# Classifying the phase transition for hydra games and Goodstein sequences*

Andreas Weiermann
Mathematical Institute
P.O. Box 80010
3508 TA Utrecht
The Netherlands
e-mail: weierman@math.uu.nl

**Abstract**

It is well known from the work of Kirby and Paris that the hydra game and the Goodstein process yield combinatorial statements which are true but unprovable in first order Peano arithmetic PA. In this note we characterize the phase transition from provability to unprovability for these assertions. As a byproduct we classify those non pointwise descent recursions (along the standard system of fundamental sequences for the ordinals below $\varepsilon_0$) which are still in the scope of the so called slow growing hierarchy.

## 1 Introduction and statement of the results

This article is part of a general investigation on combinatorial independence results for theories of mathematical interest. We mainly focus on first order Peano arithmetic PA but the results shall extend without much effort to stronger theories for which an ordinal analysis is available. The starting point of these investigations is the classification of the provably recursive functions of PA in terms of the Hardy hierarchies $(H_\alpha)_{\alpha<\varepsilon_0}$ and $(h_\alpha)_{\alpha<\varepsilon_0}$. These functions are defined recursively

$$\begin{array}{lcllcl} H_0(x) & := & x & h_0(x) & := & x, \\ H_{\alpha+1}(x) & := & H_\alpha(x+1) & h_{\alpha+1}(x) & := & h_\alpha(x+1) \\ H_\lambda(x) & := & H_{\lambda[x]}(x+1) & h_\lambda(x) & := & h_{\lambda[x]}(x) \end{array}$$

where $\lambda$ is a limit and $\lambda[x]$ is the $x$-th member of the standard fundamental sequence for $\lambda$. Recall here that the system of standard fundamental sequences is given by $((\alpha+1)\cdot\omega^\lambda)[x] := \alpha\cdot\omega^\lambda+\omega^{\lambda[x]}$ when $\lambda$ is a limit and $((\alpha+1)\cdot\omega^{\beta+1})[x] :=$

*Research supported by a Heisenberg-Fellowship of the Deutsche Forschungsgemeinschaft

$\alpha \cdot \omega^{\beta+1} + \omega^{\beta} \cdot (x+1)$. (Moreover for technical reasons we put $(\alpha+1)[x] = \alpha$ and $0[x] := 0$.) ¿From the point of view of provably recursive functions there is no essential difference with respect to the rate of growth between $H_\alpha$ and $h_\alpha$ since $h_\alpha(x) \leq H_\alpha(x) \leq h_\alpha(x+1)$ for all $x \in \mathbb{N}$. A function $F : \mathbb{N} \to \mathbb{N}$ is called provably recursive in PA iff PA $\vdash (\forall x)(\exists y)`F(x) = y'$ where $`F(x) = y'$ is an $L_{\text{PA}}$ formula (in $\Sigma_1$) for the graph of $F$. Then $F : \mathbb{N} \to \mathbb{N}$ is provably recursive in PA iff $F$ is elementary recursive in $H_\alpha$ (or $h_\alpha$) for some $\alpha < \varepsilon_0$. Therefore the assertion

$$\forall x \exists y `H_{\omega_x}(1) = y' \tag{1}$$

is true but unprovable in PA since for every $\alpha < \varepsilon_0$ the function $x \mapsto H_{\omega_x}(1)$ eventually dominates $H_\alpha$. (Recall that $\omega_1 := \omega$ and $\omega_{n+1} := \omega^{\omega_n}$.) This assertion is a prototype for an independent statement, i.e. a statement which is true but unprovable in PA. For, let us consider the hydra game formulated in terms of ordinals. As before let $\cdot[\cdot]$ denote the standard assignments of fundamental sequences for the ordinals below $\varepsilon_0$ and let $\alpha(0) = \alpha$ and $\alpha(i+1) := \alpha(i)[i]$ where it is understood that $(\alpha+1)[x] = \alpha$. Then $H_\alpha(0) \leq \mu i \alpha(i) = 0$ and thus the assertion

