THE NORM OF A REE GROUP

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Abstract. We give an explicit construction of the Ree groups of type $G_2$ as groups acting on mixed Moufang hexagons together with detailed proofs of the basic properties of these groups contained in the two fundamental papers of Tits on this subject, [7] and [8]. We also give a short proof that the norm of a Ree group is anisotropic.

1. Introduction

The finite Ree groups of type $G_2$ were introduced by Ree in [5]. In [7], Tits showed how to construct these groups over an arbitrary field $K$ of characteristic 3 having an endomorphism whose square is the Frobenius endomorphism of $K$. His result can be summarized as follows.

Theorem 1.1. Let $K$ be a field of characteristic 3 and suppose that $K$ has an endomorphism $\theta$ such that

\[ x^{\theta^2} = x^3 \]

for all $x \in K$. Let $U$ denote the set $K \times K \times K$ endowed with the multiplication:

\[ (a,b,c) \cdot (x,y,z) = (a + x, b + y + ax^\theta, c + z + ay - bx - ax^{\theta+1}), \]

and let

\[ H = \{ h_t \mid t \in K^* \}, \]

where for each $t \in K^*$, $h_t$ is the map from $U$ to itself given by the formula

\[ (a,b,c)^{h_t} = (ta, t^{\theta+1}b, t^{\theta+2}c). \]

Let

\[ N(a,b,c) = -ac^\theta + a^{\theta+1}b^\theta - a^{\theta+3}b - a^2b^2 + b^{\theta+1} + c^2 - a^{2\theta+4} \]

for all $(a,b,c) \in U$ and let $X$ denote the disjoint union of $U$ and a symbol $\infty$. Then the following hold:

(i) $U$ is a group with identity $(0,0,0)$ (which we denote by $0$) and inverses given by

\[ (a,b,c)^{-1} = (-a, -b + a^{\theta+1}, -c) \]

and $H$ is a group of automorphisms of $U$.

(ii) The map $N$ is anisotropic. This is to say, $N(a,b,c) = 0$ if and only if $(a,b,c) = 0$.

Date: September 17, 2009.
2000 Mathematics Subject Classification. 20E42, 51E12, 51E24.
Key words and phrases. generalized hexagon, Moufang polygon, Moufang set, Ree group.
(iii) Let $\omega$ be the map from $X$ to itself that interchanges $\infty$ and 0 and maps an arbitrary element $(a,b,c)$ of $U^*$ to

$$(-v/w, -u/w, -c/w),$$

where $v = a^\theta b^\theta - c^\theta + ab^2 + bc - a^{2\theta+3}$, $u = a^2 b - ac + b^\theta - a^{\theta+3}$ and $w = N(a,b,c)$. Let $U$ be identified with the permutation group of $X$ that fixes $\infty$ and acts on $X\setminus\{\infty\}$ by right multiplication. Let $H$ be identified with the permutation group of $X$ that fixes $\infty$ and acts on $X\setminus\{\infty\}$ by the formula (1.3) (and thus fixes also 0). Let $K^\dagger$ be the subgroup of $K^*$ generated by $\{N(a,b,c) \mid (a,b,c) \in U^*\}$ and let

$$H^\dagger = \{h_t \mid t \in K^\dagger\} \subset H.$$  

Then $\omega$ is a permutation of $X$ of order 2 and the subgroup $G$ of $\text{Sym}(X)$ generated by $U$ and $\omega$ has the following properties:

(I) $G$ is a 2-transitive permutation group on $X$.

(II) $U$ is a normal subgroup of the stabilizer $G_\infty$ and $G_\infty = UH^\dagger$.

(III) $G = \langle U, U^\omega \rangle$.

(IV) $H$ normalizes $G$.

(V) $\omega$ inverts every element of $H$.

(VI) If $|K| > 3$, then $G$ is simple.

Tits’ proof of Theorem 1.1 in [7] is based on the standard embedding of the split Moufang hexagon in 6-dimensional projective space; see also [10, Section 7.7]. The purpose of this note is to give an alternative proof of Theorem 1.1 in which we construct the set $X$ inside the mixed hexagon defined over the pair $(K, K^\theta)$, which we construct directly without reference to projective space.

Our motivation is threefold. First, since the Ree groups of type $G_2$ continue to be the center of lively interest (see especially [2]), we want to give a proof of Theorem 1.1 in which many of the details left to the reader in [7] are filled in. We also want to provide independent confirmation of the accuracy of the formulas occurring in Theorem 1.1. (In fact, there is a $\theta$ missing in the second term in the definition of the norm and a minus sign missing in front of the whole expression on page 12 in [7], where $\theta$ is called $\sigma$ and the norm $N$ is called $w$.) Secondly, we want to examine the fact that the map $N$, which we call the norm of $G$, is anisotropic. As in [7], this fact emerges “geometrically” in the course of our proof of Theorem 1.1; in Section 6, we give a short algebraic proof. Thirdly, we hope that the method we use to prove Theorem 1.1 can serve as a model for other calculations in Moufang polygons and in more general types of buildings.

If $|K| = 3$, then the endomorphism $\theta$ is trivial and the group $G$ is not simple; in fact, it is isomorphic to $\text{Aut}(L_2(8))$ in this case and thus has a normal subgroup of index 3 (which is simple).

If $K$ is finite, then $H^\dagger = H$ and thus $H \subset G$ (by [5, 8.4]). It is not true in general, however, that $H = H^\dagger$. We say a few words about this in Section 7. For another approach to the finite Ree groups, see [4].

We mention that there are also Ree groups of type $F_4$. The canonical reference for these groups is [8].

We would also like to bring the reader’s attention to Remark 3.11 below.

Acknowledgment: We would like to thank the referee for his many valuable suggestions and observations.


2. **The hexagon of mixed type**

Let $K$ be a field of characteristic 3 and let $\theta$ be a square root of the Frobenius endomorphism of $K$. We now begin our proof of Theorem 1.1 by constructing the mixed hexagon associated with the pair $(K, \theta)$. (See [9, 16.20 and 41.20] for the definition of a mixed hexagon.) Let $U_1, U_2, \ldots, U_6$ be six groups isomorphic to the additive group of $K$. For each $i \in [1, 6]$, let $x_i$ be an isomorphism from $K$ to $U_i$. Let $U_+$ be the group generated by the groups $U_1, U_2, \ldots, U_6$ subject to the commutator relations

$$[x_1(s), x_5(t)] = x_3(-st)$$

$$[x_2(s), x_6(t)] = x_4(st)$$

for all $s, t \in K$ and $[U_i, U_j] = 1$ for all other pairs $i, j$ such that $1 \leq i < j \leq 6$. (We are using the convention that $[a, b] = a^{-1}b^{-1}ab = (b^{-1})^a b$.) By Propositions 2.2 and 2.5 below and [9, (5.6)], every element of $U_+$ can be written uniquely as an element in the product $U_1U_2\cdots U_6$. It is easily checked that there is an automorphism $\rho$ of $U_+$ interchanging $x_i(t)$ and $x_7(t)$ for all $i \in [1, 6]$ and all $t \in K$. We will see below that the group $U'$ in Theorem 1.1 is the centralizer of $\rho$ in $U_+$.

Let $U_{i,j}$ denote the subgroup $U_{i+1}U_{i+2}\cdots U_{j}$ of $U_+$ for all $i, j$ such that $1 \leq i \leq j \leq 6$ (so $U_{i,i} = U_i$ for each $i$). For each $i \in [1, 5]$, let $W_i$ denote the set of right cosets in $U_+$ of $U_{1,6-i}$. For each $i \in [6, 10]$, let $W_i$ denote the set of right cosets in $U_+$ of $U_{12-i,6}$. Let $W$ be the disjoint union of $W_1, W_2, \ldots, W_{10}$ together with two symbols $\bullet$ and $\star$. For each $i \in [1, 9]$, let $E_i$ be the set of pairs $\{x, y\}$ such that $x \in W_i$, $y \in W_{i+1}$ and the intersection of $x$ and $y$ is non-empty. Let $E$ be the set of (unordered) 2-element subsets of $W$ consisting of $\{\bullet, \star\}$, $\{\bullet, x\}$ for all $x \in W_1$, $\{\star, y\}$ for all $y \in W_{10}$ together with all the pairs in $E_1 \cup E_2 \cup \ldots \cup E_9$. Finally, let $\Gamma$ be the graph with vertex set $W$ and edge set $E$.

