Codes in classical association schemes

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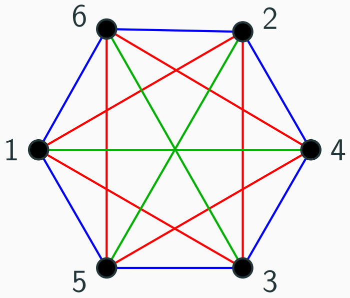
- Hermitian matrices,
- Alternating matrices,
- Quadratic forms (or cosets of alternating matrices),
- ...and connections to other objects.

Association schemes

In coding theory and related subjects, an association scheme (such as the Hamming scheme) should mainly be viewed as a "structured space" in which the objects of interest (such as codes, or designs) are living.

— Delsarte & Levenshtein, 1998

A simple example



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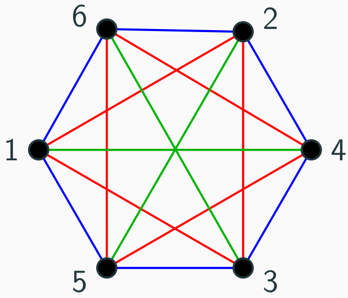

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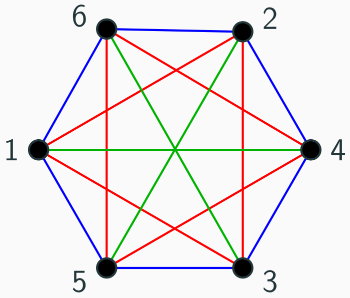

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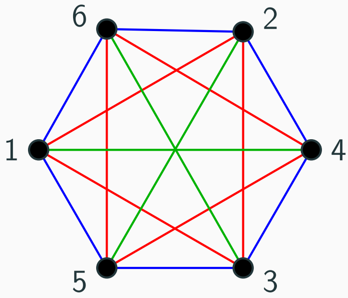

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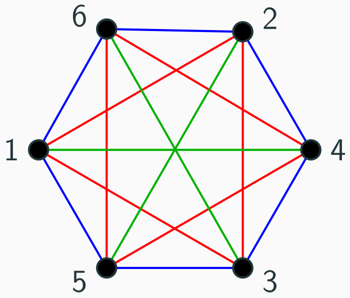

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$$D_1 D_2 = 1 \cdot D_1 + 0 \cdot D_2 + 2 \cdot D_3$$

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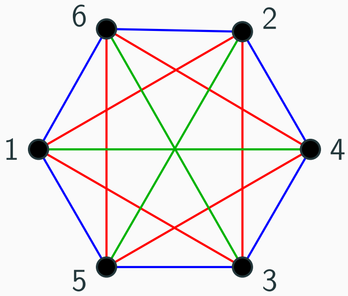

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The matrices I, D_1, D_2, D_3 generate a commutative algebra:

$$D_1 D_2 = D_2 D_1 = D_1 + 2D_3 \qquad D_1^2 = 2I + D_2$$

$$D_1 D_3 = D_3 D_1 = D_2 \qquad D_2^2 = 2I + D_2$$

$$D_2 D_3 = D_3 D_2 = D_1 \qquad D_3^2 = I$$

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Combinatorial definition

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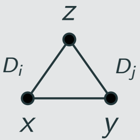

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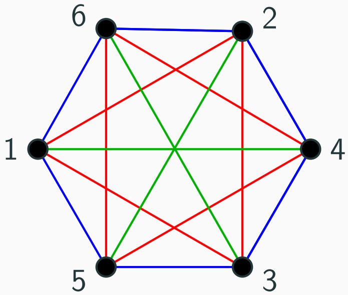
The number $P_i(k)$ is an eigenvalue of D_i whose eigenspace is the column space of E_k.

Subsets and inner distribution



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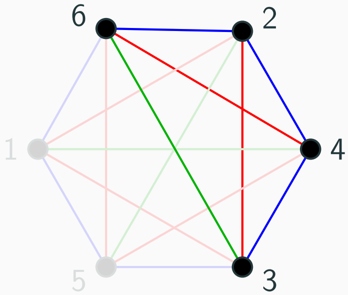

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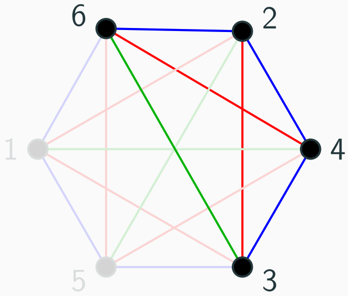

Inner distribution: $\frac{1}{4}(4, 6, 4, 2)^T$.