$$\forall x \exists i \omega_x(i) = 0, \tag{2}$$

(which formalizes that chopping of the righmost head is a winning strategy for Hercules) is independent. Moreover it is well known that (2) can be reformulated in terms of Goodstein sequences where the underlying Hardy hierarchy is now $(h_\alpha)_{\alpha<\varepsilon_0}$. For stating the Goodstein principle precisely let us define $a[i := \alpha]$ as follows. If $a < i$ then $a[i := \alpha] = a$. If $a = i^{a_1} \cdot m_1 + \cdots + i^{a_n} \cdot m_n$ where $a_1 > \ldots > a_n$ and $i > m_j > 0$ then $a[i := \alpha] := (i+1)^{a_1[i:=\alpha]} \cdot m_1 + \cdots + (i+1)^{a_n[i:=\alpha]} \cdot m_n$. Let $m_0 := m$, $m_{k+1} := (m_k)[k+2 := k+3] - 1$. Then the statement

$$\forall m \exists k m_k = 0 \tag{3}$$

is independent of PA.

In this article we give a unified treatment of refinements and alterations of (2) and (3). In particular we are going to classify the phase transition from provability to unprovability in these assertions. For a given function $f : \mathbb{N} \to \mathbb{N}$ let $\alpha^f(0) = \alpha$ and $\alpha^f(i+1) := \alpha^f(i)[f(i)]$. Then (2) states that the assertion

$$\forall x \exists i \omega_x^f(i) = 0, \tag{4}$$

is independent for $f(i) = i$. Inspired from Arai's results [1] we show that (4) is independent even for all $f$ with $f(i) \geq |i|_{H_{\varepsilon_0}^{-1}(i)}$ but PA-provable for all $f$ with $f(i) \leq |i|_{H_\alpha^{-1}(i)}$ where $\alpha < \varepsilon_0$. Moreover we classify those $f$ for which the resulting lengths of descending chains are within the scope of the elementary functions. This is the case when $f(i) \leq |i|_{|i|_h}$ and $h$ is arbitrary. (It seems to us that this observation yields a partial analysis of Girard's notion of pointwiseness.)

For any function $f$ such that $f(i) \geq 1$ let $m_0^f := m$ and $m_{k+1}^f := (m_k^f)[1 + f(n) := 1 + f(n+1)] - 1$. Then (3) states that the assertion

$$\forall m \exists k m_k^f = 0 \tag{5}$$

is independent for $f(i) = i$. Refining a result from Kent and Hodgson [4] we show that (5) is independent even for all $f$ with $f(i) \geq |i|_{H_{\varepsilon_0}^{-1}(i)}$ but PA-provable for all $f$ with $f(i) \leq |i|_{H_\alpha^{-1}(i)}$ where $\alpha < \varepsilon_0$. For any $\alpha < \varepsilon_0$ let $\mathrm{mc}(\alpha)$ be defined as follows. $\mathrm{mc}(m) := m$ and $\mathrm{mc}(\alpha) := \max\{m_1, \dots, m_n, \mathrm{mc}(\alpha_1), \dots, \mathrm{mc}(\alpha_n)\}$ where $\alpha = \omega^{\alpha_1} \cdot m_1 + \cdots + \omega^{\alpha_n} \cdot m_n$ and $\alpha_1 > \ldots > \alpha_n$. Then the assertion

$$\forall K \exists M \forall \alpha_0, \dots, \alpha_M \leq \omega_K [(\forall i \leq M \mathrm{mc}(\alpha_i) \leq f(i)) \implies \exists i < M \; \& \; \alpha_i \leq \alpha_{i+1})].$$

is independent for $f(i) \geq |i|_{H_{\varepsilon_0}^{-1}(i)}$ and PA-provable for all $f$ with $f(i) \leq |i|_{H_\alpha^{-1}(i)}$ where $\alpha < \varepsilon_0$. These results indicate that the $\mathrm{mc}(\cdot)$ operation is genuinly connected to the standard assignment of fundamental sequences.