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**Proposition 2.2.** The graph $\Gamma$ is the Moufang hexagon associated with the hexagonal system $(K/K^\theta)^6$ as defined in [9, 15.20 and 16.8].

**Proof.** Let $\tilde{U}_+$ and $\tilde{U}_1, \ldots, \tilde{U}_6$ be the groups obtained by setting $F = K^\theta$, $J = K$, $T(a, b) = 0$, $a^\# = a^2$, $N(a) = a^3$ and $a \times b = 2ab$ for all $a, b \in K$ in [9, 16.8]. By [9, 8.13], the maps $x_i(s) \mapsto x_i(s^\theta)$ for $i = 2, 4$ and $6$, $x_i(s) \mapsto x_i(-s)$ for $i = 3$ and $5$ and $x_1(s) \mapsto x_1(s)$ extend to an isomorphism $\psi$ from $U_+$ to $\tilde{U}_+$ mapping $U_i$ to $\tilde{U}_i$. 

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**Figure 1.** The graph $\Gamma$
for all $i \in [1, 6]$. The graph $\Gamma$ is precisely the graph called $G(U_+, U_1, \ldots, U_6)$ in [9, 8.1] and the Moufang hexagon associated with the hexagonal system $(K/K^\theta)^0$ is $G(U_+, \tilde{U}_1, \ldots, \tilde{U}_6)$ (see the first page of Chapter 16 in [9]). Hence the isomorphism $\psi$ induces an isomorphism from $\Gamma$ to this Moufang hexagon.

**Notation 2.3.** Let $D = \text{Aut}(\Gamma)$ and let $D^\dagger$ denote the subgroup of $D$ generated by all the root groups of $\Gamma$.

From now on, we will write $U_{ij}$ in place of $U_{i,j}$. The group $U_+$ acts faithfully by right multiplication on the elements of

$$W_1 \cup \cdots \cup W_{10}$$

and maps the set $E$ of edges of $\Gamma$ to itself. This allows us to identify $U_+$ with a subgroup of the stabilizer $D_{\bullet, \star}$ (which we continue to denote by $U_+$). Just to fix notation, we observe, for example, that

$$(2.4) \quad U_{15}x(t) = U_{15}x(t),$$

where the cosets $U_{15}$ and $U_{15}x(t)$ are vertices in the set $W_1$ and the expression on the left means the image of the vertex $U_{15}$ under the action of the element $x(t) \in U_+$.

**Proposition 2.5.** The groups $U_1, U_2, \ldots, U_6$ are the root groups of $\Gamma$ corresponding to the six roots of $\Sigma$ that contain the edge $\{\bullet, \star\}$.

**Proof.** This holds by [9, 8.2].

We mention that by [9, 35.13 and 36.1], the extension $K/K^\theta$ is an invariant of the quadrangle $\Gamma$, from which it follows that $\Gamma$ is a split Moufang hexagon if and only if the field $K$ is perfect.

3. **The Automorphisms $m_1$ and $m_6$**

Let $\Sigma$ denote the apartment of $\Gamma$ spanned by the vertices $\bullet, \star, U_{1,6-i} \in W_i$ for all $i \in [1, 5]$ and $U_{12-i,6} \in W_i$ for all $i \in [6, 10]$. Let

$$m_1 = \mu(x_1(1)) \quad \text{and} \quad m_6 = \mu(x_6(1)),$$

where the map $\mu$ is defined (with respect to the apartment $\Sigma$) as in [9, 6.1]. Both of these elements are contained in the group $D^\dagger$ and both induce reflections on $\Sigma$; $m_1$ induces the reflection fixing $\star$ and $U_1$ and $m_6$ induces the reflection fixing $\bullet$ and $U_6$. By [9, 32.12], we have

$$x_6(t)m_1 = x_2(t) \quad \text{and} \quad x_5(t)m_1 = x_3(t)$$

and

$$x_1(t)m_6 = x_5(-t) \quad \text{and} \quad x_2(t)m_6 = x_4(t)$$

for all $t \in K$. Thus the action of $m_1$ on the vertices in $W_1$ is given by

$$(3.1) \quad (U_{15}x_6(t))^{m_1} = U_{15}^{x_6(t)m_1} = U_{15}^{x_6x_2(t)} = U_{36}x_2(t)$$

— see (2.4) above — and the action of $m_6$ on the vertices in $W_{10}$ is given by

$$(3.2) \quad (U_{26}x_1(t))^{m_6} = U_{26}^{x_1(t)m_6} = U_{26}^{x_6x_5(-t)} = U_{14}x_5(-t)$$

for all $t \in K$. Similarly, we have

$$(3.3) \quad (U_{14}x_5(t))^{m_1} = U_{46}x_3(t)$$
The maps \( m_1 \) and \( m_6 \) are as in Tables 1 and 2. (For use in Section 4, we have recorded also the product \( m_1 m_6 \) in Table 3.)

**Proposition 3.5.** The maps \( m_1 \) and \( m_6 \) are as in Tables 1 and 2. (For use in Section 4, we have recorded also the product \( m_1 m_6 \) in Table 3.)

**Proof.** Let \( \xi \) denote the permutation of \( W \) given in Table 1. We claim that \( \xi \) maps edges to edges and is thus an automorphism of \( \Gamma \). To begin, we choose an edge \( e \) containing one vertex in \( W_5 \) and one vertex in \( W_6 \). Thus \( e = \{ U_1 g, U_6 g \} \) for some

\[ g = x_1(s) x_2(t) x_3(r) x_4(u) x_5(v) x_6(w) \in U_+. \]

We have

\[ U_1 g = U_1 x_2(t) x_3(r) x_4(u) x_5(v) x_6(w) \]

and hence

\[ (U_1 g) \xi = U_1 x_2(w) x_3(v) x_4(u + wt) x_5(-r) x_6(-t). \]

By (2.1), we have

\[ U_6 g = U_6 x_1(s) x_2(t) x_3(r) x_4(u) x_5(v) x_6(w) \]

\[ = U_6 x_1(s) \cdot x_6(w) x_2(t) \cdot x_3(r) x_4(u + wt) x_5(v) \]

\[ = U_6 x_1(s) \cdot x_2(t - s^9 w) x_3(r - s^2 w^6) x_4(u + wt + s^9 w^2) x_5(v + sw^9). \]

Table 1. The action of \( m_1 \) on \( \Gamma \)