Subsets and duality

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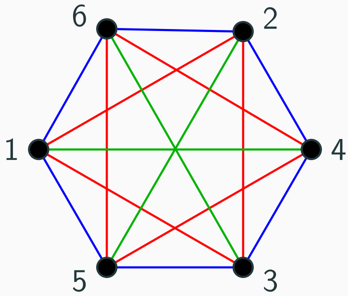
Proof: E_k has eigenvalues 0 or 1, so is positive semidefinite.

A linear program



$$Q = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & -1 & 2 \\ 2 & 1 & -1 & -2 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

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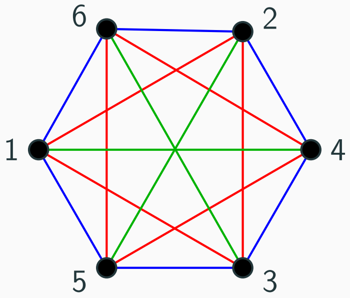

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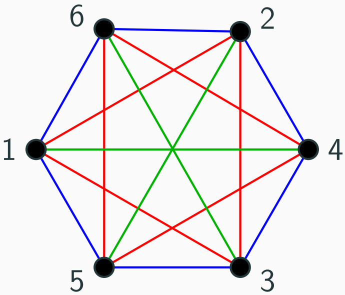

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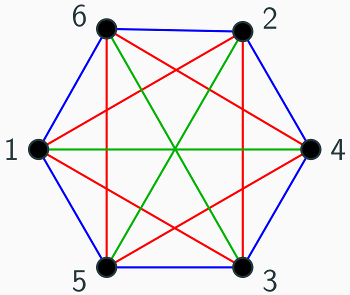

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Unique solution: $\boldsymbol{a} = (1, 0, 2, 0)^T$.

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Moreover, $cx = yb$ if and only if x and y are both optimal solutions.

Translation schemes

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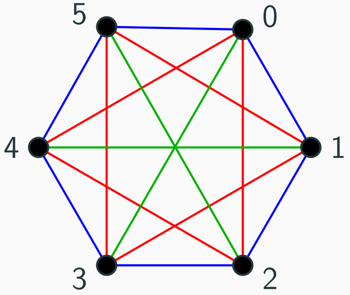

A translation scheme on $(\mathbb{Z}_6, +)$ with the partition

$X_0 = \{0\}$
$X_1 = \{1, 5\}$
$X_2 = \{2, 4\}$
$X_3 = \{3\}$.

Duality of translation schemes

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The role of the P- and the Q-numbers are swapped in the dual translation scheme.

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For additive subsets Y, we have the divisibility constraints

$$a_i \in \mathbb{Z}, \quad a'_k/|Y| \in \mathbb{Z}.$$

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All are self-dual translation schemes.

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- $b = q^2$ and $c = q$ or $c = 1/q$ and $n = \lfloor m/2 \rfloor$ in $\mathrm{Alt}(m, q)$ (Delsarte-Goethals 1975).

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If the condition does not hold, then the bound still holds for additive codes.

Bounds for d-codes in $\mathrm{Her}(n, q)$

Theorem (S. 2017).

For odd d, every d-code Y in $\mathrm{Her}(n, q)$ satisfies

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The bounds are tight, except possibly when n and d are even.

Constructions of optimal additive codes

Every Hermitian form $H : \mathbb{F}_{q^{2n}} \times \mathbb{F}_{q^{2n}} \to \mathbb{F}_{q^2}$ can be uniquely written as

$$H(x, y) = \mathrm{Tr}(y^q L(x)),$$

where

$$L(x) = \sum_{i=1}^{n} a_i x^{q^{2i}} \in \mathbb{F}_{q^{2n}}[x], \qquad a_{n-i+1} = a_i^{q^{2n-2i+1}}.$$

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- For even n and odd d, take $a_{(n-d+3)/2} = \cdots a_{n/2} = 0$.
- For even n and even d, I don't know, except when $d \in \{2, n\}$.

Constructions in the non-additive case

Theorem
(Gow-Lavrauw-Sheekey-Vanhove 2014, S. 2017).

Let n be even and let Z be a set of q^n matrices over $\mathbb{F}_{q^2}$ of size $n/2 \times n/2$ with the property that $A - B$ is nonsingular for all distinct $A, B \in Z$. Let

$$Y = \left\{ \begin{pmatrix} I & A^* \\ A & AA^* \end{pmatrix} : A \in Z \right\} \cup \left\{ \begin{pmatrix} O & O \\ O & I \end{pmatrix} \right\},$$

Then Y is an n-code in $\mathrm{Her}(n, q)$ of size $q^n + 1$.

