

Cameron-Liebler line classes

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Abstract

New examples of Cameron-Liebler line classes in $PG(3, q)$ are given with parameter $\frac{1}{2}(q^2 - 1)$. These examples have been constructed for many odd values of q using a computer search, by forming a union of line orbits from a cyclic collineation group acting on the space. While there are many equivalent characterizations of these objects, perhaps the most significant is that a set of lines \mathcal{L} in $PG(3, q)$ is a Cameron-Liebler line class with parameter x if and only if every spread \mathcal{S} of the space shares precisely x lines with \mathcal{L} . These objects are related to generalizations of symmetric tactical decompositions of $PG(3, q)$, as well as to subgroups of $P\Gamma L(4, q)$ having equally many orbits on points and lines of $PG(3, q)$. Furthermore, in some cases the line classes we construct are related to two-intersection sets in $AG(2, q)$. Since there are very few known examples of these sets for q odd, any new results in this direction are of particular interest.

1 Introduction

Cameron and Liebler [4] studied sets of lines in $PG(3, q)$ having certain nice properties, today known as *Cameron-Liebler line classes*. Such a set contains $x(q^2 + q + 1)$ lines for some integer x , and among many equivalent properties, shares with every spread of $PG(3, q)$ precisely x lines; x is called the *parameter* of the set. There are some trivial examples of Cameron-Liebler line classes; if we take all of the lines through a common point, or all of the lines in a common plane, we get a line class with parameter 1. If we take a non-incident point-plane pair, and take all of the lines through the point along with all of the lines in the plane, we get a line class having parameter 2. Also, it is easy to see that the complement of a Cameron-Liebler line class with parameter x is a line class with parameter $q^2 + 1 - x$.

While it was originally conjectured by Cameron and Liebler that only the trivial examples of these line classes would exist, this was disproved by an example due to Drudge in [7], which was shortly thereafter shown to be part of an infinite family of

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examples by Bruen and Drudge [3]. These examples exist for all odd values of q , and have $x = \frac{1}{2}(q^2 + 1)$. The first counterexample to the conjecture for an even value of q was given by Goevarts and Penttila in [9], having $q = 4$ and $x = 7$. There has been much work excluding certain parameters for Cameron-Liebler line classes; Govaerts and Storme [10] first showed that there are no Cameron-Liebler line classes in $PG(3, q)$ with parameter $2 < x \leq q$ when q is prime. Some time later, De Beule, Hallez, Storme [5] excluded parameters $2 < x \leq q/2$ for all values of q . Most recently, Metsch [11] eliminated the possibility of having $2 < x \leq q$ for any prime power q .

In this work, we detail new Cameron-Liebler line classes constructed for many odd values of q satisfying $q \equiv 1 \pmod{4}$ and $q \not\equiv 1 \pmod{3}$, having parameter $x = \frac{1}{2}(q^2 - 1)$. These new examples are made up of a union of orbits of a cyclic collineation group having order $q^2 + q + 1$. A Cameron-Liebler line class with parameter x Klein-corresponds to an x -tight set of points in $Q^+(5, q)$ [6]; we can use this model to easily test if a given set of points in $Q^+(5, q)$ corresponds to a Cameron-Liebler line class using an eigenvector condition. Among these new examples, when $q \equiv 0 \pmod{3}$ we seem to have some especially interesting behavior; in these cases, the Cameron-Liebler line class constructed gives rise to a symmetric tactical decomposition of $PG(3, q)$, which can be used to derive a set of type (m, n) in the affine plane $AG(2, q)$. Our hope is that these new Cameron-Liebler line classes will belong to an infinite family, and that there will also be an infinite family of related affine sets of type (m, n) . Affine sets of type (m, n) have previously only been found in planes of order 9, so the discovery of an infinite family of new examples would hold particular interest.

2 Cameron-Liebler line classes (tight sets of $Q^+(5, q)$)

Definition 2.1. A **Cameron-Liebler line class** \mathcal{L} is a set of lines in $PG(3, q)$ such that any line ℓ of $PG(3, q)$ is incident with

$$|\{m \in \mathcal{L} : m \text{ meets } \ell, m \neq \ell\}| = \begin{cases} (q+1)x + (q^2 - 1) & \text{if } \ell \in \mathcal{L} \\ (q+1)x & \text{if } \ell \notin \mathcal{L} \end{cases}$$

for a fixed integer x , called the *parameter* of \mathcal{L} .

There are many other equivalent characterizations of these sets of lines; for an extensive list see [13]. The lines of $PG(3, q)$ Klein-correspond to points of $Q^+(5, q)$ [6]. The Klein correspondence is a bijection (which we shall always denote by κ) between the set of lines of $PG(3, q)$ and the set points of $Q^+(5, q)$, such that lines ℓ and ℓ' of $PG(3, q)$ are incident if and only if $\kappa(\ell)$ and $\kappa(\ell')$ are collinear. In this context, we will be interested in *tight sets* of points. The notion of an i -tight set of a finite generalized quadrangle was introduced by Payne in [12], and was extended to polar spaces of higher rank by Drudge in [6].