\[ \begin{align*}
U_{15} x_6(t) & \mapsto U_{36} x_2(t) \\
U_{14} x_5(s) x_6(t) & \mapsto U_{46} x_2(t) x_3(s) \\
U_{13} x_4(r) x_5(s) x_6(t) & \mapsto U_{56} x_2(t) x_3(s) x_4(r) \\
U_{12} x_3(u) x_4(r) x_5(s) x_6(t) & \mapsto U_6 x_2(t) x_3(s) x_4(r) x_5(-u) \\
U_1 x_2(v) x_3(u) x_4(r) x_5(s) x_6(t) & \mapsto U_1 x_2(t) x_3(s) x_4(r + vt) x_5(-u) x_6(-v) \\
U_6 x_1(s) x_2(t) x_3(r) x_4(u) x_5(v) & \xrightarrow{\sigma m_0} U_{12} x_3(v) x_4(u) x_5(-r) x_6(-t) \\
& \xrightarrow{\tau p \theta} U_6 x_1(-s^{-1}) x_2(-s^{-6}) x_3(v + s^{-2} t) x_5(s^{-1} t^6 - r) \\
& \quad \cdot x_4(u - s^{-6} t^6) x_5(s^{-1} t^6 - r) \\
U_{56} x_1(s) x_2(t) x_3(r) x_4(u) & \xrightarrow{\sigma m_0} U_{13} x_4(u) x_5(-r) x_6(-t) \\
& \xrightarrow{\tau p \theta} U_{56} x_1(-s^{-1}) x_2(-s^{-6}) x_3(-s^{-1} r - s^{-2} t) x_5(s^{-1} t^6 - r) \\
& \quad \cdot x_4(u - s^{-6} t^6) \\
U_{46} x_1(s) x_2(t) x_3(r) & \xrightarrow{\sigma m_0} U_{14} x_5(-r) x_6(-t) \\
& \xrightarrow{\tau p \theta} U_{46} x_1(-s^{-1}) x_2(-s^{-6}) x_3(-s^{-1} r - s^{-2} t) x_5(s^{-1} t^6 - r) \\
U_{36} x_1(s) x_2(t) & \xrightarrow{\sigma m_0} U_{12} x_6(-t) \\
& \xrightarrow{\tau p \theta} U_{36} x_1(-s^{-1}) x_2(-s^{-6}) x_5(-s^{-1} r - s^{-2} t) x_6(-u) \\
U_{26} x_1(s) & \xrightarrow{\sigma m_0} \bullet \\
& \xrightarrow{\tau p \theta} U_{26} x_1(-s^{-1})
\end{align*} \]
If \( s = 0 \), then
\[
(U_6 \xi)^\xi = (U_6 x_2(t)x_3(r)x_4(u + wt)x_5(v))^\xi
= U_{12}x_3(v)x_4(u + wt)x_5(-r)x_6(-t)
\]
and thus \((U_1 \xi)^\xi \subset (U_6 \xi)^\xi\). Suppose, instead, that \( s \neq 0 \) and let
\[
\hat{g} = x_1(-s^{-1})x_2(w)x_3(v)x_4(u + wt)x_5(-r)x_6(-t).
\]
Note that \( \hat{g} \in (U_1 \xi)^\xi \). By (2.1) again, we have
\[
U_6 \hat{g} = U_6 x_1(-s^{-1}) \cdot x_6(-t)x_2(w) \cdot x_3(v)x_4(u)x_5(-r)
= U_6 x_1(-s^{-1}) \cdot x_2(w - s^{-\theta})x_3(v + s^{-2\theta})x_4(u - s^{-\theta}t^2)x_5(-r + s^{-1}t^\theta).
\]
Therefore \((U_6 \xi)^\xi = U_6 \hat{g}\) by Table 1 and a bit of calculation. Thus
\[
\hat{g} \in (U_1 \xi)^\xi \cap (U_6 \xi)^\xi.
\]
We conclude that \( e \xi = \{(U_1 \xi)^\xi, (U_6 \xi)^\xi\}\) is an edge of \( \Gamma \) whether \( s = 0 \) or not. It is now an easy task to check in a similar fashion that \( \xi \) maps all the remaining edges to edges; we leave this to the reader.

Next we observe that the automorphism \( \xi \) induces the same reflection of the apartment \( \Sigma \) as does \( m_1 \) and it agrees with \( m_1 \) on the set of neighbors of \( \bullet \) on and on the set of neighbors of \( U_{15} \) by (3.1) and (3.3). By [9, 3.7], it follows that \( \xi = m_1 \).

(In fact, Table 1 was created by starting with the action of \( m_1 \) on \( \Sigma \), the set of neighbors of \( \bullet \) and the set of neighbors of \( U_{15} \) and working backwards.) By (3.2), (3.4) and a similar argument, the claim holds for \( m_6 \).

Now let \( \rho \) be the automorphism of \( U_+ \) mentioned above. Thus
\[
(3.6) \quad x_i(t)^\rho = x_{7-i}(t)
\]
for all \( i \in [1,6] \) and all \( t \in K \). By [9, 7.5], there exists a unique automorphism of \( \Gamma \) that maps the apartment \( \Sigma \) to itself, interchanges \( \bullet \) and \( * \) and induces \( \rho \) on \( U_+ \).

We denote this automorphism of \( \Gamma \) also by \( \rho \). Thus, in particular, \( U_1^\rho = U_6 \) and \( U_6^\rho = U_1 \).

From now on, we set
\[
(3.7) \quad \omega = (m_1m_6)^3.
\]

**Proposition 3.8.** The automorphisms \( \rho \) and \( \omega \) commute with each other and both have order 2.

**Proof.** Since \( \rho \) has order 2 as an automorphism of \( U_+ \), it also has order 2 as an automorphism of \( \Gamma \). By [9, 6.9], \( \omega = (m_6m_1)^3 \) and by [9, 6.2], \( m_1^\rho = m_6 \) and \( m_6^\rho = m_1 \). Thus \( \omega^\rho = (m_6m_1)^3 = \omega \). Let \( d = m_1^2 \) and \( e = m_6^2 \) (so \( [m_1, d] = [m_6, e] = 1 \)). Then \( d \) and \( e \) both act trivially on the apartment \( \Sigma \) and by [9, 29.12], \( d \) centralizes \( U_1 \) and \( U_4 \) and inverts every element of \( U_i \) for all other \( i \in [1,6] \) and \( e \) centralizes \( U_3 \) and \( U_6 \) and inverts every element of \( U_i \) for all other \( i \in [1,6] \). By [9, 6.7], \( d \) and \( e \) are elements of order 2 (so \( m_1^{-1} = dm_1 \) and \( m_6^{-1} = em_6 \)) and their product (in either order) is the unique element of \( D \) acting trivially on \( \Sigma \) that centralizes \( U_2 \) and \( U_5 \) and inverts every element of \( U_i \) for all other \( i \in [1,6] \). Since \( U_i^{m_1} = U_{8-i} \) for all \( i \in [2,6] \) and \( U_i^{m_6} = U_{6-i} \) for all \( i \in [1,5] \), both \( e^{m_1} \) and \( d^{m_6} \) centralize \( U_2 \) and \( U_5 \) and invert every element of \( U_i \) for all other \( i \in [1,6] \). Thus \( e^{m_1} = ed = d^{m_6} \). It follows by repeated use of these relations that
\[
(m_1^{-1}m_6^{-1})^3 = (dm_1 \cdot em_6)^3 = (m_1m_6)^3
\]
Thus that

\[ \omega \]

Let Proposition 3.9. By (2.1) and (3.6), we have

\[ \phi \]

Then Proof.

\[ a, b, c, d, e, f \] clearly injective. Now choose

\[ x = x = x = x = x \]

commutes with \( \phi \)

\[ \rho = \rho = \rho = \rho = \rho \]

\[ 1 \]

\[ 2 \]

\[ 3 \]

\[ 4 \]

\[ 5 \]

\[ 6 \]

\[ U_{14} x_5(v) x_6(w) \]

\[ U_{13} x_4(u) x_5(v) x_6(w) \]

\[ U_{12} x_3(r) x_4(u) x_5(v) x_6(w) \]

\[ U_{11} x_2(t) x_3(r) x_4(u) x_5(v) x_6(w) \]

\[ U_0 x_1(s) x_2(t) x_3(r) x_4(u) x_5(v) \]

\[ U_{56} x_1(s) x_2(t) x_3(r) x_4(u) \]

\[ U_{46} x_1(s) x_2(t) x_3(r) \]

\[ U_{36} x_1(s) x_2(t) \]

\[ U_{26} x_1(s) \]

Table 2. The action of \( m_6 \) on \( \Gamma \)

and hence \( \omega^{-1} = (m_6 m_1)^{-3} = \omega \).

\[ \square \]

Proposition 3.9. Let \( \varphi \) be the map from \( U \) to \( U^+ \) given by

\[ \varphi(a, b, c) = x_1(a) x_2(b) x_3(c - ab + a^{\theta + 2}) x_4(c + ab) x_5(b - \alpha^{\theta + 1}) x_6(a). \]

Then \( \varphi \) is an injective homomorphism whose image is the centralizer of \( \rho \) in \( U^+ \).

