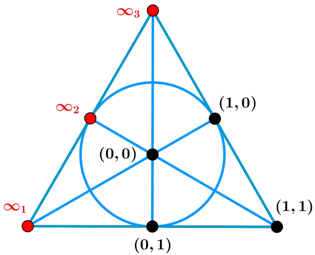
On f-Pyramidal Steiner Triple Systems

Tommaso Traetta

joint work with S. Bonvicini, M. Buratti, and G. Rinaldi

University of Perugia



Finite Geometries - Fifth Irsee Conference

September 12, 2017

f-pyramidal Steiner triple systems

A **Steiner triple system STS(v) of order v** is a set $\mathcal{S}$ of triples of $\{1, 2, \ldots, v\}$ such that every pair of elements is contained in exactly one triple.

An automorphism α of $\mathcal{S}$ is a permutation of the point-set such that $\alpha(S) = S$.
The set Aut($\mathcal{S}$) of all automorphisms of $\mathcal{S}$ is called the full automorphism group.
Each subgroup of Aut($\mathcal{S}$) is an automorphism group of $\mathcal{S}$.

An STS(v) is called f-pyramidal if there exists an automorphsm group fixing f points and acting sharply transitively on the others

0-pyramidal = sharply transitive   1-pyramidal = 1-rotational

There exists an STS(v) IFF $v \equiv 1, 3 \pmod 6$ (Kirkman 1847)
There exists a 0-pyramidal STS(v) IFF $v \equiv 1, 3 \pmod 6$ (Peltesohn 1939)

Given an f-pyramidal STS(v), the set of f fixed points form a subsystem. Therefore,

$f = 0$ or $f \equiv 1, 3 \pmod 6$, and $\underset{\text{(trivial)}}{f = v}$ or $f < \frac{v}{2}$.

f-pyramidal Steiner triple systems

A Steiner triple system STS(v) of order v is a set $\mathcal{S}$ of triples of $\{1, 2, \dots, v\}$ such that every pair of elements is contained in exactly one triple.

An automorphism α of $\mathcal{S}$ is a permutation of the point-set such that $\alpha(S) = S$.
The set Aut($\mathcal{S}$) of all automorphisms of $\mathcal{S}$ is called the full automorphism group.
Each subgroup of Aut($\mathcal{S}$) is an automorphism group of $\mathcal{S}$.

An STS(v) is called f-pyramidal if there exists an automorphsm group fixing f points and acting sharply transitively on the others

0-pyramidal = sharply transitive 1-pyramidal = 1-rotational

There exists an STS(v) IFF $v \equiv 1, 3 \pmod 6$ (Kirkman 1847)
There exists a 0-pyramidal STS(v) IFF $v \equiv 1, 3 \pmod 6$ (Peltesohn 1939)

Given an f-pyramidal STS(v), the set of f fixed points form a subsystem. Therefore,

$f = 0$ or $f \equiv 1, 3 \pmod 6$, and $\underset{\text{(trivial)}}{f = v}$ or $f < \frac{v}{2}$.

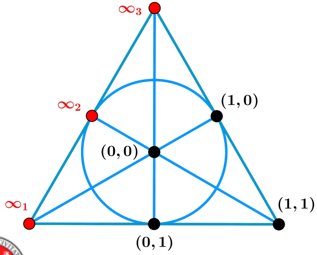
f-pyramidal Steiner triple systems

A Steiner triple system STS(v) of order v is a set $\mathcal{S}$ of triples of $\{1, 2, \ldots, v\}$ such that every pair of elements is contained in exactly one triple.

An automorphism α of $\mathcal{S}$ is a permutation of the point-set such that $\alpha(S) = S$.
The set Aut($\mathcal{S}$) of all automorphisms of $\mathcal{S}$ is called the full automorphism group.
Each subgroup of Aut($\mathcal{S}$) is an automorphism group of $\mathcal{S}$.

An STS(v) is called f-pyramidal if there exists an automorphsm group fixing f points and acting sharply transitively on the others

0-pyramidal = sharply transitive   1-pyramidal = 1-rotational



The Fano plane

$I = \{\infty_1, \infty_2, \infty_3\}$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_2$

$$\overline{G} = \{\overline{g} \mid g \in G\} \text{ where } \overline{g}(x) = \begin{cases} x + g & \text{if } x \in G, \\ x & \text{if } x \in I. \end{cases}$$

$\overline{G}$ maps lines to lines, fixes each point in I, and acts sharply transitively on G.

We say that the Fano plane is 3-pyramidal over $\overline{G}$.

There exists an STS(v) IFF $v \equiv 1, 3 \pmod 6$ (Kirkman 1847)

There exists a 0-pyramidal STS(v) IFF $v \equiv 1, 3 \pmod 6$ (Peltesohn 1939)

f-pyramidal Steiner triple systems

A Steiner triple system STS(v) of order v is a set $\mathcal{S}$ of triples of $\{1, 2, \dots, v\}$ such that every pair of elements is contained in exactly one triple.

