Commutative Semifields

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Origin

- Finite semifields are related to many important geometric and combinatorial constructs.
- Commutative semifields are of particular interest, as historically there have been few known examples.
- The problem is approached using the connection between commutative semifields and certain types of linear sets.
- We detail computational results towards classifying examples with rank 2 and 3 over the left nucleus (R2CS and R3CS).

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Finite semifields

A **semifield** $\mathbb{S}$ is a possibly non-associative algebra with an identity and no zero divisors.

The **left nucleus** $\mathbb{N}_\ell(\mathbb{S})$ (right, middle) is the set of elements $x \in \mathbb{S}$ such that, for all $y, z \in \mathbb{S}$, $x \circ (y \circ z) = (x \circ y) \circ z$.

The intersection of the left, right, and middle nuclei is called simply the **nucleus**. The intersection of the nucleus with the commutative center is called the **center** $\mathbb{Z}(\mathbb{S})$.

The **dimension** of a semifield is the (vector space) dimension over its center; the **rank** of a semifield is its dimension the left nucleus.

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Semifield spreads

Semifields can be described in terms of **spread sets** of q^k $k \times k$ matrices over $\mathbb{F}_q$ with nonsingular pairwise differences.

A spread set over $\mathbb{F}_q$ determines a spread $\mathcal{S}$ of $\mathrm{PG}(2k-1, q)$, giving a translation plane $A(\mathcal{S})$ of order q^k.

A spread $\mathcal{S}$ is a **semifield spread** if the spread set is an additive subgroup of $\mathcal{M}_k(q)$, in which case $A(\mathcal{S})$ can be coordinatized by a semifield of order q^k whose left nucleus contains $\mathbb{F}_q$.

A semifield $\mathbb{S}$ is classified according to its isotopism class $[\mathbb{S}]$, isotopic semifields coordinatise isomorphic translation planes.

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Knuth orbit and commutativity

The **Knuth orbit** of $\mathbb{S}$ is a collection of at most six isotopism classes $\{[\mathbb{S}], [\mathbb{S}^t], [\mathbb{S}^d], [\mathbb{S}^{td}], [\mathbb{S}^{dt}], [\mathbb{S}^{tdt}]\}$, t and d denote *transpose* and *dual* operations.

Commutativity of a semifield is **not** invariant under isotopism, a semifield $\mathbb{S}$ can be considered "**commutative**" if it is isotopic to a commutative semifield, i.e. if $[\mathbb{S}^d] = [\mathbb{S}]$.

Then the subspaces of the spread associated with $\mathbb{S}^{td}$ are totally isotropic with respect to a symplectic polarity of $\mathrm{PG}(2k-1, q)$, i.e. $\mathbb{S}^{td}$ is a **symplectic** semifield.

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Linear sets

Under the field reduction map,
points of $\mathrm{PG}(t-1,q^n) \to$ Desarguesian spread of $\mathrm{PG}(tn-1,q)$.

An $\mathbb{F}_q$-**linear set** $\mathcal{L}$ of $\mathrm{PG}(t-1,q^n)$ corresponds to the spread elements intersecting some subspace U of $\mathrm{PG}(tn-1,q)$. The **rank** of $\mathcal{L}$ is $\dim(U)$.

$$\begin{array}{ccc} \text{Semifield } \mathbb{S},\ |\mathbb{S}| = q^{kn} & \longleftrightarrow & \text{Spread set } \mathcal{S} \subseteq \mathrm{PG}(k^2-1,q^n) \\ \mathbb{F}_{q^k} \subseteq \mathbb{N}_\ell(\mathbb{S}),\ \mathbb{F}_q \subseteq \mathbb{Z}(\mathbb{S}) & & \mathbb{F}_q\text{-linear, disjoint from } \mathcal{S}_{k,k}(q^n) \end{array}$$

If $\mathbb{S}$ is symplectic, (WLOG) the matrices of $\mathcal{S}$ are symmetric, contained in a $\left(\frac{k(k+1)}{2}-1\right)$-dimensional subspace intersecting $\mathcal{S}_{k,k}(q^n)$ in a quadratic Veronesean $\mathcal{V}_k$.

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Rank 2 commutative semifields

Easiest place to start is with commutative semifields having rank 2 over their left nucleus (R2CS).

Cohen and Ganley showed that any R2CS of order q^{2n} with center $\mathbb{F}_q$, q odd, arises from a pair (f, g) of $\mathbb{F}_q$-linear functions such that $g^2(t) - 4tf(t)$ is a nonsquare for all $t \in \mathbb{F}_{q^n}^*$.

