An extremely sharp phase transition threshold for the slow growing hierarchy

Andreas Weiermann*

Fakulteit Bétawetenschapen
Departement Wiskunde
Postbox 80010
3508 TA Utrecht
The Netherlands,
email: weierman@math.uu.nl

Abstract. We investigate natural systems of fundamental sequences for ordinals below the Howard Bachmann ordinal and study growth rates of the resulting slow growing hierarchies. We consider a specific assignment of fundamental sequences which depends on a non negative real number ε. We show that the resulting slow growing hierarchy is eventually dominated by a fixed elementary recursive function if ε is equal to zero. We show further that the resulting slow growing hierarchy exhausts the provably recursive functions of ID_1 if ε is strictly greater than zero. Finally we show that the resulting fast growing hierarchies exhaust the provably recursive functions of ID_1 for all non negative values of ε.
Our result is somewhat surprising since usually the slow growing hierarchy along the Howard Bachmann ordinal exhausts precisely the provably recursive functions of PA. Note that the elementary functions are a very small subclass of the provably recursive functions of PA and the provably recursive functions of PA are a very small subclass of the provably recursive functions of ID_1. Thus the jump from ε equal to zero to ε greater than zero is one of the biggest jumps in growth rates for subrecursive hierarchies one might think of.

This article is part of our general research program on phase transitions in logic and combinatorics. Phase transition phenomena are ubiquitous in a wide variety of branches of mathematics and neighbouring sciences, in particular, physics (see, for example, [6]). An informal description of a 'phase transition effect' is the effect behaviour wherein 'small' changes in certain parameters of a system occasion dramatic shifts in some globally observed behaviour of the system, such shifts being marked by a 'sharp threshold point'. An everyday life example of this is the change from one material state to a different one as temperature is increased, with the 'threshold' being given by melting/boiling point. Similar phenomena occur in mathematical and computational contexts like evolutionary

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graph theory (see, e.g., [3, 10]), percolation theory (see, e.g., [9]), computational complexity theory and artificial intelligence (see, for example, [7, 11]).

The purpose of PTLC is to study Phase Transitions in Logic and Combinatorics. We are particularly interested in the transition from provability to unprovability of a given assertion by varying a threshold parameter. On the side of hierarchies of recursive functions this reduces to classifing the phase transition for the growth rates of the functions involved. In this article we are concerned with phase transitions for the slow growing hierarchy and we continue the investigations from [12–15].

From the pure logical side this article is motivated by the classical classification problem for the recursive functions and the resulting problem of comparing the slow and fast growing hierarchies. It has been claimed, for example in [3] p. 439 l.-5, that for sufficiently big prooftheoretic ordinals the slow and fast growing hierarchies will match up. The results of this paper may indicate that this claim might not be true in general.

To formulate the results precisely we introduce some notation. For an ordinal α less than the Howard Bachmann ordinal let $N\alpha$ be the number of symbols in α which are different from 0 and $+$. The idea is essentially that $N\alpha$ is the number of edges in the tree which represents the term for α.

For a limit ordinal λ let $\lambda[x] := \max\{\beta < \lambda : N\beta \leq N\lambda + x\}$. This assignment of fundamental sequences is natural and does not change, as we will show in the appendix, the growth rate of the induced fast growing hierarchy. But, as our first main theorem shows, the induced slow growing hierarchy (along the Howard Bachmann ordinal) consists of elementary functions only. This generalizes results from [4] where we showed that the resulting slow growing hierarchy along Γ_0 consists of elementary recursive functions only.

At first sight the resulting slow growing hierarchies seem always to collapse under this assignment of fundamental sequences and one may wonder how robust this phenomenon is. We prove therefore in a separate section a very surprising and extremely sharp phase transition threshold for the slow growing hierarchy. The upshot is that small changes prevent the hierarchies from collapsing. For a given real number $\varepsilon \geq 0$ let $\lambda[x]_\varepsilon := \max\{\beta < \lambda : N\beta \leq (1+\varepsilon) \cdot N\lambda + x\}$. Then, as we just said, for $\varepsilon = 0$ the resulting slow growing hierarchy is very slow growing but for any $\varepsilon > 0$ the resulting slow growing hierarchy becomes fast growing and matches up with the fast growing hierarchy at all ϵ-numbers below the Howard-Bachmann ordinal. We conjecture that within the phase transition, i.e. when in the definition of $\lambda[x]_\varepsilon$ the number ε is a function of λ and x, we may arrange other behaviours of the resulting slow growing hierarchy.

The paper is not fully self contained. The proof of the first main theorem requires basic familiarity with Buchholz style notation systems for the Howard Bachmann ordinal. (Knowledge of [4] is more then sufficient.) The proof of the second main result should be generally accessible (at least when one restricts the consideration to ordinals below ε_0.

1 Proof of the first main result

1.1 Basic concepts

We recall some basic definitions and facts from Buchholz papers on ordinal notations. Missing proofs can be found in [4].

Definition 1 *Inductive definition of a set of terms* T *and a set* $\mathrm{P} \subseteq T$.
$0 \in \mathrm{T}$,
$a \in \mathrm{T} \,\&\, i \in \{0,1\} \;\Rightarrow\; D_i a \in \mathrm{P}$,
$a_0, \ldots, a_n \in P$ *and* $n \geq 1 \Rightarrow\; (a_0, \ldots, a_n) \in \mathrm{T}$.

Notations for Section 1:
1. a, b, c, d, e range over T.
2. If $a \in P$, then we identify the one element sequence (a) with the term a.
3. The empty sequence ( ) is identified with the term 0.
4. x, y, z, i, l, m, n range over non negative integers.

Definition 2 *Recursive definition of* $a \prec b$ *for* $a, b \in \mathrm{T}$.
$a \prec b$ *holds, iff one of the following cases holds:*
1. $a = 0$ *and* $b \neq 0$,
2. $(a = D_0 a_0 \,\&\, b = D_1 b_0)$,
3. $(a = D_i a_0 \,\&\, b = D_i b_0 \,\&\, a_0 \prec b_0)$,
4. $a = (a_0, \ldots, a_m) \,\&\, b = (b_0, \ldots, b_n) \,\&\, 1 \leq m + n$ *and*
$[(m < n \,\&\, \forall i \leq n (a_i = b_i))$ *or* $(\exists k \leq \min\{m, n\} (\forall i < k (a_i = b_i) \,\&\, a_k \prec b_k))]$

Lemma 1 $(\mathrm{T}, \prec)$ *is a linear order.*

$(\mathrm{T}, \prec)$ is not a well-order. To see this, let $a_0 := D_1 0$ and $a_{n+1} := D_0 a_n$. Then $a_{i+1} \prec a_i$ for all i. In the sequel we thin out T to a smaller set OT which is well-ordered by $\prec$.

Definition 3 *Assume that* $M, N \subseteq \mathrm{T}$.
$M \preceq N \;:\Longleftrightarrow\; \forall x \in M \exists y \in N (x \preceq y)$.
$a \preceq N \;:\Longleftrightarrow\; \{a\} \preceq N$.
$M \prec a \;:\Longleftrightarrow\; \forall x \in M (x \prec a)$.
$(a_1, \ldots, a_m) + (b_1, \ldots, b_n) := (a_1, \ldots, a_i, b_1, \ldots, b_n)$
where $i \in \{1, \ldots, m\}$ *is maximal with* $b_1 \preceq a_i$.

Definition 4 *Recursive definition of* $\mathrm{K}^* a$ *and* $\mathrm{K} a$ *for* $a \in \mathrm{T}$.
$\mathrm{K}^* 0 := \emptyset$, $\quad \mathrm{K} 0 := \emptyset$,
$\mathrm{K}^* (a_0, \ldots, a_n) := \bigcup_{i \leq n} \mathrm{K}^* a_i$, $\quad \mathrm{K}(a_0, \ldots, a_n) := \bigcup_{i \leq n} \mathrm{K} a_i$,
$\mathrm{K}^* D_1 a := \mathrm{K}^* a$, $\quad \mathrm{K} D_1 a := \mathrm{K} a$,
$\mathrm{K}^* D_0 a := \{a\} \cup \mathrm{K}^* a$, $\quad \mathrm{K} D_0 a := \{D_0 a\}$.

Lemma 2 $\mathrm{K}^* c \prec a \;\Longrightarrow\; \mathrm{K} c \prec D_0 a$.

Definition 5 *Inductive definition of a set of terms* $\mathrm{OT} \subseteq \mathrm{T}$.
$0 \in \mathrm{OT}$,
$a_0, \ldots, a_n \in \mathrm{OT} \cap P$, $n \geq 1$ *and* $a_n \preceq \ldots \preceq a_0 \implies (a_0, \ldots, a_n) \in \mathrm{OT}$,
$a \in \mathrm{OT} \implies D_1 a \in \mathrm{OT}$,
$a \in \mathrm{OT}$ & $\mathrm{K}^* a \prec a \implies D_0 a \in \mathrm{OT}$.

Notations.
$1 := D_0 0$, $\omega := D_0 1$, $\Omega := D_1 0$.
$\mathrm{T}_0 := \{x \in \mathrm{T} : x \prec \Omega\}$, $\quad \mathrm{OT}_0 := \mathrm{OT} \cap \mathrm{T}_0$.