An automorphism α of $\mathcal{S}$ is a permutation of the point-set such that $\alpha(S) = S$.
The set Aut($\mathcal{S}$) of all automorphisms of $\mathcal{S}$ is called the full automorphism group.
Each subgroup of Aut($\mathcal{S}$) is an automorphism group of $\mathcal{S}$.

An STS(v) is called f-pyramidal if there exists an automorphsm group fixing f points and acting sharply transitively on the others

0-pyramidal = sharply transitive   1-pyramidal = 1-rotational

There exists an STS(v) IFF $v \equiv 1, 3 \pmod 6$ (Kirkman 1847)
There exists a 0-pyramidal STS(v) IFF $v \equiv 1, 3 \pmod 6$ (Peltesohn 1939)

Given an f-pyramidal STS(v), the set of f fixed points form a subsystem. Therefore,

$f = 0$ or $f \equiv 1, 3 \pmod 6$, and $\underset{\text{(trivial)}}{f = v}$ or $f < \frac{v}{2}$.

f-pyramidal Steiner triple systems

A Steiner triple system STS(v) of order v is a set $\mathcal{S}$ of triples of $\{1, 2, \ldots, v\}$ such that every pair of elements is contained in exactly one triple.

An automorphism α of $\mathcal{S}$ is a permutation of the point-set such that $\alpha(S) = S$.
The set Aut($\mathcal{S}$) of all automorphisms of $\mathcal{S}$ is called the full automorphism group.
Each subgroup of Aut($\mathcal{S}$) is an automorphism group of $\mathcal{S}$.

An STS(v) is called f-pyramidal if there exists an automorphsm group fixing f points and acting sharply transitively on the others

0-pyramidal = sharply transitive   1-pyramidal = 1-rotational

There exists an STS(v) IFF $v \equiv 1, 3 \pmod 6$ (Kirkman 1847)
There exists a 0-pyramidal STS(v) IFF $v \equiv 1, 3 \pmod 6$ (Peltesohn 1939)

Given an f-pyramidal STS(v), the set of f fixed points form a subsystem. Therefore,

$f = 0$ or $f \equiv 1, 3 \pmod 6$, and $\underset{\text{(trivial)}}{f = v}$ or $f < \frac{v}{2}$.

A characterization of f-pyramidal STSs

Let T be a 3-subset of a group G.
The list of differences of T is the multiset ΔT defined as follows:

$$\Delta T = \pm\{a - b \mid a, b \in T, a \neq b\}$$

The list of differences of a set $\mathcal{F}$ of triples is the multiset $\Delta\mathcal{F} = \bigcup_{T\in\mathcal{F}} \Delta T$

A partial spread Σ of G of type $\{^f2, {}^e3\}$ is a symmetric (i.e. $-\Sigma = \Sigma$) subset of G containing the zero element of G, f elements of order 2, and e elements of order 3.

A set $\mathcal{F}$ of triples of G is a $(G, \Sigma, 3, 1)$-DF whenever $\Delta\mathcal{F} = G \setminus \Sigma$
(Difference Family)

M. Buratti, G. Rinaldi, TT (2017)

There exists an f-pyramidal STS(v) under G, with $f < \frac{v}{2}$, if and only if, $|G| = v - f$, G has exactly f involutions, and there exists a $(G, \Sigma, 3, 1)$-DF where Σ is a partial spread of G of type $\{^f2, {}^e3\}$

A characterization of f-pyramidal STSs

Let T be a 3-subset of a group G.
The list of differences of T is the multiset ΔT defined as follows:

$$\Delta T = \pm\{a - b \mid a, b \in T, a \neq b\}$$

The list of differences of a set $\mathcal{F}$ of triples is the multiset $\Delta\mathcal{F} = \bigcup_{T\in\mathcal{F}} \Delta T$

A partial spread Σ of G of type $\{{}^f2, {}^e3\}$ is a symmetric (i.e. $-\Sigma = \Sigma$) subset of G containing the zero element of G, f elements of order 2, and e elements of order 3.

A set $\mathcal{F}$ of triples of G is a $(G, \Sigma, 3, 1)$-DF whenever $\Delta\mathcal{F} = G \setminus \Sigma$
(Difference Family)

M. Buratti, G. Rinaldi, TT (2017)

There exists an f-pyramidal STS(v) under G, with $f < \frac{v}{2}$, if and only if, $|G| = v - f$, G has exactly f involutions, and there exists a $(G, \Sigma, 3, 1)$-DF where Σ is a partial spread of G of type $\{{}^f2, {}^e3\}$

A characterization of f-pyramidal STSs

Let T be a 3-subset of a group G.
The list of differences of T is the multiset ΔT defined as follows:

$$\Delta T = \pm\{a - b \mid a, b \in T, a \neq b\}$$

The list of differences of a set $\mathcal{F}$ of triples is the multiset $\Delta\mathcal{F} = \bigcup_{T\in\mathcal{F}} \Delta T$

A partial spread Σ of G of type $\{^f2, {}^e3\}$ is a symmetric (i.e. $-\Sigma = \Sigma$) subset of G containing the zero element of G, f elements of order 2, and e elements of order 3.