# 2 Classifying the phase transition for hydra games and Goodstein sequences

To fix the context we first recall some well known definitions and lemmata from subrecursive hierarchy theory. We write $\alpha =_{\mathrm{NF}} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ if $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ and $\alpha \geq \alpha_1 \geq \ldots \geq \alpha_n$. Let $\alpha[x] := \omega^{\alpha_1} + \cdots + \omega^{\alpha'_n[x]}$ if $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ and $\alpha_n = {\alpha_n}' + 1$ and $\alpha[x] := \omega^{\alpha_1} + \cdots + \omega^{\alpha_n[x]}$ if $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ and $\alpha_n \in Lim$. Moreover let $(\alpha+1)[x] := \alpha$ and $0[x] := 0$.

In the sequel $f$ denotes a weakly increasing function $f : \mathbb{N} \to \mathbb{N}$ satisfying $F(x) \geq 1$.

**Definition 1 (Predecessor operations).** 1. $P_x^f 0 := 0$, $P_x^f(\alpha+1) = \alpha$, and $P_x^f \lambda := P_x^f \lambda[f(x)]$.

2. $Q_x^f 0 := 0$, $Q_x^f(\alpha+1) = \alpha$, and $Q_x^f \lambda := \lambda[f(x)]$.

When $f$ is the identity function $f(i) = i = id(i)$ then we drop the superscript in $R \in \{P, Q\}$, thus $R_x \alpha := R_x^{id} \alpha$.

**Definition 2.** For $R \in \{P, Q\}$ define

1. $\alpha \succ_f^{R,n} \beta :\iff R_{n-1}^f \ldots R_1^f \alpha > 0 \; \& \; \beta = R_n^f \ldots R_1^f \alpha$ for some $n > 0$,
2. $\alpha \succ_f^R \beta :\iff (\exists n)[\alpha \succ_f^{R,n} \beta]$,
3. $\alpha \succ_k^R \beta :\iff \alpha \succ_f^R \beta$ where $f(i) = k$ for all $i \in \mathbb{N}$,
4. $\alpha \succeq_k^R \beta :\iff \alpha \succ_k^R$ or $\alpha = \beta$.

**Definition 3.** 1. $\alpha =_{\mathrm{NF}} \omega^{\alpha_1} + \cdots + \omega^{\alpha_m}$ if $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_m}$ and $\alpha_1 \geq \ldots \geq \alpha_m$.

2. $NF(\beta, \gamma)$ if $\beta =_{\mathrm{NF}} \omega^{\beta_1} + \cdots + \omega^{\beta_m}$ and $\gamma =_{\mathrm{NF}} \omega^{\gamma_1} + \cdots + \omega^{\gamma_n}$ and $\beta_m \geq \gamma_1$.

3. $\alpha_0(\beta) := \beta$, $\alpha_{h+1}(\beta) := \alpha^{\alpha_h(\beta)}$. Moreover, $\omega_h := \omega_h(1)$.

**Lemma 1.** *Let $R \in \{P, Q\}$. Then*

1. $NF(\gamma, \beta)$ & $\beta > 0 \implies R^f_x(\gamma + \beta) = \gamma + R^f_x \beta$.
2. $NF(\gamma, \alpha)$ & $\alpha \succ^{R,n}_f \beta \implies \gamma + \alpha \succ^{R,n}_f \gamma + \beta$.
3. $\alpha > 0 \implies \alpha \succeq^Q_x 1$ & $\alpha \succ^R_f 0$.
4. $x > 0$ & $\alpha \succ^{R,m}_x \beta \implies \omega^\alpha \succ^{R,n}_x \omega^\beta$ *for some* $n \geq m$.
5. $\alpha \succ^P_x \beta \implies \alpha \succ^Q_x \beta$.
6. $\alpha > 0$ *and* $y \geq x \implies \alpha \succeq^Q_y P_x \alpha + 1$.
7. $y > 0$ & $\lambda \in Lim \implies \lambda[x+1] \succeq^Q_y \lambda[x] + 1$.
8. $\alpha \succeq^Q_x \beta \succ^{P,m}_x \gamma \implies \alpha \succ^{P,n}_x \gamma$ *for some* $n \geq m$.
9. $y > 0$ & $\lambda \in Lim \implies \lambda[x+1] \succ^P_y \lambda[x]$ & $\lambda[x+1] \succ^Q_y \lambda[x]$.
10. $\alpha \succ^R_x \beta$ & $x \leq y \implies \alpha \succ^R_y \beta$.
11. $(\forall i)[f(i) \leq g(i)]$ & $\alpha \succ^R_f \beta \implies \alpha \succ^R_g \beta$.