This is equivalent to the existence of a rank n $\mathbb{F}_q$-linear set

$$\mathcal{W} = \{(t, f(t), g(t)) : t \in \mathbb{F}_{q^n}^*\}$$

contained in the set of interior points $\mathcal{I}(\mathcal{C})$ of the conic $\mathcal{C}$ with equation $X_2^2 - 4X_0X_1 = 0$ in $\mathrm{PG}(2, q^n)$.

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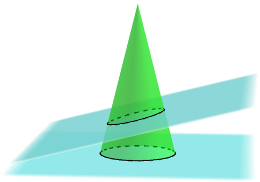
Semifield flocks and translation ovoids

These functions f and g give connections to many other important geometric objects. For example

- The functions f and g defining a R2CS determine planes
$$\{\pi_t : tX_0 + f(t)X_1 + g(t)X_2 + X_3 = 0 \mid t \in \mathbb{F}_{q^n}\}$$
in $\mathrm{PG}(3, q^n)$ forming a **semifield flock of a quadratic cone**.



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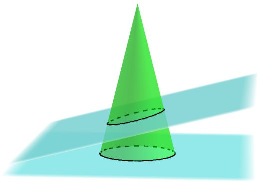

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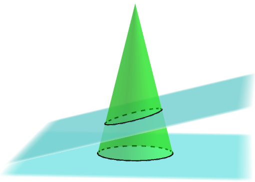

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Known R2CS examples and bounds

There are no R2CS with q even. For q odd, there are very few known examples.

- The Dickson semifields (1906).
- The Cohen-Ganley semifields (1982), which have order 3^{2n} for $n \geq 2$ and center $\mathbb{F}_3$.
- The example found by Penttila and Williams (2000), having order 3^{10} and center $\mathbb{F}_3$.

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Structure of linear sets

If the linear set $\mathcal{W}$ associated with an R2CS is contained in a line, then it must be a Dickson semifield; to find new semifields we need linear sets that contain an $\mathbb{F}_q$-subplane.

Theorem (Ball, Blokhuis, Lavrauw 2003; Lavrauw, 2006)

If there exists an $\mathbb{F}_q$-subplane in $\mathrm{PG}(2, q^n)$ contained in $\mathcal{I}(\mathcal{C})$ then there exists an $\mathbb{F}_q$-subline contained in $\ell \cap \mathcal{I}(\mathcal{C})$ with ℓ external to $\mathcal{C}$; such a subline does not exist for

$$q \geq 4n^2 - 8n + 2,$$

or for

$$q > 2n^2 - (4 - 2\sqrt{3})n + (3 - 2\sqrt{3})$$

when q is prime.

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Outline of the algorithm

This leads us to the following strategy for searching for $2k$-dimensional R2CS:

- Determine values of q for which there actually exists an $\mathbb{F}_q$-subline contained in $\mathcal{I}(\mathcal{C})$ spanning an external line to $\mathcal{C}$ (for some fixed conic $\mathcal{C}$).
- For these values, find all $\mathbb{F}_q$-sublines contained in $\mathcal{I}(\mathcal{C})$ and determine whether there are two which generate a suitable $\mathbb{F}_q$-subplane.
- Use a clique-finding algorithm to determine if subplanes can be combined to give a rank k $\mathbb{F}_q$-linear set contained in $\mathcal{I}(\mathcal{C})$.

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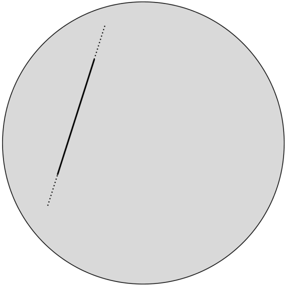
Sublines

Want to find $\mathbb{F}_q$ sublines contained in $\mathcal{I}(\mathcal{C})$ spanning a line external to $\mathcal{C}$, using the group of the conic we can fix an external line ℓ_e and a point $\boldsymbol{x}$, restrict our search for sublines of ℓ_e containing $\boldsymbol{x}$.



Subplanes and higher dimensional spaces

Looking for $\mathbb{F}_q$-subplanes in $\mathcal{I}(\mathcal{C})$, again restrict to subplanes containing $\boldsymbol{x}$.

We want **all** $\mathbb{F}_q$-sublines on $\boldsymbol{x}$, whether they span a secant or an external line to $\mathcal{C}$.

For each pair of sublines, generate points of their subplane one-by-one, check they are in $\mathcal{I}(\mathcal{C})$; if the test fails for a single point reject the pair as incompatible.