Proof. By (1.2) and (2.1) and a bit of calculation, \( \varphi \) is a homomorphism. It is clearly injective. Now choose \( a, b, c, d, e, f \in K \) and let

\[ g = x_1(a) x_2(b) x_3(c) x_4(d) x_5(e) x_6(f). \]

By (2.1) and (3.6), we have

\[ g^\rho = x_6(a) x_5(b) x_4(c) x_3(d) x_2(e) x_1(f) \]

\[ = x_5(b) x_4(c - ae) x_3(d) x_2(e) x_1(f) \]

\[ = x_2(e) x_3(d) x_4(c - ae) x_5(b) \cdot x_1(f) x_6(a) \cdot x_2(af^\theta x_3(a^\theta f^2) x_4(-a^2 f^\theta) x_5(-a^\theta f)) \]

\[ = x_5(b) x_4(d + bf) x_3(c - ae) x_5(b) \cdot x_2(af^\theta x_3(a^\theta f^2) x_4(af^\theta) x_5(a^\theta f) x_6(a)) \]

\[ = x_1(f) x_2(e) x_3(d + bf + a^\theta f^2) x_4(c - ae + a^2 f^\theta) x_5(b - a^\theta f) x_6(a). \]

Thus \( g^\rho = g \) if and only if \( a = f, \ e = b - \alpha^{\theta + 1} \) and \( c = d + ab + a^{\theta+2} \). We conclude that \( g \) commutes with \( \rho \) if and only if \( g = \varphi(a, b, d - ab) \).

\[ \square \]
\[ \ast \mapsto U_{15} \]
\[ \bullet \mapsto U_{14} \]
\[ U_{13} x_6(t) \mapsto U_{13} x_4(t) \]
\[ U_{14} x_5(s) x_6(t) \mapsto U_{12} x_3(s) x_4(t) \]
\[ U_{13} x_4(r) x_5(s) x_6(t) \mapsto U_{1} x_4(-r) x_3(s) x_4(t) \]
\[ U_{12} x_3(u) x_4(r) x_5(s) x_6(t) \mapsto U_{6} x_1(u) x_2(-r) x_3(s) x_4(t) \]
\[ U_{1} x_2(v) x_3(u) x_4(r) x_5(s) x_6(t) \xrightarrow{\ast \neq 0} U_{56} x_1(u) x_2(-r) x_3(s) x_4(t) \]
\[ U_{6} x_1(s) x_2(t) x_3(r) x_4(u) x_5(v) \xrightarrow{\ast \neq 0} U_{46} x_1(r) x_2(-u) x_3(v) \]
\[ U_{56} x_1(s) x_2(t) x_3(r) x_4(u) x_5(v) \xrightarrow{\ast \neq 0} U_{36} x_1(r) x_2(-u) \]
\[ U_{46} x_1(s) x_2(t) x_3(r) \xrightarrow{\ast \neq 0} U_{26} x_1(r) \]
\[ U_{26} x_1(s) \xrightarrow{\ast \neq 0} \ast \]

**Table 3.** The action of \( m_1 m_6 \) on \( \Gamma \)

From now on we identify \( U \) with its image in \( U_+ \) under the map \( \varphi \) in Proposition 3.9.

**Proposition 3.10.** Let \( X \) be the set of edges of \( \Gamma \) fixed by \( \rho \), let \( \infty \) denote the edge \( \{ \bullet, \ast \} \) and let \( G = \langle U, \omega \rangle \), where \( \omega \) is as in (3.7). Then the following hold:

(i) \( U \) acts regularly on \( X \setminus \{ \infty \} \).
(ii) \( G \) acts 2-transitively on \( X \).
(iii) \( G = B \cup B \omega B \), where \( B = G_\infty \).
(iv) \( U \) is a normal subgroup of the stabilizer \( G_\infty \).
(v) \( G \) acts faithfully on \( X \).

**Proof.** Since \( \rho \) interchanges the vertices \( \bullet \) and \( \ast \), all the edges in \( X \) other than \( \infty = \{ \bullet, \ast \} \) are two-element subsets containing a right coset of \( U_1 \) and a right coset of \( U_6 \). Since \( U_1 \cap U_6 = 1 \), the intersection of a right coset of \( U_1 \) and a right coset
of $U_6$ is either empty or consists of a unique element. It follows that

$$X = \{ \{U_1g, U_6g\} \mid g \in U \} \cup \{ \infty \}.$$ 

In particular, (i) holds and we can identify $U$ with $X \setminus \{ \infty \}$ via the map that sends $g \in U$ to $\{U_1g, U_6g\}$, $\{0, 0, 0\} \in U$ now denotes the edge $\{U_1, U_6\}$ itself. By Proposition 3.8, $\omega$ acts on the set $X$. Since $\omega$ interchanges the edges $\infty$ and $0$ (by Table 3) and $U$ acts transitively on $X \setminus \{ \infty \}$, we conclude that (ii) and (iii) hold. Since $U_+^*$ is normal in $D_\infty$ (by [9, 4.7 and 5.3]) and $G$ is contained in the centralizer of $\rho$, also (iv) holds.

Note that $\omega$ maps each vertex of $\Sigma$ to a vertex at distance 6 from itself. Since the elements of $U$ all fix the vertex $\bullet$ and $\Gamma$ is bipartite, it follows that the distance from $x$ to $x^g$ is even for every vertex $x$ and every $g \in \langle U, \omega \rangle$. In particular, no element of $G$ interchanges the two vertices of an edge.

For each $x \in X \setminus \{ \infty \}$, there exists a unique apartment $\Sigma_x$ of $\Gamma$ containing the edges $x$ and $\infty$. For each $(a, b, c) \in U$, we have $U_{15}\varphi(a, b, c) = U_{15}x_6(a)$ and $U_{26}\varphi(a, b, c) = U_{26}x_1(a)$ by Proposition 3.9. For each vertex $u$ adjacent to $\bullet$ or $\ast$, therefore, there exists an edge $x \in X \setminus \{ \infty \}$ such that $u \in \Sigma_x$. If an element of $G$ acts trivially on $X$, then it acts trivially on all these apartments, thus it also acts trivially on the set of all vertices adjacent to $\bullet$ or $\ast$. Hence it is itself trivial by [9, 3.7]. Thus (v) holds. 

\textbf{Remark 3.11.} The permutation group on $U$ obtained by letting $U$ act on itself by right multiplication is the same as the permutation group on $U$ obtained by letting $U^{opp}$ act on itself by left multiplication. It follows that Theorem 1.1 is equivalent to the assertion obtained by replacing the multiplication on $U$ defined in (1.2) by the opposite multiplication and in part (iii) letting $U$ act on $U = X \setminus \{ \infty \}$ by left rather than right multiplication, and this “left-handed” version of Theorem 1.1 produces the same group $G$. We have chosen to work with right cosets and to let $U_+$ act by right multiplication in order to conform to [9] and to the recent literature on Moufang sets, whereas Tits chose to work with left multiplication in [7]. This explains why the group $U$ in Theorem 1.1 is the opposite of the group $U$ in Example 3 on page 210-15 in [7].

\textbf{Proposition 3.12.} Let $H$ be as in (1.3), let $D^1$ be as in (2.3), let $D^\varphi$ denote the centralizer of $\rho$ in $D^1$, and let $T$ denote the two-point stabilizer $D^\varphi_{\infty, 0}$. Then there is a canonical isomorphism $\pi$ from $H$ to $T$ that is compatible with the map $\varphi$ in Proposition 3.9.