*Proof.* Assertion 1 follows from the definitions. Assertion 2 follows from assertion 1. Assertion 3 follows by induction on $\alpha$. Assertion 4 follows from assertion 3 by induction on $\alpha$. Assertion 5 follows by induction on $\alpha$. Assertion 6 follows by induction on $\alpha$. Indeed, if $\alpha = \beta + 1$ then $\alpha = P_x \alpha + 1$. If $\alpha$ is a limit then the i.h. yields $P_x \alpha[x] \succ^Q_y P_x \alpha[x] + 1$. Assertion 7 follows by induction on $\alpha$. Indeed, if $\lambda = \omega^\alpha + \beta$ with $\beta \in Lim$ and $\beta < \omega^{\alpha+1}$ then the i.h. yields $\beta[x+1] \succ^Q_y \beta[x]$. Thus the assertion follows from assertion 2. Now assume that $\lambda = \omega^\alpha$. If $\alpha \in Lim$ then the assertion follows from assertion 4. If $\alpha = \beta + 1$ then the assertion follows from assertion 3.

Assertion 9 follows from assertion 7 and assertion 8. Assertion 8 follows by induction on $\alpha$. The assertion is true for $\alpha = \beta$ and follows from the i.h. if $\alpha = \delta + 1$. If $\alpha \in Lim$ then the i.h. yields $\alpha[x] \succ^{P,n}_x \gamma$ for some $n \geq m$. Now notice that a descent at $x$ via $P$ starting from $P(\alpha[x])$ is also a descent starting from $P(\alpha)$.

Assertion 10 follows from assertions 7, 9 and 8 by induction on $\alpha$. Assertion 11 follows from assertion 10

□

**Corollary 1.**

1. $x > 0 \implies \omega^{\alpha+1} \succ^Q_x \omega^\alpha + \omega^\alpha$.
2. $x > 0 \implies \omega^{\alpha+1} \succ^Q_x \omega^\alpha + 1$.
3. $x > 0 \implies \omega_{h+1}(\alpha + 1) \succ^Q_x \omega_{h+1}(\alpha) + \omega_{h+1}$.

*Proof.* Assertion 1 follows from assertion 3 of Lemma 1. Assertion 2 follows from assertion 1 and assertion 3 of Lemma 1. Assertion 3 follows from assertion 2. Indeed, by induction on $i$ one shows that for $i \in \{0, \dots, i\}$ we have $\omega^{\omega_h(\alpha+1)} \succ_x^Q \omega^{\omega_{h-i}(\omega_i(\alpha)+1)}$. This is true for $i = 0$. Assume that the assertion holds for $i$ and that $i+1 \le h$. Then the i.h. yields $\omega^{\omega_h(\alpha+1)} \succ_x^Q \omega^{\omega_{h-i}(\omega_i(\alpha)+1)}$. Moreover $\omega^{\omega_{h-i}(\omega_i(\alpha)+1)} = \omega^{\omega_{h-i-1}(\omega^{\omega_i(\alpha)+1})} \succ_x^Q \omega^{\omega_{h-i-1}(\omega^{\omega_i(\alpha)})+1} = \omega^{\omega_{h-(i+1)}(\omega_{i+1}(\alpha)+1)}$. Therefore $\omega^{\omega_h(\alpha+1)} \succ_x^Q \omega^{\omega_h(\alpha)+1} \succeq_x^Q \omega^{\omega_h(\alpha)} + \omega^{\omega_h(\alpha)} \succeq_x^Q \omega^{\omega_h(\alpha)} + \omega^{\omega_h(1)}$ $\square$