Theorem 1 $(\mathrm{OT}, \prec)$ *is a well-order. The order type of* $(\mathrm{OT}_0, \prec)$ *is equal to the Howard Bachmann ordinal.*

Definition 6 $b \lhd_c a \ :\iff \ b \prec a$ & $\forall d \in \mathrm{OT}\ (b \preceq d \preceq a \Rightarrow \mathrm{K}^* b \preceq \mathrm{K}^* d \cup \mathrm{K}^* c)$ *[*$a, b, c \in \mathrm{T}$*].*

Lemma 3 *1.* $a \prec D_1 a$.
2. $a, b \in \mathrm{OT}_0$ & $a \prec b \implies \mathrm{K}^* a \preceq \mathrm{K}^* b$.
3. $b \lhd_c a$ & $\mathrm{K}^* a \prec a$ & $\mathrm{K}^* c \prec b$ & $a, b \in \mathrm{OT} \implies \mathrm{K}^* b \prec b$.
4. $b \lhd_c a \implies d + b \lhd_c d + a$ & $D_i b \lhd_c D_i a$ $(i \in \{0, 1\})$.

Proof. We prove assertion 3. Assume that $b \preceq \mathrm{K}^* b$. Choose a subterm d of b with $b \preceq \mathrm{K}^* d$ such that the length of d is minimal possible. Then $d = D_0 e$ with $\mathrm{K}^* e \prec b \preceq e \preceq a$, since $\mathrm{K}^* b \preceq \mathrm{K}^* c \cup \mathrm{K}^* a \prec a$. Then we obtain $\mathrm{K}^* b \preceq \mathrm{K}^* c \cup \mathrm{K}^* e \prec b$. Contradiction.

Definition 7 *Definition of* $\mathsf{tp}(a) \in \{0, 1, \omega, \Omega\}$ *for* $a \in \mathrm{T}$.
$\mathsf{tp}(0) := 0$.
$\mathsf{tp}(1) := 1$.
$\mathsf{tp}(\Omega) := \Omega$.
$\mathsf{tp}(a) = 1 \Rightarrow \mathsf{tp}(D_i a) := \omega$.
$\mathsf{tp}(a) = \omega \Rightarrow \mathsf{tp}(D_i a) := \omega$
$\mathsf{tp}(a) = \Omega \Rightarrow \mathsf{tp}(D_0 a) := \omega$.
$\mathsf{tp}(a) = \Omega \Rightarrow \mathsf{tp}(D_1 a) := \Omega$.
$\mathsf{tp}((a_0, \ldots, a_n)) := \mathsf{tp}(a_n)$.

Definition 8 *Recursive definition of* $a\{c\} \in \mathrm{T}$ *for* $c \in T_0$ *and* $a \in \mathrm{T}$ *with* $tp(a) = \Omega$.
$\Omega\{c\} := c$.
$(D_1 a)\{c\} := D_1 a\{c\}$.
$a = (a_0, \ldots, a_n) \Rightarrow a\{c\} := (a_0, \ldots, a_{n-1}) + a_n\{c\}$.

Lemma 4 *1.* $\mathsf{tp}(a) = \Omega$ & $c \in T_0 \Rightarrow a\{c\} \lhd_c a$.
2. $\mathsf{tp}(a) = \Omega$ & $c, d \in T_0$ & $c \prec d \Rightarrow a\{c\} \prec a\{d\}$.
3. $\mathsf{tp}(a) = \Omega$ & $c \in T_0 \Rightarrow \mathrm{K}^* a\{c\} \preceq \mathrm{K}^* a\{0\} \cup \mathrm{K}^* c$.
4. $a \in \mathrm{OT}, \mathsf{tp}(a) = \Omega$ & $c \in OT_0 \Rightarrow a\{c\} \in \mathrm{OT}$.

1.2 Refined concepts

Here we collect some technical material which is needing during the proof of the first main result.

Definition 9 *Recursive definition of Na for $a \in T$.*
$N0 := 0$.
$N(a_0, \ldots, a_n) := Na_0 + \ldots + Na_n$.
$ND_i a := 1 + Na$.

Definition 10 *Definition of $a[x]$ and $a[\![x]\!]$ for $a \in \mathrm{OT} \setminus \{0\}$.*
$a[x] := \max\{b \in \mathrm{OT} : b \prec a \;\&\; Nb \leq Na + x\}$.
$a[\![x]\!] := \max\{b \in \mathrm{OT} : b \prec a \;\&\; Nb \leq x\}$.

Lemma 5 *1.* $(a_0, \ldots, a_{n-1}, a_n)[x] = (a_0, \ldots, a_{n-1}) + a_n[x]$.
2. $a = (a_0, \ldots, a_n)$, $b = (a_1, \ldots, a_n)$, $x \geq Na_0 \implies a[\![x]\!] = a_0 + b[\![x - Na_0]\!]$.
3. $a = (a_0, \ldots, a_n)$, $x < Na_0 \implies a[\![x]\!] = a_0[\![x]\!]$.
4. $a[\![0]\!] = 0$.
5. $(D_1 a)[x] = D_1 a[x]$.
6. $x > 0 \implies (D_1 a)[\![x]\!] = D_1 a[\![x - 1]\!]$.

Definition 11 *Recursive definition of $G_a(x)$ for $a \in \mathrm{OT}_0$.*
$G_0(x) := 0$.
$G_{a+1}(x) := G_a(x) + 1$.
$G_a(x) := G_{a[x]}(x)$ *if* $\mathsf{tp}(a) = \omega$.

Lemma 6 *Let $a, b \in OT_0$.*
1. $a = (a_0, \ldots, a_n) \implies G_a(x) = G_{a_0}(x) + \cdots + G_{a_n}(x)$.
2. $a = D_0(b + 1) \implies G_a(x) = G_{D_0 b}(x) + G_{a[\![x+1]\!]}(x)$.
3. $a \preceq b \;\&\; Na \leq x + 1 \implies G_a(x) \leq G_b(x)$.

Assertion 2 motivates the definition of the assignment $\cdot[\![\cdot]\!]$. Moreover it gives a first indication why the resulting slow growing hierarchy will collapse since the second term $G_{a[\![x+1]\!]}(x)$ refers to $a[\![x + 1]\!]$ which is in general very small when compared to a. In the sequel we verify that this phenomenon also holds true in more complicated situations.

Definition 12 *Definition of $T_a(x)$ for $a \in T$.*
$T_0(x) := 1$.
$T_a(x) := T_{a[\![x]\!]}(x) + 2$.

Remark. $T_a(x)$ is defined by recursion on the cardinality of the set $\{b \prec a : Nb \leq x\}$. The asymptotic of T_a is very interesting from the analytic number theory point of view. We conjecture that sharp bounds on T_a will prove useful to obtain good upper bounds on G_a but in this article we will just prove that each G_a is elementary for $a \in \mathrm{OT}$.

Lemma 7 *1.* $a \preceq b$ & $Na \le x \Rightarrow T_a(x) \le T_b(x)$.
2. $a \preceq b \Rightarrow T_a(x) \le T_b(x)$.

Proof. 1. By induction on the cardinality of the set $\{c \prec b : Nc \le x\}$. Assume that $T_b(x) = T_{b[\![x]\!]}(x) + 2$ and $a \ne b$. Then $a \preceq b[\![x]\!]$ and the induction hypothesis yields $T_a(x) \le T_{b[\![x]\!]}(x)$ and the assertion follows.
2. If $a = 0$ then the assertion is clear. Assume that $T_a(x) = T_{a[\![x]\!]}(x) + 2$ and $T_b(x) = T_{b[\![x]\!]}(x) + 2$. Then $a[\![x]\!] \preceq b[\![x]\!]$ and the assertion follows from 1.

Definition 13 *Recursive definition of $C_x(a, g)$ for $a \in OT$ and $g, x < \omega$.*
1. $C_x(0, g) := 0$.
2. $C_x((a_0, \dots, a_n), g) := C_x(a_0, g) + \cdots + C_x(a_n, g)$.
3. $C_x(D_0 a, g) := g \cdot G_{D_0 a}(x)$.
4. $C_x(D_1 a, g) := g^{2^{T_{D_1 a}(x+1)}} \cdot (C_x(a, g) + 1)$.

Lemma 8 *If $a \in OT_0$ then $C_x(a, g) = g \cdot G_a(x)$.*

Proof by induction on Na.
1. $a = 0$. Then the assertion is obvious.
2. $a = (a_0, \dots, a_n)$. Then the induction hypothesis yields $C_x(a, g) = C_x(a_0, g) + \cdots + C_x(a_n, g) = g \cdot G_{a_0}(x) + \cdots + g \cdot G_{a_n}(x) = g \cdot G_{(a_0, \dots, a_n)}(x)$.
3. $a = D_0 b$. Then $C_x(a, g) = g \cdot G_a(x)$.

The following Lemma is a crucial tool in proving the hierarchy collapse.

