On the metric dimension of affine planes, biaffine planes and generalized quadrangles

Marcella Takáts
(Eötvös Loránd University, Budapest)
Joint work with Daniele Bartoli, Tamás Héger and György Kiss

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Let $G = (V, E)$ be a simple graph.

$d(x, y)$: distance of $x$ and $y$, $x, y \in V$

$S = \{s_1, s_2, \ldots, s_k\}$ vertex set

$x$ is resolved by $S$ if its distance list is different from all the other distance lists.
Definition (Resolving set)

The subset \( S = \{s_1, \ldots, s_k\} \subseteq V \) is a **resolving set**, if the ordered distance lists \((d(x, s_1), \ldots, d(x, s_k))\) are different for all \( x \in V \).

In other words: 

\( S = \{s_1, \ldots, s_k\} \subseteq V \) is a **resolving set** \( \iff \forall x, y \in V \ \exists z \in S : d(x, z) \neq d(y, z). \)

Definition (Metric dimension)

The **metric dimension** \( \mu(G) \) is the size of the smallest resolving set in \( G \).

Definition (Metric basis)

The **metric basis** of \( G \) is a resolving set for \( G \) of size \( \mu(G) \).
Let $G = (A \cup B, E)$ be a bipartite graph.

**split resolving set:**

$S_A \subset A$ resolves $B$

$S_B \subset B$ resolves $A$

$S_A$ and $S_B$ are called **semi-resolving sets**.
Incidence graphs of partial linear spaces (point-line incidence structures): projective planes, affine planes, biaffine planes, generalized quadrangles.
Motivation and background

Metric dimension:
- first introduced by Harary and Melter and (independently) by Slater in the 1970s
- a survey of Bailey and Cameron (2011)
- distance regular graphs: natural class of graphs to consider

**Definition (Distance regular graph)**

Given a graph $G(V, E)$ with diameter $d$, it is **distance regular** if for any $x, y \in V$, $d(x, y) = i$ the number of neighbours of $x$ at distances $i - 1, i, i + 1$ from $y$ depend only on $i$.

Interesting classes of distance regular graphs:
- $d = 2$ strongly regular graphs, distance transitive graphs (i.e. for any $x, y, x', y' \in V$ s.t. $d(x, y) = d(x', y')$ there is an automorphism $g: x^g = x', y^g = y'$).
**Motivation and background**

**Definition (Distance regular graph)**

\[ G(V, E) \text{ with diameter } d \text{ is distance regular if } \forall i : 0 \leq i \leq d \text{ for any } x, y \in V, d(x, y) = i \] the number of neighbours of \( x \) at distances \( i - 1, i, i + 1 \) from \( y \) depend only on \( i \).

- theorem of Babai (in other context) \( \Rightarrow \) primitive distance regular graphs
- method of Robert Bailey \( \Rightarrow \) imprimitive distance regular graphs, except some cases, e.g. bipartite graphs with \( d = 3, 4 \)
- 2011: Bailey asked for the metric dimension of the incidence graphs of finite projective planes
- 2012: Tamás Héger, M. T.: answered
2015: Bailey’s computer calculations on small distance regular graphs; some missing cases, e.g. the incidence graph of the Desarguesian biaffine plane of order 7 and $GQ(4, 4)$

2015: György Kiss asked for the metric dimension of the incidence graphs of other point-line geometries

2017: Daniele Bartoli, T. Héger, Gy. Kiss, M. T.: answer for affine planes, partial results for biaffine planes and generalized quadrangles
Congratulations to Daniele for Kirkman medal!
Motivation and background

Incidence graphs of partial linear spaces:
- projective planes: distance regular graphs
- $PG(2, q)$: distance transitive graphs
- biaffine planes: distance regular graphs

Resolving sets:
- nice combinatorial point-line incidence structures
- natural connection with blocking sets and (almost) double blocking sets
- thus we can use stability results for blocking sets
Notation and preliminaries

\[ G = (\mathcal{P}, \mathcal{L}, E) : \text{incidence graph of a partial linear space} \]
\[ (P, \ell) \text{ edge} \iff P \in \ell \]

\[ S = (\mathcal{P}_S \cup \mathcal{L}_S) \text{ resolving set in } \Pi \iff S \text{ is a resolving set in the incidence graph} \]

\[ d(P, \ell) = 1 \text{ or } 3, \quad d(\ell_1, \ell_2) = 2 \text{ or } 4, \quad d(P_1, P_2) = 2 \text{ or } 4 \]
Notation and preliminaries