A set $\mathcal{F}$ of triples of G is a $(G, \Sigma, 3, 1)$-DF whenever $\Delta\mathcal{F} = G \setminus \Sigma$
(Difference Family)

M. Buratti, G. Rinaldi, TT (2017)

There exists an f-pyramidal STS(v) under G, with $f < \frac{v}{2}$, if and only if, $|G| = v - f$, G has exactly f involutions, and there exists a $(G, \Sigma, 3, 1)$-DF where Σ is a partial spread of G of type $\{^f2, {}^e3\}$

A characterization of f-pyramidal STSs

Let T be a 3-subset of a group G.
The list of differences of T is the multiset ΔT defined as follows:

$$\Delta T = \pm\{a - b \mid a, b \in T, a \neq b\}$$

The list of differences of a set $\mathcal{F}$ of triples is the multiset $\Delta\mathcal{F} = \bigcup_{T\in\mathcal{F}} \Delta T$

A partial spread Σ of G of type $\{{}^f2, {}^e3\}$ is a symmetric (i.e. $-\Sigma = \Sigma$) subset of G containing the zero element of G, f elements of order 2, and e elements of order 3.

A set $\mathcal{F}$ of triples of G is a $(G, \Sigma, 3, 1)$-DF whenever $\Delta\mathcal{F} = G \setminus \Sigma$
(Difference Family)

M. Buratti, G. Rinaldi, TT (2017)

There exists an f-pyramidal STS(v) under G, with $f < \frac{v}{2}$, if and only if, $|G| = v - f$, G has exactly f involutions, and there exists a $(G, \Sigma, 3, 1)$-DF where Σ is a partial spread of G of type $\{{}^f2, {}^e3\}$

A characterization of f-pyramidal STSs

Let T be a 3-subset of a group G.
The list of differences of T is the multiset ΔT defined as follows:

$$\Delta T = \pm\{a - b \mid a, b \in T, a \neq b\}$$

The list of differences of a set $\mathcal{F}$ of triples is the multiset $\Delta\mathcal{F} = \bigcup_{T\in\mathcal{F}} \Delta T$

A partial spread Σ of G of type $\{{}^f2, {}^e3\}$ is a symmetric (i.e. $-\Sigma = \Sigma$) subset of G containing the zero element of G, f elements of order 2, and e elements of order 3.

A set $\mathcal{F}$ of triples of G is a $(G, \Sigma, 3, 1)$-DF whenever $\Delta\mathcal{F} = G \setminus \Sigma$
(Difference Family)

M. Buratti, G. Rinaldi, TT (2017)

There exists an f-pyramidal STS(v) under G, with $f < \frac{v}{2}$, if and only if, $|G| = v - f$, G has exactly f involutions, and there exists a $(G, \Sigma, 3, 1)$-DF where Σ is a partial spread of G of type $\{{}^f2, {}^e3\}$

A characterization of f-pyramidal STSs

Let T be a 3-subset of a group G.
The list of differences of T is the multiset ΔT defined as follows:

$$\Delta T = \pm\{a - b \mid a, b \in T, a \neq b\}$$

The list of differences of a set $\mathcal{F}$ of triples is the multiset $\Delta\mathcal{F} = \bigcup_{T\in\mathcal{F}} \Delta T$

A partial spread Σ of G of type $\{{}^f2, {}^e3\}$ is a symmetric (i.e. $-\Sigma = \Sigma$) subset of G containing the zero element of G, f elements of order 2, and e elements of order 3.

A set $\mathcal{F}$ of triples of G is a $(G, \Sigma, 3, 1)$-DF whenever $\Delta\mathcal{F} = G \setminus \Sigma$
(Difference Family)

M. Buratti, G. Rinaldi, TT (2017)

There exists an f-pyramidal STS(v) under G, with $f < \frac{v}{2}$, if and only if, $|G| = v - f$, G has exactly f involutions, and there exists a $(G, \Sigma, 3, 1)$-DF where Σ is a partial spread of G of type $\{{}^f2, {}^e3\}$

M. Buratti, G. Rinaldi, TT (2017)

There exists an f-pyramidal STS(v) under G, with $f < \frac{v}{2}$, if and only if, $|G| = v - f$, G has exactly f involutions, and there exists a $(G, \Sigma, 3, 1)$-DF where Σ is a partial spread of G of type $\{^f2, {}^e3\}$

How to construct an f-pyramidal STS from a difference family

Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$.

$\Sigma = \{(0,0,0), s_1 = (1,0,0), s_2 = (0,1,0), s_3 = (1,1,0), s_4 = (0,0,1), -s_4\}$.