Lemma 9 (Mainlemma) *If $a, b \in \mathrm{OT}$, $\mathrm{K}^* b \prec a$, $Nb \le x + 1$ and $g = G_{(D_0 a)[\![x+1]\!]}(x)$*
then

$$C_x(b, g) \le g^{2^{T_b(x+1)+1}}.$$

Proof by induction on Nb.
1. $b = 0$. Then the assertion is obvious.
2. $b = (b_0, \dots, b_n)$. Then $n + 1 \le x + 1$, $Nb_i \le x + 1$ and $\mathrm{K}^* b_i \subseteq \mathrm{K}^* b \prec a$ for $i = 0, \dots, n$. The induction hypothesis yields $C_x(b_i, g) \le g^{2^{T_{b_i}(x+1)+1}}$ for $i = 0, \dots, n$. Thus $C_x(b, g) = C_x(b_0, g) + \cdots + C_x(b_n, g) \le g^{2^{T_{b_0}(x+1)+1}} + \cdots + g^{2^{T_{b_n}(x+1)+1}} \le (x + 1) \cdot g^{2^{T_b(x+1)-1}} \le g^{2^{T_b(x+1)+1}}$,
since $x + 1 \le g$ and $T_{b_i}(x + 1) < T_b(x + 1)$ by Lemma 7.
3. $b = D_0 c$. Then $C_x(b, g) = g \cdot G_{D_0 c}(x)$. $\mathrm{K}^* b \prec a$ yields $b \prec D_0 a$. Thus $Nb \le x+1$ yields $b \preceq (D_0 a)[\![x + 1]\!]$ and $G_b(x) \le G_{(D_0 a)[\![x+1]\!]}(x)$ hence $C_x(b, g) \le g^2$.
4. $b = D_1 c$. Then the induction hypothesis yields $C_x(c, g) \le g^{2^{T_c(x+1)+1}}$ hence

$$\begin{aligned} C_x(b, g) &= g^{2^{T_{D_1 c}(x+1)}} \cdot (C_x(c, g) + 1) \\ &\le g^{2^{T_{D_1 c}(x+1)}} \cdot (g^{2^{T_c(x+1)+1}} + 1) \\ &\le g^{2^{T_{D_1 c}(x+1)+1}} \\ &= g^{2^{T_b(x+1)+1}} \end{aligned}$$

since $T_c(x + 1) + 1 < T_{D_1 c}(x + 1)$ because $c \prec D_1 c$ and $Nc \le x$.

1.3 Collapsing ordinals with countable cofinalities

Lemma 10 *If $x < Na$ then* $\mathrm{K}^*a[\![x]\!] \preceq \mathrm{K}^*a$.

Proof by induction on Na using Lemma 5.
1. $a = 0$. Then the assertion is obvious.
2. $a = (a_0, \ldots, a_n)$. Let $b := (a_1, \ldots, a_n)$.
2.1. $x < Na_0$. Then $a[\![x]\!] = a_0[\![x]\!]$ and the induction hypothesis yields $\mathrm{K}^*a[\![x]\!] = \mathrm{K}^*a_0[\![x]\!] \preceq \mathrm{K}^*a_0 \subseteq \mathrm{K}^*a$ by assertion 2 of Lemma 3.
2.2. $x = Na_0$. Then $a[\![x]\!] = a_0$ and $\mathrm{K}^*a_0 \subseteq \mathrm{K}^*a$.
2.3. $x > Na_0$. Then $a[\![x]\!] = a_0 + b[\![x - Na_0]\!]$. $x < Na$ yields $x - Na_0 < Nb$ and the induction hypothesis yields $\mathrm{K}^*b[\![x - Na_0]\!] \preceq \mathrm{K}^*b$. Thus $\mathrm{K}^*a[\![x]\!] = \mathrm{K}^*a_0 \cup \mathrm{K}^*b[\![x - Na_0]\!] \preceq \mathrm{K}^*a_0 \cup \mathrm{K}^*b = \mathrm{K}^*a$.
3. $a = D_0b$. Then $a[\![x]\!] \prec a \prec \Omega$, therefore $\mathrm{K}^*a[\![x]\!] \preceq \mathrm{K}^*a$ by assertion 2 of Lemma 3.
4. $a = D_1b$. Then $Na = 1 + Nb$.
4.1. $b = 0$. Then $Na = 1$ hence $x = 0$ and the assertion is obvious.
4.2. $\mathsf{tp}(b) \in \{\omega, \Omega\}$. We may assume that $x > 0$. Then $(D_1b)[\![x]\!] = D_1b[\![x-1]\!]$. $x < Na$ yields $x - 1 < Nb$. The induction hypothesis yields $\mathrm{K}^*b[\![x-1]\!] \preceq \mathrm{K}^*b$, hence $\mathrm{K}^*a[\![x]\!] \preceq \mathrm{K}^*a$.
4.3. $b = c + 1$. Then $a[\![x]\!] = D_1c \cdot y + (D_1c)[\![z]\!]$ where $ND_1c > z$. The induction hypothesis yields $\mathrm{K}^*(D_1c)[\![z]\!] \preceq \mathrm{K}^*D_1c$ hence $\mathrm{K}^*a[\![x]\!] \preceq \mathrm{K}^*a$.

Lemma 11 *If $a, b \in OT$, $\mathsf{tp}(b) = \omega$ and $b[x] \preceq a \preceq b$ then* $\mathrm{K}^*b[x] \leq \mathrm{K}^*a$.

Proof by induction on Nb.
1. $b = (b_0, \ldots, b_n)$. Then we have $b[x] = (b_0, \ldots, b_{n-1}) + b_n[x]$ by assertion 1 of Lemma 5. $b[x] \preceq a \preceq b$ yields $a = (b_0, \ldots, b_{n-1}) + c$ for some c with $b_n[x] \preceq c \preceq b_n$. The induction hypothesis yields $\mathrm{K}^*b_n[x] \preceq \mathrm{K}^*c$ hence $\mathrm{K}^*b[x] \preceq \mathrm{K}^*a$.
2. $b = D_0c$. Then $b[x] \preceq a \preceq b \prec \Omega$ and $\mathrm{K}^*b[x] \preceq \mathrm{K}^*a$.
3. $b = D_1c$.
3.1. $\mathsf{tp}(c) = \omega$. Then $b[x] = D_1c[x] \preceq a \preceq D_1c$. Thus $a = D_1d + e$ for some d with $c[x] \preceq d \preceq c$. The induction hypothesis yields $\mathrm{K}^*c[x] \preceq \mathrm{K}^*d$ hence $\mathrm{K}^*b[x] \preceq \mathrm{K}^*a$.
3.2. $c = d + 1$. Then $(D_1c)[x] = D_1d \cdot y + (D_1d)[\![z]\!]$ with $z < ND_1d$ and $y > 0$. Lemma 10 yields $\mathrm{K}^*(D_1d)[\![z]\!] \leq \mathrm{K}^*D_1d$. $D_1d \cdot y + (D_1d)[\![z]\!] \preceq a \preceq D_1c$ yields $a = D_1d + e$ for some $e \prec D_1c$ hence $\mathrm{K}^*b[x] \preceq \mathrm{K}^*a$.

Lemma 12 *Assume that $b \in OT$ and $\mathsf{tp}(b) = \omega$.*
1. $b[x] \lhd_x b$.
2. $\mathrm{K}^*b[x] \prec b[x]$.
3. $D_0b[x] \in OT$.

Proof. Assertions 2 and 3 follow from assertion 1. Assertion 1 itself follows from Lemma 11.

Lemma 13 *Assume $D_0b \in \mathrm{OT}$. If $x < \omega$ and $\mathsf{tp}(b) = \omega$ then $D_0b[x] \in \mathrm{OT}$ and* $(D_0b)[x] = D_0b[x]$.

Proof. Lemma 12 yields $\mathrm{K}^*b[x] \prec b[x]$, hence $D_0b[x] \in OT$. By definition we have $D_0b[x] \preceq (D_0b)[x]$. Assume now that $(D_0b)[x] = D_0c + d$ for some $d \prec D_0(c+1)$. If $d \neq 0$ then $D_0(c+1)$ would be a better choice for $(D_0b)[x]$ than D_0c+d. Hence $d = 0$. We have $N(D_0b)[x] = 1 + Nb + x = 1 + Nc$, thus $Nc = Nb + x$. $c \prec b$ yields $c \preceq b[x]$ hence $D_0c \leq D_0b[x]$. Therefore $D_0b[x] = (D_0b)[x]$.

Lemma 14 *Assume that* $a, b \in OT$, $\mathsf{tp}(b) = \omega$, $\mathrm{K}^*b \prec a$ *and* $g = G_{(D_0a)[\![x+1]\!]}(x)$. *Then* $C_x(b[x], g) \leq C_x(b, g)$.

Proof by induction on b.
1. $b = (b_0, \ldots, b_n)$ with $\mathsf{tp}(b_n) = \omega$.
Then $\mathrm{K}^*b_n \prec a$ and the induction hypothesis yields $C_x(b_n[x], g) \leq C_x(b_n, g)$. Then $C_x(b[x], g) = C_x(b_0, g) + \cdots + C_x(b_n[x], g) \leq C_x(b_0, g) + \cdots + C_x(b_n, g) = C_x(b, g)$.
2. $b = D_0c$.
Then $b[x] \prec \Omega$ and Lemma 8 yields $C_x(b[x], g) = g \cdot G_{b[x]}(x) = g \cdot G_b(x) = C_x(b, g)$.
3. $b = D_1c$ where $\mathsf{tp}(c) = \omega$.
We have $\mathrm{K}^*c \subseteq \mathrm{K}^*b \prec a$ and the induction hypothesis yields $C_x(c[x], g) \leq C_x(c, g)$. Therefore Lemma 7 yields

$$\begin{aligned} C_x(b[x], g) &= C_x(D_1c[x], g) \\ &= g^{2^{T_{D_1c[x]}(x+1)}} \cdot (C_x(c[x], g) + 1) \\ &\leq g^{2^{T_{D_1c}(x+1)}} \cdot (C_x(c, g) + 1) \\ &= C_x(b, g). \end{aligned}$$

4. $b = D_1c$ where $c = d + 1$.
In this critical case we have $b[x] = D_1d + (D_1c)[\![x+1]\!] = D_1d \cdot y + (D_1c)[\![z]\!]$ where $z < ND_1c$ and $y > 0$. Lemma 10 yields $\mathrm{K}^*(D_1c)[\![z]\!] \preceq \mathrm{K}^*D_1c \preceq \mathrm{K}^*b \prec a$. Thus $\mathrm{K}^*(D_1c)[\![x+1]\!] \prec a$. Hence Lemma 9 yields

$$\begin{aligned} C_x(b[x], g) &= C_x(D_1d, g) + C_x((D_1c)[\![x+1]\!], g) \\ &\leq g^{2^{T_{D_1d}(x+1)}} \cdot (C_x(d, g) + 1) + g^{2^{T_{(D_1c)[\![x+1]\!]}(x+1)+1}} \\ &\leq g^{2^{T_{D_1c}(x+1)}} \cdot (C_x(c, g) + 1) \\ &= C_x(b, g) \end{aligned}$$

since $T_{D_1c}(x+1) > T_{(D_1c)[\![x+1]\!]}(x+1) + 1$ and $T_{D_1c}(x+1) \geq T_{D_1d}(x+1)$.