\textbf{Proof.} Let $g \in D^1_{\infty, 0}$. Thus $g$ acts trivially on the apartment $\Sigma$. By [9, 15.20 and 33.16] and the isomorphism described in the proof of Proposition 2.2, there exist $a, u \in K^*$ such that $x_1(s)^g = x_1(a^2u^{-\theta}s)$ and $x_6(s)^g = x_6(a^{-\theta}u^2s)$ for all $s \in K$. By [9, 33.5], the centralizer of $\langle U_1, U_6 \rangle$ in $D^\varphi_{\infty, 0}$ is trivial. By (3.6), therefore, $g$ commutes with $\rho$ (and hence is contained in $T$) if and only if $a^2u^{-\theta} = a^{-\theta}u^2$. Since the maps $x \mapsto x^{2+\theta}$ and $x \mapsto x^{2-\theta}$ from $K^*$ to $K^*$ are inverses of each other, we conclude that $a = u$ and the map $g \mapsto a^{2-\theta}$ is an isomorphism from $T$ to $K^*$. Now let $t = a^{2-\theta}$, so $x_1(s)^g = x_1(ts)$ and $x_6(s)^g = x_6(ts)$ for all $s \in K$. By the commutator relations (2.1), it follows that $x_2(s)^g = x_2(t^{\theta+1}s)$, $x_3(t)^g = x_3(t^{\theta+2}s)$, $x_4(s)^g = x_4(t^{\theta+2}s)$ and $x_5(s)^g = x_5(t^{\theta+1}s)$. By Proposition 3.9, therefore, $(a, b, c)^g = (a, b, c)^{\lambda t}$, where $\lambda t$ is as in (1.3). 

\hfill \Box
From now on we identify $H$ with the two point stabilizer $T$ via the map $\pi$ in Proposition 3.12.

4. The formula (1.5)

In this section we show that the norm $N$ defined in (1.4) is anisotropic and that the automorphism $\omega$ satisfies (1.5). We will do this by computing explicitly the action of $\omega$ on $X$ using Table 3.

For each $g = (a,b,c) \in U$, we have

$$(4.1) \quad U_1 g = U_1 x_2(b) x_3(c - ab + a^{\theta+2}) x_4(c + ab) x_5(b - a^{\theta+1}) x_6(a)$$

by Proposition 3.9 and

$$(4.2) \quad U_1 g \cap U = \{ g \}$$

by Proposition 3.10(i).

**Lemma 4.3.** Suppose that $U_1 x_2(\bar{v}) x_3(\bar{u}) x_4(\bar{r}) s_5(\bar{s}) x_6(\bar{t}) = U_1 g$ for some $g \in U$. Then $g = (\bar{t}, \bar{v}, \bar{r} - \bar{v}\bar{t})$.

**Proof.** This holds by (4.1) and (4.2). \qed

We now fix $g = (a,b,c) \in U^*$ and let $u, v$ and $w = N(a,b,c)$ be as in Theorem 1.1(iii). Observe first that the following curious identity holds:

$$(4.4) \quad w = av + bu + c^2.$$ 

Let $m = m_1 m_6$ (so $\omega = m^3$), let $\alpha$ denote the vertex $U_1 g$, let $\beta = \alpha^m$ and let $\gamma = \beta^m$. Our goal is to show that $w \neq 0$ and that

$$(4.5) \quad (a,b,c)^\omega = (-v/w, -u/w, -c/w).$$

**Lemma 4.6.** Suppose that $w \neq 0$ and that

$$\alpha^\omega = U_1 x_2(\bar{v}) x_3(\bar{u}) x_4(\bar{r}) x_5(\bar{s}) x_6(\bar{t}).$$

Then (4.5) holds if and only if

$$(4.7) \quad \bar{t} = -v/w;$$

$$(4.8) \quad \bar{v} = -u/w; \quad \text{and}$$

$$(4.9) \quad \bar{r} = -c/w + (-v/w)(-u/w).$$

**Proof.** Since $\omega$ maps $X \setminus \{\infty, 0\}$ to itself, we have $\alpha^\omega = U_1 e$ for some $e \in U^*$. The claim holds, therefore, by Lemma 4.3. \qed

To begin, we assume that

$$(4.10) \quad b \neq 0,$$

so by Table 3 applied to (4.1), we have

$$\beta = \alpha^m = U_1 x_2(\bar{v}) x_3(\bar{u}) x_4(\bar{r}) x_5(\bar{s}) x_6(\bar{t}),$$
where
\[ \hat{v} = b^{-1} c^\theta - a^\theta b^{-1} + a^{2\theta+3} b^{-1} - c - ab; \]
\[ \hat{u} = b - a^{\theta+1} + b^{-\theta+2} c + a^{2\theta+4} b^{-\theta} + ab^{-\theta+1} c - a^{\theta+2} b^{-\theta} + b^{-1} c - a; \]
\[ \hat{r} = b^{-2} c^\theta - a^{\theta+2} b^{-\theta} + a^{2\theta+3} b^{-2} + b^{-1} c - a; \]
\[ \hat{s} = -b^{-\theta} c + ab^{-\theta+1} - a^{\theta+2} b^{-\theta}; \] and
\[ \hat{t} = b^{-1}. \] (4.11)

It is straightforward to check that the following identities hold:
\[ w = b^\theta \hat{v} - \hat{v} (\hat{v} - c); \] (4.12)
\[ b^\theta \hat{r} = \hat{v} - c; \] (4.13)
\[ b^\theta \hat{s} = -a - b^{-1} (\hat{v} + c); \] and
\[ \hat{v} = -b^{-1} v. \] (4.15)

Next we assume that
\[ v \neq 0. \] (4.16)

Thus also \( \hat{v} \neq 0 \) (by (4.15)), so by a second application of Table 3, we have
\[ \gamma = \beta^m = U_1 x_2 (\hat{v}) x_3 (\hat{u}) x_4 (\hat{r}) x_5 (\hat{s}) x_6 (\hat{t}), \]
where
\[ \hat{v} = \hat{u}^\theta \hat{v}^{-1} - \hat{r}; \] (4.17)
\[ \hat{u} = \hat{s} + \hat{v}^2 \hat{v}^{-\theta}; \] (4.18)
\[ \hat{r} = \hat{u}^\theta \hat{v}^{-2} + \hat{v}^{-1} \hat{r} + \hat{t}; \] (4.19)
\[ \hat{s} = -\hat{u} \hat{v}^{-\theta}; \text{ and} \]
\[ \hat{t} = \hat{v}^{-1}. \] (4.20)

Note that
\[ \hat{v} = \hat{u}^\theta \hat{v}^{-1} - \hat{r} \]
\[ = \hat{v}^{-1} b^{-1} \cdot (b \hat{u}^\theta - b \hat{r} \hat{v}) \]
\[ = \hat{v}^{-1} b^{-1} \cdot (b \hat{u}^\theta - \hat{v} (\hat{v} - c)) \]
\[ = \hat{v}^{-1} b^{-1} \cdot w \]
\[ = -w/v \]
by (4.17)
by (4.13)
by (4.12)
by (4.15).

In particular, we have
\[ \hat{v} = b^{-1} \hat{v}^{-1} w \] (4.21)
as well as
\[ \hat{v} = -w v^{-1} \] (4.22)
and \( \hat{u}^\theta \hat{v}^{-1} = \hat{r} + b^{-1} \hat{v}^{-1} w \), so
\[ b^2 \hat{u}^{2\theta} \hat{v}^{-2} = b^2 \hat{r}^2 - b \hat{r} \hat{v}^{-1} w + \hat{v}^{-2} w^2. \] (4.23)
Moreover,
\[
\tilde{r} = \hat{u}^\theta \hat{v}^{-2} + \hat{v}^{-1} \tilde{r} + \tilde{\ell} \quad \text{by (4.19)}
\]
\[
= \hat{u}^\theta \hat{v}^{-2} + \hat{v}^{-1} \tilde{r} + b^{-1} \quad \text{by (4.11)}
\]
\[
= (\hat{v} - \tilde{r})\hat{v}^{-1} + b^{-1} \quad \text{by (4.17)},
\]
hence
\[
(4.24) \quad \tilde{r} = b^{-1} \hat{v}^{-2} w - \hat{v} \hat{v}^{-1} + b^{-1}
\]
by (4.21), and thus
\[
(4.25) \quad b \tilde{r} w = \hat{v}^{-2} w^2 - b \hat{v} \hat{v}^{-1} w + w.
\]
We record also that
\[
(4.26) \quad b^2 \hat{s}^\theta \hat{v} = -ab\hat{v} - \hat{v}^2 - bc
\]
by (4.14).