$\mathcal{F} = \Big\{\{(0,0,0),(1,0,1),(1,1,2)\}\Big\}$

Σ is a partial spread of G of type $\{^32, {}^23\}$

$\Delta\mathcal{F} = \pm\{(1,0,1),(0,1,1),(1,1,2)\} = G \setminus \Sigma$

$\Rightarrow \quad \mathcal{F}$ is a $(G, \Sigma, 3, 1)$-DF $\quad \Rightarrow \quad \exists$ a 3-pyramidal STS(15)

The set of all the distinct translates of triples in $\mathcal{F} \cup \Sigma^+ \cup \{\infty_1, \infty_2, \infty_3\}$ is a 3-pyramidal STS(15), where

$\Sigma^+ = \Big\{\{\infty_1, (0,0,0), s_1\}, \{\infty_2, (0,0,0), s_2\}, \{\infty_3, (0,0,0), s_3\}, \{(0,0,0), s_4, -s_4\}\Big\}$

A characterization of f-pyramidal STSs

M. Buratti, G. Rinaldi, TT (2017)

There exists an f-pyramidal STS(v) under G, with $f < \frac{v}{2}$, if and only if, $|G| = v - f$, G has exactly f involutions, and there exists a $(G, \Sigma, 3, 1)$-DF where Σ is a partial spread of G of type $\{^f2, {}^e3\}$

How to construct an f-pyramidal STS from a difference family

Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$.

$\Sigma = \{(0,0,0), s_1 = (1,0,0), s_2 = (0,1,0), s_3 = (1,1,0), s_4 = (0,0,1), -s_4\}$.

$\mathcal{F} = \Big\{\{(0,0,0),(1,0,1),(1,1,2)\}\Big\}$

Σ is a partial spread of G of type $\{^32, {}^23\}$

$\Delta\mathcal{F} = \pm\{(1,0,1),(0,1,1),(1,1,2)\} = G \setminus \Sigma$

$\Rightarrow \quad \mathcal{F}$ is a $(G,\Sigma,3,1)$-DF $\quad \Rightarrow \quad \exists$ a 3-pyramidal STS(15)

The set of all the distinct translates of triples in $\mathcal{F} \cup \Sigma^+ \cup \{\infty_1, \infty_2, \infty_3\}$ is a 3-pyramidal STS(15), where

$\Sigma^+ = \Big\{\{\infty_1,(0,0,0),s_1\},\{\infty_2,(0,0,0),s_2\},\{\infty_3,(0,0,0),s_3\},\{(0,0,0),s_4,-s_4\}\Big\}$

A characterization of f-pyramidal STSs

M. Buratti, G. Rinaldi, TT (2017)

There exists an f-pyramidal STS(v) under G, with $f < \frac{v}{2}$, if and only if, $|G| = v - f$, G has exactly f involutions, and there exists a $(G, \Sigma, 3, 1)$-DF where Σ is a partial spread of G of type $\{^f2,\ ^e3\}$

How to construct an f-pyramidal STS from a difference family

Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$.

$\Sigma = \{(0,0,0), s_1 = (1,0,0), s_2 = (0,1,0), s_3 = (1,1,0), s_4 = (0,0,1), -s_4\}$.

$\mathcal{F} = \Big\{\{(0,0,0),(1,0,1),(1,1,2)\}\Big\}$

Σ is a partial spread of G of type $\{^32,\ ^23\}$

$\Delta\mathcal{F} = \pm\{(1,0,1),(0,1,1),(1,1,2)\} = G \setminus \Sigma$

$\Rightarrow$ $\mathcal{F}$ is a $(G,\Sigma,3,1)$-DF $\Rightarrow$ $\exists$ a 3-pyramidal STS(15)

The set of all the distinct translates of triples in $\mathcal{F} \cup \Sigma^+ \cup \{\infty_1, \infty_2, \infty_3\}$ is a 3-pyramidal STS(15), where

$\Sigma^+ = \Big\{\{\infty_1,(0,0,0),s_1\},\{\infty_2,(0,0,0),s_2\},\{\infty_3,(0,0,0),s_3\},\{(0,0,0),s_4,-s_4\}\Big\}$

A characterization of f-pyramidal STSs

M. Buratti, G. Rinaldi, TT (2017)

There exists an f-pyramidal STS(v) under G, with $f < \frac{v}{2}$, if and only if, $|G| = v - f$, G has exactly f involutions, and there exists a $(G, \Sigma, 3, 1)$-DF where Σ is a partial spread of G of type $\{^f2,\ ^e3\}$

How to construct an f-pyramidal STS from a difference family

Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$.

$\Sigma = \{(0,0,0), s_1 = (1,0,0), s_2 = (0,1,0), s_3 = (1,1,0), s_4 = (0,0,1), -s_4\}$.

