On the metric dimension of affine planes, biaffine planes and generalized quadrangles

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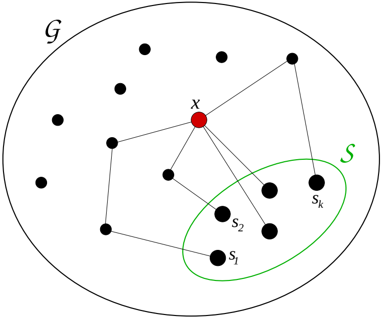
Let $G = (V, E)$ be a simple graph.
$d(x, y)$: distance of x and y, $x, y \in V$
$S = \{s_1, s_2, \ldots, s_k\}$ vertex set



$$d(x, s_1)$$
$$d(x, s_2)$$
$$\ldots$$
$$d(x, s_k)$$

x is resolved by S if its distance list is different from all the other distance lists

Definition (Resolving set)

The subset $S = \{s_1, \ldots, s_k\} \subset V$ is a resolving set, if the ordered distance lists $(d(x, s_1), \ldots, d(x, s_k))$ are different for all $x \in V$.

In other words:
$S = \{s_1, \ldots, s_k\} \subset V$ is a resolving set $\iff$
$\forall x, y \in V \;\; \exists z \in S \colon d(x, z) \neq d(y, z)$.

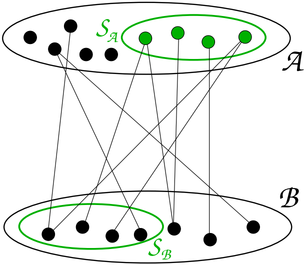
Definition (Metric dimension)

The metric dimension $\mu(G)$ is the size of the smallest resolving set in G.

Definition (Metric basis)

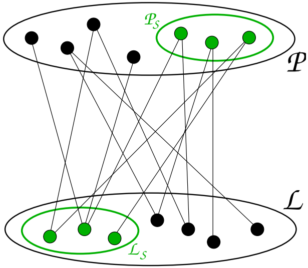
The metric basis of G is a resolving set for G of size $\mu(G)$.

Let $G = (A \cup B, E)$ be a bipartite graph.
split resolving set:
$S_A \subset A$ resolves B
$S_B \subset B$ resolves A



S_A and S_B are called semi-resolving sets.

Incidence graphs of partial linear spaces
(point-line incidence structures):
projective planes, affine planes, biaffine planes,
generalized quadrangles



Motivation and background

metric dimension:

- first introduced by Harary and Melter and (independently) by Slater in the 1970s
- a survey of Bailey and Cameron (2011)
- distance regular graphs: natural class of graphs to consider

Definition (Distance regular graph)

$G(V, E)$ with diameter d is distance regular if $\forall i : 0 \le i \le d$ for any $x, y \in V$, $d(x, y) = i$ the number of neighbours of x at distances $i-1, i, i+1$ from y depend only on i.

- interesting classes of distance regular graphs:
  $d = 2$ strongly regular graphs, distance transitive graphs
  (i.e. for any $x, y, x', y' \in V$ s. t. $d(x, y) = d(x', y')$
  $\exists$ automorphism g: $x^g = x'$, $y^g = y'$)

Motivation and background

Definition (Distance regular graph)

$G(V,E)$ with diameter d is distance regular if $\forall i: 0 \leq i \leq d$ for any $x, y \in V$, $d(x,y) = i$ the number of neighbours of x at distances $i-1, i, i+1$ from y depend only on i.

- theorem of Babai (in other context) $\Longrightarrow$ primitive distance regular graphs
- method of Robert Bailey $\Longrightarrow$ imprimitive distance regular graphs, except some cases, e.g. bipartite graphs with $d = 3, 4$
- 2011: Bailey asked for the metric dimension of the incidence graphs of finite projective planes
- 2012: Tamás Héger, M. T.: answered

Motivation and background

- 2015: Bailey's computer calculations on small distance regular graphs;
  some missing cases, e.g. the incidence graph of the Desarguesian biaffine plane of order 7 and $GQ(4,4)$
- 2015: György Kiss asked for the metric dimension of the incidence graphs of other point-line geometries
- 2017: Daniele Bartoli, T. Héger, Gy. Kiss, M. T.:
  answer for affine planes, partial results for biaffine planes and generalized quadrangles

Congratulations to Daniele
for Kirkman medal!