The vertex \(\hat{\alpha}^\omega = \gamma^m\) lies on an edge contained in \(X\). Hence \(\hat{\alpha}^\omega \in W_5\) (where \(W_5\) is as in Figure 1). It follows that \(\hat{v} \neq 0\) since otherwise \(\gamma^m \in W_7\) by Table 3. By (4.22), we conclude that
\[
w \neq 0
\]
and by a final application of Table 3, we have
\[
\hat{\alpha}^\omega = \gamma^m = U_1 x_2(\hat{v}) x_3(\hat{u}) x_4(\tilde{r}) x_5(\hat{s}) x_6(\tilde{\ell}),
\]
where
\[
\hat{v} = \hat{u}^\theta \hat{v}^{-1} - \tilde{r};
\]
\[
\hat{u} = \hat{s} + \hat{u}^2 \hat{v}^{\theta};
\]
\[
\tilde{r} = \hat{u}^\theta \hat{v}^{-2} + \tilde{v}^{-1} \tilde{r} + \tilde{\ell};
\]
\[
\hat{s} = -\hat{u}^{\theta}; \quad \text{and}
\]
\[
\tilde{\ell} = \hat{v}^{-1}.
\]
We now observe that \(\tilde{\ell} = \tilde{v}^{-1} = -v/w\) by (4.22), so (4.7) holds. Furthermore,
\[
-b \hat{v} w = -b(\hat{u}^\theta \hat{v}^{-1} - \tilde{r}) w
\]
\[
= -b^2 \hat{u}^\theta \hat{v} + b \tilde{r} w \quad \text{by (4.21)}
\]
\[
= -b^2 (\hat{s} + \hat{u}^2 \hat{v}^{\theta}) \hat{v} + b \tilde{r} w \quad \text{by (4.18)}
\]
\[
= -b^2 \hat{s} \hat{v} - b^2 \hat{u}^{2 \theta} \hat{v}^{-2} + b \tilde{r} w.
\]
Applying (4.23), (4.25) and (4.26) to the three terms in this last expression, we find that
\[
-b \hat{v} w = ab\hat{v} + \hat{v}^2 + c \hat{v} - b^2 \hat{r}^2 + w
\]
\[
= ab\hat{v} + \hat{v}^2 + c \hat{v} - (\hat{v} - c)^2 + w \quad \text{by (4.13)}
\]
\[
= -av - c^2 + w \quad \text{by (4.15)}
\]
\[
= bu \quad \text{by (4.4)}. 
\]
Thus (4.8) holds. Finally, we have

\[ w\ddot{r} = w(\tilde{u}^\theta \tilde{v}^{-2} + \tilde{v}^{-1} \tilde{r} + \tilde{t}) \]
\[ = w\tilde{v}^{-1}(\tilde{v} - \tilde{r}) + w\tilde{t} \]
\[ = -\tilde{v}^{-1}(u + w\tilde{r}) + w\tilde{t} \quad \text{by (4.8)} \]
\[ = uvw^{-1} + v\tilde{r} + w\tilde{t} \quad \text{by (4.22)} \]
\[ = uvw^{-1} + v(\hat{b}^{-1} \hat{v}^{-2}w - \hat{v}^{-1} \hat{r} + \hat{b}^{-1}) + w\hat{v}^{-1} \quad \text{by (4.24)} \]
\[ = uvw^{-1} + (-\hat{v}^{-1}w + \hat{b} \hat{r} - \hat{v}) + w\hat{v}^{-1} \quad \text{by (4.15)} \]
\[ = uvw^{-1} - c \quad \text{by (4.13)}, \]

so also (4.9) holds. By Lemma 4.6, it follows that (4.5) holds. We conclude that \( w \neq 0 \) and that the identity (1.5) holds for all “generic” points in \( U^* \), i.e. for all \( g = (a, b, c) \) in \( U^* \) satisfying (4.10) and (4.16).

Next we consider the case that \( b \neq 0 \) but \( v = 0 \). By (4.15), we have \( \hat{v} = 0 \) as well and hence

\[ \beta = \alpha^m = U_1 x_3(\hat{u}) x_4(\tilde{r}) x_5(\hat{s}) x_6(\hat{t}). \]

It follows from Table 3 that

\[ \gamma = \beta^m = U_{56} x_1(\hat{u}) x_2(\tilde{r}) x_3(\hat{s}) x_4(\hat{t}). \]

If \( \hat{u} = 0 \), it would follow from Table 3 that \( \alpha^{\omega} = \gamma^m \in W_3 \cup W_9 \). This is impossible since the vertex \( \alpha^{\omega} \) lies on an edge contained in \( X \). We conclude that \( \hat{u} \neq 0 \). It follows from (4.12) (with \( \hat{v} = 0 \)) that \( w \neq 0 \). From Table 3 we now obtain

\[ \alpha^{\omega} = \gamma^m = U_1 x_2(\hat{v}) x_3(\hat{u}) x_4(\tilde{r}) x_5(\hat{s}), \]

where

\[ \hat{v} = -\hat{t} + \hat{u}^\theta \hat{r}^2; \]
\[ \hat{u} = -\hat{u}^{-1} \hat{s} + \hat{u}^{-2} \hat{r}^\theta; \]
\[ \tilde{r} = \hat{u}^{-\theta} \hat{r}; \text{ and} \]
\[ \hat{s} = -\hat{u}^{-1}. \]

Remembering that \( v = \hat{v} = 0 \), we calculate that

\[ \tilde{r} = \hat{u}^{-\theta} \hat{r} \]
\[ = w^{-1}b \cdot \hat{r} \quad \text{by (4.12)} \]
\[ = -w^{-1}c \quad \text{by (4.13)} \]

and

\[ \hat{v} = -\hat{t} + \hat{u}^{-\theta} \hat{r}^2 \]
\[ = -b^{-1} + w^{-1}b \cdot (-b^{-1}c)^2 \quad \text{by (4.11), (4.12) and (4.13)} \]
\[ = -b^{-1}w^{-1} \cdot (w - c^2) \]
\[ = -b^{-1}w^{-1} \cdot bu \quad \text{by (4.4)} \]
\[ = -w^{-1}u. \]

By Lemma 4.6, we conclude that (4.5) holds.
We can thus assume from now on that $b = 0$, so
\[ \alpha = U_1 x_3(c + a^{\theta+2}) x_4(c) x_5(-a^{-1}) x_6(a) \]
as well as
\begin{align*}
(4.27) & \quad v = -c^\theta - a^{2\theta+3} = -z^\theta; \\
(4.28) & \quad u = -ac - a^{\theta+3}; \\
(4.29) & \quad w = -ac^\theta + c^2 - a^{2\theta+4} = c^2 - az^\theta,
\end{align*}
where
\[ z = c + a^{\theta+2}. \]

From Table 3 we now obtain
\[ \beta = \alpha^m = U_{56} x_1(z) x_2(-c) x_3(-a^{\theta+1}) x_4(a). \]

Note that $a$ and $c$ cannot both be 0, since otherwise $g = (a, 0, c) = 0 \in U$.

