Classifying the phase transition threshold for Ackermannian functions

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Abstract

It is well known that the Ackermann function can be defined via diagonalization from an iteration hierarchy (of Grzegorczyk type) which is built on a start function like the successor function. In this paper we study for a given start function g iteration hierarchies with a sublinear modulus h of iteration. In terms of g and h we classify the phase transition for the resulting diagonal function from being primitive recursive to being Ackermannian.

1 Introduction

This paper is part of a general program on phase transitions in logic and combinatorics. In general terms phase transition is a type of behavior wherein small changes of a parameter of a system cause dramatic shifts in some globally observed behavior of the system, such shifts being usually marked by a sharp 'threshold point'. (An everyday life example of such thresholds are ice melting and water boiling temperatures.) This kind of phenomena nowadays occur throughout many mathematical and computational disciplines: statistical physics, evolutionary graph theory, percolation theory, computational complexity, artificial intelligence etc.

The last few years have seen an unexpected series of achievements that bring together independence results in logic, analytic combinatorics and Ramsey Theory. These achievements can be intuitively described as phase transitions from provability to unprovability of an assertion by varying a threshold parameter [12, 15, 16, 20]. Another face of this phenomenon is the transition from slow-growing to fast-growing computable functions [14, 17].

In this paper we investigate phase transition phenomena which are related to natural subclasses of the recursive functions. In particular we take a closer look at the Grzegorczyk hierarchy from the phase transition perspective. Assume

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that we have given two functions $g, h : \mathbb{N} \to \mathbb{N}$. Define

$$\begin{aligned} B(g,h)_0(x) &:= g(x), \\ B(g,h)_{k+1}(x) &:= B(g,h)_k^{h(x)}(x) \quad \text{i.e. } h(x) \text{ many iterations}, \\ B(g,h)_\omega(x) &:= B(g,h)_x(x). \end{aligned}$$

We recall that Ackermann's function is defined as $\mathrm{Ack}(n) = B(g,h)_\omega(n)$ where $g(x) = x+1$ and $h = \mathrm{Id}$, and that $\mathrm{A}_i(n) = B(g,h)_i(n)$ is called the i-th approximation of the Ackermann function. We use the term "Ackermannian" to mean "eventually faster than every primitive recursive function". There is no "smallest" Ackermannian function; if $B : \mathbb{N} \to \mathbb{N}$ is Ackermannian, then so is $B/2$ or $B^{1/2}$, etc. If the composition $f \circ g$ is Ackermannian and one of $\{f, g\}$ is primitive recursive, then the other is Ackermannian. It is also important to note that there are functions $B : \mathbb{N} \to \mathbb{N}$ which are neither Ackermannian nor bounded by any primitive recursive function.

For an unbounded function $g : \mathbb{N} \to \mathbb{N}$ define the inverse function $g^{-1} : \mathbb{N} \to \mathbb{N}$ by $g^{-1}(m) := \min\{n : g(n) \geq m\}$. Let us remark that although Ack is not primitive recursive, its inverse Ack^{-1} is primitive recursive.

To avoid trivialities we assume that for some positive $\varepsilon > 0$ we have $g(x) \geq x + \varepsilon$ and we assume that h is weakly increasing and unbounded. Now, fixing g, one may ask for which h the function $B(g,h)_\omega$ becomes Ackermannian. Similarly, fixing h, one may ask for which g the function $B(g,h)_\omega$ becomes Ackermannian. So in contrast to the situations previously considered the phase transition depends on two order parameters and we will indicate that the phase transition has a rich structure.

2 Iteration hierarchies for $g(x) := x + 1$

In this section we fix $g(x) := x + 1$. This particular case was considered and partially solved in [7]. This result was later on improved in [4]. The results given in these two papers were rather indirect and involved the phase transition for the Kanamori McAloon result for pairs. Nevertheless, they have independent interest since they show how regressive Ramsey functions are intrinsically related to parameterized iteration hierarchies. The following yields a rather sharp threshold on the behavior of such function hierarchies. Using the notation of [7, 4] we denote $B(g, x^{1/t})$, where $t \in \mathbb{N}$ is a constant, by (f_t).