LP bounds

Theorem (S. 2017).

For even d, every d-code Y in $\mathrm{Her}(n, q)$ satisfies

$$|Y| \leq q^{n(n-d+1)} \frac{q^n(q^{n-d+1}+(-1)^n)-(-1)^n(q^{n-d+2}-(-1)^n)}{q^{n-d+1}(q+1)}.$$

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For $d = n$, this is $|Y| \leq q^{2n-1} - q^n + q^{n-1}$ (Thas 1992).

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Some numbers for 2-codes in $\mathrm{Her}(2,q)$:

q	Largest add. code	Largest code	LP	SDP
2	4	5	6	5
3	9	15	21	17
4	16	24	52	43
5	25	47	105	89

The unique 2-code in $\mathrm{Her}(2,3)$ of size 15

For every of the 15 pairs of matrices over $\mathbb{F}_9$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \theta^3 \\ \theta^2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \theta^2 \\ \theta^3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & \theta^{-2} \\ \theta^{-3} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \theta^{-3} \\ \theta^{-2} & 0 \end{bmatrix}$$

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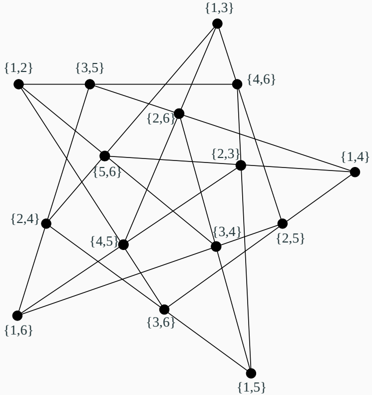

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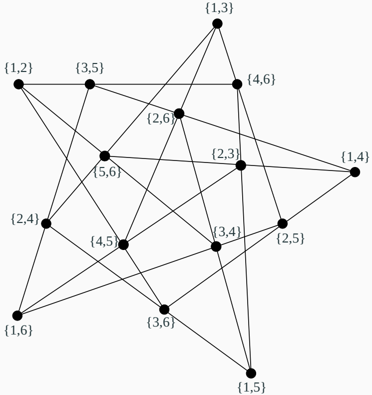

The Cremona-Richmond configuration.

Partial spreads in the Hermitian polar space

Partial spread in $H(2n-1,q^2)$: Collection of n-dimensional subspaces in $H(2n-1,q^2)$ with pairwise trivial intersection.

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The correspondence is: $Y \mapsto \{\langle O \mid I \rangle\} \cup \{\langle I \mid M \rangle \,:\, M \in Y\}$.

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For even n, several bounds have been obtained by (De Beule-Klein-Metsch-Storme 2008, Ihringer 2014, M. Schmidt 2016, Ihringer-Sin-Xiang 2018).

Bounds for d-codes in $\mathrm{Alt}(m, q)$

Theorem (Delsarte-Goethals 1975).

Every d-code Y in $\mathrm{Alt}(m, q)$ satisfies

$$|Y| \leq \begin{cases} q^{m((m-1)/2-d+1)} & \text{for odd } m \\ q^{(m-1)(m/2-d+1)} & \text{for even } m. \end{cases}$$

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This bound is tight when m is odd.

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For odd q and $n > 2$, no nontrivial d-codes in $\mathrm{Alt}(2n, q)$ meeting the LP bound are known to exist.

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- For $m = 8$, there at least three different inner distributions:

$$(1, 0, 0, 85, 170), \quad (1, 0, 1, 80, 174), \quad (1, 0, 2, 75, 178).$$

Two (nonclassical) association schemes

Sym(m, q): $m \times m$ symmetric matrices over $\mathbb{F}_q$

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Qua(m, q): cosets of $m \times m$ alternating matrices over $\mathbb{F}_q$

The group $\mathbb{F}_q^\times \times \mathrm{GL}_m(\mathbb{F}_q)$ acts on $\mathrm{Sym}(m, q)$ and $\mathrm{Qua}(m, q)$ by

$$((\lambda, L), S) \mapsto \lambda \cdot LSL^T$$
$$((\lambda, L), [A]) \mapsto [\lambda \cdot LAL^T].$$

In each case there is one orbit for each odd rank and two orbits for each nonzero even rank.

These orbits define two translation association association schemes with $m + \lfloor m/2 \rfloor + 1$ classes.