**Definition 4.** 1. $G_x 0 := 0$, $G_x(\alpha+1) = G_x\alpha + 1$, and $G_x\lambda := 1 + G_x\lambda[x]$.

2. $g_x 0 := 0$, $g_x(\alpha+1) = g_x\alpha + 1$, and $g_x\lambda := g_x\lambda[x]$.

3. $H_0^f(x) := x$, $H_{\alpha+1}^f(x) := H_\alpha^f(x+1)$, and $H_\lambda^f(x) := H_{\lambda[f(x)]}^f(x+1)$

4. $h_0^f(x) := x$, $h_{\alpha+1}^f(x) := h_\alpha^f(x+1)$, and $h_\lambda^f(x) := h_{\lambda[f(x)]}^f(x)$

If $f(i) = i$ then we suppress the superscript $f$ when we refer to $g^f, G^f, h^f, H^f$.

**Lemma 2.** *1. Let $F \in \{g, G, h, H\}$. Then $F_\alpha(x) < F_\alpha(x+1)$.*

*2. Let $F \in \{g, h\}$. If $\alpha[x] < \beta < \alpha$ then $F_\alpha(x+1) < F_\beta(x)$.*

*3. Let $F \in \{G, H\}$. If $\alpha[x] < \beta < \alpha$ and $y \ge 1$ then $F_\alpha(y) < F_\beta(y)$.*

*4. $\alpha < \beta \in Lim$ & $\mathrm{mc}(\alpha) \le x \implies \alpha < \beta[x]$.*

*5. Let $F \in \{g, G, h, H\}$. If $\alpha < \beta$ and $\mathrm{mc}(\alpha) \le x$ then $F_\alpha(x) < F_\beta(x)$.*

**Lemma 3.** *1. $\mu i(P_x^i\alpha = 0) = g_x\alpha$ where the upper index denotes iteration.*

*2. $\mu i \ge x(P_i^f P_{i-1}^f \dots P_1^f \alpha = 0) = h_\alpha^f(x)$.*

*3. $\alpha < \beta$ & $\mathrm{mc}(\alpha) \le x+1 \implies \alpha \le \beta[x]$.*

*4. $\#\{\alpha < \beta : \mathrm{mc}(\alpha) \le x+1\} \le G_x\beta$.*

*5. If $\alpha =_{\mathrm{NF}} \omega^{\alpha_1} + \dots + \omega^{\alpha_n}$ then $g_x(\alpha) = (x+1)^{g_x(\alpha_1)} + \dots + (x+1)^{g_x(\alpha_n)}$.*

*6. $g_x\alpha$ is the result of replacing in the Cantor Normal form of $\alpha$ the symbol $\omega$ by $x+1$.*

*Proof.* Assertions 1 and 2 are proved by induction on $\alpha$. Assertion 3 is proved by induction on $\beta$. Assertion 4 is proved by induction on $\alpha$ using assertion 3. Assertion 5 is proved by induction on $\alpha$. Assertion 6 follows from assertion 3. $\square$

Let $|0| := 1$ and for $i \ge 1$ let $|i|$ be the least natural number larger than or equal to $\log_2(i+1)$. Let $|i|_h := i$ if $h = 0$ and $|i|_{h+1} := ||i|_h|$

**Lemma 4.** *Let $h \ge 0$ and $f(i) := |i|_h$ for $i \in \mathbb{N}$. Let $\alpha$ be given and $\beta := \omega_{h+1}(\alpha) + \omega_{h+1}$. Then there exists an $i \ge H_\alpha(1)$ such that $\beta \succ_f^{P,i} \omega_{h+1}(0)$.*