1.4 Collapsing ordinals with uncountable cofinalities

Lemma 15 $\mathsf{tp}(a) = \Omega \;\&\; a\{0\} \prec b \prec a \;\Rightarrow\; Na\{0\} < Nb$.

Proof by induction on Na.
1. $a = \Omega$. Then $a\{0\} = 0$ and the assertion is obvious.

2. $a = (a_0, \ldots, a_n)$. $a\{0\} \prec b \prec a$ yields $b = (a_0, \ldots, a_{n-1}, c)$ for some c with $a_n\{0\} \prec c \prec a_n$. The induction hypothesis yields $Na_n\{0\} < Nc$ hence $Na\{0\} < Nb$.
3. $a = D_1c$. $a\{0\} = D_1c\{0\} \prec b \prec D_1c$ yields $b = D_1d + e$ for some $e \prec D_1(d+1)$ and $c\{0\} \preceq d \prec c$. The induction hypothesis yields $Nc\{0\} \leq Nd$. If $e \neq 0$ then $Nb \geq Nd + Ne + 1 > Na$. If $e = 0$ then $c\{0\} \prec d \prec c$. The induction hypothesis yields $Nc\{0\} < Nd$ hence the assertion.

Lemma 16 *Assume that* $D_0a, b, c \in OT$, $\mathsf{tp}(a) = \mathsf{tp}(c) = \Omega$, $b \preceq c \preceq a$, $\mathrm{K}^*b \prec a$ *and* $Nb \leq Nc + x$. *Then* $b \preceq c\{(D_0a)[\![x+1]\!]\}$.

Proof by induction on Nb.
1. $b = 0$. Then the assertion is obvious.
2. $b = (b_0, \ldots, b_m)$. Assume that $c = (c_0, \ldots, c_n)$.
2.1. $b_0 = c_0, \ldots, b_m = c_m$ and $m < n$. Then $b \preceq (c_0, \ldots, c_m) \preceq c\{(D_0a)[\![x+1]\!]\}$.
2.2. $\exists i \leq \min\{m, n\}[b_0 = c_0, \ldots, b_{i-1} = c_{i-1}, b_i \prec c_i]$.
If $i < n$ then $b \prec (c_0, \ldots, c_i) \preceq c\{(D_0a)[\![x+1]\!]\}$.
Assume $i = n$ and $b_n, \ldots, b_m \prec c_n]$.
2.2.1. $m = n$. $Nb \leq Nc + x$ yields $Nb_n \leq Nc_n + x$. The induction hypothesis yields $b_n \preceq c_n\{(D_0a)[\![x+1]\!]\}$ hence $b \preceq c\{(D_0a)[\![x+1]\!]\}$.
2.2.2. $m > n$. $Nb \leq Nc + x$ yields $Nb_n, \ldots, Nb_m \leq Nc_n + x - 1$ The induction hypothesis yields for $x > 0$ that $b_n, \ldots, b_m \preceq c_n\{(D_0a)[\![x]\!]\} \prec c_n\{(D_0a)[\![x+1]\!]\}$. For $x = 0$ we obtain $b_n, \ldots, b_m \preceq c_n\{0\} = c_n\{(D_0a)[\![x]\!]\}$ by Lemma 15. Since $c_n\{(D_0a)[\![x+1]\!]\} \in P$ we obtain $(b_n, \ldots, b_m) \prec c_n\{(D_0a)[\![x+1]\!]\}$ hence $b \prec c\{(D_0a)[\![x+1]\!]\}$ holds for $x \geq 0$.
3. $b = D_0c$. $\mathrm{K}^*b \prec a$ yields $\mathrm{K}^*c \cup \{c\} \prec a$ hence $b \prec D_0a$, thus $b \preceq (D_0a)[\![Nb]\!]$.
3.1. $c = \Omega$. Then $Nc = 1$ and $b \preceq (D_0a)[\![x+1]\!] = c\{(D_0a)[\![x+1]\!]\}$.
3.2. $\Omega \prec c$. Then $b \preceq \Omega \preceq c[0] \preceq c\{(D_0a)[\![x+1]\!]\}$.
4. $b = D_1d$.
4.1. $c = (c_0, \ldots, c_n)$ with $n \geq 1$.
Then $b = D_1d \preceq c_0 \preceq c\{0\} \leq c\{(D_0a)[\![x+1]\!]\}$.
4.2. $c = D_1e$. $Nb \leq Nc + x$ yields $Nd \leq Ne + x$. The induction hypothesis yields $d \preceq e\{(D_0a)[\![x+1]\!]\}$ since $e \prec D_1e \preceq a$. Thus $b = D_1d \preceq D_1e\{(D_0a)[\![x+1]\!]\} = c\{(D_0a)[\![x+1]\!]\}$.

Corollary 1 *Assume that* $\mathsf{tp}(a) = \Omega$, $b \preceq a$, $\mathrm{K}^*b \prec b$ *and* $Nb \leq Na + x$. *Then* $b \leq a\{(D_0a)[\![x+1]\!]\}$.

Proof. Put $c = a$ in Lemma 16.

Lemma 17 *Assume that* $D_0a \in \mathrm{OT}$ *and* $\mathsf{tp}(a) = \Omega$. *Let* $z < ND_0a$. *Then* $(D_0a)[\![z]\!] = D_0a\{0\}$ *if* $z = ND_0a[0]$ *and* $(D_0a)[\![z]\!] = (D_0a\{0\})[\![z]\!]$ *else.*

Proof. It suffices to show $(D_0a)[\![z]\!] \preceq D_0a\{0\}$. Assume for a contradiction that $D_0a\{0\} \prec (D_0a)[\![z]\!] \prec D_0a$. Then $(D_0a)[\![z]\!] = D_0b + d$ for some b with $a\{0\} \preceq b \prec a$. Lemma 15 yields $Nb \geq Na\{0\}$. If $d \neq 0$ then $N(D_0b+d) \geq ND_0a\{0\}+1 =$

ND_0a. This contradicts $z < ND_0a$. Hence $d = 0$ and $a\{0\} \prec b \prec a$. Lemma 15 yields $Nb > Na\{0\}$ hence $ND_0b \geq ND_0a$. This contradicts $z < ND_0a$.

Lemma 18 *Assume that $D_0a \in \mathrm{OT}$ and $\mathsf{tp}(a) = \Omega$. Let $z < N(D_0a)$. Let $d_0a(0, z) := (D_0a\{0\})[\![z]\!]$ and $d_0a(y + 1, z) := D_0a\{d_0a(y, z)\}$. Then $d_0a(y, z) \prec d_0a(y + 1, z)$ and $d_0a(y, z) \in OT$. Moreover $(D_0a)[\![x]\!] = d_0a(y, z')$ where y and z' are chosen such that $(Na + 1) \cdot y + z' = x$ and $z' < Na + 1$.*

Proof. By induction on y we show $d_0a(y, z) \prec d_0a(y + 1, z)$.
Assume first that $y = 0$.
Then $d_0a(0, z) = (D_0a\{0\})[\![z]\!]$ and $d_0a(1, z) = D_0a\{(D_0a\{0\})[\![z]\!]\}$. If $z = 0$ then $d_0a(0, z) < d_0a(1, z)$ is obvious. Assume that $z \neq 0$. Lemma 10 yields $\mathrm{K}^*(D_0a\{0\})[\![z]\!] \preceq \mathrm{K}^*(D_0a\{0\}) \preceq \mathrm{K}^*a\{0\} \cup \{a\{0\}\} \prec a\{(D_0a\{0\})[\![z]\!]\}$. Lemma 2 yields $\mathrm{K}(D_0a\{0\})[\![z]\!] \prec D_0a\{(D_0a\{0\})[\![z]\!]\}$ hence $(D_0a\{0\})[\![z]\!] \prec D_0a\{(D_0a\{0\})[\![z]\!]\}$.

Now assume that $y = y' + 1$.
The induction hypothesis yields $d_0a(y', z) \prec d_0a(y' + 1, z)$ hence $a\{d_0a(y', z)\} \prec a\{d_0a(y' + 1, z)\}$ thus $d_0a(y, z) \prec d_0a(y + 1, z)$.