- $\Pi$: partial linear space (projective, affine, biaffine plane, generalized quadrangle)
- $S = \mathcal{P}_S \cup \mathcal{L}_S$: set in the incidence graph of $\Pi$
- $PQ$: line joining two distinct points $P$ and $Q$
- $[P]$: set of lines through a point $P$
- $[\ell]$: set of points on a line $\ell$
- $P$ is incident with a line $\ell \leftrightarrow P$ blocks $\ell$ and $\ell$ covers $P$
- blocking set, covering set
- inner points, inner lines; outer points, outer lines
- tangent line $\longleftrightarrow$ 1-covered point
- skew line $\longleftrightarrow$ not covered point
Lemma

Let \( S = \mathcal{P}_S \cup \mathcal{L}_S \), \( \ell \) be a line in \( \Pi \). If \( |[\ell] \cap \mathcal{P}_S| \geq 2 \) then \( \ell \) is resolved by \( S \).

Dually, let \( P \) be a point in \( \Pi \). If \( |[P] \cap \mathcal{L}_S| \geq 2 \) then \( P \) is resolved by \( S \).

- Points and lines in \( S \) are resolved (trivial)
- (At least) 2-secants are resolved
- (At least) 2-covered points are resolved

We have to distinguish:
- tangents and skew lines (to \( \mathcal{P}_S \))
- 1-covered points and not covered points (by \( \mathcal{L}_S \))

"Almost" double blocking sets: resolving sets for lines

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Proposition (T. Héger, M. Takáts, 2012)

The metric dimension of a projective plane of order $q \geq 23$ is $4q - 4$.

- List of the metric basises (resolving sets of size $4q - 4$) if $q \geq 23$.

Proposition (T. Héger, P. Szilárd)

The metric dimension of any projective plane of order $q \geq 13$ is $4q - 4$. 
Proposition

$S = \mathcal{P}_S \cup \mathcal{L}_S$ is a resolving set in a finite projective plane if and only if the following properties hold for $S$:

- **P1** There is at most one outer line skew to $\mathcal{P}_S$.
- **P1’** There is at most one outer point not covered by $\mathcal{L}_S$.
- **P2** Through every inner point there is at most one outer line tangent to $\mathcal{P}_S$.
- **P2’** On every inner line there is at most one outer point that is 1-covered by $\mathcal{L}_S$. 

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Example

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And we need 2 more objects in addition:
2 lines or 1 point and 1 line
surprisingly many (more than 30) different types
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On the metric dimension of affine planes, biaffine planes and generalized quadrangles.
purely combinatorial methods, works for all projective planes
open question: metric dimension if $q$ is small ($q \leq 13$)

Fano plane:

$\mu(\text{PG}(2, 2)) = 5$

$\mu(\text{PG}(2, 4)) = 10$
construction: hyperoval
purely combinatorial methods
we can deduce it from the projective case
incidence graph is not distance regular
note: $d(\ell_1, \ell_2) = 2$ or $4$ in the incidence graph
$d$ is a covered direction,
if $\mathcal{L}_S$ contains a line with direction $d$
Proposition

\( S = \mathcal{P}_S \cup \mathcal{L}_S \) is a resolving set for an affine plane if and only if the following properties hold for \( S \):

- **A1** There is at most one not covered outer point.
- **A1'** On every inner line, there is at most one 1-covered outer point.
- **A2** For each covered direction \( d \), there is at most one outer skew line with direction \( d \). There is at most one outer skew line having a not covered direction.
- **A2'** For each inner point, there is at most one tangent line having not covered direction.
**Proposition**

Let $S = \mathcal{P}_S \cup \mathcal{L}_S$ be a resolving set for the affine plane $\Pi$, and suppose that there is a direction $d \in \ell_\infty$ that contains at least two lines of $\mathcal{L}_S$. Let $\overline{\mathcal{P}_S} = \mathcal{P}_S \cup ([\ell_\infty] \setminus \{d\})$. Then $\overline{S} = (\overline{\mathcal{P}_S}, \mathcal{L}_S)$ is a resolving set for $\overline{\Pi}$.

