The category of Moufang sets

Tom De Medts

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Abstract

We make the class of Moufang sets into a category in such a way that it is compatible with other categories (such as the category of quadratic Jordan division algebras). This note is not intended for publication as it is.

1 Another notation for Moufang sets

We start by recalling the definition of a Moufang set.

Definition 1.1. A *Moufang set* is a set X together with a collection of subgroups $(U_x)_{x \in X}$, such that each U_x is a subgroup of $\mathrm{Sym}(X)$ fixing x and acting regularly (i.e. sharply transitively) on $X \setminus \{x\}$, and such that each U_x permutes the set $\{U_y \mid y \in X\}$ by conjugation. The group $G := \langle U_x \mid x \in X \rangle$ is called the *little projective group* of the Moufang set; the groups U_x are called *root groups*.

By the results of [1], every Moufang set can be obtained from a single group U (written additively, although not necessarily commutative) and a permutation $\tau \in \mathrm{Sym}(U^*)$, where $U^* := U \setminus \{0\}$; we denote this Moufang set as $\mathbb{M} = \mathbb{M}(U, \tau)$. The disadvantage of this notation is that the permutation τ is not uniquely determined by the Moufang set, and in particular, every μ_a (for each $a \in U^*$) can serve as the permutation τ, i.e. $\mathbb{M} = \mathbb{M}(U, \mu_a)$ for all $a \in U^*$. We therefore prefer to introduce the new notation

$$\mathbb{M} = \mathbb{M}(U, (\mu_a)_{a \in U^*}) = \mathbb{M}(U, \boldsymbol{\mu}),$$

where $\boldsymbol{\mu}$ denotes the collection of all the μ-maps $(\mu_a)_{a \in U^*}$. Note that $\boldsymbol{\mu}$ is uniquely determined by the Moufang set.

2 A bad attempt

It is common to define isomorphisms between two Moufang sets in a general setting, as follows.

Definition 2.1. Let $(X,(U_x)_{x\in X})$ and $(Y,(V_y)_{y\in Y})$ be two Moufang sets. A bijection β from X to Y is called an *isomorphism* of Moufang sets, if the induced map $\chi_\beta : \mathrm{Sym}(X) \to \mathrm{Sym}(Y) : g \mapsto \beta^{-1}g\beta$ maps each root group U_x isomorphically onto the corresponding root group $V_{x\beta}$.

However, when we try to generalize this to arbitrary homomorphisms between Moufang sets, we run into problems. The natural try would be to say that a map β from X to Y is a homomorphisms of Moufang sets, if

$$U_x\beta \subseteq \beta U_{x\beta} \quad \text{for all } x \in X\,. \tag{2.1}$$

But by the properties of Moufang sets, this can only be true if β is injective! Indeed, suppose that $x, w \in X$ are two distinct elements such that $x\beta = w\beta$, and choose a third element $z \in X$ with $x\beta \neq z\beta$. Let g be the unique element in U_x mapping w to z; then $wg\beta = z\beta$. But equation (2.1) implies $wg\beta \in w\beta U_{x\beta} = x\beta U_{x\beta} = \{x\beta\}$, so $z\beta = wg\beta = x\beta$ after all, a contradiction.

3 A better attempt

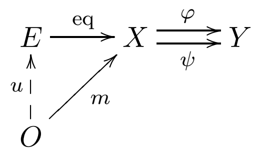
It turns out to be better to restrict to "algebraic homomorphisms", i.e. homomorphisms preserving the chosen elements 0 and ∞ and restricting to group homomorphisms of U_∞ (or equivalently, of U). Observe that already the notation $(X,(U_x)_{x\in X}) = \mathbb{M}(U,\boldsymbol{\mu})$ implies the existence of two distinguished elements $0,\infty$ in X which we would like to leave untouched.

Definition 3.1. Let $\mathbb{M}_1 = \mathbb{M}(U,\boldsymbol{\mu})$ and $\mathbb{M}_2 = \mathbb{M}(V,\boldsymbol{\nu})$ be two Moufang sets. A *homomorphism* from $\mathbb{M}_1$ to $\mathbb{M}_2$ is a group homomorphism $\varphi : U \to V$ such that $\mu_a\varphi = \varphi\nu_{a\varphi}$ for all $a \in U^*$.

It is straightforward to check that $\mathcal{C}$, where $\mathrm{ob}(\mathcal{C})$ is the class of Moufang sets, and where $\mathrm{hom}(\mathcal{C})$ is the class of homormorphisms between Moufang sets, forms a category, which we will call the *category of Moufang sets* and denote by **MSet**. Composition of homomorphisms of **MSet** is given by composition of the corresponding group homomorphisms.

The category **MSet** is of course not complete, since there are no products in general. It does admit equalizers, however.

Definition 3.2. Let $\mathcal{C}$ be a category, let $X, Y \in \mathrm{ob}(\mathcal{C})$ and $\varphi,\psi \in \mathrm{hom}(X,Y)$. An *equaliser* for (X,Y,φ,ψ) consists of an object E and a morphism $\mathrm{eq} : E \to X$ satisfying $\varphi \circ \mathrm{eq} = \psi \circ \mathrm{eq}$, and such that, given any other object O and morphism $m : O \to X$, if $\varphi \circ m = \psi \circ m$, then there exists a unique morphism $u : O \to E$ such that $\mathrm{eq} \circ u = m$.



It is clear from this universal property that if an equalizer exists, it is unique up to isomorphism.