By induction on y we show $d_0a(y, z) \in \mathrm{OT}$.
Assume $y = 0$.
Then $d_0a(y, z) = (D_0a\{0\})[\![z]\!] \in \mathrm{OT}$.
Assume $y = y' + 1$.
Then $d_0a(y, z) := D_0a\{d_0a(y', z)\}$. The induction hypothesis yields $d_0a(y', z) \in \mathrm{OT}$ hence $a\{d_0a(y', z)\} \in \mathrm{OT}$.
We have to show $\mathrm{K}^*a\{d_0a(y', z)\} \prec a\{d_0a(y', z)\}$ and compute $\mathrm{K}^*a\{d_0a(y', z)\} \preceq \mathrm{K}^*a\{0\} \cup \mathrm{K}^*d_0a(y', z)$. Lemma 4 yields $a\{0\} \lhd_0 a$ hence $\mathrm{K}^*a\{0\} \prec a\{0\}$ since $\mathrm{K}^*a \prec a$.
We first consider the case $y' = 0$. If $z = 0$ then $d_0a(y', z) = 0$ hence $\mathrm{K}^*a\{d_0a(y', z)\} = \mathrm{K}^*a\{0\} \prec a\{0\} = a\{d_0a(y', z)\}$. If $z > 0$ then $(D_0a\{0\})[\![z]\!] \neq 0$ and Lemma 10 yields $\mathrm{K}^*d_0a(y', z) \preceq \mathrm{K}^*D_0a\{0\} \preceq \mathrm{K}^*a\{0\} \cup \{a\{0\}\} \preceq a\{0\} \prec a\{(D_0a\{0\})[\![z]\!]\}$.
Now assume that $y' > 0$. We already have shown that $\{d_0a(y' - 1, z)\} \prec d_0a(y', z)$. The induction hypothesis yields $d_0a(y', z) \in \mathrm{OT}$ hence $\mathrm{K}^*a\{d_0a(y' - 1, z)\} \prec a\{d_0a(y' - 1, z)\}$ hence $\mathrm{K}^*a\{d_0a(y', z)\} = \mathrm{K}^*D_0a\{d_0a(y' - 1, z)\} \preceq \mathrm{K}^*a\{0\} \cup \mathrm{K}^*\{d_0a(y' - 1, z)\} \cup \{d_0a(y' - 1, z)\} \preceq \{d_0a(y' - 1, z)\} \prec d_0a(y', z)$.

Now we prove $(D_0a)[\![x]\!] = d_0a(y, z')$ by induction on x where $(Na+1)\cdot y+z' = x$ and $z' < Na+1$. If $x < N(D_0a)$ then the assertion follows from Lemma 17. Now assume that $x \geq N(D_0a)$. The choice of y and z' and the definition of $(D_0a)[\![x]\!]$ yield $(D_0a)[\![x]\!] \geq d_0a(y, z')$ since $Nd_0a(y, z') = x$.
Now assume that $(D_0a)[\![x]\!] = D_0b + c$ with $b \prec a$ and $c \prec D_0(b + 1)$. Then $c = 0$ since otherwise $D_0(b + 1)$ would be a better choice than $D_0b + c$ for $(D_0a)[x]$. We have $b \preceq a$, $\mathsf{tp}(a) = \Omega$, $\mathrm{K}^*b \prec b$ and $Nb \leq x = Na + 1 + x - Na - 1$. The induction hypothesis yields $(D_0a)[\![x - Na - 1]\!] = d_0a(y - 1, z)$. Lemma 1 yields $b \preceq a\{(D_0a)[\![x - Na]\!]\}$ hence $D_0b \preceq D_0a\{(D_0a)[\![x - Na - 1]\!]\} = d_0a(y, z) \in \mathrm{OT}$.

Corollary 2 *Assume that* $\mathsf{tp}(a) = \Omega$. *Then* $D_0a\{(D_0a)[\![x+1]\!]\} \in \mathrm{OT}$ *and* $(D_0a)[x] = D_0a\{(D_0a)[\![x+1]\!]\}$.

Proof. Lemma 18 yields $(D_0a)[x] = (D_0a)[\![Na+1+x]\!] = D_0a\{(D_0a)[\![x+1]\!]\} \in \mathrm{OT}$.

Definition 14 *Recursive definition of a nominal form* $\mathcal{C}_x(a,g)$ *for* $a \in OT$ *with* $\mathsf{tp}(a) = \Omega$ *and* $g < \omega$.
1. $\mathcal{C}_x(\Omega, g) := \star$.
2. $\mathcal{C}_x((a_0,\dots,a_{n-1},a_n),g) := C_x(a_0,g) + \dots + C_x(a_{n-1},g) + \mathcal{C}_x(a_n,g)$.
3. $\mathcal{C}_x(D_1a,g) := g^{2^{T_{D_1a}(x+1)}} \cdot (\mathcal{C}_x(a,g)+1)$.

If $\mathcal{C}$ is a nominal form then $\mathcal{C}[\star := c]$ denotes the result of replacing every occurrence of $\star$ in $\mathcal{C}$ by c.

Lemma 19 *If* $a \in \mathrm{OT}$, $\mathsf{tp}(a) = \Omega$ *and* $c \in OT_0$ *then* $C_x(a[c],g) \le \mathcal{C}_x(a,g)[\star := g \cdot G_c(x)]$.

Proof by induction on Na.
1. $a = \Omega$. Then Lemma 8 yields $C_x(a[c],g) = C_x(c,g) = g \cdot G_c(x) = \star[\star := g \cdot G_c(x)]$.
2. $a = (a_0,\dots,a_n)$. Then the induction hypothesis yields $C_x(a[c],g) = C_x(a_0,g) + \dots + C_x(a_{n-1},g) + C_x(a_n[c],g) \le C_x(a_0,g) + \dots + C_x(a_{n-1},g) + \mathcal{C}_x(a_n,g)[\star := g \cdot G_c(x)] = \mathcal{C}_x(a,g)[\star := g \cdot G_c(x)]$
3. $a = D_1b$.
Then assertion 2 of Lemma 7 and the induction hypothesis yields

$$\begin{aligned} C_x(a[c],g) &= C_x(D_1(b[c]),g) \\ &= g^{2^{T_{D_1b[c]}(x+1)}} \cdot (C_x(b[c],g)+1) \\ &\le g^{2^{T_{D_1b}(x+1)}} \cdot (\mathcal{C}_x(b,g)[\star := g \cdot G_c(x)]+1) \\ &= \mathcal{C}_x(a,g)[\star := g \cdot G_c(x)]. \end{aligned}$$

Lemma 20 *If* $a \in \mathrm{OT}$ *and* $\mathsf{tp}(a) = \Omega$ *then* $\mathcal{C}_x(a,g)[\star := g^2] \le C_x(a,g)$.

Proof by induction on Na.
1. $a = \Omega$. Then $\mathcal{C}_x(a,g)[\star := g^2] = g^2$ and $C_x(D_10,g) = g^{2^{T_{D_10}(x+1)}} \cdot (0+1) \ge g^2$.
2. $a = (a_0,\dots,a_n)$
Then the induction hypothesis yields $\mathcal{C}_x(a,g)[\star := g^2] = C_x(a_0,g) + \dots + C_x(a_{n-1},g) + \mathcal{C}_x(a_n,g)[\star := g^2] \le C_x(a_0,g) + \dots + C_x(a_{n-1},g) + C_x(a_n,g) = C_x(a,g)$.
3. $a = D_1b$. Then the induction hypothesis yields

$$\begin{aligned} \mathcal{C}_x(a,g)[\star := g^2] &= g^{2^{T_{D_1b}(x+1)}} \cdot (\mathcal{C}_x(b,g)[\star := g^2]+1) \\ &\le g^{2^{T_{D_1b}(x+1)}} \cdot (C_x(b,g)+1) \\ &= C_x(a,g). \end{aligned}$$

1.5 Putting things together

Theorem 2 *Let* $D_0a \in OT_0$ *and* $g := G_{(D_0a)[\![x+1]\!]}(x)$*. Let* $D_0b \in OT$ *and assume that* $\mathrm{K}^*D_0b \prec a$*. Then*

$$G_{D_0b}(x) \leq 1 + C_x(b, g).$$

Proof by induction on D_0b.
1. $G_{D_00}(x) = 1 \leq 1 + C_x(0, g)$.
2. $b = c+1$. Then $G_{D_0b}(x) = G_{D_0c+(D_0b)[\![x+1]\!]}(x) = G_{D_0c}(x) + G_{(D_0b)[\![x+1]\!]}(x) \leq 1 + C_x(c, g) + g = 1 + C_x(c+1, g)$ since $C_x(1, g) = g$ and $G_{(D_0b)[\![x+1]\!]}(x) \leq G_{(D_0a)[\![x+1]\!]}(x)$ by assertion 3 of Lemma 6.
3. $tp(b) = \omega$.
Then the induction hypothesis, Lemma 13 and Lemma 14 yield
$G_{D_0b}(x) = G_{(D_0b)[x]}(x) = G_{D_0(b[x])}(x) \leq 1 + C_x(b[x], g) \leq 1 + C_x(b, g)$
4. $tp(b) = \Omega$.
Then the induction hypothesis, Lemma 2, Lemma 19 and Lemma 20 yield

$$\begin{aligned}
G_{D_0b}(x) &= G_{D_0b[(D_0b)[\![x+1]\!]]}(x) \\
&\leq 1 + C_x(b\{(D_0b)[\![x+1]\!]\}, g) \\
&\leq 1 + \mathcal{C}_x(b, g)[\star := g \cdot G_{(D_0b)[\![x+1]\!]}(x)](x) \\
&\leq 1 + \mathcal{C}_x(b, g)[\star := g^2] \\
&\leq 1 + C_x(b, g).
\end{aligned}$$

Lemma 21 *Let* $U_x := \{a \in T : Na \leq x\}$ *and* $\#U_x$ *be the cardinality of* U_x*. Then* $\#U_x \leq 4^{4^{4^x}}$.

Proof. By induction on x. Obviously $\#U_0 = 1$ and $\#U_{x+1}$ is less than or equal to one plus the cardinality of the Cartesian product $\{0, 1, 2\} \times U_x \times \cdots U_x \times U_x$ with $x+2$ factors. For, if $a \in T$ then a is either of the form $(a_0, \ldots, a_n)$ with $n \leq x$ and $a_i \in T_x$ or a is of the form D_0b or D_1b for $b \in T_x$. Hence, arguing by induction, $\#U_{x+1} \leq 3 \cdot (\#U_x)^x \leq 3 \cdot (4^{4^{4^x}})^x \leq 4^{4^{4^{x+1}}}$.