$\mathcal{F} = \left\{\{(0,0,0),(1,0,1),(1,1,2)\}\right\}$

Σ is a partial spread of G of type $\{^32,\ ^23\}$

$\Delta\mathcal{F} = \pm\{(1,0,1),(0,1,1),(1,1,2)\} = G \setminus \Sigma$

$\Rightarrow\ \ \mathcal{F}$ is a $(G, \Sigma, 3, 1)$-DF $\ \Rightarrow\ \ \exists$ a 3-pyramidal STS(15)

The set of all the distinct translates of triples in $\mathcal{F} \cup \Sigma^+ \cup \{\infty_1, \infty_2, \infty_3\}$ is a 3-pyramidal STS(15), where

$\Sigma^+ = \left\{\{\infty_1, (0,0,0), s_1\}, \{\infty_2, (0,0,0), s_2\}, \{\infty_3, (0,0,0), s_3\}, \{(0,0,0), s_4, -s_4\}\right\}$

M. Buratti, G. Rinaldi, TT (2017)

There exists an f-pyramidal STS(v) under G, with $f < \frac{v}{2}$, if and only if, $|G| = v - f$, G has exactly f involutions, and there exists a $(G, \Sigma, 3, 1)$-DF where Σ is a partial spread of G of type $\{{}^{f}2,\ {}^{e}3\}$

How to construct an f-pyramidal STS from a difference family

Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$.

$\Sigma = \{(0,0,0), s_1 = (1,0,0), s_2 = (0,1,0), s_3 = (1,1,0), s_4 = (0,0,1), -s_4\}$.

$\mathcal{F} = \Big\{\{(0,0,0),(1,0,1),(1,1,2)\}\Big\}$

Σ is a partial spread of G of type $\{{}^{3}2,\ {}^{2}3\}$

$\Delta\mathcal{F} = \pm\{(1,0,1),(0,1,1),(1,1,2)\} = G \setminus \Sigma$

$\Rightarrow\ \mathcal{F}$ is a $(G, \Sigma, 3, 1)$-DF $\ \Rightarrow\ \exists$ a 3-pyramidal STS(15)

The set of all the distinct translates of triples in $\mathcal{F} \cup \Sigma^+ \cup \{\infty_1, \infty_2, \infty_3\}$ is a 3-pyramidal STS(15), where

$\Sigma^+ = \Big\{\{\infty_1,(0,0,0),s_1\},\{\infty_2,(0,0,0),s_2\},\{\infty_3,(0,0,0),s_3\},\{(0,0,0),s_4,-s_4\}\Big\}$

Given $f \equiv 1, 3 \pmod 6$, determine all v for which there exists an f-pyramidal STS(v)

The case $f = 1$: 1-pyramidal STS(v)

Some history

G	There exists a 1–pyramidal STS(v)
cyclic	iff $v \equiv 3, 9 \pmod{24}$, [Phelps, Rosa (1981)]
abelian	iff $v \equiv 1, 3, 9, 19, 27, 33, 51, 57 \pmod{72}$, [Buratti (2001)]
dicyclic	iff $v \equiv 9 \pmod{24}$, [Mishima (2008)]

Given $f \equiv 1, 3 \pmod 6$, determine all v for which there exists an f-pyramidal STS(v)

The case $f = 1$: 1-pyramidal STS(v)

Some history

G	There exists a 1–pyramidal STS(v)
cyclic	iff $v \equiv 3, 9 \pmod{24}$, [Phelps, Rosa (1981)]
abelian	iff $v \equiv 1, 3, 9, 19, 27, 33, 51, 57 \pmod{72}$, [Buratti (2001)]
dicyclic	iff $v \equiv 9 \pmod{24}$, [Mishima (2008)]

Given $f \equiv 1, 3 \pmod 6$, determine all v for which there exists an f-pyramidal STS(v)

The case $f = 1$: 1-pyramidal STS(v)

Some history

G	There exists a 1–pyramidal STS(v)
cyclic	iff $v \equiv 3, 9 \pmod{24}$, [Phelps, Rosa (1981)]
abelian	iff $v \equiv 1, 3, 9, 19, 27, 33, 51, 57 \pmod{72}$, [Buratti (2001)]
dicyclic	iff $v \equiv 9 \pmod{24}$, [Mishima (2008)]

A 1–pyramidal STS(v) is reverse, namely, it has an involution fixing exactly one point.

Doyen (1972), Rosa (1972), Teirlinck (1973)

There exists a reverse STS(v) IFF $v \equiv 1, 3, 9, 19 \pmod{24}$.

Given $f \equiv 1, 3 \pmod 6$, determine all v for which there exists an f-pyramidal STS(v)

The case $f = 1$: 1-pyramidal STS(v)

Some history

G	There exists a 1–pyramidal STS(v)
cyclic	iff $v \equiv 3, 9 \pmod{24}$, [Phelps, Rosa (1981)]
abelian	iff $v \equiv 1, 3, 9, 19, 27, 33, 51, 57 \pmod{72}$, [Buratti (2001)]
dicyclic	iff $v \equiv 9 \pmod{24}$, [Mishima (2008)]
arbitrary	only if $v \equiv 1, 3, 9, 19 \pmod{24}$, [Doyen (1972), Rosa (1972), Teirlinck (1973)]

Given $f \equiv 1,3 \pmod 6$, determine all v for which there exists an f-pyramidal STS(v)

The case $f = 1$: 1-pyramidal STS(v)

Some history

G	There exists a 1–pyramidal STS(v)
cyclic	iff $v \equiv 3, 9 \pmod{24}$, [Phelps, Rosa (1981)]
abelian	iff $v \equiv 1, 3, 9, 19, 27, 33, 51, 57 \pmod{72}$, [Buratti (2001)]
dicyclic	iff $v \equiv 9 \pmod{24}$, [Mishima (2008)]
arbitrary	only if $v \equiv 1, 3, 9, 19 \pmod{24}$, [Doyen (1972), Rosa (1972), Teirlinck (1973)]

$\Downarrow$

The existence of a 1–pyramidal STS(v) remains open for $v \equiv 25, 43, 49, 67 \pmod{72}$.