Claim 2.1. *For every $t > 0$ and $n > \max(\{4, 3^t, t^t\})$ it holds that*

$$(f_t)_{i+t^2+2t+1}(n) > A_i(n).$$

Proof. *See Claim* 16 *in [4]* $\square$

Claim 2.2. *For every $i \in \mathbb{N}$ and for every $n \in \mathbb{N}$ such that:*

1. $n > i + (\lg \lg n)^2 + 2 \lg \lg n + 1$

2. $\mathrm{Ack}(\lg\lg n) > \mathrm{A}_i(n)$

it holds for $h(n) = n^{\frac{1}{\mathrm{Ack}^{-1}(n)}}$ *that*

$$B(g,h)_{i+(\lg\lg n)^2+2\lg\lg n+1}(n) > \mathrm{A}_i(n).$$

Proof. To show that, we examine two cases. First, if it holds that $B(g,h)_{i+(\lg\lg n)^2+2\lg\lg n+1}(n) \geq \mathrm{Ack}(\lg\lg n)$, then we are done by demand 2. Otherwise, we may fix $t := \lg\lg n$ and we have that for all $y \in [o \ldots \mathrm{Ack}(t))$ it holds that $y^{\frac{1}{t}} < y^{\frac{1}{\mathrm{Ack}^{-1}(y)}}$. Thus, since $B(g,h)_{i+t^2+2t+1}$ is monotone, it holds that $B(g,h)_{i+t^2+2t+1}(n) \geq (f_t)_{i+t^2+2t+1}(n)$ which by Claim 2.1 is larger than $A_i(n)$.

□

We remark that the choice of $t = \lg\lg n$ is arbitrary and any α^{-1}, such that α is a monotone increasing primitive recursive function and $\alpha(x) > x^x$ for large enough x, would do the job.

Theorem 1. *Let* $g(x) := x+1$ *Let* $h_\alpha(x) := x^{\frac{1}{B(g,id)_\alpha^{-1}(x)}}$. *Then* $B(g,h_\alpha)_\omega$ *is Ackermannian iff* $\alpha = \omega$.

Proof. The 'if' direction is in fact the claim that if $h_\alpha(x) = x^{\frac{1}{\mathrm{Ack}^{-1}(x)}}$, then $B(g,h_\alpha)_\omega$ eventually grows faster than any primitive recursive function. This direction is a direct corollary of Claim 2.2, since it is clear that for every $i \in \mathbb{N}$ there exists some $x_0 \in \mathbb{N}$ such that for all $x > x_0$ it holds that $B(g,h_\alpha)_x(x) \leq B(g,h_\alpha)_{i+(\lg\lg x)^2+2\lg\lg x+1}(x)$ which by Claim 2.2 is larger than $\mathrm{A}_i(x)$. In other words, for every primitive recursive function f, $B(g,h_\alpha)_x(x)$ eventually dominates f.

The 'only if' direction is the claim that if h_α is of the form $h_\alpha(x) = x^{\frac{1}{\mathrm{A}_i^{-1}(x)}}$ for some $i \in \mathbb{N}$, then $B(g,h_\alpha)_\omega(x)$ is not Ackermannian in terms of x. Note this implies the same for any h_α of the form $h_\alpha(x) = x^{\frac{1}{\beta^-1(x)}}$ where β is a non-decreasing primitive recursive function. To show this direction, we again refer to [4]. Theorem 2.1 in [4] states that the h_α-regressive Ramsey number $R^{\mathrm{reg}}_{h_\alpha}(k)$ where $h_\alpha(k) = k^{\frac{1}{\beta^-1(k)}}$ is bounded by $\beta(k)$. Namely, is primitive recursive in k. On the other hand, Corollary 2.6 in [4] asserts that if $B(g,h_\alpha)_\omega(k)$ is Ackermannian in k, then the lower bound for $R^{\mathrm{reg}}_h(k)$ for $h = k^{\frac{1}{\beta^-1(k)}}$ where $\beta^-1(k) = k^{\frac{1}{\left(\frac{\lg k\,\mathrm{A}_i^{-1}(k)}{\lg k+\mathrm{A}_i^{-1}(k)}\right)}}$ is also Ackermannian, but this would be a contradiction to Theorem 2.1 of [4] as clearly $\beta(k) \leq 2^{\mathrm{A}_i(k)}$ and thus bounded by a primitive recursive function. □