Motivation and background

Incidence graphs of partial linear spaces:

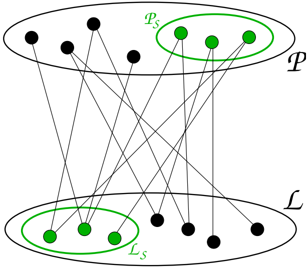
- projective planes: distance regular graphs
- $PG(2,q)$: distance transitive graphs
- biaffine planes: distance regular graphs

Resolving sets:

- nice combinatorial point-line incidence structures
- natural connection with blocking sets and (almost) double blocking sets
- thus we can use stability results for blocking sets

Notation and preliminaries

$G = (\mathcal{P}, \mathcal{L}, E)$: incidence graph of a partial linear space
(P, ℓ) edge $\Leftrightarrow P \in \ell$



$S = (\mathcal{P}_S \cup \mathcal{L}_S)$ resolving set in $\Pi \Longleftrightarrow$
S is a resolving set in the incidence graph

$d(P, \ell) = 1$ or 3, $d(\ell_1, \ell_2) = 2$ or 4, $d(P_1, P_2) = 2$ or 4

Notation and preliminaries

- Π: partial linear space (projective, affine, biaffine plane, generalized quadrangle)
- $S = \mathcal{P}_S \cup \mathcal{L}_S$: set in the incidence graph of Π
- PQ: line joining two distinct points P and Q
- $[P]$: set of lines through a point P
- $[\ell]$: set of points on a line ℓ
- P is incident with a line $\ell \leftrightarrow P$ blocks ℓ and ℓ covers P
- blocking set, covering set
- inner points, inner lines; outer points, outer lines
- tangent line $\longleftrightarrow$ 1-covered point
- skew line $\longleftrightarrow$ not covered point

Lemma

Let $S = \mathcal{P}_S \cup \mathcal{L}_S$, ℓ be a line in Π. If $|[\ell] \cap \mathcal{P}_S| \geq 2$ then ℓ is resolved by S.
Dually, let P be a point in Π. If $|[P] \cap \mathcal{L}_S| \geq 2$ then P is resolved by S.

- Points and lines in S are resolved (trivial)
- (At least) 2-secants are resolved
- (At least) 2-covered points are resolved

We have to distinguish:

- tangents and skew lines (to $\mathcal{P}_S$)
- 1-covered points and not covered points (by $\mathcal{L}_S$)

"Almost" double blocking sets: resolving sets for lines

Projective planes

Proposition (T. Héger, M. Takáts, 2012)

The metric dimension of a projective plane of order $q \geq 23$ is $4q-4$.

- List of the metric basises (resolving sets of size $4q-4$) if $q \geq 23$.

Proposition (T. Héger, P. Szilárd)

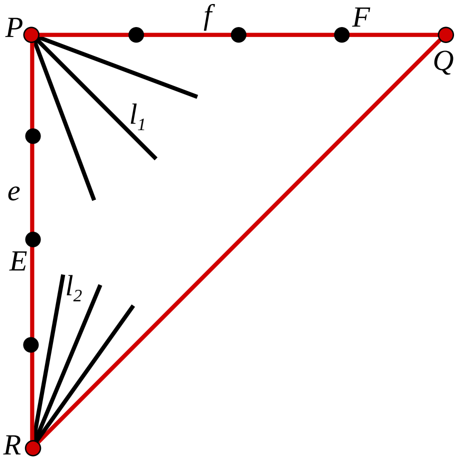
The metric dimension of any projective plane of order $q \geq 13$ is $4q-4$.

Proposition

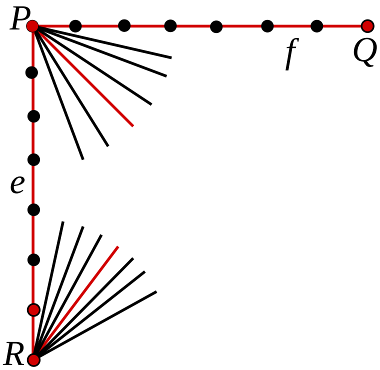
$S = \mathcal{P}_S \cup \mathcal{L}_S$ is a resolving set in a finite projective plane if and only if the following properties hold for S:

- *P1 There is at most one outer line skew to $\mathcal{P}_S$.*
- *P1' There is at most one outer point not covered by $\mathcal{L}_S$.*
- *P2 Through every inner point there is at most one outer line tangent to $\mathcal{P}_S$.*
- *P2' On every inner line there is at most one outer point that is 1-covered by $\mathcal{L}_S$.*

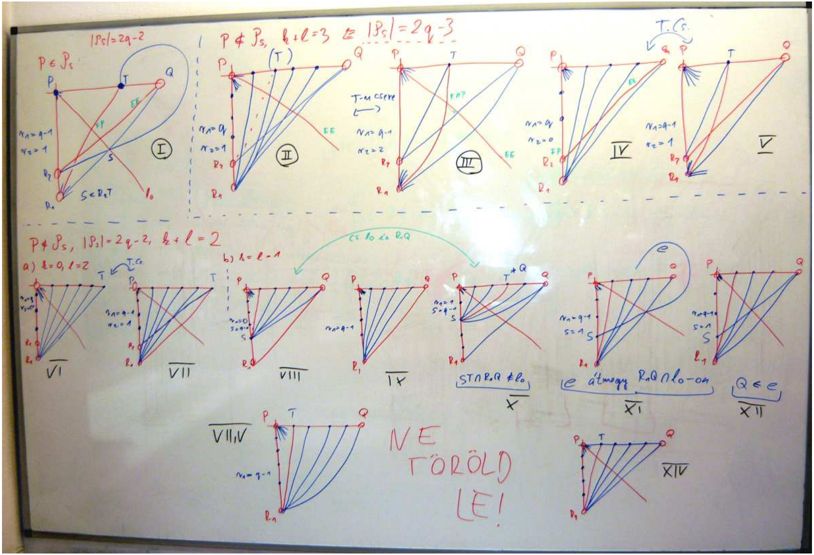
Example

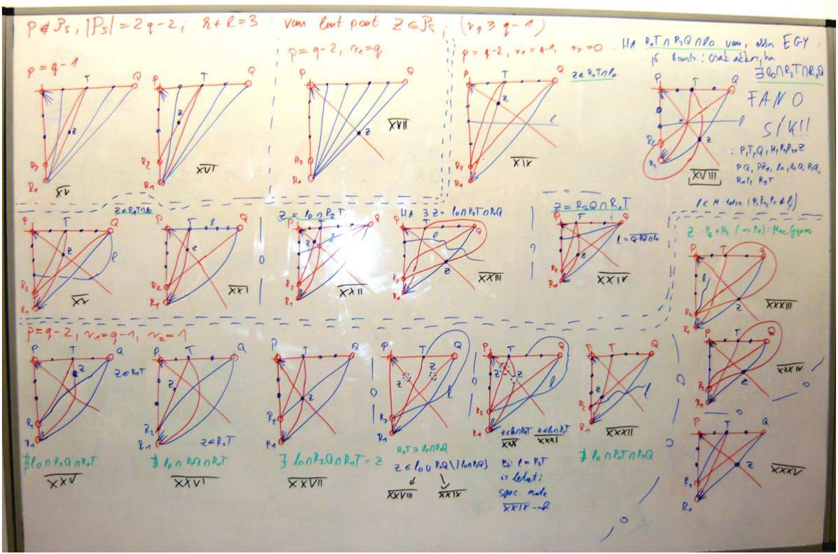


Projective planes



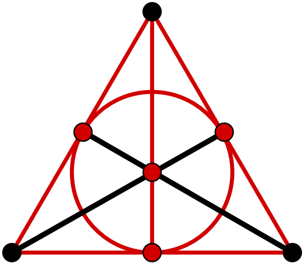
And we need 2 more objects in addition:
2 lines or 1 point and 1 line
surprisingly many (more than 30) different types


NE TÖRÖLD LE!