Theorem 3 *Let* $p(x) := 4^{4^{4^x}}$*. If* $a \in \mathrm{OT}_0$ *and* $Na \leq x$ *then* $G_a(x) \leq p(p(T_a(x+1)))$.

Proof. By induction on Na.
1. $a = 0$. Then the assertion is obvious.
2. $a = (a_0, \ldots, a_n)$
Then the induction hypothesis yields $G_a(x) = G_{a_0}(x) + \cdots + G_{a_n}(x) \leq p^2(T_{a_0}(x+1)) + \cdots + p^2(T_{a_n}(x+1)) \leq p^2(T_a(x+1))$.
2. $a = D_0b$. Let $g := (D_0a)[\![x+1]\!]$. Then the induction hypothesis and Lemma 9 yield

$$G_a(x) \leq 1 + C_x(b, g)$$

$$\leq g^{2^{T_b(x+1)+1}}$$
$$\leq (p^2(T_{a[\![x+1]\!]}(x+1)))^{2^{4^{4^{4^x}}}}$$
$$\leq p^2(T_a(x+1)).$$

Corollary 3 *Let* $4_0(x) := x$ *and* $4_{n+1}(x) := 4^{4_n(x)}$. *If* $a \in \mathrm{OT}_0$ *and* $Na \leq x$ *then* $G_a(x) \leq 4_9(x)$.

Proof. By Lemma 21 and Theorem 3.

We find it an interesting open question to decide whether G_a can be majorized eventually by a double (or triple) exponential function. Another interesting open question is whether our first main result extends to the proof-theoretic ordinal of the theory $ID_{<\omega}$. Then the usual first subrecursively inaccessible ordinal would not be subrecursively inaccessible for the assignment of fundamental sequences considered in this section. The corresponding phase transition would then even sharper than the one obtained in this paper.

2 The second main result

2.1 Preliminaries

We collect here some folklore material which proves useful in proofs later on. In this section we denote ordinals by small Greek letters. The idea is to indicate that the results of this chapter are independent of the notation system OT to a large extent. In particular, for $\varepsilon > 0$ the resulting slow growing hierarchy, when restricted to the segment of ordinals below ε_0 will exactly exhaust all provably recursive functions of PA. Therefore this section can be read by readers without knowledge of higher ordinal notations.

Definition 15 *For a given real number* $\varepsilon \geq 0$ *let* $\lambda[x]_\varepsilon := \max\{\beta < \lambda : N\beta \leq (1+\varepsilon)\cdot N\lambda + x\}$. *Any such system will be called a norm based assignment of fundamental sequences. If* $\varepsilon = 0$ *we call* $\cdot[\cdot]_\varepsilon$ *the standard norm based assignment.*

Definition 16
Let $\cdot[\cdot]$ *be an assignment of fundamental sequences. With respect* $\cdot[\cdot]$ *we define certain ordinal relations as follows:*

1. $\alpha \succ_x \beta$
   $:\Longleftrightarrow (\exists n > 0)(\exists \gamma_0, \ldots, \gamma_n)[\alpha = \gamma_0 \;\&\; \beta = \gamma_n \;\&\; (\forall i < n)[\gamma_{i+1} = \gamma_i[x]]]$.
2. $\alpha \preceq_x \beta :\Longleftrightarrow \alpha \prec_x \beta \vee \alpha = \beta$.
3. $\beta \sqsupseteq_x m :\Longleftrightarrow (\exists \alpha)[\beta \succeq_x \alpha \;\&\; N\alpha \geq m]$.

The following lemma provides some useful properties for investigating the growth rate of pointwise hierarchies.

Lemma 22 *Let $\cdot[\cdot]$ be a norm based assignment of fundamental sequences and let the slow growing hierarchy (G_α) be defined with respect to $\cdot[\cdot]$. Then we have:*

1. $\alpha \succeq_x \beta \;\Rightarrow\; G_\alpha(x) \geq G_\beta(x)$.
2. $G_\beta(x) \geq N\beta$.
3. $\beta \sqsupseteq_x m \;\Rightarrow\; G_\beta(x) \geq m$.

Proof. Straightforward. □

Lemma 23 *Assume that $\cdot[\cdot]$ is a norm based assignment. Let $\lambda \in Lim$. Then $N\lambda + x \leq N\lambda[x]$. If further $\lambda[x] + 1 < \alpha < \lambda$ then $\lambda[x] + 1 \leq \alpha[0]$.*

Proof. The first claim is obvious. Assume that $\alpha[0] < \lambda[x] + 1 < \alpha < \lambda$. Then $N\lambda[x] + 1 > N\alpha[0] \geq N\alpha > N\lambda[x] + 1$. Contradiction. □

Corollary 4 *Assume that $\cdot[\cdot]$ is a norm based assignment. Let $\succ_y$ be defined with respect to $\cdot[\cdot]$.*

1. *Let $\lambda \in Lim$. Then $\lambda[x+1] \succeq_y \lambda[x] + 1$.*
2. *Assume that $\alpha \succ_x \beta$ and $\delta = \omega^\gamma + \alpha$ where $\alpha < \omega^{\gamma+1}$. Then $\delta \succ_x \omega^\gamma + \beta$.*
3. *Assume that $\alpha \succ_x$. Then $\omega^\alpha \succ_x \omega^\beta$.*

Proof. Assertion 1) follows from Lemma 23. The assertions 2) are 3) are proved by induction on β with the use of 1). □

The following lemma shows that monotonicity for the indices of the assignment of fundamental sequences yields the expected monotonicity for the induced assignments and pointwise hierarchies.

Lemma 24 *Let $\varepsilon, \varepsilon'$ be real numbers with $1 \leq \varepsilon \leq \varepsilon'$.*

Let (G_α) be defined with respect to $\cdot[\cdot]_\varepsilon$ and let (G'_α) be defined with respect to $\cdot[\cdot]_\varepsilon$. Let $\succeq_y$ be defined with respect to $\cdot[\cdot]$. Then the following holds:

1. *If $\lambda \in Lim$ then $\lambda[x]'_\varepsilon \succeq_y \lambda[x]_\varepsilon$.*
2. *$G_\alpha(x) \leq G'_\alpha(x)$ for any $\alpha \in \mathrm{OT}$ and $x < \omega$.*

Proof. Straightforward. □

We are going to show that for $\varepsilon > 0$ the pointwise hierarchies consists in fact of fast growing functions. For this purpose we recall some basic facts from hierarchy theory.

Definition 17 (The Hardy-Hierarchy) *With regard to the a given norm based system $\cdot[\cdot]$ of fundamental sequences we define recursively numbertheoretic functions H_α as follows.*

1. *$H_0(x) := x$.*
2. *$H_{\alpha+1}(x) := H_\alpha(x+1)$.*

3. $H_\lambda(x) := H_{\lambda[x]}(x)$ *if* λ *is a limit.*

Lemma 25 *Let* $\succeq_x$ *be defined with respect to a given norm based assignment* $\cdot[\cdot]$*. Then* $\alpha \succeq_x \beta$ *yields* $H_\alpha(y) \geq H_\beta(y)$ *for all* $y \geq x$*. Furthermore each function* H_α *is strictly monotonic increasing.*

Lemma 26 *Let* $\varepsilon, \varepsilon'$ *be real numbers with* $1 \leq \varepsilon \leq \varepsilon'$*. Let* (H_α) *be defined with respect to* $\cdot[\cdot]_\varepsilon$ *and let* (H'_α) *be defined with respect to* $\cdot[\cdot]_\varepsilon$*. Let* $\succeq_y$ *be defined with respect to* $\cdot[\cdot]$*. Then* $H_\alpha(x) \leq H'_\alpha(x)$ *for any* α *and* $x < \omega$*.*

Proof. Straightforward. □

Lemma 27 *Let* (H_α) *be defined with respect to a norm based assignment Then* (H_α) *is a fast growing hierarchy.*

Proof. This is postponed into the appendix. Using techniques from [14] the proof is straightforward.

2.2 Putting things together

If f is an operation on natural numbers we write $f(m/n)$ for $f(l)$ where l is the largest integer less than or equal to $m/n := m \cdot n^{-1}$. In the sequel we show the fast growingness of (G_α) when defined with respect to $\cdot[\cdot]$ for $\varepsilon > 0$ by a straight forward but tedious calculation.

