$

$$Aut(\mathcal{C}) = \{(A, B, \sigma) \in GL(m, q) \times GL(n, q) \times Aut(\mathbb{F}_q) : AC^\sigma B = \mathcal{C}\}$$

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$$L(\mathcal{C})^* \times R(\mathcal{C})^* \times \{id\} \subseteq Aut(\mathcal{C})$$

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If $m \leq n$ then $L(\mathcal{C})$ is a finite field. Hence $|L(\mathcal{C})| \leq q^m$.
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First Examples of linear MRD-codes

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$$\mathcal{G}_k = \{f(x) = a_0x + a_1x^q + \dots + a_{k-1}x^{q^{(k-1)}} : a_0, a_1, \dots, a_{k-1} \in \mathbb{F}_{q^n}\},$$

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$\mathcal{C}_{\mathcal{G}_{k,s}}$	Generalised Gabidulin MRD-code	$[n \times n, kn, n - k + 1]_{\mathbb{F}_q}$
$\mathcal{C}_{\mathcal{G}_{k,s}}(U)$	Generalised Gabidulin MRD-code	$[n \times m, kn, n - k + 1]_{\mathbb{F}_q}$
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$$\mathcal{L}_{n,q} = \{f(x) = a_0x + a_1x^q + \dots a_{n-1}x^{q^{(n-1)}} : a_0, a_1, \dots, a_{n-1} \in \mathbb{F}_{q^n}\}$$

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Left idealiser of $\mathcal{C}$

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Finite presemifields

J. De La Cruz, Kiermaier, Wassermann, Willem: Algebraic structures of MRD codes, *Adv. Math. Commun.* (2016)

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Generalized Twisted Gabidulin Codes

Sheekey, 2016

$$\mathcal{H}_{k,s}(\eta, h) = \{f(x) = a_0x + a_1x^{q^s} + \cdots + a_{k-1}x^{q^{s(k-1)}} + \eta a_0^{q^h} x^{q^{sk}} : a_i \in \mathbb{F}_{q^n}\}$$
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K. Otal, F. Özbudak: Explicit Construction of Some Non-Gabidulin Linear Maximum Rank Distance Codes, *Adv. Math. Commun.* (2016)

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Horlemann-Trautmann, Marshall: New Criteria for MRD and Gabidulin Codes and some Rank-Metric Code Constructions, arXiv:1507.08641v3 (2016)

MRD-codes and maximum scattered spaces

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Maximum Scattered Space of V

Maximum scattered spaces-Maximum scattered linear sets

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R. Calderbank, W.M. Kantor: The geometry of two-weight codes, *Bull. London Math. Soc.* (1986)

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- $U_1 = \{(x, x^{q^s}) : x \in \mathbb{F}_{q^n}\}, 1 \leq s \leq n-1, \gcd(s, n) = 1.$
- $U_2 = \{(x, \delta x^{q^s} + x^{q^{n-s}}) : x \in \mathbb{F}_{q^n}\}, N_{q^n/q}(\delta) \neq 0, 1, \gcd(s, n) = 1.$

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$$k = 2, n = 6, 8 \quad \Rightarrow \quad \mathcal{H}_L \subset \mathcal{M}_L$$

MRD-codes and Maximum Scattered Spaces

Classification results

- $\mathcal{C}$ MRD-code, $[3 \times 3, 6, 2]_{\mathbb{F}_q}$, $\mathbb{F}_{q^3}$ -linear

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For $q = 2$ - **Horlemann-Trautmann, Marshall:** New criteria for MRD and Gabidulin codes and some rank-metric code constructions, arXiv:1507.08641v3 (2015)

MRD-codes and the Segre Variety

Lunardon, JCTA (2017)

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$$\mathcal{C} \subset \text{End}(\mathbb{F}_{q^n}, \mathbb{F}_q) \text{ linear MRD-code, } [n \times n, kn, n - k + 1]_{\mathbb{F}_q}$$

$$\updownarrow$$

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MRD-codes and the Segre Variety

Lunardon, JCTA (2017)

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$$P(\mathcal{C}) \Rightarrow P(\mathcal{C})^\Phi = P(\mathcal{C}^T)$$

$$P(\mathcal{C}) \Rightarrow P(\mathcal{C})^\tau = P(\mathcal{C}^\perp)$$

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- Understand the relationship between Sheekey's construction and Lunardon's approach.

Thank you

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