The symmetric representation of lines in PG($\mathbb{F}^3 \otimes \mathbb{F}^3$)

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Finite Geometries: Fifth Irsee Conference

11 September 2017

Motivation

Tensors have many applications:

- complexity theory (e.g. Strassen's algorithm for matrix multiplication);
- quantum information, entanglement, coding;
- data analysis (chemistry, biology, physics ...);
- signal processing (e.g. source separation) ...
- ... and finite geometry, e.g. finite semifields.

Motivation (continued)

An expression

$$\tau = \sum_{i=1}^{r} v_{1i} \otimes \cdots \otimes v_{mi}$$

is a decomposition of $\tau \in V_1 \otimes \cdots \otimes V_m$ into “fundamental tensors”.

Questions:

- Existence — what is the smallest possible r (the rank of τ)?
- Uniqueness.
- Algorithms (including approximation, ‘noisy’ data) ...

Another question: orbits

$\mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_m)$ acts on rank-1 tensors in $V = V_1 \otimes \cdots \otimes V_m$:

$$(v_1 \otimes \cdots \otimes v_m)^{(g_1,\ldots,g_m)} = v_1^{g_1} \otimes \cdots \otimes v_m^{g_m}.$$

If some V_i are equal then one can also permute components; in particular, if $V_1 = \cdots = V_m$ then all of Sym_m acts via

$$(v_1 \otimes \cdots \otimes v_m)^{\pi} = v_{\pi(1)} \otimes \cdots \otimes v_{\pi(m)}.$$

How many "different" tensors are there under the full stabiliser of the set of rank-1 tensors?

Application: for $m = 3$, orbits of certain tensors correspond to isotopism classes of presemifields (Lavrauw, 2013).

Existing results

$\underline{m=2}$ $V(a,K) \otimes V(b,K) \cong a \times b$ matrices over K, i.e.

$$v \otimes w = vw^T = \begin{bmatrix} v_1 w_1 & \dots & v_1 w_b \\ \vdots & \ddots & \vdots \\ v_a w_1 & \dots & v_a w_b \end{bmatrix},$$

so rank $=$ matrix rank, and there is one orbit for each rank.

$\underline{m=3}$ is much harder, often depends on field and/or dimensions:

- $K = \mathbb{F}_p$ (Brahana, Thrall, 1930's); $p = 2$ (Glynn et al. 2006, Havlicek et al. 2012);
- $K = \mathbb{C}$ (Thrall–Chanler, Kac, Kaśkiewicz–Weyman, Nurmiev, Parfenov, Buczyński–Landsberg; 1930's–2010's);
- $K^2 \otimes K^3 \otimes K^r$, $r \geqslant 1$, K algebraically closed, real or finite (Lavrauw–Sheekey, 2014–2017).

Geometry, subspaces

Consider $m = 2$ again, and take $K = \mathbb{F}_q$.

Projectively, the rank-1 tensors in $V = \mathbb{F}_q^a \otimes \mathbb{F}_q^b$ correspond to the Segre variety $S_{a,b}(\mathbb{F}_q) \subset \mathrm{PG}(V) \cong \mathrm{PG}(ab-1, q)$, i.e. the image of

$$\begin{aligned} \mathrm{PG}(\mathbb{F}_q^a) \times \mathrm{PG}(\mathbb{F}_q^b) &\to \mathrm{PG}(V) \\ (\langle v \rangle, \langle w \rangle) &\mapsto \langle v \otimes w \rangle. \end{aligned}$$

The setwise stabiliser in $\mathrm{GL}(V)$ of the rank-1 tensors induces a subgroup $G \leqslant \mathrm{PGL}(V)$, the setwise stabiliser of $S_{a,b}(\mathbb{F}_q)$.

What are the orbits of subspaces of $\mathrm{PG}(V)$ under this group?