**Proposition**

Let $S = \mathcal{P}_S \cup \mathcal{L}_S$ be a resolving set for an arbitrary affine plane $\Pi$ of order $q$. If $|S| \leq 3q - 4$ then $|\mathcal{L}_S| \geq 2q - 3$. 
Let \( \Pi \) be an arbitrary affine plane of order \( q \geq 13 \). Then the metric dimension of \( \Pi \) is \( 3q - 4 \).

List of the metric basises: derived from the projective metric basises.
We need 1 more object in addition: only 4 different types
Definition

$B_q$: biaffine plane of order $q$
derived from an affine plane of order $q$ by
removing a parallel class of lines
Biaffine planes

- $q^2$ points, $q^2$ lines
- each point is incident with $q$ lines, each line is incident with $q$ points
- for a non-incident $(P, \ell)$:
  - there is exactly one line through $P$ not intersecting $\ell$,
  - there is exactly one point lying on $\ell$ not collinear with $P$
- $q$ parallel classes, $q$ non-adjacency classes, each containing $q$ elements
- uniquely embeddable into a projective plane of order $q$
- incidence graph is distance regular
Biaffine planes are also called flag-type elliptic semiplanes (due to Dembowski)

semiplane:

- \( \forall 2 \) points are connected with \( \leq 1 \) line
- for a non-incident \((P, \ell)\):
  - there is at most one line through \( P \) not intersecting \( \ell \),
  - there is at most one point lying on \( \ell \) not collinear with \( P \)
- every vertex has degree \( \geq 3 \) in the incidence graph

elliptic: incidence graph is regular
From a projective plane throw out a

- whole line (line and the points incident with it)
- whole pencil (point and the lines incident with it)

Flag-type: the deleted point and line are incident
Antiflag-type: not incident
Let $S = \mathcal{P}_S \cup \mathcal{L}_S$ be a vertex set of a biaffine plane.

Notation

- **$d$** is a **covered direction**, if $\mathcal{L}_S$ contains a line with direction $d$.
- **$C$** is a **blocked non-adjacency class**, if $\mathcal{P}_S$ contains a point from $C$.
- For a line $\ell$, **$C(\ell)$**: parallel class containing $\ell$
- For a point $P$, **$C(P)$**: non-adjacency class containing $P$
Proposition

$S = \mathcal{P}_S \cup \mathcal{L}_S$ is a resolving set for a biaffine plane if and only if the following properties hold for $S$:

- **B1** For each blocked class $C$, there is at most one uncovered outer point in $C$; furthermore, there is at most one outer uncovered point in the union of unblocked classes.

- **B1’** On each inner line, there is at most one 1-covered point lying in an unblocked class.

- **B2** For each covered direction $\vec{d}$, there is at most one skew outer line with direction $\vec{d}$; furthermore, there is at most one outer skew line having an uncovered direction.

- **B2’** On each inner point, there is at most one tangent line with uncovered direction.
Biaffine planes

Lower bound:

**Proposition**

Let $S$ be a resolving set for $B_q$. Then $|\mathcal{P}_S| \geq q - |S|/(q - 1)$ and $|\mathcal{L}_S| \geq q - |S|/(q - 1)$.

**Proposition**

For any biaffine plane $B_q$ of order $q$, we have $\mu(B_q) \geq 2q - 2$. 
Upper bound:

**Proposition**

\[ \text{If } q \geq 4 \text{ then } \mu(B_q) \leq 3q - 6. \]

We give a construction of a resolving set of size \(3q - 6\).
Sharpness of the bounds

Notation:
Let $\tau(\Pi)$: size of the smallest blocking set in a finite plane $\Pi$

Construction:

- $\Pi_q$ projective plane, $P$ point, $\ell$ line
- $\Pi_q \setminus [\ell]$: affine plane, $\Pi_q \setminus [P]$: dual affine plane
- $(\Pi_q \setminus [P])^*$: dual of $\Pi_q \setminus [P]$
- Let $\mathcal{B}$: blocking set in $\Pi_q \setminus [\ell]$, $\mathcal{C}$: covering set in $\Pi_q \setminus [P]$, assume that $P \in \ell$.
- Then $B_{\ell,P} := \Pi_q \setminus ([\ell] \cup [P])$ is a biaffine plane.
- $\mathcal{B} \cup \mathcal{C}$: resolving set in $\mathcal{B}$; moreover,
- for any point $Q \in \mathcal{B}$ and any line $r \in \mathcal{C}$, $(\mathcal{B} \setminus \{Q\}) \cup (\mathcal{C} \setminus \{r\})$: resolving set for $B_{\ell,P}$; hence

$$\mu(B_{\ell,P}) \leq \tau((\Pi_q \setminus [P])^*) + \tau(\Pi_q \setminus [\ell]) - 2$$
\[ \mu(B_{\ell,P}) \leq \tau((\Pi_q \setminus [P])^*) + \tau(\Pi_q \setminus [\ell]) - 2 \]

Let \( A_q \): affine plane of order \( q \)

- **General bound:** \( \tau(A_q) \geq q + \sqrt{q} + 1 \)
  - its sharpness is wide open
- **Recent result:** \( \exists A_q \) (Hall plane) such that
  \[ \tau(A_q) \leq 4q/3 + 5\sqrt{q}/3 \]
  (De Beule, Héger, Szőnyi, Van de Voorde)

"Conjecture:" There exists a non-Desarguesian biaffine plane \( B \)
such that \( \mu(B) \ll 3q \).

No general bound?
Desarguesian biaffine planes

$BG(2, q)$: derived from $AG(2, q)$

$P_S$: almost blocking set in $B_q$ \Rightarrow

we can use stability results

**Definition**

For a point $P \in B_q$ and a point-set $\mathcal{X}$, let the index of $P$ with respect to $\mathcal{X}$, $\text{ind}_{\mathcal{X}}(P)$, be the number of skew lines through $P$ to $\mathcal{X}$.

**Result (Blokhuis–Brouwer)**

Let $\mathcal{B}$ be a blocking set of $PG(2, q)$. Then each essential point of $\mathcal{B}$ is incident with at least $2q + 1 - |\mathcal{B}|$ tangents to $\mathcal{B}$.
Desarguesian biaffine planes

Result (Szőnyi–Weiner)

Let \( B \) be a set of points in \( PG(2, q) \), \( q = p \) prime, with at most \( \frac{3}{2}(q + 1) - \varepsilon \) points. Suppose that the number \( \delta \) of skew lines to \( B \) is less than \( \left(\frac{2}{3}(\varepsilon + 1)\right)^2 / 2 \). Then there is a line that contains at least \( q - \frac{2\delta}{q+1} \) points of \( B \).

Result (Szőnyi–Weiner)

Let \( B \) be a set of points in \( PG(2, q) \), \( q = p^h \), \( h \geq 2 \). Denote the number of skew lines to \( B \) by \( \delta \) and suppose that \( \delta \leq \frac{1}{100} pq \). Assume that \( |B| < \frac{3}{2}(q + 1 - \sqrt{2\delta}) \). Then \( B \) can be extended to a blocking set by adding at most

\[
\frac{\delta}{2q + 1 - |B|} + \frac{1}{100}
\]

points to it.

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Desarguesian biaffine planes

Theorem (Lower bound)

Suppose that $S = \mathcal{P}_S \cup \mathcal{L}_S$ is a resolving set for $BG(2, q)$, $q = p^h$, $p$ prime. Assume that

(i) $h = 1$ and $q = p \geq 17$, or

(ii) $h \geq 2$ and $p \geq 400$.

Then $|S| > 3q - 9\sqrt{q}$. 

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Result (Metsch; Szőnyi–Weiner)

Let $\mathcal{B}$ be a point set in $\text{PG}(2, q)$. Pick a point $P$ not from $\mathcal{B}$ and assume that through $P$ there pass exactly $r$ lines meeting $\mathcal{B}$ (that is containing at least 1 point of $\mathcal{B}$). Then the total number of lines meeting $\mathcal{B}$ is at most

$$1 + rq + (|\mathcal{B}| - r)(q + 1 - r).$$

Equivalent formulation of the above result:

Result

Let $\delta$ denote the number of skew lines to a point set $\mathcal{B}$ in $\text{PG}(2, q)$. Then for any point $P \notin \mathcal{B}$,

$$\text{ind}_\mathcal{B}(P)^2 - (2q + 1 - |\mathcal{B}|)\text{ind}_\mathcal{B}(P) + \delta \geq 0$$

meaning: the index of a point is either small or large
Theorem (General lower bound)

The metric dimension of $BG(2, q)$ is at least $\frac{8q}{3} - 7$. 
For Your Interest:
For $r \in \mathbb{R}$, let $r^+ := \max\{0, r\}$.