*Proof.* Let $L := H_\alpha(1)$. Then $L = \mu i \geq 1(P_{i-1}P_{i-2}\dots P_1\alpha = 0)$. By definition we have $f(i) \geq 1$ for all $i$. Further we obtain

$$\begin{array}{rcl} \beta & = & \omega_{h+1}(\alpha) + \omega_{h+1} \\ & \succ_1^P & \omega_{h+1}(\alpha) + P_1\omega_{h+1} \\ & \succ_1^P & \omega_{h+1}(\alpha) + P_1P_1\omega_{h+1} \\ & \succ_1^P & \dots \\ & \succ_1^P & \omega_{h+1}(\alpha) + P_1^{2_h(2)}\omega_{h+1} = \omega_{h+1}(\alpha) \end{array}$$

since $\mu i(P_1^i\omega_{h+1} = 0) = g_1\omega_{h+1} = 2_h(2)$.

Hence there exists an $i_0 \geq 2_h(2)$ such that $\beta \succ_f^{P,i_0} \omega_{h+1}(\alpha)$. For $i \geq i_0$ we have $2 \leq f(i)$. Further we obtain

$$\begin{array}{rcl} \omega_{h+1}(\alpha) & \succeq_2^Q & \omega_{h+1}(P_1\alpha + 1) \\ & \succ_2^Q & \omega_{h+1}(P_1\alpha) + \omega_{h+1} \\ & \succ_2^P & \omega_{h+1}(P_1\alpha) + P_2\omega_{h+1} \\ & \succ_2^P & \omega_{h+1}(P_1\alpha) + P_2P_2\omega_{h+1} \\ & \succ_2^P & \dots \\ & \succ_2^P & \omega_{h+1}(\alpha) + P_2^{3_h(3)}\omega_{h+1} = \omega_{h+1}(P_1\alpha) \end{array}$$

since $\mu i(P_2^i\omega_{h+1} = 0) = g_2\omega_{h+1} = 3_h(3)$.

Hence there exists an $i_1 \geq 3_h(3)$ such that $\beta \succ_f^{P,i_0+i_1} \omega_{h+1}(P_1\alpha)$. For $i \geq i_1$ we have $3 \leq f(i)$. By iteration we obtain for any $k < L$ that there exists an $i_k \geq (k+2)_h(k+2)$ such that $\beta \succ_f^{P,i_0+i_1+\dots+i_k} \omega_{h+1}(P_kP_{k-1}\dots P_1\alpha) = \omega_{h+1}(0)$. The assertion follows from $i_0 + i_1 + \dots + i_{L-1} \geq L$. □

The following lemma is inspired by Arai [1].

**Lemma 5.** *Let $f(i) = |i|_{H_{\varepsilon_0}^{-1}(i)}$, $m \geq 2$ and $\alpha := \omega_{m+1}(\omega_m) + \omega_{m+1}$. Then $\alpha \succ_f^{P,k} 0$ for some $k \geq H_{\omega_m}(1)$.*

*Proof.* Let $f_m(i) := |i|_m$. If $i \leq H_{\omega_m}(1) =: l$ then $H_{\varepsilon_0}^{-1}(i) \leq H_{\varepsilon_0}^{-1}(l) \leq m$. Thus $|i|_{H_{\varepsilon_0}^{-1}(i)} \geq f_m(i)$ for $i \leq l$. By Lemma 4 there exists a $\delta > 0$ such that $\alpha \succ_{f_m}^{P,l} \delta$. Hence there exists some $k \geq l$ such that $\alpha \succ_f^{P,k} 0$ since $f(i) \geq f_m(i)$ for $i \leq l$. □

Let $T$ denote a standard primitive recursive Kleene predicate for the enumeration of the partial recursive functions. Let $U$ be the corresponding result function. Within the language of PA the $T$ predicate is $\Sigma_1$. Let $\Phi_e(m) := U(\mu nT(e,m,n)$. Our applications are based on the following classical result over the provably recursive functions of PA

**Theorem 1.** *If $\mathrm{PA} \vdash \forall x \exists y T(e,x,y)$ then there exists an $\alpha < \varepsilon_0$ such that $\Phi_e$ is primitive recursive in and bounded by $H_\alpha$.*

Moreover it is well known that $H_{\varepsilon_0}$ eventually dominates every function $H_\alpha$ for $\alpha < \varepsilon_0$.