Given $f \equiv 1, 3 \pmod 6$, determine all v for which there exists an f-pyramidal STS(v)

1-pyramidal STS(v) with $v \equiv 43, 67 \pmod{72}$

Theorem [1]

A 1–pyramidal STS(v) does not exists IFF both (1) and (2) hold:

1. $\frac{v-1}{6}$ is square free;
2. all prime divisors of $\frac{v-1}{6}$ are $\equiv 2 \pmod 3$.

There exists a 1–pyramidal $STS(43)$.
There is NO 1–pyramidal $STS(67)$.

[1] S. Bonvicini, M. Buratti, G. Rinaldi, T. T., *Des. Codes Cryptogr.* (2011)

Given $f \equiv 1, 3 \pmod 6$, determine all v for which there exists an f-pyramidal STS(v)

The case $f = 1$ and $v \equiv 25, 49 \pmod{72}$

Construction [Buratti (2001)]

The orbits of the triples below under $SL_2(3)$ yields a 1–pyramidal STS(25).

$$\left\{\infty, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right\} \qquad \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}\right\}$$

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\right\} \qquad \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}\right\}$$

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right\} \qquad \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}\right\}$$

Given $f \equiv 1, 3 \pmod 6$, determine all v for which there exists an f-pyramidal STS(v)

The case $f = 1$ and $v \equiv 25, 49 \pmod{72}$

Construction [1]

The orbits of the triples below under the binary octahedral group O yields a 1-pyramidal STS(49).

$$\{\infty, 1, -1\} \qquad \{1, -j, k\}$$

$$\{1, \tfrac{1}{2}(-1+i-j+k), \tfrac{1}{2}(-1-i+j-k)\} \qquad \{1, \tfrac{1}{\sqrt{2}}(j-k), \tfrac{1}{2}(-1-i+j+k)\}$$

$$\{1, \tfrac{1}{2}(-1+i+j+k), \tfrac{1}{2}(-1-i-j-k)\} \qquad \{1, \tfrac{1}{\sqrt{2}}(i+k), \tfrac{1}{\sqrt{2}}(1+i)\}$$

$$\{1, \tfrac{1}{2}(-1+i+j-k), -\tfrac{1}{\sqrt{2}}(j+k)\} \qquad \{1, \tfrac{1}{2}(1+i+j+k), -\tfrac{1}{\sqrt{2}}(1+j)\}$$

$$\{1, \tfrac{1}{2}(1+i-j-k), \tfrac{1}{\sqrt{2}}(1-k)\} \qquad \{1, \tfrac{1}{\sqrt{2}}(i-k), -\tfrac{1}{\sqrt{2}}(1+k)\}$$

[1] S. Bonvicini, M. Buratti, G. Rinaldi, T. T., *Des. Codes Cryptogr.* (2011)

Given $f \equiv 1, 3 \pmod 6$, determine all v for which there exists an f-pyramidal STS(v)

1-pyramidal STS(v) with $v \equiv 25, 49 \pmod{72}$

Theorem [1]

There exists a 1–pyramidal STS($72t + 25$) over G in cases (1) and (2).

	t	G
(1)	even	$SL_2(3) \times \mathbb{Z}_n$
(2)	$\equiv 3 \pmod 4$	$O \times \mathbb{Z}_n$
(3)	$\equiv 1 \pmod 4$	some open cases

There exists a 1-pyramidal STS($72t + 49$) over G in cases (4) and (5).

	t	G
(4)	odd	$SL_2(3) \times \mathbb{Z}_n$
(5)	$\equiv 0 \pmod 4$	$O \times \mathbb{Z}_n$
(6)	$\equiv 2 \pmod 4$	some open cases

O is the binary octahedral group.

[1] S. Bonvicini, M. Buratti, G. Rinaldi, T. T., *Des. Codes Cryptogr.* (2011)

Given $f \equiv 1, 3 \pmod 6$, determine all v for which there exists an f-pyramidal STS(v)

1-pyramidal STSs: open cases

The existence of a 1–pyramidal STS(v) is undecided if simultaneously[1]

1. $v - 1 = (p^3 - p)n \equiv 0 \pmod{96}$, p prime;
2. $n \not\equiv 0 \pmod 4$;
3. the odd part of $v - 1$ is square free and all prime divisors are $\not\equiv 1 \pmod 6$.