Existing results

$a = b = 2$ (Lavrauw–Sheekey, 2014) The setwise stabiliser of

$$S_{2,2}(\mathbb{F}_q) = Q^+(3,q) \subset \mathrm{PG}(3,q)$$

has 4 orbits on lines of $\mathrm{PG}(3,q)$.

$a = 2$, $b = 3$ (Lavrauw–Sheekey, 2017) 7 orbits on lines/solids of $\mathrm{PG}(\mathbb{F}_q^2 \otimes \mathbb{F}_q^3)$, 11 orbits on planes.

$a = b = 3$ (Lavrauw–Sheekey, 2015) 14 orbits on lines of $\mathrm{PG}(\mathbb{F}_q^3 \otimes \mathbb{F}_q^3)$.
Planes, etc. will be harder.

Proofs. Appropriate contraction spaces of tensors in $\mathbb{F}_q^2 \otimes \mathbb{F}_q^k \otimes \mathbb{F}_q^k$, $k \in \{2,3\}$, are in one-to-one correspondence with tensor orbits.

Symmetric subspace representatives

Now for $n = a = b$ consider the Veronese variety

$$\mathcal{V}_n(\mathbb{F}_q) \subset S_{n,n}(\mathbb{F}_q),$$

i.e. the image of the map induced by

$$\begin{aligned} \mathbb{F}_q^n &\to \mathbb{F}_q^n \otimes \mathbb{F}_q^n \\ v &\mapsto v \otimes v. \end{aligned}$$

If we think of elements of $S_{n,n}(\mathbb{F}_q)$ as being represented by $n \times n$ matrices, then elements of $\mathcal{V}_n(\mathbb{F}_q)$ are symmetric matrices.

Symmetric subspace representatives (continued)

Recall: $G \leqslant \mathrm{PGL}(\mathbb{F}_q^n \otimes \mathbb{F}_q^n)$, stabiliser of $S_{n,n}(\mathbb{F}_q) \subset \mathrm{PG}(n^2-1,q)$.

Two questions:

1. Which G-subspace orbits have symmetric representatives, i.e. which are represented in $\langle \mathcal{V}_n(\mathbb{F}_q) \rangle \leqslant \mathrm{PG}(n^2-1,q)$?
2. Which of those orbits split under the action of the setwise stabiliser of $\langle \mathcal{V}_n(\mathbb{F}_q) \rangle$?

$n = 2$

- $\mathcal{V}_2(\mathbb{F}_q)$ is a conic in $Q^+(3,q) = S_{2,2}(\mathbb{F}_q) \subset \mathrm{PG}(3,q)$.
- Line orbits in $\langle \mathcal{V}_2(\mathbb{F}_q) \rangle \cong \mathrm{PG}(2,q)$ under the setwise stabiliser $\mathrm{PGL}(2,q) \leqslant \mathrm{PGL}(3,q)$ of the conic? (There are 3.)

$n = 3$: line orbits in $\langle \mathcal{V}_3(\mathbb{F}_q) \rangle$

For $n = 3$ we have $\mathcal{V}_3(\mathbb{F}_q) \subset \mathrm{PG}(5, q) \leqslant \mathrm{PG}(\mathbb{F}_q^3 \otimes \mathbb{F}_q^3)$, represented by symmetric 3×3 matrices.

Setwise stabiliser $H \leqslant \mathrm{PGL}(6, q)$, $H \cong \mathrm{PGL}(3, q)$, induced by $g \in \mathrm{GL}(3, q)$ acting on symmetric 3×3 matrices A via $A^g = gAg^\top$.

