Cameron-Liebler sets in different settings

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Kloster Irsee, September 13, 2017

Outline

Introductory remarks

- Many links with Erdős-Ko-Rado problems.
- Algebraic techniques: of great importance for Cameron-Liebler problems.

OUTLINE

Definition

Cameron-Liebler line sets in $\mathrm{PG}(3, q)$: arise from attempt of Cameron and Liebler to classify collineation groups of $\mathrm{PG}(n, q)$, $n \geq 3$, that have equally many orbits on lines and on points.

Definition

Spread of PG$(3, q)$: set of $q^2 + 1$ lines partitioning point set of $\mathrm{PG}(3, q)$.

Classical example: Regular spread.

DEFINITION

- A: incidence matrix of points and lines of PG$(3, q)$.
- χ: characteristic vector of set of lines of PG$(3, q)$.

Equivalent definitions for CL sets in PG(3, q)

Definition

Cameron-Liebler set of lines $\mathcal{L}$ of $\mathrm{PG}(3,q)$:

- There exists integer x such that $\mathcal{L}$ shares x lines with every spread of $\mathrm{PG}(3,q)$.
- There exists integer x such that $\mathcal{L}$ shares x lines with every regular spread of $\mathrm{PG}(3,q)$.
- There exists integer x such that for every line ℓ of $\mathrm{PG}(3,q)$:

$$|\{m \in \mathcal{L} \setminus \{\ell\} | m \text{ meets } \ell\}| = x(q+1) + (q^2-1)\chi(\ell).$$

Equivalent definitions for CL sets in PG(3, q)

Definition

Cameron-Liebler set of lines $\mathcal{L}$ of PG(3, q):

- There exists integer x such that for every incident point-plane pair (P, π) of PG(3, q):

$$|\text{Star}(P) \cap \mathcal{L}| + |\text{Line}(\pi) \cap \mathcal{L}| = x + (q+1)|\text{Pencil}(P, \pi) \cap \mathcal{L}|.$$

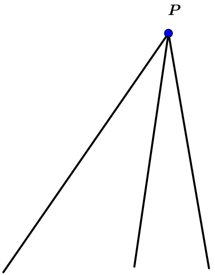
- There exists integer x such that for every pair of disjoint lines ℓ and m of PG(3, q):

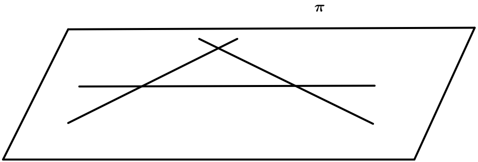
$$|\{n \in \mathcal{L} | n \text{ meets } \ell \text{ and } m\}| = x + q(\chi(\ell) + \chi(m)).$$

- $\chi \in \text{row}(A)$ ($\Leftrightarrow \chi \in \ker(A^T)^\perp$).

Classical examples

- Set L_P of lines through point P: CL-set with $x = 1$.
- Set L_π of lines in plane π: CL-set with $x = 1$.



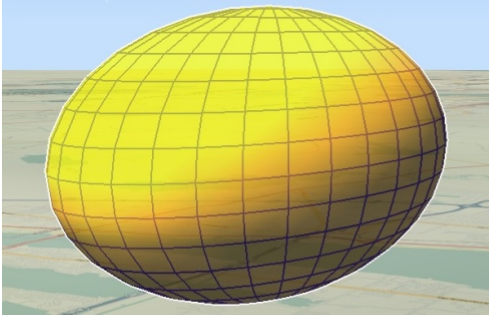


CLASSICAL EXAMPLES

- $L_P \cup L_\pi$, with $P \notin \pi$: CL-set with $x = 2$.
- Complement of preceding examples.