Suppose that $c = -a^{\theta+2}$, or equivalently, that $z = 0$. Then $a \neq 0$, and Table 3 tells us that
\begin{align*}
(4.27) & \quad v = 0, \\
(4.28) & \quad u = 0 \quad \text{and} \quad w = a^{2\theta+4} \neq 0 \quad \text{and by Lemma 4.6,} \\
(4.29) & \quad \text{we conclude once again that (4.5) holds.}
\end{align*}

Suppose, finally, that $c \neq -a^{\theta+2}$, or equivalently, that $z \neq 0$. From Table 3 we obtain
\[ \gamma = \beta^m = U_1 x_3(\tilde{v}) x_4(\tilde{u}) x_5(\tilde{r}) x_6(\tilde{s}), \]
where
\begin{align*}
(4.30) & \quad \tilde{v} = -a + z^{-\theta}c^2; \\
(4.31) & \quad \tilde{u} = -a^{\theta+1} + z^{-2}c^\theta; \\
(4.32) & \quad \tilde{r} = -a^{-1}c; \quad \text{and} \\
(4.33) & \quad \tilde{s} = -z^{-1}.
\end{align*}

It follows from (4.29) and (4.30) that
\[ w = z^\theta \tilde{v}. \]

Observe that $\tilde{v} \neq 0$, since it would otherwise follow from Table 3 again that $\alpha^\omega = \gamma^m \in W_7$, which is impossible. Therefore $w \neq 0$ also in this last case. By one final application of Table 3, we obtain
\[ \alpha^\omega = \gamma^m = U_1 x_3(\ddot{v}) x_4(\ddot{u}) x_5(\ddot{r}) x_6(\ddot{s}) \]
where
\begin{align*}
\ddot{v} & = a^\theta \ddot{u}^{-1} - \ddot{r}; \\
\ddot{u} & = s + \ddot{u}^2 \ddot{v}^{-\theta}; \\
\ddot{r} & = a^\theta \ddot{u}^{-2} + \ddot{u}^{-1} \ddot{r}; \\
\ddot{s} & = -\ddot{u} \ddot{v}^{-\theta}; \quad \text{and} \\
\ddot{t} & = \ddot{v}^{-1}. \]
By (4.27) and (4.33), we have
\[ \ddot{t} = \tilde{v}^{-1} = z^\theta / w = -v/w. \]
Furthermore,
\[
\ddot{v} = \tilde{u} \tilde{v}^{-1} - \ddot{r}
\]
\[
= (z^{-1} a^{\theta+1} + z^{-2} c^\theta) \cdot (-v/w) - z^{-\theta} c
\]
\[ = w^{-1} \cdot \left( (z^{-\theta} a^{\theta+3} + z^{-2\theta} c^3) \cdot z^\theta - z^{-\theta} cw \right)
\]
\[ = w^{-1} \cdot (a^{\theta+3} + z^{-\theta} c^3 - z^{-\theta} c^3 + ac)
\]
by (4.31), (4.32) and (4.34)
\[ = w^{-1} \cdot (a^{\theta+3} + z^{-\theta} c^3 - z^{-\theta} c^3 + ac)
\]
by (4.27)
\[ = -u/w
\]
by (4.28)
and
\[
\ddot{r} = \tilde{u} \tilde{v}^{-2} + \tilde{r}^{-1} \ddot{r}
\]
\[ = (-v/w) \cdot (\tilde{u} \tilde{v}^{-1} + \ddot{r})
\]
\[ = (-v/w) \cdot \left( (-u/w) - \ddot{r} \right)
\]
\[ = (-v/w)(-u/w) + (v/w) \cdot z^{-\theta} c
\]
\[ = -c/w + (-v/w)(-u/w)
\]
by (4.35)
by (4.27).

By Lemma 4.6, we conclude that (4.5) holds also in this last case.

This completes the proof that \( w \neq 0 \) and that the identity (1.5) holds for every \( g = (a, b, c) \) in \( U^* \).

5. Properties (I)–(VI)

By Proposition 3.8, \( \omega \) is a permutation of \( X \) of order 2. To conclude our proof of Theorem 1.1, it thus remains only to show that (I)–(VI) hold. By Proposition 3.10(ii) and (v), (I) holds. For each \( x \in X \), there exists \( g \in G \) mapping \( \infty \) to \( x \); let \( U_x = U^g \). If \( g_1, g_2 \) are two elements of \( G \) mapping \( \infty \) to the same element of \( X \), then \( g_1 g_2^{-1} \in G_\infty \) and thus \( U^{g_1} = U^{g_2} \) (by Proposition 3.10(iv)). By Proposition 3.10(i), it follows that \( (X, (U_x)_{x \in X}) \) is a Moufang set (as defined, for example, in [1, 2.1]). Let \( G^1 = \langle U_x \mid x \in X \rangle \) and let \( \mu \) be as in [1, 3.1]. Thus for each \( a \in U^* \), \( \mu(a) \) is the unique element of \( U_0 a U_0 = U_0 a U_0 \) that interchanges \( \infty \) and \( 0 \). (Note that this is not the same \( \mu \) as in the definition of \( m_1 \) and \( m_6 \) at the beginning of Section 3 above.) By [1, 3.1(ii)], we have
\[ G^1_\infty = U \cdot \mu(a) \mu(b) \mid a, b \in U^* \).

Proposition 5.2. The following hold:

(i) \( G^1_\infty = U H^\dagger \), where \( H^\dagger \) is as defined in (1.6).
(ii) \( \omega \in \langle U, U^\omega \rangle \) (so \( G = G^\dagger \)).

Proof. We have \( \langle \mu(a) \mu(b) \mid a, b \in U^* \rangle = H^\dagger \) by [3, 6.12(ii)], whose proof depends only on knowing that the norm \( N \) is anisotropic. By (5.1), therefore, (i) holds. At the conclusion of the proof of [3, 6.12(ii)], it is observed that \( \omega = \mu(0, 0, 1) \). Hence (ii) holds.

By Propositions 3.10(iv) and 5.2, (II) and (III) hold. Let
\[ t \cdot (a, b, c) = (a, b, c)^{h_t} \]
for each \((a, b, c) \in U\) and each \(t \in K^*\). By (1.5), we have
\[
(5.4) \quad \omega(t \cdot (a, b, c)) = t^{-1} \cdot \omega(a, b, c)
\]
for all \((a, b, c) \in U\) and all \(t \in K^*\). Thus (V) holds. Since \(H\) normalizes \(U\), it follows that \(H\) also normalizes \(U^\omega\). Hence (IV) follows from (III).

Suppose, finally, that \(|K| > 3\). Let \(K^\dagger\) be as in (1.6). Thus, in particular, \((K^*)^2 = N(0, 0, K^*) \subset K^\dagger\). Since \(|K| > 3\), it follows that we can choose \(t \in K^\dagger\) such that \(t^{θ+1} ≠ 1\). Thus \(t^θ ≠ 1\), so also \(t^{θ+2} ≠ 1\). We have
\[
[h_t, (a, 0, 0)] = ((1 - t)a, (t - 1)t^{θ}a^{θ+1}, 0),
\]
\[
[h_t, (0, b, 0)] = (0, (1 - t^{θ+1})b, 0)
\]
and \(h_t, (0, 0, c] = (0, 0, (1 - t^{θ+2})c)\) for all \(a, b, c \in K\). Hence \(U \subset [G, G]\). By Proposition 3.10(iii), \((G_∞, ⟨ω⟩)\) is a BN-pair (as defined in [6, 2.1]). The group \(U\) is nilpotent. By [6, 2.8] and Proposition 3.10(iv) and (v), it follows that \(G\) is simple. Thus (VI) holds.

6. A more elementary reason why the norm is anisotropic

In this section we give a short algebraic proof that the norm \(N\) defined in (1.4) is anisotropic. Let
\[
(6.1) \quad Ω(a, b, c) = (-v, -uw^θ, -cw^{θ+1})
\]
for all \((a, b, c) \in U\), where, as in (1.4) and (1.5),
\[
v = a^θb^θ - c^θ + ab^2 + bc - a^{2θ+3},
u = a^2b - ac + b^θ - a^{θ+3}
\]
and \(w = N(a, b, c) = -ac^θ + a^{θ+1}b^θ - a^{θ+3}b - a^{2θ+2} + b^{θ+1} + c^2 - a^{2θ+4}\). Note that
\[
(6.2) \quad N(t \cdot (a, b, c)) = t^{2θ+4}N(a, b, c)
\]
for all \((a, b, c) \in U\), where \(t \cdot (a, b, c)\) is as in (5.3), and
\[
(6.3) \quad N((a, b, c)^{-1}) = N(a, b, c)
\]
for all \((a, b, c) \in U^*\), where \((a, b, c)^{-1}\) is as in Theorem 1.1(i).

Our proof rests on the observation that
\[
(6.4) \quad N(Ω(a, b, c)) = N(a, b, c)^{2θ+3}
\]
for all \((a, b, c) \in U\). This can be checked simply by plugging the definitions of \(v, u\) and \(w\) into (6.1). (That this identity ought to hold follows from [3, 6.18] and (5.4).)