**Lemma 6.** *1.* $\mathrm{PA} \nvdash (\forall K)(\exists M) P_M \ldots P_1 \omega_K = 0$.

*2.* $\mathrm{PA} \nvdash (\forall K)(\exists M) Q_M \ldots Q_1 \omega_K = 0$.

*3.* $\mathrm{PA} \nvdash (\forall K)(\exists M)(\forall \alpha_1, \ldots, \alpha_M < \omega_K)[\forall i \leq M \mathrm{mc}(\alpha_i) \leq i \implies (\exists i < M)[\alpha_i \leq \alpha_{i+1}]$.

*Proof.* Assertions 2 follows from assertion 1. Assertion 3 follows from assertion 2. We may thus concentrate on proving assertion 1. Assume that $\mathrm{PA} \vdash (\forall K)(\exists M) P_M \ldots P_1 \omega_K = 0$. Then $\mathrm{PA} \vdash (\forall K)(\exists M) P_M \ldots P_1 \omega_{2 \cdot K+1} = 0$. Hence, by Theorem 1 there exists an $\alpha < \varepsilon_0$ such that for all $K$ there is an $M \leq H_\alpha(K)$ such that $P_M \ldots P_1 \omega_{2\cdot K+1} = 0$. Thus for all $K$ we would have $h_{\omega_{2\cdot K+1}}(1) \leq H_\alpha(K)$. But

$$\begin{array}{lll} \omega_{2\cdot K+1} & \succ_1^P & \omega_{2K} \\ & \succ_2^P & \omega_{2K-1} \\ & \ldots & \\ & \succ_K^P & \omega_{K+1} \\ & \succ_{K+1}^P & \omega_K. \end{array}$$

Hence for all $K$ we would have $H_{\varepsilon_0}(K) \leq h_{\omega_K}(K+1) \leq h_{\omega_{2K+1}}(1) \leq H_\alpha(K)$. Contradiction. □

**Theorem 2 (The phase transition for the hydra game. Part I).** *Let* $f(i) = |i|_{H_{\varepsilon_0}^{-1}(i)}$

*1.* $\mathrm{PA} \nvdash (\forall K)(\exists M) P_M^f P_{M-1}^f \ldots P_1^f \omega_K = 0$.

*2.* $\mathrm{PA} \nvdash (\forall K)(\exists M) Q_M^f Q_{M-1}^f \ldots Q_1^f \omega_K = 0$.

*3.* $\mathrm{PA} \nvdash (\forall K)(\exists M)(\forall \alpha_1, \ldots, \alpha_M < \omega_k)[\forall i \leq M(\mathrm{mc}(\alpha)_i \leq f(i)) \implies \exists i, j(0 \leq i < j \leq M \mathbin{\&} \alpha_i \leq \alpha_j)]$

*Proof.* This follows from Lemma 5 and Lemma 6. □

**Theorem 3 (The phase transition for the hydra game. Part II).** *Let* $f_\alpha(i) := |i|_{H_\alpha{}^{-1}(i)}$. *Assume that* $\alpha < \varepsilon_0$.

*1.* $\mathrm{PA} \vdash (\forall K)(\exists M)(\forall \alpha_1, \ldots, \alpha_M < \omega_k)[\forall i \leq M(\mathrm{mc}(\alpha)_i \leq f_\alpha(i)) \implies \exists i, j(0 \leq i < j \leq M \mathbin{\&} \alpha_i \leq \alpha_j)]$

*2.* $\mathrm{PA} \vdash (\forall K)(\exists M)[Q_M^{f_\alpha} \ldots Q_1^{f_\alpha} \omega_K = 0]$

*3.* $\mathrm{PA} \vdash (\forall m)(\exists i) m_{f_\alpha, i} = 0$.