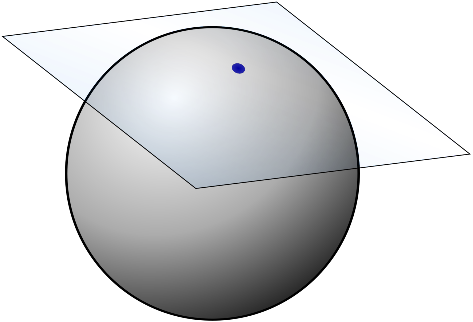
CL-sets of Bruen and Drudge

Elliptic quadric $Q^{-}(3, q)$



CL-SETS OF BRUEN AND DRUDGE

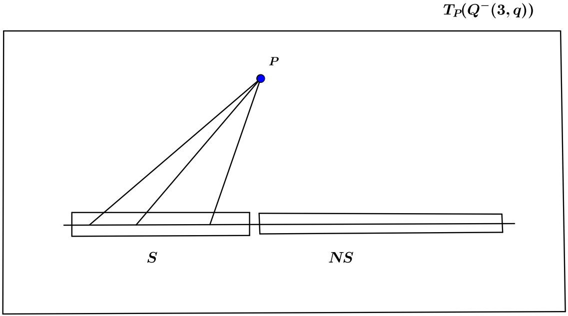
Tangent plane to point of $Q^{-}(3, q)$



CL-sets of Bruen and Drudge

- $Q^-(3, q)$, q odd: quadratic form Q.
- $v \notin Q^-(3, q)$:
  - set S: $Q(v)$ non-zero square.
  - set NS: $Q(v)$ non-square.

CL-sets of Bruen and Drudge



CL-sets of Bruen and Drudge

- S_0: set of skew lines to $Q^-(3;q)$.
- $S_{1,S}$: set of tangent lines through $P \in Q^-(3,q)$ and points of S in $T_P(Q^-(3,q))$.
- $S_{1,NS}$: set of tangent lines through $P \in Q^-(3,q)$ and points of NS in $T_P(Q^-(3,q))$.
- $S_0 \cup S_{1,S}$ and $S_0 \cup S_{1,NS}$: CL-set with $x = (q^2+1)/2$.

OTHER EXAMPLES

- Derivation construction of Gavrilyuk, Matkin, Penttila, and Cossidente, Pavese: $x = (q^2 + 1)/2$.
- CL-set $\mathcal{L}$ with $x = (q^2 - 1)/2$ by De Beule, Demeyer, Metsch, Rodgers and Feng, Momihara, Xiang.
- Plane π contains no lines of $\mathcal{L}$: $\mathcal{L}$ union Line(π): CL-set with $x = (q^2 + 1)/2$.

Modular equality of Gavrilyuk-Metsch

Theorem (Gavrilyuk, Metsch)

Cameron-Liebler line set $\mathcal{L}$ with parameter x. Then for every plane π and every point P:

$$\binom{x}{2} + n(n-x) \equiv 0 \pmod{q+1},$$

where n is number of lines of $\mathcal{L}$ through P in plane π.

Theorem (Gavrilyuk, Metsch)

There is no Cameron-Liebler set in $PG(3,q)$, q odd, with parameter x, where $x \equiv 3 \pmod{4}$.

EQUIVALENT DEFINITIONS FOR CL K-SETS

DEFINITION (RODGERS, STORME, VANSWEEVELT)

Let $\mathcal{L}$ be set of k-spaces in $\mathrm{PG}(2k+1,q)$ with characteristic function χ. Then the following are equivalent:

- Integer x for which $|\mathcal{L} \cap \mathcal{S}| = x$ for every k-spread in $\mathrm{PG}(2k+1,q)$.
- Integer x for which $|\mathcal{L} \cap \mathcal{S}| = x$ for every regular k-spread in $\mathrm{PG}(2k+1,q)$.
- $\chi \in \mathrm{row}(A)$ ($\Leftrightarrow \chi \in \ker(A^T)^{\perp}$).
- Integer x such that for every k-space π, number of elements of $\mathcal{L}$ disjoint from π is $(x-\chi(\pi))q^{(k+1)k}$.
- Integer x such that $\chi - \frac{x}{q^{k+1}+1}j$ is eigenvector of K for eigenvalue $\tau = -q^{(k+1)k}$.

NOTATION

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q = \prod_{i=0}^{k} \frac{q^{n+1-i}-1}{q^{i+1}-1}.$$

(Number of k-spaces in PG(n,q))

Erdős-Ko-Rado results

Theorem (Rodgers, Storme, Vansweevelt)

Cameron-Liebler k-set of $PG(2k+1, q)$ with parameter $x = 1$ is either:

- *all k-spaces through one point P.*
- *all k-spaces in one hyperplane H.*

Theorem

Largest Erdős-Ko-Rado sets of k-spaces in $PG(2k+1, q)$ have size $\begin{bmatrix} 2k+1 \\ k+1 \end{bmatrix}_q$ and are equal to either: all k-spaces through one point P or all k-spaces in one hyperplane H.