3 Slow growing iteration hierarchies

For the rest of this section let $F_0(x) := 2^x$ and $F_{k+1}(x) := F_k^x(x)$. Then F_k primitive recursive. Let $2_l(x) := F_0^l(x)$ and $|x|$ be the binary length of x adjusted so that $|2^x| = x$. Thus $|x|$ is a numbertheoretic logarithm function with respect to base two. Let $|x|_{l+1} := ||x||_l$ where $|x|_0 := x$. Then $|\cdot|_l$ is the l-th iterate of $|\cdot|$ so that $|2_l(x)|_l = x$.

Further let $F(x) := F_x(x)$. Then F is Ackermannian and hence not primitive recursive. The following results classifies slow growing iteration hierarchies for a rather large class of order parameters.

Theorem 2. *Let $1 \geq \varepsilon > 0$ and let d be a natural number. Let $g_0(x) := x + \varepsilon$. Define recursively $g_{k+1}(x) := 2^{g_k(|x|)}$. Define $h(d)_l(x) := |x|_l^{\frac{1}{F_d^{-1}(|x|_l)}}$ and*

$$B(d, l)_k(x) := B(g_l, h(d)_l)_k(x).$$

Let $C := \max\{2_l(F_d(2^{k+2})), 2_l(F_d(\frac{1}{\varepsilon}))\}$. Then for all $x \geq C$ and all $i \leq |x|_l^{\frac{1}{F_d^{-1}(|x|_l)}}$ we have

$$B(d, l)_k^i(x) \leq 2_l(|x|_l + \varepsilon \cdot |x|_l^{\frac{2^{k+1}}{F_d^{-1}(|x|_l)}} \cdot i).$$

Proof. Without loss of generality let $\varepsilon = 1$. We prove the claim by main induction on k. If $k = 0$ then $B(l)_0^i(x) = g_l^i(x)$. We prove the claim by subsidiary induction on i. Assume first that $i = 1$. We prove the claim by another subsidiary induction on l. Assume $l = 0$. Then for $x \geq C$:

$$\begin{aligned} B(d, 0)_0^1(x) = g_0(x) &= x + 1 \\ &\leq 2_0(|x|_0 + |x|_0^{\frac{2^1}{F_d^{-1}(|x|_0)}}). \end{aligned}$$

Assume now $l > 0$. Then the induction hypothesis for $l - 1$ yields for $x \geq C$:

$$\begin{aligned} B(d, l)_0^1(x) &= g_l(x) \\ &= 2^{g_{l-1}(|x|)} \\ &\leq 2^{2_{l-1}(||x||_{l-1} + ||x||_{l-1}^{\frac{2}{F_d^{-1}(||x||_{l-1})}})} \\ &= 2_l(|x|_l + |x|_l^{\frac{2}{F_d^{-1}(|x|_l)}}). \end{aligned}$$