Theorem 4 *Assume* $k \geq 4$*. Let* $\cdot[\cdot] := \cdot[\cdot]_{1/k}$ *and let* $\sqsupseteq_0, \succeq_0$ *and* (H_α) *be defined with respect to* $\cdot[\cdot]$ *If* $\gamma =_{NF} \delta + \omega^{\omega^\alpha} \cdot k^k$ *then* $\gamma \sqsupseteq_0 H_\alpha(N\delta/k)$*.*

Proof. By induction on α. In the following calculations we frequently make use of assertions 1),2) and 3) of Corollary 4.
We may assume that $\alpha > 0$.
Case 1. $\alpha = \beta + 1$.
Then $\delta + \omega^{\omega^{\beta+1}} \cdot k^k \succeq_0 \delta + \omega^{\omega^{\beta+1}} \cdot (k^k - 1) + \omega^{\omega^\beta + \gamma}$ for some $\gamma < \omega^{\beta+1}$ with $N\gamma \geq (N\beta + 1) \cdot k^{k-1} + 1$. Let $\xi_0 := \delta + \omega^{\omega^{\beta+1}} \cdot (k^k - 1) + \omega^{\omega^\beta + \gamma}$.
If $\gamma < \omega$ then $\xi_0 \succeq_0 \delta + \omega^{\omega^{\beta+1}} \cdot (k^k - 1) + \omega^{\omega^\beta + (1+N\beta)\cdot k^{k-1}+1} =: \xi_1$. If $\gamma \geq \omega$ then $\xi_0 \succeq_0 \delta + \omega^{\omega^{\beta+1}} \cdot (k^k - 1) + \omega^{\omega^\beta + \omega} \succeq_0 \xi_1$. In both cases $\xi_0 \succeq_0 \xi_1$. We have $\xi_1 \succeq_0 \delta + \omega^{\omega^{\beta+1}} \cdot (k^k - 1) + \omega^{\omega^\beta + (1+N\beta)\cdot k^{k-1}} \cdot 2 =: \xi_2$ since $N\xi_1 \cdot 1/k \geq (3+N\beta)\cdot(k^k-1)/k + (2+N\beta+1+N\beta\cdot k^{k-1})\cdot k \geq (3+N\beta)\cdot k^{k-1} + N\beta \cdot k^{k-2} \geq N\omega^{\omega^\beta + (1+N\beta)\cdot k^{k-1}}$. Similarly we obtain

$$\begin{aligned}
\xi_2 &= \delta + \omega^{\omega^{\beta+1}} \cdot (k^k - 1) + \omega^{\omega^\beta + (1+N\beta)\cdot k^{k-1}} \cdot 2 \\
&\succeq_0 \delta + \omega^{\omega^{\beta+1}} \cdot (k^k - 1) + \omega^{\omega^\beta + (1+N\beta)\cdot k^{k-1}} + \omega^{\omega^\beta + (1+N\beta)\cdot k^{k-1} - 1} \cdot 2
\end{aligned}$$

$$
\begin{aligned}
&\succeq_0 \dots \\
&\succeq_0 \delta + \omega^{\omega^{\beta+1}} \cdot (k^k - 1) + \omega^{\omega^{\beta} + (1+N\beta)\cdot k^{k-1}} + \omega^{\omega^{\beta} + (1+N\beta)\cdot k^{k-1} - 1} + \cdots \\
&\quad + \omega^{\omega^{\beta}+2} + \omega^{\omega^{\beta}+1} =: \xi_3
\end{aligned}
$$

We have $N\xi_3 \cdot (1/k) \geq (1/k) \cdot (N\delta + (3+N\beta) \cdot (k^{k-1} - 1)) + (1/k) \cdot \sum_{i=1}^{2+N\beta+(1+N\beta)\cdot k^{k-1}} i \geq (2 + N\beta) \cdot k^k$ hence the induction hypothesis yields

$$
\begin{aligned}
\xi_3 \;&=\; \delta + \omega^{\omega^{\beta+1}} \cdot (k^k - 1) + \omega^{\omega^{\beta} + (1+N\beta)\cdot k^{k-1}} + \omega^{\omega^{\beta} + (1+N\beta)\cdot k^{k-1} - 1} + \cdots \\
&\quad + \omega^{\omega^{\beta}+2} + \omega^{\omega^{\beta}+1} \\
&\succeq_0 \delta + \omega^{\omega^{\beta+1}} \cdot (k^k - 1) + \omega^{\omega^{\beta} + (1+N\beta)\cdot k^{k-1}} + \omega^{\omega^{\beta} + (1+N\beta)\cdot k^{k-1} - 1} + \cdots \\
&\quad + \omega^{\omega^{\beta}+2} + \omega^{\omega^{\beta}} \cdot k^k \\
&\sqsupseteq_0 H_\beta((N(\delta + \omega^{\omega^{\beta+1}} \cdot (k^k - 1) + \dots))/k) \geq H_\beta(N\delta/k + 1) = H_\alpha(N\delta/k).
\end{aligned}
$$

Case 2. $\alpha \in Lim$. We have

$$
\begin{aligned}
\delta + \omega^{\omega^{\alpha}} \cdot k^k &\succeq_0 \delta + \omega^{\omega^{\alpha}} \cdot (k^k - 1) + \omega^{\omega^{\alpha[N\delta/k + (2+N\alpha)\cdot k^{k-1}]}} \\
&\succeq_0 \delta + \omega^{\omega^{\alpha}} \cdot (k^k - 1) + \omega^{\omega^{\alpha[N\delta/k + (1+N\alpha)\cdot k^{k-1} + 1]}} \\
&\succeq_0 \delta + \omega^{\omega^{\alpha}} \cdot (k^k - 1) + \omega^{\omega^{\alpha[N\delta/k + (1+N\alpha)\cdot k^{k-1}] + 1}} =: \eta_0 \\
&\succeq_0 \delta + \omega^{\omega^{\alpha}} \cdot (k^k - 1) + \omega^{\omega^{\alpha[N\delta/k + (1+N\alpha)\cdot k^{k-1}]} \cdot 2} =: \eta_1
\end{aligned}
$$

since $N\eta_0 \cdot 1/k \geq N\delta/k + (2+N\alpha)k^k + (1+N\alpha) \cdot k^{k-2} \geq N[\omega^{\alpha[N\delta/k + (1+N\alpha)\cdot k^{k-1}]}$
Further

$$
\begin{aligned}
\eta_1 &\succeq_0 \delta + \omega^{\omega^{\alpha}} \cdot (k^k - 1) + \omega^{\omega^{\alpha[N\delta/k + (1+N\alpha)\cdot k^{k-1}]}} + \omega^{\omega^{\alpha[N\delta/k + (1+N\alpha)\cdot k^{k-1} - 1]} + 1} \\
&\succeq_0 \delta + \omega^{\omega^{\alpha}} \cdot (k^k - 1) + \omega^{\omega^{\alpha[N\delta/k + (1+N\alpha)\cdot k^{k-1}]}} + \omega^{\omega^{\alpha[N\delta/k + (1+N\alpha)\cdot k^{k-1} - 1]} \cdot 2} \\
&\succeq_0 \dots \\
&\succeq_0 \delta + \omega^{\omega^{\alpha}} \cdot (k^k - 1) + \omega^{\omega^{\alpha[N\delta/k + (1+N\alpha)\cdot k^{k-1}]}} \\
&\quad + \omega^{\omega^{\alpha[N\delta/k + (1+N\alpha)\cdot k^{k-1} - 1]} + \dots + \omega^{\alpha[1]} + \omega} \\
&\succeq_0 \delta + \omega^{\omega^{\alpha}} \cdot (k^k - 1) + \omega^{\omega^{\alpha[N\delta/k + (1+N\alpha)\cdot k^{k-1}]}} \\
&\quad + \omega^{\omega^{\alpha[N\delta/k + (1+N\alpha)\cdot k^{k-1} - 1]} + \dots + \omega^{\alpha[1]} + N\delta/k \cdot k^{k-1}}
\end{aligned}
$$

Let $\eta_2 := \omega^{\alpha[N\delta/k + (1+N\alpha)\cdot k^{k-1} - 1]} + \ldots + \omega^{\alpha[1]}$. Then

$$
\begin{aligned}
\eta_2 \;&=\; \delta + \omega^{\omega^{\alpha}} \cdot (k^k - 1) + \omega^{\eta_2 + N\delta/k \cdot k^{k-1}} \\
&\succeq_0 \delta + \omega^{\omega^{\alpha}} \cdot (k^k - 1) + \omega^{\eta_2 + N\delta/k \cdot k^{k-1} - 1} + \omega^{\eta_2 + N\delta/k \cdot k^{k-1} - 1} \\
&\succeq_0 \delta + \omega^{\omega^{\alpha}} \cdot (k^k - 1) + \omega^{\eta_2 + N\delta/k \cdot k^{k-1} - 1} + \omega^{\omega^{\alpha[N\delta/k + (1+N\alpha)\cdot k^{k-1}]}}
\end{aligned}
$$

$$
\begin{aligned}
&\succeq_0 \delta + \omega^{\omega^{\alpha}} \cdot (k^k - 1) + \omega^{\eta_2 + N\delta/k \cdot k^{k-1} - 1} \\
&\quad + \omega^{\omega^{\alpha[N\delta/k+(1+N\alpha)\cdot k^{k-1}-1]} + \omega^{\alpha[N\delta/k+(1+N\alpha)\cdot k^{k-1}-2]} + \cdots \omega^{\alpha[1]} + N\delta/k \cdot k^{k-1} - 1} \\
&\quad + \cdots + \\
&\quad + \omega^{\omega^{\alpha[N\delta/k]} + \cdots + N\delta/k \cdot k^{k-1} - 1} \\
&\succeq_0 \delta + \omega^{\omega^{\alpha}} \cdot (k^k - 1) + \cdots + \omega^{\omega^{\alpha[N\delta/k]}+1} =: \eta_3
\end{aligned}
$$

The induction hypothesis yields

$$
\begin{aligned}
\eta_3 &= \ldots + \omega^{\omega^{\alpha[N\delta/k]}+1} \\
&\succeq_0 \omega^{\omega^{\alpha[N\delta/k]}} \cdot k^k \\
&\sqsupseteq_0 H_{\alpha[N\delta/k]}(N\delta/k) = H_\alpha(k)
\end{aligned}
$$

since $N\eta_3 \geq N\delta/k \cdot k^{k-1} \cdot k^{k-1} + 1/2 \cdot (N\alpha+1)^2 \cdot (k^{k-1})^2 \geq (2 + N\alpha(k+1)/k + N\delta/k)k^k$.