Cameron-Liebler k-sets with parameter $x = 2$

Theorem (Rodgers, Storme, Vansweevelt)

Cameron-Liebler k-set of $PG(2k+1, q)$ with parameter $x = 2$ is union of:

- *all k-spaces through one point P,*
- *all k-spaces in one hyperplane H,*

with $P \notin H$.

Cameron-Liebler 2-sets in PG$(5, q)$ and EKR-sets

Theorem

Let $\mathcal{S}$ be set of planes of PG$(5, q)$ pairwise intersecting non-trivially. Then

- *either* $|\mathcal{S}| = \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q$ *and $\mathcal{S}$ consists of all planes through fixed point P or all planes in fixed hyperplane H of PG$(5, q)$.*
- *(Blokhuis, Brouwer and Szőnyi) If* $|\mathcal{S}| > q^5 + 2q^4 + 3q^3 + 2q^2 + q + 1$, *then $\mathcal{S}$ is contained in a preceding example.*

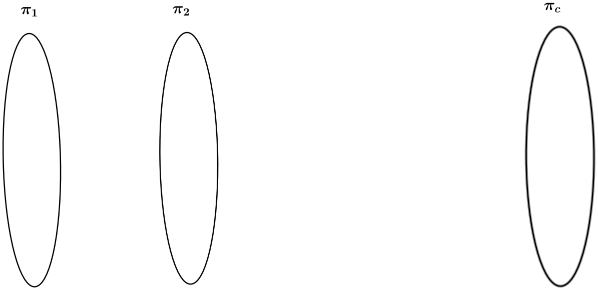
Erdős-Ko-Rado results

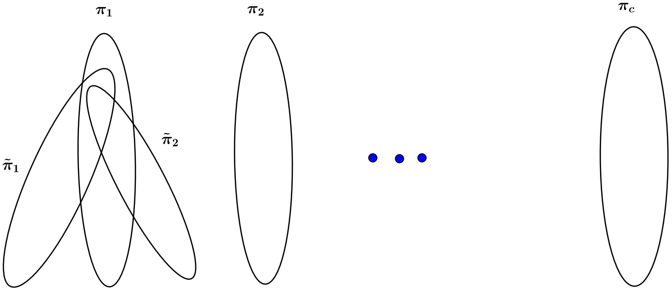
Theorem

Largest Erdős-Ko-Rado sets of k-spaces in $PG(2k+1, q)$ have size $\begin{bmatrix} 2k+1 \\ k+1 \end{bmatrix}_q$ and are equal to either: all k-spaces through one point P or all k-spaces in one hyperplane H.

Theorem (Blokhuis, Brouwer, Szőnyi)

Let $\mathcal{L}$ be Erdős-Ko-Rado set of k-spaces in $PG(2k+1, q)$, with $k+1 < q\log q - q$ and $|\mathcal{L}| > \frac{1}{2}q^{(k+1)k}$, then all k-spaces of $\mathcal{L}$ pass through one point P or lie in one hyperplane H.


π_1
π_2
π_c


π_1
π_2
π_c
$\tilde{\pi}_1$
$\tilde{\pi}_2$

Results for general k

Theorem (Metsch)

Let $\mathcal{L}$ be Cameron-Liebler 2-set in $PG(5, q)$, with parameter x. Then it is not possible that $3 \leq x < q/3$.

Theorem (Metsch)

Let $\mathcal{L}$ be Cameron-Liebler k-set in $PG(2k+1, q)$, $3 < k < q \log q - q - 1$, $q \geq q_0$, with parameter x. Then it is not possible that $3 \leq x < q/5$.