Now consider the case $1 \leq i < |x|_l^{\frac{1}{F_d^{-1}(|x|_l)}}$. Then we obtain

$$\begin{aligned} B(d, l)_0^{i+1}(x) &= B(d, l)_0(B(d, l)_0^i(x)) \\ &\leq B(d, l)_0(2_l(|x|_l + |x|_l^{\frac{2}{F_d^{-1}(|x|_l)}} \cdot i)) \\ &= 2_l(|(2_l(|x|_l + |x|_l^{\frac{2}{F_d^{-1}(|x|_l)}} \cdot i))|_l + \varepsilon) \\ &\leq 2_l(|x|_l + |x|_l^{\frac{2}{F_d^{-1}(|x|_l)}} \cdot i + 1) \\ &= 2_l(|x|_l + |x|_l^{\frac{2}{F_d^{-1}(|x|_l)}} \cdot (i + 1)) \end{aligned}$$

since by assumption $x \geq 2_l(F_d(\frac{1}{\varepsilon}))$.

Now assume that $k > 0$. We prove the claim by subsidiary induction on i.

If $i = 1$ then

$$\begin{aligned}
B(d,l)_k(x) &= B(d,l)_{k-1}^{|x|_l^{\frac{1}{F_d^{-1}(|x|_l)}}}(x) \\
&\leq 2_l\big(|x|_l + |x|_l^{\frac{2^k}{F_d^{-1}(|x|_l)}} \cdot |x|_l^{\frac{1}{F_d^{-1}(|x|_l)}}\big) \\
&\leq 2_l\big(|x|_l(x) + |x|_l^{\frac{2^{k+1}}{F_d^{-1}(|x|_l)}}\big).
\end{aligned}$$

If $1 \leq i < |x|_l^{\frac{1}{F_d^{-1}(|x|_l)}}$ then we obtain

$$\begin{aligned}
B(d,l)_k^{i+1}(x) &= B(d,l)_k(B(d,l)_k^i(x)) \\
&\leq B(d,l)_k(2_l(|x|_l + |x|_l^{\frac{2^{k+1}}{F_d^{-1}(|x|_l)}} \cdot i)).
\end{aligned}$$

Now set $y := 2_l(|x|_l + |x|_l^{\frac{2^{k+1}}{F_d^{-1}(|x|_l)}} \cdot i)$. Then we obtain from the main induction hypothesis and $i < |x|_l^{\frac{1}{F_d^{-1}(|x|_l)}}$ that

$$\begin{aligned}
B(d,l)_k^{i+1}(x) &= B(d,l)_{k-1}^{|y|_l^{\frac{1}{F_d^{-1}(|y|_l)}}}(y) \\
&\leq 2_l\Big(|y|_l + |y|_l^{\frac{2^k}{F_d^{-1}(|y|_l)}} \cdot |y|_l^{\frac{1}{F_d^{-1}(|y|_l)}}\Big) \\
&= 2_l\big(|x|_l + |x|_l^{\frac{2^{k+1}}{F_d^{-1}(|x|_l)}} \cdot i + (|x|_l + |x|_l^{\frac{2^{k+1}}{F_d^{-1}(|x|_l)}})^{\frac{2^k+1}{F_d^{-1}(|x|_l + |x|_l^{\frac{2^{k+1}}{F_d^{-1}(|x|_l)}} \cdot i)}}\big).
\end{aligned}$$

The claim would now follow from

$$(|x|_l + |x|_l^{\frac{2^{k+1}}{F_d^{-1}(|x|_l)}})^{\frac{2^k+1}{F_d^{-1}(|x|_l + |x|_l^{\frac{2^{k+1}}{F_d^{-1}(|x|_l)}} \cdot i)}} \leq |x|_l^{\frac{2^{k+1}}{F_d^{-1}(|x|_l)}}.$$

Since $F_d^{-1}(|x|_l + |x|_l^{\frac{2^{k+1}}{F_d^{-1}(|x|_l)}} \cdot i) \geq F_d^{-1}(|x|_l)$ this would follow from

$$|x|_l + |x|_l^{\frac{2^{k+1}}{F_d^{-1}(|x|_l)}} \leq |x|_l^{\frac{2^{k+1}}{2^{k+1}}}.$$

This finally follows from the assumption that $x \geq 2_l(F_d(2^{k+2}))$.