Definition 2.2. A set of points \mathcal{T} in $Q^+(5, q)$ is said to be **i -tight** if

$$|P^\perp \cap \mathcal{T}| = \begin{cases} i(q+1) + q^2 & \text{if } P \in \mathcal{T} \\ i(q+1) & \text{if } P \notin \mathcal{T}. \end{cases}$$

It is easy to see that a Cameron-Liebler line class of $PG(3, q)$ with parameter x Klein-corresponds to an x -tight set of $Q^+(5, q)$.

We now describe the model of the Klein quadric we will be using. Let $F = GF(q)$, $E = GF(q^3)$, and T be the relative trace function from E to F . We consider the quadric $Q^+(5, q)$ to have the underlying vector space $V = E^2$ considered as a vector space over F , and equipped with the quadratic form $Q : (x, y) \rightarrow T(xy)$. The polar form of Q is given by $f((u_1, u_2), (v_1, v_2)) = T(u_1v_2) + T(u_2v_1)$. This form is nondegenerate, and it can be seen that $\pi_1 = \{(x, 0) : x \in E^*\}$ and $\pi_2 = \{(0, y) : y \in E^*\}$ are totally singular planes in the quadric.

We will be using the orbits of a cyclic group to construct our Cameron-Liebler line classes. Take $\mu \in E^*$ with $|\mu| = q^2 + q + 1$. Define the map g on $Q^+(5, q)$ by $g : \langle(x, y)\rangle \rightarrow \langle(\mu x, \mu^{-1}y)\rangle$. Now g is a projective isometry, and the group $C = \langle g \rangle$ can be seen to have $|C| = q^2 + q + 1$. This group also stabilizes the planes π_1 and π_2 .

Theorem 2.3. *If $q \not\equiv 1 \pmod{3}$, the group C acts semi-regularly on the points of $Q^+(5, q)$.*

Proof. Notice that $g^i \langle(x, y)\rangle = \langle(x, y)\rangle$ implies that $\mu^i \in F$. But $(q^2 + q + 1, q - 1) = (q - 1, 3) = 1$, since $q \not\equiv 1 \pmod{3}$. Thus we have that $\mu^i = 1$, and so g^i is the identity map. \square

This model of the Klein quadric and this cyclic group are very nice to work with algebraically; they were used in [15] to construct cyclic parallelisms. For us, since each orbit has size $q^2 + q + 1$, taking x orbits gives the right amount of points to have an x -tight set, so we will seek to combine these orbits in an appropriate way so that the proper conditions are satisfied.

3 New line classes

The main tool we will use to find tight sets of $Q^+(5, q)$ is an eigenvector condition given in [1].

Theorem 3.1. *Let A be the collinearity matrix of $Q^+(5, q)$ and let \mathcal{L} be an x -tight set with characteristic vector χ . Then*

$$\left(\chi - \frac{x}{q^2 + 1} \mathbf{j} \right) \tag{3.1}$$

is an eigenvector for A with eigenvalue $q^2 - 1$, where \mathbf{j} is the vector consisting of all ones.

This theorem follows directly from our characterization of i -tight sets.

We construct new examples of Cameron-Liebler line classes using the computational software MAGMA [2]. We begin by taking a distinguished plane π in $PG(3, q)$ along with a distinguished point $P \notin \pi$. Thus $\text{line}(\pi) \cup \text{star}(P)$ is a Cameron-Liebler line class of parameter 2, and the complement of this set is a Cameron-Liebler line class with parameter $q^2 - 1$. Working in the Klein quadric $Q^+(5, q)$ as detailed in the previous section, we can assume that $\text{line}(\pi)$ Klein-corresponds to $\pi_1 = \{(x, 0) : x \in E^*\}$

and $\text{star}(P)$ corresponds to $\pi_2 = \{(0, y) : y \in E^*\}$. Defining our group C as before, and requiring that $q \not\equiv 1 \pmod{3}$, we have π_1 and π_2 as orbits of C , as well as $q^2 - 1$ other orbits each having size $q^2 + q + 1$. Our wish is to split these other orbits in half in such a way that we obtain two Cameron-Liebler line classes \mathcal{L}_1 and \mathcal{L}_2 each having parameter $\frac{1}{2}(q^2 - 1)$.

The difficulty in searching for these line classes is twofold. First, constructing the collinearity matrix A of $Q^+(5, q)$ is very time consuming. Second, combining orbits through brute force is not at all computationally efficient, and searching the eigenspace of A corresponding to $(q^2 - 1)$ for eigenvectors of the appropriate form is not much better. To circumvent this first problem, we avoid constructing the entire collinearity matrix of $Q^+(5, q)$. Instead, we use the following result [8]:

Lemma 3.2. *Suppose A can be partitioned as*

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \cdots & A_{kk} \end{bmatrix} \quad (3.2)$$

with each A_{ii} square, $1 \leq i \leq k$, and each A_{ij} having constant row sum b_{ij} . Then any eigenvalue of the matrix $B = (b_{ij})$ is also an eigenvalue of A .