Proposition 3.3. *The category* **MSet** *admits equalizers.*

Proof. Let $\mathbb{M}_1 = \mathbb{M}(U, \boldsymbol{\mu}), \mathbb{M}_2 = \mathbb{M}(V, \boldsymbol{\nu}) \in \mathrm{ob}(\mathbf{MSet})$ and let $\varphi, \psi \in \mathrm{hom}(\mathbb{M}_1, \mathbb{M}_2)$. Define the set $E := \{a \in U \mid a\varphi = a\psi\}$; it is clear that E is a subgroup of U. For each $a \in E^*$, let ρ_a be the restriction of μ_a to E. For all $a, b \in E^*$, we have

$$(a\rho_b)\varphi = a\mu_b\varphi = a\varphi\nu_{b\varphi} = a\psi\nu_{b\psi} = a\mu_b\psi = (a\rho_b)\psi\,,$$

hence $a\rho_b \in E^*$ as well; this shows that $\mathbb{M}_0 := (E, \boldsymbol{\rho})$ is a Moufang set. It is now clear that $\mathbb{M}_0$ together with the inclusion map $\mathbb{M}_0 \hookrightarrow \mathbb{M}_1$ is an equalizer for $(\mathbb{M}_1, \mathbb{M}_2, \varphi, \psi)$. □

We will now focus on the Moufang sets arising from quadratic Jordan division algebras.

Definition 3.4. Let $(J, U, 1)$ and $(J', U', 1')$ be two quadratic Jordan algebras (in the sense of McCrimmon [3]), possibly defined over different fields. Then a vector space homomorphism $\varphi : J \to J'$ is called a *homotopy* from J to J', if there is a vector space homomorphism $\psi : J \to J'$ such that $U_a\varphi = \psi U'_{a\varphi}$ for all $a \in J$; see [2, (93'), p.59]. Let **Jor** be the category of quadratic Jordan algebras with homotopies as morphisms, and let **DJor** be the subcategory of quadratic Jordan division algebras.

Proposition 3.5. *Let F be the map from* $\mathrm{ob}(\mathbf{DJor})$ *to* $\mathrm{ob}(\mathbf{MSet})$ *which maps each quadratic Jordan division algebra $(J, U, 1)$ to its corresponding Moufang set $\mathbb{M}(J)$ as defined in [1]. Then F induces a faithful covariant functor from* **DJor** *to* **MSet**.

Proof. Let $J, J' \in \mathrm{ob}(\mathbf{DJor})$ and let $\varphi \in \mathrm{hom}(J, J')$. Let $\mathbb{M} := \mathbb{M}(J) = \mathbb{M}(U, \boldsymbol{\mu})$ and $\mathbb{M}' := \mathbb{M}(J') = \mathbb{M}(U', \boldsymbol{\mu}')$; then $U = (J, +)$ and $U' = (J', +)$, and hence φ induces a group homomorphism $F(\varphi)$ from U to U' by forgetting the vector space structure of J and J'; we will also denote this group homomorphism by φ. We have to check whether $F(\varphi) \in \mathrm{hom}(\mathbb{M}, \mathbb{M}')$, i.e. whether $\mu_a\varphi = \varphi\mu'_{a\varphi}$ for all $a \in U^*$.

Recall that the fact that $\varphi \in \hom(J, J')$ translates to $h_a\varphi = \psi h'_{a\varphi}$ for all $a \in U^*$. Taking b in place of a and substituting this back yields $h_a\varphi = h_b\varphi h'^{-1}_{b\varphi} h'_{a\varphi}$, or equivalently,

$$h_b^{-1} h_a \varphi = \varphi h'^{-1}_{b\varphi} h'_{a\varphi}$$

for all $a, b \in U^*$. We now use the fact that $h_a = \mu_1^{-1}\mu_a$ to get

$$\mu_b^{-1}\mu_a\varphi = \varphi \mu'^{-1}_{b\varphi}\mu'_{a\varphi}$$

for all $a, b \in U^*$. Applying this equation on $(-b)$ and using the fact that $(-b)\mu_b^{-1} = b$ for all $b \in U^*$, we get

$$b\mu_a\varphi = (b\varphi)\mu'_{a\varphi}$$

for all $a, b \in U^*$, i.e. $\mu_a\varphi = \varphi\mu'_{a\varphi}$ for all $a \in U^*$. This proves that that F is a covariant functor from **DJor** to **MSet**.

Since for given $J, J' \in \mathrm{ob}(\mathbf{DJor})$, every $\varphi \in \hom(J, J')$ is mapped by F onto the corresponding element in $\hom(\mathbb{M}(J), \mathbb{M}(J'))$, it is clear that the restriction of F to $\hom(J, J')$ is injective, i.e. F is a faithful functor. □

Remark 3.6. The functor F is not full. Indeed, suppose that J and J' are Jordan algebras defined over some (big) field k with prime field $\mathbb{F}$, then J and J' are also Jordan algebras over $\mathbb{F}$, and there could be $\mathbb{F}$-homotopies from J to J' which do not preserve the k-vector space structure of J. On the other hand, note that every group homomorphism from $(J, +)$ to $(J', +)$ is an $\mathbb{F}$-vector space homomorphism. If we denote the category of quadratic Jordan division algebras over prime fields (i.e. over finite fields or over $\mathbb{Q}$) as **DJorP**, then the functor F decomposes as

$$\mathbf{DJor} \xrightarrow{P} \mathbf{DJorP} \xrightarrow{Q} \mathbf{MSet}\,,$$

where P is a forgetful functor and Q is a fully faithful functor.

References

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