Theorem (Lavrauw–P., 2017+)

- 11 *of the* 14 *line orbits in* $PG(\mathbb{F}_q^3 \otimes \mathbb{F}_q^3)$ *are represented in* $\langle \mathcal{V}_3(\mathbb{F}_q) \rangle$*;*
- 4 *of these* 11 *orbits split into (exactly) two H-orbits;*
- *different orbits split depending on whether q is even or odd;*
- *unique H-orbit of constant rank-3 lines.*

Some data

Tensor orbit[1]	$\sharp$ symmetric line-orbit representatives	splits?	rank distribution
o_5	$\frac{1}{2}q(q+1)(q^2+q+1)$		$[2, q-1, 0]$
o_6	$(q+1)(q^2+q+1)$		$[1, q, 0]$
o_8	$q^4(q^2+q+1)$	yes	$[1, 1, q-1]$
o_9	$q(q^3-1)(q+1)$		$[1, 0, q]$
o_{10}	$\frac{1}{2}q(q^3-1)$		$[0, q+1, 0]$
o_{12}	$q^2(q^2+q+1)$	q even	$[0, q+1, 0]$
o_{13}	$q^3(q^3-1)(q+1)$	yes	$[0, 2, q-1]$
o_{14}	$\frac{1}{6}q^3(q^3-1)(q^2-1)$	q odd	$[0, 3, q-2]$
o_{15}	$\frac{1}{2}q^3(q^3-1)(q^2-1)$	q odd	$[0, 1, q]$
o_{16}	$q^2(q^3-1)(q+1)$	q even	$[0, 1, q]$
o_{17}	$\frac{1}{3}q^3(q^3-1)(q^2-1)$		$[0, 0, q+1]$

[1] In the notation of Lavrauw–Sheekey (2015).

More data: line stabilisers in $H = \mathrm{PGL}(3,q)$

Tensor orbit	Common line orbit	Additional line orbit
o_5	$E_q^2 : C_{q-1}^2 : C_2$	
o_6	$E_q^{1+2} : C_{q-1}^2$	
o_8	$C_{q-1} \times \mathrm{O}^{\pm}(2,q)$, $q \equiv \pm 1(4)$	$C_{q-1} \times \mathrm{O}^{\mp}(2,q)$, $q \equiv \pm 1(4)$
	$E_q \times C_{q-1}$, q even	$C_{q-1} \times \mathrm{SL}(2,q)$, q even
o_9	$E_q^2 : C_{q-1}$	
o_{10}	$E_q^2 : \mathrm{O}^-(2,q)$	
o_{12}	$\mathrm{GL}(2,q)$, q odd	
	$E_q^2 : \mathrm{GL}(2,q)$, q even	$E_q^2 : E_q : C_{q-1}$, q even
o_{13}	$C_{q-1} \times C_2$, q odd	$C_{q-1} \times C_2$
	$E_q : C_{q-1}$, q even	E_q, q even
o_{14}	$C_2^2 : \mathrm{Sym}_3$, $q \equiv 1(4)$	$C_2^2 : C_2$, $q \equiv 1(4)$
	$C_2^2 : C_2$, $q \equiv 3(4)$	$C_2^2 : \mathrm{Sym}_3$, $q \equiv 3(4)$
	Sym_3, q even	
o_{15}	C_2^2, q odd	C_2^2
	C_2, q even	
o_{16}	$E_q : C_{q-1}$, q odd	
	$E_q^2 : C_{q-1}$, q even	E_q^2, q even
o_{17}	C_3	

Historical context

Lines in $\langle \mathcal{V}_3(\mathbb{F}_q) \rangle$ correspond to pencils of conics in $\mathrm{PG}(2, q)$, i.e. pairs of ternary quadratic forms over $\mathbb{F}_q$.

For q odd, these were classified by Dickson (1908). Our proof has advantages, e.g. don't need to treat $q \equiv 0, 1, 2 \pmod 3$ separately.

For q even, Campbell (1927) gives a list of inequivalent pencils of conics, but does not attempt a full classification. We fill this gap.[2]

Indeed, our approach is largely field independent. We also obtain the classification for algebraically closed fields and the real numbers, and in particular recover Jordan's (1906–1907) classification for $\mathbb{C}$ and $\mathbb{R}$.

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[2] The complete classification is stated without proof in Hirschfeld's *Projective geometries of finite fields* (Oxford UP, 1998).

Und nun zur Kneipe !