Outline

Finite classical polar spaces

- Hyperbolic quadric $Q^+(2n+1,q)$:
  $X_0X_1 + X_2X_3 + \cdots + X_{2n}X_{2n+1} = 0$,
- Elliptic quadric $Q^-(2n+1,q)$:
  $f(X_0,X_1) + X_2X_3 + \cdots + X_{2n}X_{2n+1} = 0$, $f(X_0,X_1)$ irreducible homogeneous quadratic equation over $\mathbb{F}_q$,
- Parabolic quadric $Q(2n,q)$:
  $X_0^2 + X_1X_2 + X_3X_4 + \cdots + X_{2n-1}X_{2n} = 0$,
- Hermitian variety $H(n,q^2)$: $X_0^{q+1} + \cdots + X_n^{q+1} = 0$,
- Symplectic polar space $W(2n+1,q)$: defined by totally isotropic spaces under symplectic polarity.

Cameron-Liebler sets in finite classical polar spaces

- Hyperbolic quadric $Q^+(2n+1, q)$: points, lines, ..., n-spaces,
- Elliptic quadric $Q^-(2n+1, q)$: points, lines, ..., $(n-1)$-spaces,
- Parabolic quadric $Q(2n, q)$: points, lines, ..., $(n-1)$-spaces,
- Hermitian variety $H(n, q^2)$: points, lines, ..., $\lfloor (n-1)/2 \rfloor$-spaces,
- Symplectic polar space $W(2n+1, q)$: points, lines, ..., n-spaces.

Generators on hyperbolic quadric

- Set of generators Ω on $Q^+(2n+1,q)$ can be partitioned into two equivalence classes Ω_1 and Ω_2:

$$\Pi_1 \sim \Pi_2 \Leftrightarrow \dim(\Pi_1 \cap \Pi_2) \equiv n \pmod 2.$$

- Ω_1: **Latin generators.**
  Ω_2: **Greek generators.**
- For $Q^+(4n+1,q)$, generators of Ω_1 pairwise intersect (EKR-set).
- For $Q^+(4n+1,q)$, generators of Ω_2 pairwise intersect (EKR-set).

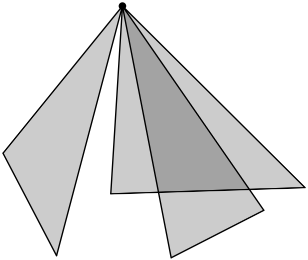
Generators in finite classical polar spaces

- Finite classical polar space **rank** N: generators have dimension $N-1$.
- Finite classical polar space: parameters (q, q^e):
  1. q = size finite field,
  2. $q^e + 1$ = number of generators through codimension one space to generator.
- Example: $Q^+(2n+1, q)$: parameters (q, q^0). ($(n-1)$-space on $Q^+(2n+1, q)$ lies in two generators.)

Erdős-Ko-Rado problem in finite classical polar spaces

Problem:

- What are largest sets of generators in finite classical polar space P, pairwise intersecting non-trivially?
- All generators of P through fixed point (point-pencil = p.-p.).



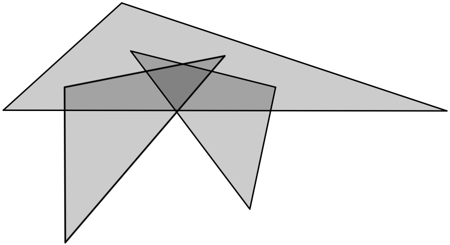
Erdős-Ko-Rado problem in finite classical polar spaces

Problem:

- What are largest sets of generators in finite classical polar space P, pairwise intersecting non-trivially?

- Sometimes different largest Erdős-Ko-Rado sets.
- $W(5,q)$, q odd: Largest Erdős-Ko-Rado sets of planes:
  - all planes of $W(5,q)$, q odd, through given point,
  - all planes of $W(5,q)$, q odd, intersecting given plane π in line, including π.