**Lemma (Szőnyi-Weiner Lemma)**

Let $u, v \in \text{GF}(q)[X, Y]$. Suppose that the term $X^{\deg(u)}$ has non-zero coefficient in $u(X, Y)$. For $y \in \text{GF}(q)$, let $k_y := \deg \gcd(u(X, y), v(X, y))$, where $\gcd$ denotes the greatest common divisor of the two polynomials in $\text{GF}(q)[X]$. Then for any $y \in \text{GF}(q)$,

$$\sum_{y' \in \text{GF}(q)} (k_{y'} - k_y)^+ \leq (\deg u(X, Y) - k_y)(\deg v(X, Y) - k_y).$$
Summary

Upper bound:
- If $q \geq 4$ then $\mu(B_q) \leq 3q - 6$.

Lower bound:
- For any $B_q$ we have $\mu(B_q) \geq 2q - 2$.
- For $BG(2, q)$, $q = p^h$, $p$ prime, if (i) $h = 1$ and $q = p \geq 17$, or (ii) $h \geq 2$ and $p \geq 400$, then $\mu(BG(2, q)) > 3q - 9\sqrt{q}$.
- $\mu(BG(2, q)) \geq 8q/3 - 7$. 

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Generalized quadrangles

$GQ(s, 1)$: grid
Metric dimension of grid graphs are known:

**Theorem (Cáceres et al.)**

Let $G_{n,m}$ be an $n \times m$ grid, with $n \geq m \geq 1$. The metric dimension of $G_{n,m}$ is given by

$$
\mu(G_{n,m}) = \begin{cases} 
\left\lfloor \frac{2(n+m-1)}{3} \right\rfloor, & \text{if } m \leq n \leq 2m - 1, \\
n - 1, & \text{if } n \geq 2m.
\end{cases}
$$

**Corollary**

The metric dimension of $GQ(s, 1)$ is $\varphi(s)$, with

$$
\varphi(s) = \begin{cases} 
4r + 1, & \text{if } s = 3r, \\
4r + 2, & \text{if } s = 3r + 1, \\
4r + 3, & \text{if } s = 3r + 2.
\end{cases}
$$
Proposition

The metric dimension of any $GQ(q, q)$ is at least
$\max\{6q - 27, 4q - 7\}$. 
Generalized quadrangles: $GQ(q, q)$

**Proposition**

There exists a semi-resolving set of size $4q$ for the points of $W(q)$.

**Construction:**

$a_1, a_2, a_3$: three pairwise skew lines of $W(q)$ ⇒ define a hyperbolic quadric $\mathcal{H}$ in $\text{PG}(3, q)$.

$a_4$: a line of $W(q)$ which has empty intersection with $\mathcal{H}$.

$\mathcal{P}_S = [a_1] \cup [a_2] \cup [a_3] \cup [a_4]$: semi-resolving set of size $4q + 4$ for the points of $W(q)$.

Deleting one point from each line $a_1, a_2, a_3, a_4$ ⇒ the remaining points: semi-resolving set of size $4q$ for the points of $W(q)$. 

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If \( q \) is even
then \( W(q) \) is self-dual,
hence the dual of a semi-resolving set for the points
is a semi-resolving set for the lines.

**Corollary**

*If \( q \) is even then the metric dimension of \( W(q) \) is at most \( 8q \).*
Proposition

If \( q \) is odd then there is a semi-resolving set of size \( 5q - 4 \) for the lines of \( W(q) \), which contains exactly \( q - 3 \) points, all incident with the same line.

Corollary

If \( q \) is odd then the metric dimension of \( W(q) \) is at most \( 8q - 1 \).
Generalized quadrangles

Theorem

The metric dimension of $W(q)$ satisfies the inequalities

$$\max\{6q - 27, 4q - 7\} \leq \mu(W(q)) \leq 8q$$
Thanks for your attention!