Classifying the phase transition threshold for unordered regressive Ramsey numbers

Florian Pelupessy and Andreas Weiermann
Vakgroep Zuivere Wiskunde en Computeralgebra
Krijgslaan 281 Gebouw S22
9000 Ghent
Belgium
pelupessy@cage.ugent.be
weiermann@cage.ugent.be

Abstract. Following ideas of Richer (2000) we introduce the notion of unordered regressive Ramsey numbers or unordered Kanamori-McAloon numbers. We show that these are of Ackermannian growth rate. For a given number-theoretic function $f$ we consider unordered $f$-regressive Ramsey numbers and classify exactly the threshold for $f$ which gives rise to the Ackermannian growth rate of the induced unordered $f$-regressive Ramsey numbers. This threshold coincides with the corresponding threshold for the standard regressive Ramsey numbers. Our proof is based on an extension of an argument from a corresponding proof in a paper by Kojman,Lee,Omri and Weiermann 2007.

Key words: regressive Ramsey numbers, Ackermann function, unordered canonical Ramsey theorem, Kanamori McAloon theorem

1 Introduction

There exist three basic and well known infinitary Ramsey principles which give rise to independence results for PA: Ramsey’s theorem, the canonical Ramsey theorem (Erdős Rado) and the regressive Ramsey theorem (Kanamori McAloon) [8,14,13,12]. The latter two principles make essential use of a pre-existing order on the natural numbers in order to speak about min-colourings, max-colourings and min-homogeneous sets, etc. The three basic infinitary Ramsey-principles give rise to finitary Ramsey principles which can be formulated in the language of arithmetic: the finite Ramsey principle with a suitable largeness condition (i.e. the Paris-Harrington principle [14]), the Kanamori McAloon principle [8] and the finite canonical Ramsey principle with a suitable largeness condition. It turns out that all these finite versions make essential use of the standard $<$-relation and one might wonder if it is possible to find strong principles which do not depend so intrinsically on the less than relation.

An interesting approach to this question can be obtained from a recent paper by Richer [17] about unordered canonical Ramsey numbers and their asymptotic classification. It is quite natural to extend Richer’s approach to the context of
strong Ramsey principles and in this paper we do this for the Kanamori McAloon Ramsey theorem.

We consider the phase transition for unordered regressive Ramsey numbers. For simplicity we limit ourselves to only colourings of pairs (graphs) and expect that the result extends to higher dimensions (hyper graphs) using appropriate bounds from the fast growing hierarchy. We also expect that our results generalize to the unordered canonical Ramsey theorem with a suitable largeness condition. This and other related questions will be investigated jointly in a bigger research project with A. Bovykin, L. Carlucci, G. Lee et al.

This contributes to a general research program of the second author about phase transitions in logic and combinatorics (see, for example, [19,20,21,22,23] for more information).

We identify the natural numbers with their corresponding sets of predecessors and use $C(u, v)$ instead of $C(\{u, v\})$ for a colouring $C : [R]^2 \to \mathbb{N}$. Here $[R]^2$ denotes the set of pairs of unequal elements from $R$. No ordering of $u, v$ is implied here and when known we will give the relative order of the two elements. Denote the collection of subsets of size $m$ of a given set $R$ with $P_m(R)$. Given a number-theoretic function $f$ let us call a colouring $C$ of pairs $f$-regressive if $C(u, v) \leq f(u)$ for all $u, v$ with $u < v$. Call an ordered set $(H, \prec)$ min$_\prec$-homogeneous for $C$ if $C(x, y) = C(x, z)$ for all $x, y, z$ in $H$ with $x \prec y, x \prec z$.

Then given $m$ there exists a least number $R := uKM_f(m)$ such that for all $f$-regressive colourings $C : [R]^2 \to \mathbb{N}$ there exists an $H \in P_m(R)$ and a linear ordering $\prec$ on $H$ such that $H$ is min$_\prec$-homogeneous for $C$.

The class of primitive recursive functions is the smallest class of functions $\mathbb{N}^d \to \mathbb{N}$ which contains the constant functions, projections and successor function and is closed under composition and recursion. We call a function Ackermannian if it eventually dominates every primitive recursive function. Define the Ackermann function $A$ as follows:

$$A_0(i) := i + 1$$
$$A_{n+1}(i) := (A_n)^{(i)}(i)$$
$$A(i) := A_1(i)$$

The Ackermann function is Ackermannian. It is easy to see that for constant functions $f$ the function $uKM_f$ is primitive recursive and so in between the constant function and the identity function there will be phase transition from being primitive recursive to Ackermannian of $uKM_f$. Roughly speaking, for $f(i) = \sqrt[i]{7}$, the function $uKM_f$ is Ackermannian whereas for $f(i) = \log(i)$ the function $uKM_f$ is still elementary recursive. In a final step we let $k$ in $\sqrt[i]{7}$ depend on $i$. We show that function $uKM_f$ is still primitive recursive for any $d$ if $f(i) \leq A_{n-1}^{(1)}(\sqrt[i]{7})$ but becomes Ackermannian if $f(i) \geq A_{n-1}^{(1)}(\sqrt[i]{7})$.