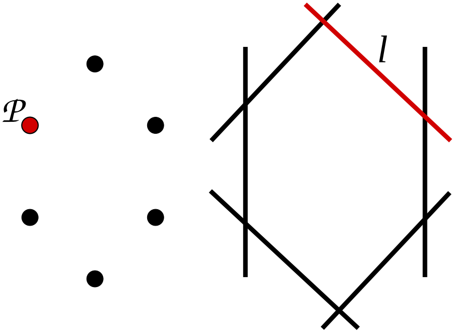

FANO SÍK!!
p = q − 1

- purely combinatorial methods, works for all projective planes
- open question: metric dimension if q is small ($q \leq 13$)

Fano plane:
$\mu(\mathrm{PG}(2,2)) = 5$



$\mu(\mathrm{PG}(2,4)) = 10$
construction: hyperoval



Affine planes

- purely combinatorial methods
- we can deduce it from the projective case
- incidence graph is not distance regular
- note: $d(\ell_1, \ell_2) = 2$ or 4 in the incidence graph
- d is a covered direction,
  if $\mathcal{L}_S$ contains a line with direction d

Affine planes

Proposition

$S = \mathcal{P}_S \cup \mathcal{L}_S$ is a resolving set for an affine plane if and only if the following properties hold for S:

- *A1 There is at most one not covered outer point.*
- *A1' On every inner line, there is at most one 1-covered outer point.*
- *A2 For each covered direction d, there is at most one outer skew line with direction d. There is at most one outer skew line having a not covered direction.*
- *A2' For each inner point, there is at most one tangent line having not covered direction.*

Affine planes

Proposition

Let $S = \mathcal{P}_S \cup \mathcal{L}_S$ be a resolving set for the affine plane Π, and suppose that there is a direction $d \in \ell_\infty$ that contains at least two lines of $\mathcal{L}_S$. Let $\overline{\mathcal{P}_S} = \mathcal{P}_S \cup ([\ell_\infty] \setminus \{d\})$. Then $\overline{S} = (\overline{\mathcal{P}_S}, \mathcal{L}_S)$ is a resolving set for $\overline{\Pi}$.

Proposition

Let $S = \mathcal{P}_S \cup \mathcal{L}_S$ be a resolving set for an arbitrary affine plane Π of order q. If $|S| \leq 3q - 4$ then $|\mathcal{L}_S| \geq 2q - 3$.

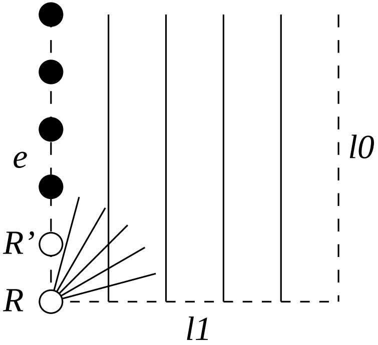
Affine planes

Theorem

Let Π be an arbitrary affine plane of order $q \geq 13$. Then the metric dimension of Π is $3q-4$.

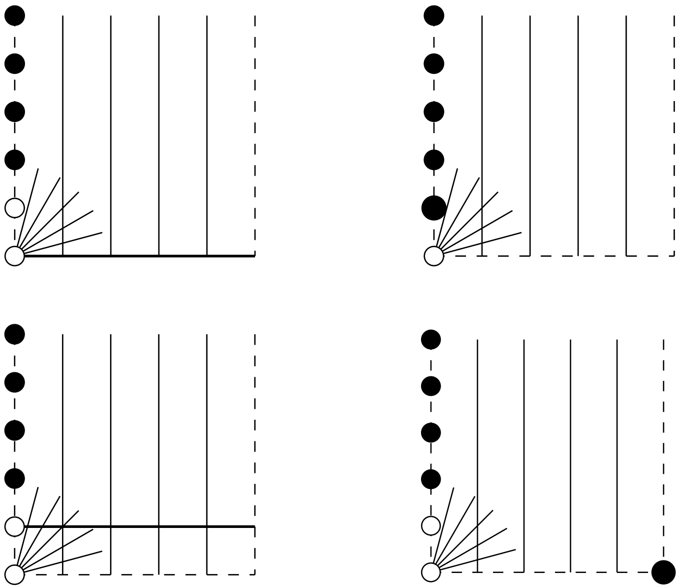
- List of the metric basises:
  derived from the projective metric basises.