$W(5, q)$, q ODD



RESULTS FOR FINITE CLASSICAL POLAR SPACES

Polar space	Maximum size	Classification
$Q^-(2n+1, q)$	$(q^2+1)\cdots(q^n+1)$	p.-p.
$Q(4n, q)$	$(q+1)\cdots(q^{2n-1}+1)$	p.-p.
$Q(4n+2, q)$, $n \geq 2$	$(q+1)\cdots(q^{2n}+1)$	p.-p., Latins $Q^+(4n+1, q)$
$Q(6, q)$	$(q+1)(q^2+1)$	p.-p., Latins $Q^+(5, q)$, base
$Q^+(4n+1, q)$	$(q+1)\cdots(q^{2n}+1)$	all Latins
Latins $Q^+(4n+3, q)$, $n \geq 2$	$(q+1)\cdots(q^{2n}+1)$	p.-p.
Latins $Q^+(7, q)$	$(q+1)(q^2+1)$	p.-p., meeting Greek in plane
$W(4n+1, q)$, $n \geq 2$, q odd	$(q+1)\cdots(q^{2n}+1)$	p.-p.
$W(4n+1, q)$, $n \geq 2$, q even	$(q+1)\cdots(q^{2n}+1)$	p.-p., Latins $Q^+(4n+1, q)$
$W(5, q)$, q odd	$(q+1)(q^2+1)$	p.-p., base,
$W(5, q)$, q even	$(q+1)(q^2+1)$	p.-p., base, Latins $Q^+(5, q)$
$W(4n+3, q)$	$(q+1)\cdots(q^{2n+1}+1)$	p.-p.
$H(2n, q^2)$	$(q^3+1)(q^5+1)\cdots(q^{2n-1}+1)$	p.-p.
$H(4n+3, q^2)$	$(q+1)(q^3+1)\cdots(q^{4n+1}+1)$	p.-p.
$H(4n+1, q^2)$, $n \geq 2$	$< \|\Omega\|/(q^{2n+1}+1)$	?
$H(5, q^2)$	$q(q^4+q^2+1)+1$	base

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CL-sets of generators in classical polar spaces

Definition

Let $\mathcal{L}$ be set of k-spaces in $\mathrm{PG}(2k+1,q)$ with characteristic function χ. Then following are equivalent:

- Some integer x for which $|\mathcal{L} \cap \mathcal{S}| = x$ for every k-spread in $\mathrm{PG}(2k+1,q)$.
- Some integer x for which $|\mathcal{L} \cap \mathcal{S}| = x$ for every regular k-spread in $\mathrm{PG}(2k+1,q)$.
- $\chi \in \mathrm{row}(A)$ ($\Leftrightarrow \chi \in \ker(A^T)^\perp$).
- Some integer x such that for every k-space π, number of elements of $\mathcal{L}$ disjoint from π is $(x-\chi(\pi))q^{(k+1)k}$.
- Some integer x such that $\chi - \frac{x}{q^{k+1}+1}j$ is eigenvector of K for eigenvalue $\tau = -q^{(k+1)k}$.

EIGENSPACES OF GRAPH Γ_j

- Γ_j: $\Pi_1 \sim_j \Pi_2$ if and only if

$$\dim(\Pi_1 \cap \Pi_2) = \dim \Pi_1 - j = \dim \Pi_2 - j.$$

- A_j: symmetric matrix: real eigenvalues.
- $\mathbb{R}\Omega = V_0 \perp V_1 \perp \cdots \perp V_N$.
- $V_0, \ldots, V_N$ are eigenspaces for all graphs $\Gamma_0, \ldots, \Gamma_N$.
-

$$V_0 = \langle (1, \ldots, 1) \rangle.$$

ROW(A)

THEOREM

$\mathbb{R}\Omega = V_0 \perp V_1 \perp \cdots \perp V_N$.

Let A be point-generator incidence matrix of $\mathcal{P}$, then

$$row(A) = V_0 \perp V_1.$$

Type I

Type I: $Q^-(2d+1,q)$, $Q(4n,q)$, $Q^+(4n+1,q)$, $W(4n-1,q)$ and $H(n,q^2)$.

Let P be finite classical polar space of type I, let A be point-generator incidence matrix.
Let $\mathcal{L}$ be set of generators of P with characteristic vector χ and let

$$x = \frac{|\mathcal{L}|}{\prod_{i=0}^{d-2}(q^{e+i}+1)}.$$

Then following statements are equivalent:

TYPE I

Type I: $Q^-(2d+1,q)$, $Q(4n,q)$, $Q^+(4n+1,q)$, $W(4n-1,q)$ and $H(n,q^2)$.