\section{Upper Bounds}

We use results from the Kanamori-McAloon principle to derive upper bounds on the unordered case.
Theorem 1 (uKM). For every $m$ there exists an $R$ such that for every $f$-regressive colouring $C : [R]^2 \to \mathbb{N}$ there exists an $H \in P_m(R)$ with linear order $\preceq \subseteq H^2$ which is $\min_{\preceq}$-homogeneous for $C$.

Proof. Let $KM_f(m)$ be the minimal $\bar{R}$ such that for every $f$-regressive colouring $C : [\bar{R}]^2 \to \mathbb{N}$ there exists an $H \in P_m(R)$ which is $\min_{\prec}$-homogeneous for $C$. (see [8] for proof of existence of such $\bar{R}$)

For given $f$ take $R = KM_f(m)$ and an $f$-regressive colouring $C$. Then there exists $H \in P_m(R)$ which is $\min$-homogeneous for $C$, hence taking for $\prec$ the ordering $<$ suffices to make $H$ $\min_{\prec}$-homogeneous for $C$. Notation: $uKM_f(m) :=$ the smallest such $R$. The proof of this theorem also delivers upper bounds on $uKM_f$.

Theorem 2. $uKM_f$ is primitive recursive if $f$ is:

1. a constant function,
2. $i \mapsto \log i$,
3. $i \mapsto A^{\sqrt{i}}(i)$.

Proof. Because $uKM_f(m) \leq KM_f(m)$ and the primitive recursive functions are closed under bounded search it suffices that $KM_f$ is primitive recursive. For proof of that for all three cases see [2].

3 Lower Bounds

For the lower bound it is not possible to easily transfer earlier results. We modify the proofs for $KM_f$ from [2] and [10] to suit the problem that allowing differing orderings on $H$ gives. As a preliminary we begin with some results about primitive recursive functions.

For $s > 0$, define the following sequence:

$$ A^s_0(i) := i + 1 $$

$$ A^s_{n+1}(i) := (A^s_n)(\lfloor \sqrt{s} \rfloor)(i) $$

$$ A^s(i) := A^s_i(i) $$

Note that for $s = 1$ this is the Ackermann function.

For $R^2_0(i)$ we take the minimal $R$ such that for every colouring $C : [R]^2 \to c$ there exists $Y \in P_1(R)$ such that $Y$ is $C$-homogeneous. ($C$ is constant on $[Y]^2$).

Lemma 1. 1. The Ackermann function is Ackermannian.

2. If the composition of two non-decreasing functions is Ackermannian and one of those is primitive recursive, then the other is Ackermannian.

3. $(i, c) \mapsto R^2_0(i)$ is primitive recursive.

Proof. A proof of the first two statements can be found in [1] and [11], for a proof of the latter one see [7].
We now give a lower bound for $uKM_f$ which should ensure that it is Ackermannian. This proof rests on two ideas, namely the use of the particular colourings similar to proofs of the ordered $KM_f$ and increasing the 'space' in the $D$-homogeneous set to solve the problem caused by allowing any linear order to determine $\min_{\prec}$-homogeneity. Fix $s \in \mathbb{N}$.

**Lemma 2 (lower bound for roots).** Let $f : i \mapsto \sqrt{i}$, then:

$$uKM_f(R^2_c(m + 4)) \geq A^{s_c}_{c+1}(m)$$

for all $c, m \in \mathbb{N}$.

**Proof.** Take $k = R^2_c(m + 4)$ and $R = uKM_f(k)$. Define a colouring $C$ on $R$ as follows for $x < y$:

$$C(x, y) = \begin{cases} 0 & \text{if } A^{s_c}_{c+1}(x) \leq y \\ l & \text{else} \end{cases}$$

where $l$ is such that for the smallest $p$ for which $A^{s_p}_{p+1}(x) > y$ we have $A^{s_p(l)}(x) \leq y < A^{s_p(l+1)}(x)$.

Taking $p$ for $x, y$ as above, define colouring $D$ of $R$ for $x < y$:

$$D(x, y) = \begin{cases} 0 & \text{if } A^{s_c}_{c+1}(x) \leq y \\ p & \text{else} \end{cases}$$

Note that $C$ is $f$-recessive (because $A^{s_p(l(f(x)))}_{p+1}(x) = A^{s_p}_{p+1}(x)$). Let $H \in P_k(R)$ with order $\prec$ be $\min_{\prec}$-homogeneous for $C$, then by definition of $k$ there exists $Y \in P_{m+4}(H)$ which is $D$-homogeneous. Enumerate such a $Y$ with a strictly $\prec$-increasing sequence $Y = \{y_1, \ldots, y_m, x, y, z, z'\}$. Then we have the following cases for the relative $\prec$-ordering of $x, y, z, z'$:

1. $x \prec y, x \prec z$
   
   **Claim:** $A^{s_c}_{c+1}(x) \leq y$.
   
   Assume for a contradiction that $A^{s_c}_{c+1}(x) > y$, then by definition of $C$ we get $C(x, y) = l \neq 0$. Hence (by $\min_{\prec}$-homogeneity of $H$) $C(x, y) = C(x, z) = l$. By definition of $D$ and $D$-homogeneity of $Y$ we also get $D(y, z) = D(x, y) = p \neq 0$. So the definition of $C$ gives us:

   $$A^{s_p(l)}(x) \leq y < A^{s_p(l+1)}(x)$$

   and

   $$A^{s_p(l)}(x) \leq z < A^{s_p(l+1)}(x),$$

   that of $D$ delivers:

   $$A^{s_p}_{p}(y) \leq z.$$

   Combining these inequalities, taking note that $A^{s_p}_{p}$ is increasing, we get the contradiction:

   $$z < A^{s_p(l+1)}(x) = A^{s_p}_{p}(A^{s_p(l)}(x)) \leq A^{s_p}_{p}(y) \leq z$$
2. \( z \prec x, z \prec y \)
   Claim: \( A_{c+1}^s(x) \leq z \).
   Assume \( A_{c+1}^s(x) > z \), then by definition and min.-homogeneity of \( C \) we have \( C(x, z) = C(y, z) = l \neq 0 \), by definition and homogeneity of \( D \) we get: \( D(x, y) = D(x, z) = p \). This gives us inequalities:
   \[
   A_p^{s(l)}(x) \leq z < A_p^{s(l+1)}(x),
   \]
   \[
   A_p^{s(l)}(y) \leq z < A_p^{s(l+1)}(y)
   \]
   and
   \[
   A_p^s(x) \leq y.
   \]
   Combining these we get:
   \[
   z < A_p^{s(l+1)}(x) = A_p^{s(l)}(A_p(x)) \leq A_p^{s(l)}(y) \leq z
   \]

3. \( y \prec x, y \prec z \), we distinguish two possibilities:
   (a) \( y \prec z' \)
   Claim: \( A_{c+1}^s(y) \leq z \).
   Assume \( A_{c+1}^s(y) > z \), then \( C(y, z) = C(y, z') = l \neq 0 \) and \( D(z, z') = D(y, z') = p \). So we have inequalities:
   \[
   A_p^{s(l)}(y) \leq z < A_p^{s(l+1)}(y),
   \]
   \[
   A_p^{s(l)}(y) \leq z < A_p^{s(l+1)}(y)
   \]
   and
   \[
   A_p^s(y) \leq z'.
   \]
   Combining these:
   \[
   z' < A_p^s(A_p^{s(l)}(y)) \leq A_p^s(y) \leq z'
   \]
   (b) \( z' \prec y \)
   Claim: \( A_{c+1}^s(x) \leq z' \).
   Assume \( A_{c+1}^s(x) > z' \), then \( C(x, z') = C(y, z') = l \neq 0 \) and \( D(x, y) = D(x, z') = p \). So we have:
   \[
   A_p^{s(l)}(x) \leq z' < A_p^{s(l+1)}(x),
   \]
   \[
   A_p^{s(l)}(y) \leq z' < A_p^{s(l+1)}(y)
   \]
   and
   \[
   A_p^s(x) \leq y.
   \]
   Combining these:
   \[
   z' < A_p^s(A_p^s(x)) \leq A_p^{s(l)}(y) \leq z'
   \]
Examining the cases above allows us to conclude $A^s_{c+1}(x) \leq z'$. But then:

$$A^s_{c+1}(m) \leq A^s_{c+1}(y_m) \leq A^s_{c+1}(x) \leq z' \in Y \subseteq H \subseteq R.$$ 

So we finally have:

$$A^s_{c+1}(m) \leq R$$

For this lower bound to result in Ackermann functions $uKM_f$ the $A^s$ have to be Ackermannian as well:

**Lemma 3.** $A_n(i) \leq A^s_{n+2i^2+1}(i)$ for any $i \geq 4^s$.

**Proof.** See [2], corollary 4.3.

**Theorem 3.** $uKM_f$ is Ackermannian for $f = f_x : i \rightarrow \sqrt[i]{i}$ and for $f : i \rightarrow A^{-1}(\sqrt[i]{i})$.

**Proof.** For the first assertion combine lemmas 1, 2 and 3. For the second we claim that

$$N(i) := uKM_f(R^2_{i+2i^2+1}(4^i + 3)) > A(i)$$

for all $i$. Assume for contradiction that $N(i) \leq A(i)$ for some $i$. Then for $l \leq N(i)$ we have $A^{-1}(l) \leq i$, so $\sqrt[l]{l} \leq A^{-1}(\sqrt[l]{l})$. Hence:

$$uKM_f(R^2_{i+2i^2+1}(4^i + 3)) \geq uKM_{f_x}(R^2_{i+2i^2+1}(4^i + 3))$$

$$\geq A^i_{i+2i^2+1}(4^i)$$

$$> A(i)$$

Where the first inequality is a consequence if the definition of $uKM_f$, the second of lemma 2 and the third of lemma 3. The resulting inequality contradicts with our assumption.

Now the claim with lemma 1 implies that $uKM_f$ is Ackermannian.

**References**


   http://www.lix.polytechnique.fr/~leegy/Publi/kmjams.pdf