Affine planes



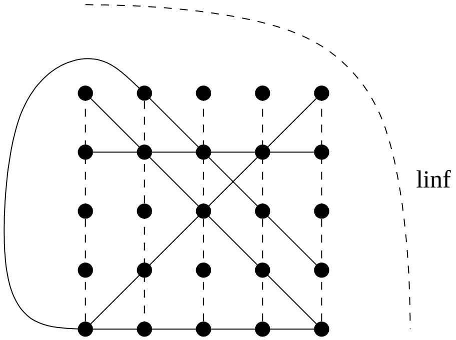
We need 1 more object in addition:
only 4 different types

Affine planes

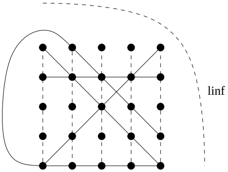


Definition

B_q: biaffine plane of order q
derived from an affine plane of order q by removing a parallel class of lines



Biaffine planes



- q^2 points, q^2 lines
- each point is incident with q lines,
  each line is incident with q points
- for a non-incident (P, ℓ):
  there is exactly one line through P not intersecting ℓ,
  there is exactly one point lying on ℓ not collinear with P
- q parallel classes, q non-adjacency classes,
  each containing q elements
- uniquely embeddedable into a projective plane of order q
- incidence graph is distance regular

Biaffine planes

Biaffine planes are also called flag-type elliptic semiplanes (due to Dembowski)

semiplane:

- $\forall 2$ points are connected with ≤ 1 line
- for a non-incident (P, ℓ):
  there is at most one line through P not intersecting ℓ,
  there is at most one point lying on ℓ not collinear with P
- every vertex has degree ≥ 3 in the incidence graph

elliptic: incidence graph is regular

From a projective plane throw out a

- whole line (line and the points incident with it)
- whole pencil (point and the lines incident with it)

Flag-type: the deleted point and line are incident

Antiflag-type: not incident

Biaffine planes

Let $S = \mathcal{P}_S \cup \mathcal{L}_S$ be a vertex set of a biaffine plane.

Notation

- d is a covered direction,
  if $\mathcal{L}_S$ contains a line with direction d.
- C is a blocked non-adjacency class,
  if $\mathcal{P}_S$ contains a point from C.
- for a line ℓ, $C(\ell)$: parallel class containing ℓ
- for a point P, $C(P)$: non-adjacency class containing P

Proposition

$S = \mathcal{P}_S \cup \mathcal{L}_S$ is a resolving set for a biaffine plane if and only if the following properties hold for S:

- *B1 For each blocked class C, there is at most one uncovered outer point in C; furthermore, there is at most one outer uncovered point in the union of unblocked classes.*
- *B1' On each inner line, there is at most one 1-covered point lying in an unblocked class.*
- *B2 For each covered direction d, there is at most one skew outer line with direction d; furthermore, there is at most one outer skew line having an uncovered direction.*
- *B2' On each inner point, there is at most one tangent line with uncovered direction.*

Lower bound:

Proposition

Let S be a resolving set for B_q. Then $|\mathcal{P}_S| \geq q - |S|/(q-1)$ and $|\mathcal{L}_S| \geq q - |S|/(q-1)$.

Proposition

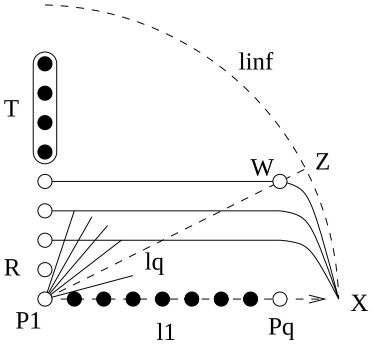
For any biaffine plane B_q of order q, we have $\mu(B_q) \geq 2q - 2$.

Upper bound:

Proposition

If $q \geq 4$ *then* $\mu(B_q) \leq 3q - 6$.

We give a construction of a resolving set of size $3q - 6$.



Sharpness of the bounds

Notation:

Let $\tau(\Pi)$: size of the smallest blocking set in a finite plane Π

Construction:

- Π_q projective plane, P point, ℓ line
- $\Pi_q \setminus [\ell]$: affine plane, $\Pi_q \setminus [P]$: dual affine plane
  $(\Pi_q \setminus [P])^*$: dual of $\Pi_q \setminus [P]$
- Let $\mathcal{B}$: blocking set in $\Pi_q \setminus [\ell]$, $\mathcal{C}$: covering set in $\Pi_q \setminus [P]$, assume that $P \in \ell$.
- Then $B_{\ell,P} := \Pi_q \setminus ([\ell] \cup [P])$ is a biaffine plane.
- $\mathcal{B} \cup \mathcal{C}$: resolving set in B; moreover,
- for any point $Q \in \mathcal{B}$ and any line $r \in \mathcal{C}$,
  $(\mathcal{B} \setminus \{Q\}) \cup (\mathcal{C} \setminus \{r\})$: resolving set for $B_{\ell,P}$; hence

$$\mu(B_{\ell,P}) \leq \tau((\Pi_q \setminus [P])^*) + \tau(\Pi_q \setminus [\ell]) - 2$$