For ℓ occuring first in a compatible pair, define a graph on the sublines occuring second with adjacency given by compatibility.

A rank k $\mathbb{F}_q$-linear set will correspond to a $\begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q - 1$ clique in such a graph, so we search for cliques of the appropriate size.

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Looking for $\mathbb{F}_q$-subplanes in $\mathcal{I}(\mathcal{C})$, again restrict to subplanes containing $\boldsymbol{x}$.

We want **all** $\mathbb{F}_q$-sublines on $\boldsymbol{x}$, whether they span a secant or an external line to $\mathcal{C}$.

For each pair of sublines, generate points of their subplane one-by-one, check they are in $\mathcal{I}(\mathcal{C})$; if the test fails for a single point reject the pair as incompatible.

For ℓ occuring first in a compatible pair, define a graph on the sublines occuring second with adjacency given by compatibility.

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Results

	Number of sublines on $\boldsymbol{x} \in \ell_e$		
q	$n = 3$	$n = 4$	$n = 5$
3	12	120	1200
5	12	600	15072
7	24	912	52080
9	0	1040	91880
11	0	744	115572
13	0	504	102340
17	-	72	≥ 1
19	-	80	≥ 1
23	-	0	≥ 1
25	-	0	≥ 1
27	-	0	≥ 1
29	-	0	≥ 1
31	-	-	≥ 1

8-dimensional

An 8-dimensional R2CS is equivalent to a rank 4 linear set contained in $\mathcal{I}(\mathcal{C})$ for some conic $\mathcal{C}$ in $\mathrm{PG}(2, q^4)$.

While the bound tells us we could have $q < 30$, $q = 19$ is the largest value for which there is a suitable subline.

There are subplanes only when $q = 3$; have 237 graphs to test, the largest containing 204 vertices. Obtain 174 total cliques of size 12 all giving equivalent rank 4 linear sets (Cohen-Ganley R2CS).

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Searching for suitable $\mathbb{F}_q$-linear sublines in $\mathrm{PG}(2, q^5)$ is more computationally difficult; as q starts to grow, we cannot search exhaustively.

We can complete the search for $q = 3$, finding two nonequivalent rank 5 linear sets (Cohen–Ganley and the Penttila–Williams examples).

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A 10-dimensional R2CS with center $\mathbb{F}_3$ is either a Dickson semifield, of Cohen–Ganley type, or Penttila–Williams.

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Rank 3 commutative semifields

The situation for semifields having rank 3 over the left nucleus is more complicated.

For the case of semifields 6-dimensional over center $\mathbb{F}_q$, we need rank 6 $\mathbb{F}_q$-linear sets in $\mathrm{PG}(5, q^2)$ disjoint from the secant variety of the quadratic Veronesean $\mathcal{V}$.

There are no known examples of 6-dimensional R3CS when q is even. When q is odd, we have

- the twisted fields (Albert, 1961);
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A bound on q

Due to a recent result of Marino and Pepe (2016), we do have a bound on q.

They show that, when $q^2 > 2 \cdot 3^8$ (so $q > 114$) the list of R3CS from the previous slide is complete.

Furthermore, new examples for $q < 114$ must correspond to a $\mathbb{F}_q$-linear set consisting of $q^3 + q^2 + q + 1$ lines passing through a common point (a cone with base $\mathrm{PG}(3, q)$).

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Searching for linear sets

To extend our work on R2CS to the case of R3CS, we adapted our algorithms to the new setting of the quadratic Veronesean.

Searching for a suitable $\mathbb{F}_3$-linear set in $\mathrm{PG}(5,9)$, we worked in the quotient space wrt a point $\mathbf{x}$. Our task was then to find a rank 4 linear set (subgeometry) contained in a set $\mathcal{U}$ of "allowed" points.

We build such a set by adding a point at a time, forming a $\mathbb{F}_q$-basis for the linear set, and removing orbits of points from $\mathcal{U}$ as we went.

After a few months(!) of computer time:

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A 6-dimensional rank 3 commutative semifield with center $\mathbb{F}_3$ is isotopic to either a twisted field, or one of the examples given by BH, LMPT, ZP.

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Final remarks

Our techniques worked very well for classifying the 8-dimensional R2CS, but the 10-dimensional classification is much more difficult.

- Can we improve the bound on q in terms of n?
- Can we impose stronger structural requirements on the linear set $\mathcal{W}$?
- Is it even possible to have subplanes contained in $\mathcal{I}(\mathcal{C})$ when $q \not\equiv 0 \bmod 3$?

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