- For each fixed generator π of P, number of generators of $\mathcal{L}$ disjoint from π equals

$$(x-(\chi)_\pi)q^{(d-1)(d-2)/2+e(d-1)}.$$

- The vector $\chi - \frac{x}{q^{i+e-1}+1}j$ is contained in eigenspace of K for eigenvalue $-q^{(d-1)(d-2)/2+e(d-1)}$.
  (K = generator disjointness matrix)
- $\chi \in \text{row}(A)$ ($\Leftrightarrow \chi \in (\ker(A))^\perp$).

TYPE I

Type I: $Q^-(2d+1,q), Q(4n,q), Q^+(4n+1,q), W(4n-1,q)$ and $H(n,q^2)$.

If P has spread S, then also following two equivalences:

- $|\mathcal{L} \cap S| = x$ for every spread S of P.
- $|\mathcal{L} \cap S| = x$ for every spread $S \in C$ of P, with C class of spreads which is union of orbits under group acting transitively on pairs of disjoint generators of P.

TYPE II

Type II: $Q^+(2d-1,q)$, d even.

Similar definitions, but restrict to $\mathcal{G}$: one class of generators of $\mathcal{Q}^+(2d-1,q)$, d even.

Type III

Type III: $Q(4n+2, q)$, all q, and $W(4n+1, q)$, q even.

Let P be finite classical polar space of type III, let B be incidence matrix of generators and hyperbolic classes of P. Then

$$\text{row}(B) = V_0 \perp V_1 \perp V_d.$$

TYPE III

Type III: $Q(4n+2,q)$, all q, and $W(4n+1,q)$, q even.

Let B be incidence matrix of generators and hyperbolic classes of P, let K be generator disjointness matrix of P.
Let $\mathcal{L}$ be set of generators of P with characteristic vector χ and let

$$x = \frac{|\mathcal{L}|}{\prod_{i=1}^{d-2}(q^i+1)}.$$

Then following statements are equivalent:

TYPE III

Type III: $Q(4n+2,q)$, all q, and $W(4n+1,q)$, q even.

- For each fixed generator π of P, number of generators of $\mathcal{L}$ disjoint from π equals

$$(x-(\chi)_\pi)q^{(d-1)(d-2)/2}.$$

- The vector $\chi - \frac{x}{q^d+1}j$ is contained in eigenspace of K for eigenvalue $-q^{d(d-1)/2}$.
- $\chi \in \text{row}(B)$ ($\Leftrightarrow \chi \in (\ker(B))^\perp$).

Type III

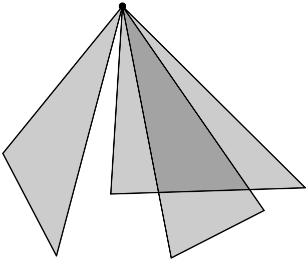
Type III: $Q(4n+2, q)$, all q, and $W(4n+1, q)$, q even.

If P has spread S, then also following two equivalences:

- $|\mathcal{L} \cap S| = x$ for every spread S of P.
- $|\mathcal{L} \cap S| = x$ for every spread $S \in C$ of P, with C class of spreads which is union of orbits under group acting transitively on pairs of disjoint generators of P.

Examples of Cameron-Liebler sets

All generators of P through fixed point (point-pencil = p.-p.) is Cameron-Liebler set with $x = 1$.



Examples of Cameron-Liebler sets

- **Partial ovoid** = set of points sharing at most one point with every generator.
- **Ovoid** = set of points sharing one point with every generator.
- Partial ovoid of size x defines Cameron-Liebler set with parameter x.

Characterization theorems

Theorem

Let $\mathcal{P}$ be finite classical polar space of type I of rank d with parameter e, and let $\mathcal{L}$ be Cameron-Liebler set of $\mathcal{P}$ with parameter x.
The number of elements of $\mathcal{L}$ meeting a generator π in a $(d-i-1)$-space equals

$$\left((x-1)\begin{bmatrix} d-1 \\ i-1 \end{bmatrix}_q + q^{i+e-1}\begin{bmatrix} d-1 \\ i \end{bmatrix}_q\right) q^{(i-1)(i-2)/2+(i-1)e}, \pi \in \mathcal{L}$$

$$x\begin{bmatrix} d-1 \\ i-1 \end{bmatrix}_q q^{(i-1)(i-2)/2+(i-1)e}, \pi \notin \mathcal{L}.$$