$\square$

4 Fast growing iteration hierarchies

In this section we show that replacing the functions h_l from Theorem 2 by slightly faster growing functions yields Ackermannian growth. Let us recall the definition of the Ackermann hierarchy from Section 1. We put $A_0(x) := x+1$ and $A_{k+1}(x) := A_k^x(x)$. Thus, if we put $\mathrm{Ack}(x) := A_x(x)$, then Ack is Ackermann's function which eventually dominates every primitive recursive function.

Theorem 3. *Let $1 \geq \varepsilon > 0$ and let d be a natural number. Let $g_0(x) := x + \varepsilon$. Define recursively $g_{k+1}(x) := 2^{g_k(|x|)}$. Define $h_0(x) := \sqrt[d]{x}$ and $h_{k+1}(x) := h_k(|x|)$ and*

$$B(l)_k(x) := B(g_l, h_l)_k(x).$$

Then we have

$$B(l)_{d+d+d+i+1}(2_l(x^d)) \geq 2_l((A_i(x))^d).$$

Proof. An induction on l yields that $g_l(x) = 2_l(\varepsilon + |x|_l)$. By induction on i one verifies $B(l)_0^i(x) = 2_l(\varepsilon \cdot i + |x|_l)$.

Now we claim

$$B(l)_k^i(x) \geq 2_l(\varepsilon \cdot i \cdot |x|_l^{\frac{k}{d}} + |x|_l) \tag{1}$$

for $i, k \geq 1$ and x sufficiently large. We now present a proof of the claim by main induction on k and subsidiary induction on i. Assume that $k = 1$. Then we obtain for $i = 1$ that $B(l)_1^1(l)(x) = B(l)_0^{(|x|_l)^{\frac{1}{d}}}(x) \geq 2_l(\varepsilon \cdot |x|_l^{\frac{1}{d}} + |x|_l)$. Assuming the claim for i we obtain

$$\begin{aligned}
& B(l)_1^{i+1}(x) \\
= \;& B(l)_1^1(B(l)_1^i(x)) \\
\geq \;& B(l)_1^1(2_l(\varepsilon \cdot i \cdot |x|_l^{\frac{1}{d}} + |x|_l)) \\
\geq \;& 2_l(\varepsilon \cdot (|2_l(\varepsilon \cdot i \cdot |x|_l^{\frac{1}{d}} + |x|_l)|_l)^{\frac{1}{d}} + |2_l(\varepsilon \cdot i \cdot |x|_l^{\frac{1}{d}} + |x|_l)|_l) \\
\geq \;& 2_l(\varepsilon \cdot |x|_l^{\frac{1}{d}} + \varepsilon \cdot i \cdot |x|_l^{\frac{1}{d}} + |x|_l)
\end{aligned}$$

Assuming the claim for k we show it for $k+1$ as follows: First let $i = 1$. Then

$$\begin{aligned}
& B(l)_{k+1}(x) \\
= \;& B(l)_k^{(|x|_l)^{\frac{1}{d}}}(x) \\
\geq \;& 2_l(\varepsilon \cdot |x|_l^{\frac{1}{d}} \cdot |x|_l^{\frac{k}{d}} + |x|_l) \\
= \;& 2_l(\varepsilon \cdot |x|_l^{\frac{k+1}{d}} + |x|_l)
\end{aligned}$$

Now we assume the assertion for i and we show it for $i+1$:

$$\begin{aligned}
& B(l)_{k+1}^{i+1}(x) \\
= \;& B(l)_{k+1}(B(l)_{k+1}^i(x)) \\
\geq \;& B(l)_{k+1}(2_l(\varepsilon \cdot i \cdot |x|_l^{\frac{k+1}{d}} + |x|_l)) \\
\geq \;& 2_l(\varepsilon \cdot (|2_l(\varepsilon \cdot i \cdot |x|_l^{\frac{k+1}{d}} + |x|_l)|_l)^{\frac{k+1}{d}} + |2_l(\varepsilon \cdot i \cdot |x|_l^{\frac{k+1}{d}} + |x|_l)|_l) \\
\geq \;& 2_l(\varepsilon \cdot |x|_l^{\frac{k+1}{d}} + \varepsilon \cdot i \cdot |x|_l^{\frac{k+1}{d}} + |x|_l)
\end{aligned}$$

The claim yields $B(l)_{3\cdot d}(x) \geq 2_l(|x|_l^2)$ for $x \geq C$ for C a suitable constant depending on l and ε.