Sharpness of the bounds

$$\mu(B_{\ell,P}) \leq \tau((\Pi_q \setminus [P])^*) + \tau(\Pi_q \setminus [\ell]) - 2$$

Let $\mathcal{A}_q$: affine plane of order q

- General bound: $\tau(\mathcal{A}_q) \geq q + \sqrt{q} + 1$
  its sharpness is wide open
- Recent result: $\exists\ \mathcal{A}_q$ (Hall plane) such that
  $\tau(\mathcal{A}_q) \leq 4q/3 + 5\sqrt{q}/3$
  (De Beule, Héger, Szőnyi, Van de Voorde)

"Conjecture:" There exist a non-Desarguesian biaffine plane B such that $\mu(B) \ll 3q$.

No general bound?

Desarguesian biaffine planes

$\mathrm{BG}(2,q)$: derived from $\mathrm{AG}(2,q)$
$\mathcal{P}_S$: almost blocking set in $B_q \Rightarrow$
we can use stability results

Definition

For a point $P \in B_q$ and a point-set $\mathcal{X}$, let the index of P with respect to $\mathcal{X}$, $\mathrm{ind}_{\mathcal{X}}(P)$, be the number of skew lines through P to $\mathcal{X}$.

Result (Blokhuis–Brouwer)

Let $\mathcal{B}$ be a blocking set of $\mathrm{PG}(2,q)$. Then each essential point of $\mathcal{B}$ is incident with at least $2q+1-|\mathcal{B}|$ tangents to $\mathcal{B}$.

Desarguesian biaffine planes

Result (Szőnyi–Weiner)

Let $\mathcal{B}$ be a set of points in $\mathrm{PG}(2,q)$, $q=p$ prime, with at most $\frac{3}{2}(q+1)-\varepsilon$ points. Suppose that the number δ of skew lines to $\mathcal{B}$ is less than $\left(\frac{2}{3}(\varepsilon+1)\right)^2/2$. Then there is a line that contains at least $q-\frac{2\delta}{q+1}$ points of $\mathcal{B}$.

Result (Szőnyi–Weiner)

Let $\mathcal{B}$ be a set of points in $\mathrm{PG}(2,q)$, $q=p^h$, $h\geq 2$. Denote the number of skew lines to $\mathcal{B}$ by δ and suppose that $\delta\leq\frac{1}{100}pq$. Assume that $|\mathcal{B}|<\frac{3}{2}(q+1-\sqrt{2\delta})$. Then $\mathcal{B}$ can be extended to a blocking set by adding at most

$$\frac{\delta}{2q+1-|\mathcal{B}|}+\frac{1}{100}$$

points to it.

Theorem (Lower bound)

Suppose that $S = \mathcal{P}_S \cup \mathcal{L}_S$ *is a resolving set for* $\mathrm{BG}(2, q)$, $q = p^h$, p *prime. Assume that*

- (i) $h = 1$ *and* $q = p \geq 17$, *or*
- (ii) $h \geq 2$ *and* $p \geq 400$.

Then $|S| > 3q - 9\sqrt{q}$.

Desarguesian biaffine planes

Result (Metsch; Szőnyi–Weiner)

Let $\mathcal{B}$ *be a point set in* $\mathrm{PG}(2,q)$. *Pick a point* P *not from* $\mathcal{B}$ *and assume that through* P *there pass exactly* r *lines meeting* $\mathcal{B}$ *(that is containing at least* 1 *point of* $\mathcal{B}$*). Then the total number of lines meeting* $\mathcal{B}$ *is at most*

$$1 + rq + (|\mathcal{B}| - r)(q + 1 - r).$$

Equivalent formulation of the above result:

Result

Let δ *denote the number of skew lines to a point set* $\mathcal{B}$ *in* $\mathrm{PG}(2,q)$. *Then for any point* $P \notin \mathcal{B}$,
$\mathrm{ind}_{\mathcal{B}}(P)^2 - (2q + 1 - |\mathcal{B}|)\mathrm{ind}_{\mathcal{B}}(P) + \delta \geq 0$

meaning: the index of a point is either small or large

Desarguesian biaffine planes

Theorem (General lower bound)

The metric dimension of $\mathrm{BG}(2, q)$ *is at least* $8q/3 - 7$.