The nice thing about applying this lemma is that we can easily construct an eigenvector of A from an eigenvector of B . By using the point orbits of a group which we assume stabilizes the Cameron-Liebler line class we wish to construct to partition the points of $Q^+(5, q)$, we can construct a matrix B more easily than constructing A , and an appropriate eigenvector of B corresponding to the eigenvalue $(q^2 - 1)$ will give rise to a tight point set of $Q^+(5, q)$.

Let $q \equiv 1 \pmod{4}$ and $q \not\equiv 1 \pmod{3}$. Assume we have an $\frac{1}{2}(q^2 - 1)$ -tight set \mathcal{L}_1 of $Q^+(5, q)$, disjoint from π_1 and π_2 , which is stabilized by C as well as the maps $\sigma : (x, y) \mapsto (x^q, y^q)$ and $\theta : (x, y) \mapsto (x, \omega^4 y)$, where $\langle \omega \rangle = F^*$. We should notice that $\mathcal{L}_2 = Q^+(5, q) \setminus (\pi_1 \cup \pi_2 \cup \mathcal{L}_1)$ is also a $\frac{(q^2 - 1)}{2}$ -tight set. By using the orbits of the group $G = C \langle \sigma \rangle \langle \theta \rangle$ (which has order $\frac{3}{4}(q - 1)(q^2 + q + 1)$) to partition the points of $Q^+(5, q)$, we obtain a matrix B that is $\frac{(q+1)}{3} \times \frac{(q+1)}{3}$ when $q \equiv 2 \pmod{3}$, and $(\frac{q}{3} + 1) \times (\frac{q}{3} + 1)$ when $q \equiv 0 \pmod{3}$. We have applied this technique for all prime powers $q < 200$ satisfying $q \equiv 1 \pmod{4}$ and $q \not\equiv 1 \pmod{3}$, and in each case we are able find eigenvectors corresponding to $\frac{(q^2 - 1)}{2}$ -tight sets (one unique example up to isomorphism for each q). In each case, G ends up being the full stabilizer of the constructed set.

4 Affine two-intersection sets

A set of type (m, n) in a projective or affine plane is a set \mathcal{H} of points such that every line of the plane contains either m or n points of \mathcal{H} ; we require that $m < n$, and we want both values to occur. For projective planes, there are many examples of these types of

sets with q both even and odd. However, the situation is quite different for affine planes. When q is even, we obtain a set of type $(0, 2)$ in $AG(2, q)$ from a hyperoval of the corresponding projective plane, and similarly a set of type $(0, n)$ from a maximal arc. Examples of sets of type (m, n) in affine planes of odd order, on the other hand, are extremely scarce. The only previously known examples exist in affine planes of order 9 [14], where we have sets of type $(3, 6)$ containing either 36 or 45 points (these are complementary cases). These examples were found by a computer search.

A Cameron-Liebler line class \mathcal{L}_1 of parameter 40 in $PG(3, 9)$ constructed in the previous section induces a symmetric tactical decomposition on the space having four classes of points and lines. The four line classes are $\text{line}(\pi)$, $\text{star}(P)$, \mathcal{L}_1 , \mathcal{L}_2 , where $\mathcal{L}_2 = Q^+(5, q) \setminus (\pi_1 \cup \pi_2 \cup \mathcal{L}_1)$, and the four point classes are π , P , and two others, \mathcal{P}_1 and \mathcal{P}_2 . \mathcal{P}_1 and \mathcal{P}_2 are obtained as follows: each point in the space, excluding P and the points of π , lies on either 30 or 60 lines of \mathcal{L}_1 . We define \mathcal{P}_1 to be the set of points on 30 lines of \mathcal{L}_1 , and \mathcal{P}_2 to be the set of points on 60 lines of \mathcal{L}_1 . Now, if we take a plane π' of $PG(3, 9)$ not equal to π , and not containing P , π' contains precisely one line of $\text{line}(\pi)$, no lines of $\text{star}(P)$, and either 30 or 60 lines of \mathcal{L}_1 , so 60 or 30 lines of \mathcal{L}_2 . WLOG we may assume that π' contains 30 lines of \mathcal{L}_1 and 60 lines of \mathcal{L}_2 . π' also contains $(q+1)$ points of π , and, under our assumptions, 30 points of \mathcal{P}_1 and 60 points of \mathcal{P}_2 . In fact, this is a symmetric tactical decomposition of π' having 3 classes on points and lines. Finally, by taking $\pi' \cap \pi$ to be the line at ∞ , we derive the affine plane $AG(2, 9)$. It can be easily verified that $\pi' \cap \mathcal{P}_1$ is a set of size 30 in $AG(2, 9)$ that is of type $(3, 6)$. As the sets of type (m, n) in $AG(2, 9)$ were completely classified in [14], this set is not new. However, following this same procedure with our Cameron-Liebler line class of parameter 3280 in $PG(3, 81)$ yields an affine set of type $(36, 45)$ in $AG(2, 81)$, which is new. Our hope is that, if related Cameron-Liebler line classes exist in $PG(3, 3^{2e})$ for all e , they will always give rise to sets of type (m, n) in $AG(2, 3^{2e})$.

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