Classification results

Type	Polar space	condition	classification
I	$\mathcal{Q}^-(2d+1, q)$	$x \leq q+1$	x point-pencils or generators of $\mathcal{Q}(2d, q)$ ($x = q+1$)
I	$\mathcal{Q}(4n, q)$	$x \leq 2$	x point-pencils or generators of $\mathcal{Q}^+(4n-1, q)$ ($x = 2$)
I	$\mathcal{Q}^+(4n+1, q)$	$x = 1$	point-pencil
I	$\mathcal{H}(2d-1, q^2)$	$x = 1$	point-pencil
I	$\mathcal{H}(2d, q^2)$	$x \leq q+1$	x point-pencils or generators of $\mathcal{H}(2d-1, q^2)$ ($x = q+1$)
I	$\mathcal{W}(4n+3, q)$, q even	$x \leq 2$	x point-pencils generators of $\mathcal{Q}^+(4n+3, q)$ ($x = 2$)
I	$\mathcal{W}(4n+3, q)$, q odd	$x \leq 2$	x point-pencils
II	one class of $\mathcal{Q}^+(4n+3, q)$	$x = 1$	point-pencil
II	one class of $\mathcal{Q}^+(7, q)$ that admits a spread	$x \leq \frac{\sqrt{3}-1}{2}q$	x point-pencils or x base-solids
III	$\mathcal{Q}(4n+2, q)$, $n \geq 2$	$x = 1$	point-pencil or hyperbolic class
III	$\mathcal{Q}(6, q)$	$x = 1$	point-pencil, hyperbolic class or base plane
III	$\mathcal{W}(4n+1, q)$, q even, $n \geq 2$	$x = 1$	point-pencil or hyperbolic class
III	$\mathcal{W}(5, q)$, q even	$x = 1$	point-pencil, hyperbolic class or base plane
	$\mathcal{W}(4n+1, q)$, $n \geq 2$	no characterisation known	

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Classification results

Theorem (De Beule and De Boeck)

Let $\mathcal{P}$ be finite classical polar space of rank $d \geq 3$ and parameter $e \geq 1$, embedded in projective space over $\mathbb{F}_q$ and let $\mathcal{S}$ be set of generators of $\mathcal{P}$ such that

(I) *for every $i = 0, \ldots, d$, number of elements of $\mathcal{S}$ meeting generator π in $(d-i-1)$-space equals*

$$\begin{cases} \left(\begin{bmatrix} d-1 \\ i-1 \end{bmatrix}_q + q^i \begin{bmatrix} d-1 \\ i \end{bmatrix}_q \right) q^{\binom{i-1}{2}+ie-1} & \text{if } \pi \in \mathcal{S} \\ (q^{e-1}+1) \begin{bmatrix} d-1 \\ i-1 \end{bmatrix}_q q^{\binom{i-1}{2}+(i-1)e} & \text{if } \pi \notin \mathcal{S} \end{cases};$$

Classification results

Theorem (De Beule and De Boeck)

(ii) *for every point P of $\mathcal{P}$ there is generator $\pi \notin \mathcal{S}$ through P;*

(iii) *for every point P of $\mathcal{P}$ and every generator $\pi \notin \mathcal{S}$ through P, there are either $(q^{e-1}+1)\begin{bmatrix} d-2 \\ j \end{bmatrix}_q q^{\binom{j}{2}+je}$ generators of $\mathcal{L}$ through P meeting τ in $(d-j-2)$-space, for all $j = 0, \ldots, d-2$, or there are no generators of $\mathcal{L}$ through P meeting τ in $(d-j-2)$-space, for all $j = 0, \ldots, d-2$.*

Then $\mathcal{S}$ is set of generators of classical polar space of rank d and with parameter $e-1$ embedded in $\mathcal{P}$.