P- and Q-numbers

The character group of Sym(m, q) can be identified with Qua(m, q) and Qua(m, q) and Sym(m, q) are dual to each other. In particular, Sym(m, q) is self-dual for odd q.

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Theorem (S. 2015, 2017).

The P- and Q-numbers of $\mathrm{Sym}(m, q)$ and $\mathrm{Qua}(m, q)$ can be expressed as linear combinations of generalised Krawtchouk polynomials.

Special cases (Bachoc-Serra-Zemor 2017) and recurrence relations (Feng-Wang-Ma-Ma 2008) were known before.

A nice form of the duality

$\mathrm{Sym}(m, q)$	$\mathrm{Qua}(m, q)$
$A_s = a_{2s^+} + a_{2s^-} + a_{2s-1}$	$A'_r = a'_{2r^+} + a'_{2r^-} + a'_{2r-1}$
$B_s = a_{2s^+} + a_{2s^-} + a_{2s+1}$	$B'_r = a'_{2r^+} + a'_{2r^-} + a'_{2r+1}$
$C_s = q^{-s}(a_{2s^+} - a_{2s^-}) \quad (q \text{ odd})$	$C'_r = q^{-r}(a_{2r^+} - a_{2r^-})$
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A nice form of the duality

Sym(m,q) | Qua(m,q)

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Then

$$\boldsymbol{A}' = Q_{m+1}\boldsymbol{A}, \qquad \boldsymbol{B}' = q^m\, Q_m\, \boldsymbol{C}, \qquad \boldsymbol{C}' = Q_m \boldsymbol{B},$$

where Q_m is the Q-matrix of Alt(m,q).

Bounds in Sym(m, q)

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Theorem (S. 2017).

For odd d, every d-code Y in Sym(m, q) satisfies

$$|Y| \le \begin{cases} q^{m(m-d+2)/2} & \text{for even } m-d, \\ q^{(m+1)(m-d+1)/2} & \text{for odd } m-d. \end{cases}$$

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The cases $m - d \in \{1, 2\}$ were first obtained by (Gow 2014).

Some numbers for Sym($m, 2$)

(m, d)	Largest add. code	Largest code	LP bound
$(3, 2)$	16	$= 22$	24
$(4, 2)$	256	≥ 320	384
$(5, 4)$	64	≥ 96	196

The constructions are from (M. Schmidt 2016).

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The optimal 2-code in Sym$(3, 2)$: Take the zero matrix together with the 21 nonalternating matrices of rank 2.

Bounds in Qua(m, q) for even q

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Theorem (S. 2017).

Let q be even and let Y be a d-code in Qua(m, q). Then

$$|Y| \le \begin{cases} q^{m(m-d+2)/2} & \text{for odd } m \text{ and odd } d, \\ q^{(m+1)(m-d+1)/2} & \text{for even } m \text{ and odd } d, \\ q^{(m-1)(m-d+2)/2} & \text{for even } m \text{ and even } d, \\ q^{m(m-d+1)/2} & \text{for odd } m \text{ and even } d. \end{cases}$$

These bounds are tight. If d is odd and equality holds, then the inner distribution of Y is determined.

Applications to coding theory

$$\begin{aligned} \mathrm{Qua}(m,q) &\cong \mathrm{GRM}(2,m)/\,\mathrm{GRM}(1,m) \\ \text{inner distribution} &\Leftrightarrow \text{distance distribution} \end{aligned}$$

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type	rank	minimum weight of coset
parabolic	$2s+1$	$q^{m-1}(q-1) - q^{m-s-1}$
elliptic	$2s$	$q^{m-1}(q-1) - q^{m-s-1}$
hyperbolic	$2s$	$(q^{m-1} - q^{m-s-1})(q-1)$

Elliptic and hyperbolic d-codes

Theorem (S. 2017).

Let Y be an elliptic $(2d)$-code in $\mathrm{Qua}(2n, q)$. Then

$$|Y| \leq q^{2n(n-d+1/2)}.$$

This bounds is tight, and if equality holds, then the inner distribution of Y is determined.

The same bound holds for additive hyperbolic d-codes.

Codes and their distance distributions

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For $q = 2$: (Berlekamp 1970), (Kasami 1971)

For odd q: (Feng & Luo 2008), (Luo & Feng 2008), (Y. Liu & Yan 2013), (X. Liu & Luo 2014a), (X. Liu & Luo 2014b), (Y. Liu, Yan & Ch. Liu 2014), (Zheng, Wang, Zeng & Hu 2014), ...

Codes in classical association schemes

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