By induction on i we see that $B(l)^i_{3\cdot d}(x) \geq 2_l((|x|_l)^{2^i})$

We claim now that

$$B(l)_{d\cdot 3+i+1}(2_l(x^d)) \geq 2_l((A_i(x))^d)$$

for $x \geq C$. Proof by induction on i. For $i = 0$ we find

$$\begin{array}{rc} & B(l)_{3\cdot d+1}(2_l(x^d)) \\ = & B(l)^x_{3\cdot d}(2_l(x^d)) \\ = & 2_l((|2_l(x^d)|_l)^{2^x}) \\ \geq & 2_l((A_0(x))^d) \end{array}$$

Assuming the claim for i we obtain it for $i+1$ as follows:

$$\begin{array}{rc} & B(l)_{3\cdot d+1+i}(2_l(x^d)) \\ = & B(l)^x_{3\cdot d+i}(2_l(x^d)) \\ \geq & 2_l((A_i^x(x))^d) \\ = & 2_l((A_{i+1}(x))^d). \end{array}$$

□

Theorem 4. *Let $1 \geq \varepsilon > 0$. Let $g_0(x) := x + \varepsilon$. Define recursively $g_{k+1}(x) := 2^{g_k(|x|)}$. Define $h^\star_l(x) := \sqrt[\mathrm{Ack}^{-1}(x)]{|x|_l}$. Let*

$$B(l)^\star_k(x) := B(g_l, h^\star_l)_k(x)$$

and

$$B(l)^\star(x) := B(l)^\star_x(x).$$

Then we have

$$B(l)^\star(2_l((4\cdot d+1)^d)) > \mathrm{Ack}(d).$$

Proof. Assume for a contradiction that $\mathrm{Ack}(d) \geq B^\star(l)(2_l((4\cdot d+1)^d))$. Then for any $i \leq B^\star(l)_{4\cdot d+1}(2_l((4\cdot d+1)^d))$ we have $\mathrm{Ack}^{-1}(i) \leq d$ hence $i^{\frac{1}{d}} \leq i^{\frac{1}{\mathrm{Ack}^{-1}(i)}}$ and therefore

$$\begin{array}{rcl} B^\star(l)(2_l((4\cdot d+1)^d)) & \geq & B(d,l)_{4\cdot d+1}(2_l((4\cdot d+1)^d)) \\ & > & 2_l(A_d(4\cdot d+1))^d \\ & > & \mathrm{Ack}(d). \end{array}$$

Contradiction! □

It seems plausible that Theorems 2, 3 and 4 hold for all start functions g_l such where $x + \varepsilon \leq g_0(x) \leq x + |x|^c$ and the same functions $h(d)_l$ and $h(l)^\star$.

For the record let us consider the situation when one starts with an exponential or double exponential function. This leads easily to Ackermannian growth

Theorem 5. *1. Let $g(x) := 2^x$ and $h(x) = |x|_k$. Then $B(g,h)$ is Ackermannian.*

2. Let $g(x) := 2^{2^x}$ and $h(x) := \log^\star(x)$. Then $B(g,h)$ is Ackermannian.

Proof. 1. By induction on k one easily shows $B(g,h)_k(2_k(x)) \geq 2_k(A_k(x))$.
2. By induction on k one easily shows $B(g,h)_k(2_k(x)) \geq 2_{A_k(x)}(A_k(x))$.

□

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