[1] S. Bonvicini, M. Buratti, G. Rinaldi, T. T., *Des. Codes Cryptogr.* (2011)

Given $f \equiv 1, 3 \pmod 6$, determine all v for which there exists an f-pyramidal STS(v)

1-pyramidal STSs: open cases

The existence of a 1–pyramidal STS(v) is undecided if simultaneously[1]

1. $v - 1 = (p^3 - p)n \equiv 0 \pmod{96}$, p prime;
2. $n \not\equiv 0 \pmod 4$;
3. the odd part of $v - 1$ is square free and all prime divisors are $\not\equiv 1 \pmod 6$.

The first two open cases:

n	p	v	Admissible groups
2	23	24289	"extension of $PGL_2(23)$ by $\mathbb{Z}_2$"
1	47	103777	$SL_2(47)$

[1] S. Bonvicini, M. Buratti, G. Rinaldi, T. T., *Des. Codes Cryptogr.* (2011)

Given $f \equiv 1, 3 \pmod 6$, determine all v for which there exists an f-pyramidal STS(v)

1-pyramidal STSs: open cases

The existence of a 1–pyramidal STS(v) is undecided if simultaneously[1]

1. $v - 1 = (p^3 - p)n \equiv 0 \pmod{96}$, p prime;
2. $n \not\equiv 0 \pmod 4$;
3. the odd part of $v - 1$ is square free and all prime divisors are $\not\equiv 1 \pmod 6$.

Undecided cases with n e p small:

n	p (< 1000)	Admissible groups
1	47, 353, 383, 479, 641...	$SL_2(p)$
$\equiv 1, 3 \pmod 4$		.. $SL_2(p) \times \mathbb{Z}_n$ ..
2	23, 47, 137, 263, 353, 383, 479, 641, 983...	G
$\equiv 2 \pmod 4$		.. $G \times \mathbb{Z}_{\frac{n}{2}}$ ..

G = "extension of $PGL_2(p)$ by $\mathbb{Z}_2$"

[1] S. Bonvicini, M. Buratti, G. Rinaldi, T. T., *Des. Codes Cryptogr.* (2011)

Given $f \equiv 1, 3 \pmod 6$, determine all v for which there exists an f-pyramidal STS(v)

1-pyramidal STSs: open cases

The existence of a 1–pyramidal STS(v) is undecided if simultaneously[1]

1. $v - 1 = (p^3 - p)n \equiv 0 \pmod{96}$, p prime;
2. $n \not\equiv 0 \pmod 4$;
3. the odd part of $v - 1$ is square free and all prime divisors are $\not\equiv 1 \pmod 6$.

Undecided cases with n e p small:

n	$p\ (< 1000)$	Admissible groups
1 $\equiv 1, 3 \pmod 4$	$47, 353, 383, 479, 641 \ldots$	$SL_2(p)$ .. $SL_2(p) \times \mathbb{Z}_n$ ..
2 $\equiv 2 \pmod 4$	$23, 47, 137, 263, 353, 383, 479, 641, 983 \ldots$	G .. $G \times \mathbb{Z}_{\frac{n}{2}}$ ..

G = "extension of $PGL_2(p)$ by $\mathbb{Z}_2$"

[1] S. Bonvicini, M. Buratti, G. Rinaldi, T. T., *Des. Codes Cryptogr.* (2011)

Given $f \equiv 1,3 \pmod 6$, determine all v for which there exists an f-pyramidal STS(v)

The case $f = 3$: 3-pyramidal STSs

Necessarily, $v \geq 7$ and $v \equiv 1,3 \pmod 6$, that is, $v \equiv 1,3,7,9,13,15,19,21 \pmod{24}$

Theorem [1]

There exists a 3-pyramidal STS(v) ($v \geq 7$) if and only if either $v \equiv 7,9,15 \pmod{24}$ or $v \equiv 3,19 \pmod{48}$

v	Existence	Group
$24n+1$	No	—
$24n+3$	Yes $\iff$ n is even	$\mathbb{Z}_4 \times \mathbb{Z}_{6n}$
$24n+7$	Yes	$\mathbb{Z}_2^2 \times \mathbb{Z}_{6n+1}$
$24n+9$	Yes	$\mathbb{D}_6 \times \mathbb{Z}_{4n+1}$
$24n+13$	No	—
$24n+15$	Yes	$\mathbb{Z}_2^2 \times \mathbb{Z}_3 \times \mathbb{Z}_{2n+1}$
$24n+19$	Yes $\iff$ n is even	$\mathbb{Z}_4 \times \mathbb{Z}_{6n+4}$
$24n+21$	No	—

There exists an abelian 3-pyramidal STS(v) ($v \geq 7$) if and only if $v \equiv 7,15 \pmod{24}$ or $v \equiv 3,19 \pmod{48}$

[1] M. Buratti, G. Rinaldi, TT, *Ars Math. Contemp.* (2017)

Given $f \equiv 1, 3 \pmod 6$, determine all v for which there exists an f-pyramidal STS(v)

The case $f = 3$: 3-pyramidal STSs

Necessarily, $v \geq 7$ and $v \equiv 1, 3 \pmod 6$, that is, $v \equiv 1, 3, 7, 9, 13, 15, 19, 21 \pmod{24}$