Lemma 28 $\omega^{\omega^{\omega^{\alpha+1}}} \cdot k^k + \omega^{\omega^{\omega^{\alpha}+\omega}} \succeq_x \omega^{\omega^{\omega^{\alpha+1}}} \cdot k^k + \omega^{\omega^{\omega^{\alpha}+x}} \cdot k^k$

Proof. We obtain

$$
\begin{aligned}
&\omega^{\omega^{\omega^{\alpha+1}}} \cdot k^k + \omega^{\omega^{\omega^{\alpha}+\omega}} \\
\succeq_x\ &\omega^{\omega^{\omega^{\alpha+1}}} \cdot k^k + \omega^{\omega^{\omega^{\alpha}+x+k^{k-1}}} \\
\succeq_0\ &\omega^{\omega^{\omega^{\alpha+1}}} \cdot k^k + \omega^{\omega^{\omega^{\alpha}+x+k^{k-1}-1}} \cdot 2 \\
\succeq_0\ &\omega^{\omega^{\omega^{\alpha+1}}} \cdot k^k + \ldots + \omega^{\omega^{\omega^{\alpha}+x+1}} \\
\succeq_0\ &\omega^{\omega^{\omega^{\alpha+1}}} \cdot k^k + k^k + \omega^{\omega^{\omega^{\alpha}+x+k^{k-1}-1}} + \ldots + \omega^{\omega^{\omega^{\alpha}+x}} \cdot k^k
\end{aligned}
$$

Theorem 5 *Assume $k \geq 4$. Let $\cdot[\cdot] := \cdot[\cdot]_{1/k}$. Assume that (G_α) is defined with respect to $\cdot[\cdot]$ and that (H_α) is defined with respect to the standard norm based assignment. Then $G_{\omega^{\omega^{\omega^{\alpha+1}}} \cdot k^k + \omega^{\omega^{\omega^{\alpha}+\omega}}}(x) \geq H_{\omega^\alpha}(x)$.*

Proof. This follows from assertion 2 of Lemma 22 and Lemma 28

Corollary 5 *Let $\varepsilon > 0$ and assume that the hierarchy (G_α) is defined with respect to $\cdot[\cdot]_\varepsilon$. Then (G_α) is fast growing.*

Proof. This follows from Theorem 5

Appendix

We stick to the notational conventions of Section 1. In this appendix we first describe the standard system of fundamental sequences in terms of the norms function and show that it gives rise to a normed Bachmann system. Second, we define the standard Hardy hierarchy (H^*_a) along OT and compare it with (H_a). For an intermediate calculation we introduce in addition a fast growing (as shown in [1]) hierarchy (A_a) (which looks slow growing at first sight).

Definition 18 *For $a \in \mathrm{OT}$ with $\mathsf{tp}(a) = \omega$ and $x < \omega$ we define a non negative integer $p(a + x)$ as follows.*

1. $a = (a_0, \ldots, a_{n-1}, a_n) \Rightarrow p(a + x) := Na_0 + \ldots + Na_{n-1} + p(a_n + x)$.
2. $a = D_i(b + 1) \Rightarrow p(a + x) := (Nb + 1) \cdot (x + 1)$.
3. $a = D_i b \,\&\, \mathsf{tp}(b) = \omega \Rightarrow p(a + x) := 1 + p(b + x)$.
4. $a = D_0 b \,\&\, \mathsf{tp}(b) = \Omega \Rightarrow p(a + x) := (Nb + 1) \cdot (x + 1)$.

Definition 19 *Definition of $a\{\{x\}\}$ for $a \in \mathrm{OT}_0$ with $\mathsf{tp}(a) = \omega$ and $x < \omega$.*
$a\{\{x\}\} := \max\{b \in \mathrm{OT} : b \prec a \,\&\, Nb \le p(a + x)\}$.

Lemma 29 *The structure $\langle \mathrm{OT}_0, \cdot\{\{\cdot\}\}, N\rangle$ is a normed Bachmann system.*

Proof. This follows from Theorem 5 of [5].

Lemma 30 *Characterization of $a\{\{x\}\}$ for $a \in \mathrm{OT}_0$ with $\mathsf{tp}(a) = \omega$ and $x < \omega$.*

1. $a = (a_0, \ldots, a_{n-1}, a_n) \Rightarrow a\{\{x\}\} = (a_0, \ldots, a_{n-1}) + a_n\{\{x\}\}$.
2. $a = D_0(b + 1) \Rightarrow a\{\{x\}\} = D_0 b \cdot (x + 1)$.
3. $a = D_0 b \,\&\, \mathsf{tp}(b) = \omega \Rightarrow a\{\{x\}\} = D_0 b\{\{x\}\}$.
4. $a = D_0 b \,\&\, \mathsf{tp}(b) = \Omega \Rightarrow a\{\{x\}\} = D_0 b_x$ *where* $b_0 := b\{0\}$ *and* $b_{y+1} := b\{D_0 b_y\}$.

Lemma 30 shows that $\cdot\{\{\cdot\}\}$ coincides with Buchholz usual definition of fundamental sequences for the limits below the Howard Bachmann ordinal.

In the sequel a, b, c, d range over OT_0.

Lemma 31 *1.* $N(a\{\{0\}\}) = N(a) + 1$.
2. $N(\omega \cdot a\{0\}) \le N(\omega \cdot a) + 1$.
3. $N(a\{\{x\}\}) \le Na \cdot (x + 1)$.
4. $N(\omega^i \cdot a) \le Na \cdot (i + 1)$.

Definition 20

1. (a) $H^*_0(x) := x$,
   (b) $H^*_{a+1}(x) := H^*_a(x + 1)$,
   (c) $H^*_a(x) := H^*_{a\{\{x\}\}}(x)$ *if* $\mathsf{tp}(a) = \omega$.
2. (a) $A_0(x) := x$,
   (b) $A_a(x) := \max\{A_b(x) + 1 : b \prec a \,\&\, Nb \le Na + x\}$.

Lemma 32 *1.* $NF(a, b) \Rightarrow H_{a+b}(x) = H_a(H_b(x))$.
2. $H_{\omega \cdot 10}(x) \ge 10 \cdot x$.

3. $H_{\omega^2\cdot a+\omega\cdot a+\omega\cdot(k+11)}(x) \geq H_{\omega^2\cdot a+\omega\cdot a+\omega\cdot(k+1}(10\cdot x)$.
4. $a \prec b \;\&\; Na \leq Nb + x \;\Rightarrow\; H_a(x) < H_b(x)$.
5. $A_a(x) \leq H_a(x) \leq H_a^*(x)$.

Definition 21 *1.* $a \trianglerighteq_x^1 b :\iff a \succeq b \;\&\; Na \geq Nb + x$.
2. $a \trianglerighteq_x b :\iff (\exists a_0, \ldots, a_n)[a_0 = a \;\&\; a_n = b \;\&\; (\forall i < n)[a_i \trianglerighteq_x a_{i+1}]]$.
3. $a \geq_x k :\iff (\exists b)[a \trianglerighteq_x b \;\&\; Nb \geq k]$.

Lemma 33 *1.* $a \trianglerighteq_x b \;\Rightarrow\; A_a(x) \geq A_b(x)$.
2. $A_a(x) \geq Na$.
3. $a \geq_x k \;\Rightarrow\; A_a(x) \geq k$.
4. $\omega \cdot a \geq_{10\cdot x} Na \cdot (10 \cdot x - 1)$.

Lemma 34 *1.* $x \geq 2 \;\Rightarrow\; \omega^2 \cdot a + \omega \cdot a + \omega \cdot k + \omega \geq_x H_a^*(k + x)$.
2. $x \geq 2 \;\Rightarrow\; H_{\omega^2\cdot a+\omega\cdot a+\omega\cdot k+\omega}(x) \geq H_a^*(k + x)$.
3. $c = \omega^{\omega+d} \;\Rightarrow\; (\forall a \prec c)(\exists b \prec c)(\forall x)[H_a^*(x) \leq H_b(x)]$

Proof of the first assertion by induction on a.
1. $a = 0$.

$$\begin{aligned}\omega \cdot k + \omega &\trianglerighteq_{10\cdot x} \omega \cdot k + 10 \cdot x + 1\\ &\trianglerighteq_{10\cdot x} \cdots\\ &\trianglerighteq_{10\cdot x} k \cdot (10 \cdot x + 1) + 10 \cdot x + 1\\ &\geq_{10\cdot x} k + x = H_0^*(k + x).\end{aligned}$$

2. $a = b + 1$. Then the induction hypothesis yields

$$\begin{aligned}&\omega^2 \cdot a + \omega \cdot a + \omega \cdot k + \omega\\ \trianglerighteq_{10\cdot x}\ &\omega^2 \cdot b + \omega \cdot b + \omega^2 + \omega \cdot (k + 1) + \omega\\ \geq_{10\cdot x}\ &H_b^*(k + x + 1) = H_a^*(k + x).\end{aligned}$$

3. $\mathsf{tp}(a) = \omega$. Then the induction hypothesis yields

$$\begin{aligned}&\omega^2 \cdot a + \omega \cdot a + \omega \cdot k + \omega\\ \trianglerighteq_{10\cdot x}\ &\omega^2 \cdot a + Na \cdot (10 \cdot x - 1) + \omega \cdot k + \omega\\ \trianglerighteq_{10\cdot x}\ &\omega^2 \cdot a\{\{x\}\} + \omega \cdot a\{\{x\}\} + \omega \cdot k + \omega\\ \geq_{10\cdot x}\ &H_{a\{\{x\}\}}^*(k + x) = H_a^*(k + x).\end{aligned}$$

The second assertion follows from assertion 3 of the Lemma 33 and assertion 5 of Lemma 32 and the last assertion follows from the second assertion.

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