Now fix \((a, b, c) \in U^*\) such that \(w = 0\).

**Lemma 6.5.** \(v = 0\).

**Proof.** By (6.1) and (6.4), we have
\[
N(-v, 0, 0) = N(Ω(a, b, c)) = 0.
\]
By (1.4), on the other hand, \(N(-v, 0, 0) = -v^{2θ+4}\). □

**Lemma 6.6.** \(a ≠ 0\).
Proof. Suppose \( a = 0 \). Since \((a,b,c) \neq 0\) and \(w = 0\), we have \(c \neq 0\). By (6.2), the norm of \(e^{b-2} \cdot (0,b,c)\) is zero. We can thus assume that \(c = 1\). It follows by (1.4) that \(b \neq 1\), but Lemma 6.5 implies that \(b = 1\).

By (6.2) and Lemma 6.6, we can assume from now on that \(a = 1\). Hence \(v = 0\) means that

\[(6.7) \quad b^\theta - c^\theta + b^2 + bc - 1 = 0\]

and \(w - v = 0\) means that

\[(6.8) \quad b^{\theta+1} + b^2 - b - bc + c^2 = 0.\]

By (6.3) and Lemma 6.5, we also have \(v(-1,-b+1,-c) = v((1,b,c)^{-1}) = 0\) and thus

\[(6.9) \quad b^\theta + c^\theta - b^2 - b - 1 + bc - c = 0.\]

Adding (6.7) and (6.9), we find that

\[(6.10) \quad b^\theta + b - 1 = -bc - c.\]

Multiplying this last equation by \(b\) and comparing with (6.8), we obtain

\[(6.11) \quad c(c - b^2 + b) = 0.\]

Assume first that \(c = 0\). Then by (6.7), we have \(b^\theta + b^2 - 1 = 0\) whereas by (6.10), we have \(b^\theta + b - 1 = 0\). We find \(b^2 = b\) and thus \(b \in \{0,1\}\), contradicting the equality \(b^\theta + b - 1 = 0\).

Hence \(c \neq 0\), and it follows from (6.11) that \(c = b^2 - b\). By (6.7), we now obtain

\[b^{2\theta} = b^3 - 1 - b^\theta;\]

from (6.10) on the other hand, we get

\[b^3 - 1 = -b^\theta.\]

Combining the last two equations, we obtain \(b^{2\theta} = b^\theta\), but then \(c^\theta = 0\) and hence \(c = 0\) after all. With this contradiction, we conclude that the norm \(N\) is anisotropic.

7. The subgroup \(H^\dagger\)

If \(K\) is finite, then \(|K|\) is an odd power of 3, from which it follows that \(K^*\) is generated by \((K^*)^2 = N(0,0,K^*)\) and \(-1 = N(0,1,1)\), so \(K^\dagger = K^*\) and \(H^\dagger = H\). This is [5, 8.4]. It is not necessarily true, however, that \(H^\dagger = H\) if \(K\) is infinite. In this section we illustrate this with an example. As Tits suggests in [8, 1.12], we only need to modify what he does there slightly.

Let \(F\) be an odd degree extension of the field with three elements and let \(K\) be the field of quotients of the polynomial ring \(F[s,t]\) in two variables \(s\) and \(t\). Since \(|F|\) is an odd power of 3, there exists a unique endomorphism \(\theta\) of \(K\) mapping \(F\) to \(F\), \(t\) to \(s\) and \(s\) to \(t^3\) whose square is the Frobenius endomorphism. (In what follows, the reader may wish to think of \(s\) as being formally equal to \(t^{\sqrt{3}}\).)

**Proposition 7.1.** The group \(K^\dagger \cap F(t)\) is generated by \((F(t)^*)^2\) and all irreducible polynomials in \(F[t]\) of even degree.
Thus $f/e$ show that no product in $F$ on the fact that the norm is anisotropic), $\nu$.

Corollary 7.2. $\nu$

Proof. Since $F$ is finite, we have $F^* \subset K^\perp$. Let $f \in F[t]$ be an irreducible polynomial of even degree over $F$ and let $\alpha$ be a root of $f$ in some splitting field $L$. Then $L = F(\alpha)$ and $[L : F] = \deg(f) = 2d$ for some $d$. Thus $L$ contains an element $\beta$ whose square is $-1$. Since $[L : F(\beta)] = d$, there are non-zero polynomials $p, q \in F[t]$ of degree at most $d$ such that $p + \beta q$ is the minimal polynomial of $\alpha$ over $F(\beta)$. Thus $p + \beta q$ divides $f$. Hence also $p - \beta q$ divides $f$. Since the polynomial $p + \beta q$ is irreducible over $F(\beta)$, it follows that it is relatively prime to the polynomial $p - \beta q$. Thus $f/e$ equals the product of these two polynomials for some $e \in F^*$. Hence $f = e(p^2 + q^2) = eN(0, p^{\theta-1}, q) \in K^\perp$.

Since $h^{-1} = h \cdot h^{-2}$ for all $h \in F[t]^*$ and $(K^*)^2 \subset K^\perp$, it will now suffice to show that no product in $F[t]$ of distinct irreducible polynomials of odd degree is contained in $K^\perp$. Let $g \in F[t]$ be such a product, let $F_1$ be the splitting field of $g$ over $F$ and let $K_1 = F_1(s, t)$. The extension $F_1/F$ is of odd degree by the choice of $g$, so $\theta$ has a unique extension to an endomorphism of $K_1$ (which we continue to call $\theta$) whose square is the Frobenius map. Let $c$ be an arbitrary root of $g$ in $F_1$ and let $d = c^\theta$. We define a valuation $\nu$ on $K_1$ with values in $\mathbb{Z}[\sqrt{3}]$. First we declare the degree of a monomial $e(s - d)^m(t - c)^n$ (for $e \in F_1^\times$) to be $n + m\sqrt{3}$. If $p \in F_1[s, t]^*$, we write $p$ as a sum of monomials in the variables $t - c$ and $s - d$ and define $\nu(p)$ to be the minimum of the degrees of these monomials (minimum with respect to the natural ordering of $\mathbb{Z}[\sqrt{3}]$ as a subset of $\mathbb{R}$). Finally we set $\nu(p/q) = \nu(p) - \nu(q)$ for all $p, q \in F_1[s, t]^*$. Then $\nu$ is a well defined valuation on $K_1$. Since $g$ is a product of distinct irreducibles, $c$ is a simple root of $g$. Since the variable $s$ does not occur in $g$, we conclude that $\nu(g) = 1$. Since

$$(e(s - d)^m(t - c)^n)^\theta = e^\theta(t^3 - c^3)^m(s - d)^n = e^\theta(t - c)^{3m}(s - d)^n,$$

for all $e \in F_1$ and all $m, n \geq 0$, it follows that $\nu(u^\theta) = \sqrt{3} \cdot \nu(u)$ for all $u \in K_1^\times$.

Now let $w = N(a, b, c)$ for $a, b, c \in K_1$. By [3, 9.3] (whose proof depends only on the fact that the norm is anisotropic), $\nu(w)$ is equal to the minimum of $(2\sqrt{3} + 4)\nu(a)$, $(\sqrt{3} + 1)\nu(b)$ and $2\nu(c)$. Since $(\sqrt{3} + 1)^2 = 2\sqrt{3} + 4$ and $(\sqrt{3} + 1)(\sqrt{3} - 1) = 2$, it follows that $\nu(K_1) = (\sqrt{3} + 1)\mathbb{Z}[\sqrt{3}]$. Since $\nu(g) = 1 \notin (\sqrt{3} + 1)\mathbb{Z}[\sqrt{3}]$, we conclude that $g \notin K_1^\perp$. Hence $g \not\in K^\perp$.

Corollary 7.2. $K^*/K^\perp$ is infinite.

Proof. There are infinitely many pairwise non-proportional irreducible polynomials of odd degree in $F[t]$. By Proposition 7.1, these polynomials have pairwise distinct images in $K^*/K^\perp$. \hfill \Box

References


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