CLASSIFICATION RESULTS

Type	Polar space	condition	classification
I	$\mathcal{Q}^-(2d+1, q)$	$x \leq q+1$	x point-pencils or generators of $\mathcal{Q}(2d, q)$ ($x = q+1$)
I	$\mathcal{Q}(4n, q)$	$x \leq 2$	x point-pencils or generators of $\mathcal{Q}^+(4n-1, q)$ ($x = 2$)
I	$\mathcal{Q}^+(4n+1, q)$	$x = 1$	point-pencil
I	$\mathcal{H}(2d-1, q^2)$	$x = 1$	point-pencil
I	$\mathcal{H}(2d, q^2)$	$x \leq q+1$	x point-pencils or generators of $\mathcal{H}(2d-1, q^2)$ ($x = q+1$)
I	$\mathcal{W}(4n+3, q)$, q even	$x \leq 2$	x point-pencils generators of $\mathcal{Q}^+(4n+3, q)$ ($x = 2$)
I	$\mathcal{W}(4n+3, q)$, q odd	$x \leq 2$	x point-pencils
II	one class of $\mathcal{Q}^+(4n+3, q)$	$x = 1$	point-pencil
II	one class of $\mathcal{Q}^+(7, q)$ that admits a spread	$x \leq \frac{\sqrt{3}-1}{2} q$	x point-pencils or x base-solids
III	$\mathcal{Q}(4n+2, q)$, $n \geq 2$	$x = 1$	point-pencil or hyperbolic class
III	$\mathcal{Q}(6, q)$	$x = 1$	point-pencil, hyperbolic class or base plane
III	$\mathcal{W}(4n+1, q)$, q even, $n \geq 2$	$x = 1$	point-pencil or hyperbolic class
III	$\mathcal{W}(5, q)$, q even	$x = 1$	point-pencil, hyperbolic class or base plane
	$\mathcal{W}(4n+1, q)$, $n \geq 2$	no characterisation known	

Outline

1. Cameron-Liebler sets in finite projective spaces
2. Cameron-Liebler sets in finite classical polar spaces
3. Cameron-Liebler sets for ordered q-tuples

- F_q^d: set of d-tuples over $\{1, \ldots, q\}$.
- $$x \sim_i y \Leftrightarrow d_H(x, y) = d - i.$$

  (x and y differ in i positions)
- **Spread:** set of q d-tuples, pairwise at distance d.
- **Classical example:**

$$\begin{matrix} 1 & \cdots & 1 \\ \vdots & \vdots & \vdots \\ q & \cdots & q \end{matrix}$$

Cameron-Liebler sets for ordered q-tuples

Definition

Cameron-Liebler set with parameter x in F_q^d: set of d-tuples sharing x d-tuples with every spread.

Classical example:

$$\begin{matrix} 1 & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots \\ x & * & \cdots & * \end{matrix}$$

Cameron-Liebler sets for ordered q-tuples

Theorem

For every x, with $1 \leq x \leq q$, there exists Cameron-Liebler set with parameter x in F_q^d.
Up to equivalence, equivalent to

$$\begin{matrix} 1 & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots \\ x & * & \cdots & * \end{matrix}$$

Symmetry group of F_q^d implies: spread contains precisely x elements of Cameron-Liebler set with parameter x, so $q - x$ elements of spread not in Cameron-Liebler set.

Cameron-Liebler sets for ordered q-tuples

Proof:

- Up to equivalence:

$$\begin{matrix} 1 & \cdots & 1 \\ \vdots & \vdots & \vdots \\ x & \cdots & x \end{matrix}$$

  in Cameron-Liebler set with parameter x.

Cameron-Liebler sets for ordered q-tuples

Backward induction:

- Up to equivalence:

$$\begin{matrix} 1 & \cdots & 1 & * \\ \vdots & \vdots & \vdots & \vdots \\ x & \cdots & x & * \end{matrix}$$

  in Cameron-Liebler set with parameter x (by finding spread with $q - x$ other elements not in Cameron-Liebler set).

Cameron-Liebler sets for ordered q-tuples

Backward induction:

- Up to equivalence:

$$\begin{array}{cccc} 1 & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots \\ x & * & \cdots & * \end{array}$$

  in Cameron-Liebler set with parameter x (by finding spread with $q - x$ other elements not in Cameron-Liebler set).

Open problems

- Modular equality for Cameron-Liebler k-sets in $\mathrm{PG}(2k+1, q)$?
- Cameron-Liebler sets of generators in $\mathcal{W}(4n+1, q)$, $n \geq 2$?
- Classification results for Cameron-Liebler sets with parameter x in finite classical polar spaces?
- Cameron-Liebler sets in other settings?

Thank you very much for your attention