Background

For Your Interest:
For $r \in \mathbb{R}$, let $r^+ := \max\{0, r\}$.

Lemma (Szőnyi-Weiner Lemma)

Let $u, v \in \mathrm{GF}(q)[X,Y]$. Suppose that the term $X^{\deg(u)}$ has non-zero coefficient in $u(X,Y)$. For $y \in \mathrm{GF}(q)$, let $k_y := \deg\gcd\left(u(X,y), v(X,y)\right)$, where $\gcd$ denotes the greatest common divisor of the two polynomials in $\mathrm{GF}(q)[X]$. Then for any $y \in \mathrm{GF}(q)$,

$$\sum_{y' \in \mathrm{GF}(q)} \left(k_{y'} - k_y\right)^+ \leq (\deg u(X,Y) - k_y)(\deg v(X,Y) - k_y).$$

Desarguesian biaffine planes

Summary

Upper bound:

- If $q \geq 4$ then $\mu(B_q) \leq 3q - 6$.

Lower bound:

- For any B_q we have $\mu(B_q) \geq 2q - 2$.
- For $\mathrm{BG}(2, q)$, $q = p^h$, p prime, if (i) $h = 1$ and $q = p \geq 17$, or (ii) $h \geq 2$ and $p \geq 400$, then $\mu(\mathrm{BG}(2, q)) > 3q - 9\sqrt{q}$.
- $\mu(\mathrm{BG}(2, q)) \geq 8q/3 - 7$.

Generalized quadrangles

$\mathrm{GQ}(s,1)$: grid
Metric dimension of grid graphs are known:

Theorem (Cáceres et al.)

Let $G_{n,m}$ be an $n \times m$ grid, with $n \geq m \geq 1$. The metric dimension of $G_{n,m}$ is given by

$$\mu(G_{n,m}) = \begin{cases} \left\lfloor \frac{2(n+m-1)}{3} \right\rfloor, & \text{if } m \leq n \leq 2m-1, \\ n-1, & \text{if } n \geq 2m. \end{cases}$$

Corollary

The metric dimension of $\mathrm{GQ}(s,1)$ is $\varphi(s)$, with

$$\varphi(s) = \begin{cases} 4r+1, & \text{if } s = 3r, \\ 4r+2, & \text{if } s = 3r+1, \\ 4r+3, & \text{if } s = 3r+2. \end{cases}$$

Generalized quadrangles: $\mathrm{GQ}(q, q)$

Proposition

The metric dimension of any $\mathrm{GQ}(q, q)$ *is at least* $\max\{6q - 27, 4q - 7\}$.

Generalized quadrangles: $\mathrm{GQ}(q, q)$

Proposition

There exists a semi-resolving set of size $4q$ for the points of $W(q)$.

Construction:

a_1, a_2, a_3: three pairwise skew lines of $W(q) \Rightarrow$ define a hyperbolic quadric $\mathcal{H}$ in $\mathrm{PG}(3, q)$.

a_4: a line of $W(q)$ which has empty intersection with $\mathcal{H}$.

$\mathcal{P}_S = [a_1] \cup [a_2] \cup [a_3] \cup [a_4]$: semi-resolving set of size $4q + 4$ for the points of $W(q)$.

Deleting one point from each line $a_1, a_2, a_3, a_4 \Rightarrow$ the remaining points: semi-resolving set of size $4q$ for the points of $W(q)$.

Generalized quadrangles: $\mathrm{GQ}(q, q)$

- If q is even
- then $W(q)$ is self-dual,
- hence the dual of a semi-resolving set for the points
- is a semi-resolving set for the lines.

Corollary

If q is even then the metric dimension of $W(q)$ is at most $8q$.

Generalized quadrangles: $\mathrm{GQ}(q,q)$

Proposition

If q is odd then there is a semi-resolving set of size $5q-4$ for the lines of $W(q)$, which contains exactly $q-3$ points, all incident with the same line.

Corollary

If q is odd then the metric dimension of $W(q)$ is at most $8q-1$.

Generalized quadrangles

Theorem

The metric dimension of $W(q)$ *satisfies the inequalities*
$\max\{6q-27, 4q-7\} \leq \mu(W(q)) \leq 8q$

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Thanks for your attention!