Theorem [1]

There exists a 3-pyramidal STS(v) ($v \geq 7$) if and only if either $v \equiv 7, 9, 15 \pmod{24}$ or $v \equiv 3, 19 \pmod{48}$

v	Existence	Group
$24n+1$	No	—
$24n+3$	Yes $\iff$ n is even	$\mathbb{Z}_4 \times \mathbb{Z}_{6n}$
$24n+7$	Yes	$\mathbb{Z}_2^2 \times \mathbb{Z}_{6n+1}$
$24n+9$	Yes	$\mathbb{D}_6 \times \mathbb{Z}_{4n+1}$
$24n+13$	No	—
$24n+15$	Yes	$\mathbb{Z}_2^2 \times \mathbb{Z}_3 \times \mathbb{Z}_{2n+1}$
$24n+19$	Yes $\iff$ n is even	$\mathbb{Z}_4 \times \mathbb{Z}_{6n+4}$
$24n+21$	No	—

There exists an abelian 3-pyramidal STS(v) ($v \geq 7$) if and only if $v \equiv 7, 15 \pmod{24}$ or $v \equiv 3, 19 \pmod{48}$

[1] M. Buratti, G. Rinaldi, TT, *Ars Math. Contemp.* (2017)

Given $f \equiv 1, 3 \pmod 6$, determine all v for which there exists an f-pyramidal STS(v)

The case $f = 3$: 3-pyramidal STSs

Necessarily, $v \geq 7$ and $v \equiv 1, 3 \pmod 6$, that is, $v \equiv 1, 3, 7, 9, 13, 15, 19, 21 \pmod{24}$

Theorem [1]

There exists a 3-pyramidal STS(v) ($v \geq 7$) if and only if either $v \equiv 7, 9, 15 \pmod{24}$ or $v \equiv 3, 19 \pmod{48}$

v	Existence	Group
$24n + 1$	No	–
$24n + 3$	Yes $\iff$ n is even	$\mathbb{Z}_4 \times \mathbb{Z}_{6n}$
$24n + 7$	Yes	$\mathbb{Z}_2^2 \times \mathbb{Z}_{6n+1}$
$24n + 9$	Yes	$\mathbb{D}_6 \times \mathbb{Z}_{4n+1}$
$24n + 13$	No	–
$24n + 15$	Yes	$\mathbb{Z}_2^2 \times \mathbb{Z}_3 \times \mathbb{Z}_{2n+1}$
$24n + 19$	Yes $\iff$ n is even	$\mathbb{Z}_4 \times \mathbb{Z}_{6n+4}$
$24n + 21$	No	–

There exists an abelian 3-pyramidal STS(v) ($v \geq 7$) if and only if $v \equiv 7, 15 \pmod{24}$ or $v \equiv 3, 19 \pmod{48}$

[1] M. Buratti, G. Rinaldi, TT, *Ars Math. Contemp.* (2017)

Given $f \equiv 1, 3 \pmod 6$, determine all v for which there exists an f-pyramidal STS(v)

The case $f = 3$: 3-pyramidal STSs

Necessarily, $v \geq 7$ and $v \equiv 1, 3 \pmod 6$, that is, $v \equiv 1, 3, 7, 9, 13, 15, 19, 21 \pmod{24}$

Theorem [1]

There exists a 3-pyramidal STS(v) ($v \geq 7$) if and only if either $v \equiv 7, 9, 15 \pmod{24}$ or $v \equiv 3, 19 \pmod{48}$

v	Existence	Group
$24n+1$	No	–
$24n+3$	Yes $\iff$ n is even	$\mathbb{Z}_4 \times \mathbb{Z}_{6n}$
$24n+7$	Yes	$\mathbb{Z}_2^2 \times \mathbb{Z}_{6n+1}$
$24n+9$	Yes	$\mathbb{D}_6 \times \mathbb{Z}_{4n+1}$
$24n+13$	No	–
$24n+15$	Yes	$\mathbb{Z}_2^2 \times \mathbb{Z}_3 \times \mathbb{Z}_{2n+1}$
$24n+19$	Yes $\iff$ n is even	$\mathbb{Z}_4 \times \mathbb{Z}_{6n+4}$
$24n+21$	No	–

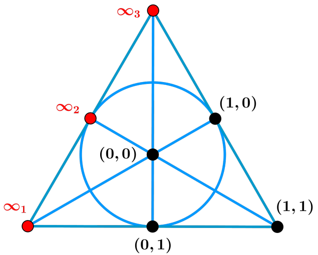
There exists an abelian 3-pyramidal STS(v) ($v \geq 7$) if and only if $v \equiv 7, 15 \pmod{24}$ or $v \equiv 3, 19 \pmod{48}$

[1] M. Buratti, G. Rinaldi, TT, *Ars Math. Contemp.* (2017)

On f-Pyramidal Steiner Triple Systems

Tommaso Traetta
joint work with S. Bonvicini, M. Buratti, and G. Rinaldi

University of Perugia



Finite Geometries - Fifth Irsee Conference
September 12, 2017