*Proof.* We only proof the first assertion. The other ones follow immediately. We may assume that $\alpha = \omega 6\beta$ and $\beta \geq 2$. Then

$$H_{\omega^\beta \cdot 2}(i) > 2 \cdot i + 2 \tag{6}$$

for all $i$. Moreover we have for every $i$ that $H_\alpha^{-1}(H_{\alpha\cdot 2}(i)) = H_\alpha(i)$ since $H_\alpha(H_\alpha(i)) = H_{\alpha\cdot 2}(i)$. Let $K$ be given. Let $\tilde{2}_1(i) = 2^i - 1$ and $\tilde{2}_{l+1}(i) := 2^{\tilde{2}_l(i)} - 1$. Then $|\tilde{2}_k(i)|_k = i$ and

$$i_k(1) \leq \tilde{2}_{2k}(i) \tag{7}$$

for every $i \geq 2$ and every $k \geq 1$. Put $M := \tilde{2}_{H_{\alpha\cdot 2}(K)} + H_{\alpha\cdot 2}(K)$. Then $M < \tilde{2}_{H_{\alpha\cdot 2}(K)+1}$. As abbreviation let $h := H_{\alpha\cdot 2}(K)$. Assume for a contradiction that there exists a sequence $\alpha_0, \ldots, \alpha_M < \omega_K$ such that $\mathrm{mc}(\alpha_i) \leq f_\alpha(i)$.

For every $i \in \{h, \ldots, M\}$ we then obtain Assume that $\alpha_0 > \ldots > \alpha_n$ and

$$\begin{aligned} \mathrm{mc}(\alpha_i) &\leq f_\alpha(i) \\ &\leq |M|_{H_\alpha^{-1}(h)} \\ &\leq \tilde{2}_{h+1-H_\alpha(K)}(1) \end{aligned}$$

Hence

$$\begin{aligned} M - h + 1 &= card\{h, M\} \\ &\leq card\{\alpha < \omega_K : \mathrm{mc}(\alpha) \leq \tilde{2}_{h+1-H_\alpha(K)}(1)\} \\ &\leq (\tilde{2}_{h+1-H_\alpha(K)}(1) + 1)_K(1) \\ &\leq \tilde{2}_{2\cdot K}(\tilde{2}_{h+1-H_\alpha(K)}(1) + 1) \\ &\leq (\tilde{2}_{h-H_\alpha(K)+2\cdot K+2}(1) \\ &< \tilde{2}_{H_{\omega^\alpha\cdot 2}(K)} \\ &\leq M - h \end{aligned}$$

Contradiction!

∎

**Theorem 4 (Classifying pointwiseness).** *For every $h \in \mathbb{N}$ let $f_h(i) := |i|_{|i|_h}$. Let $\alpha < \varepsilon_0$. Then $x \mapsto \mu i \geq x(Q_i^f \ldots Q_x^f \alpha = 0)$ is elementary recursive.*

*Proof.* This is proved like theorem 3. ∎

Our treatment of the Goodsteinsequences is in the style of Cichon [2]

**Lemma 7 (Cichon).** $G_x P_x \alpha = P_x G_x \alpha$.

*Proof.* By induction on $\alpha$. ∎

**Definition 5.** Definition of $a[i := \alpha]$ for $\omega \geq \alpha \geq i \geq 2$.

1. If $a < i$ then $a[i := \alpha] = a$.

2. If $a = i^{a_1} \cdot m_1 + \cdots + i^{a_n} \cdot m_n$ where $a_1 > \ldots > a_n$ and $i > m_j > 0$ then $a[i := \alpha] := (i+1)^{a_1[i:=\alpha]} \cdot m_1 + \cdots + (i+1)^{a_n[i:=\alpha]} \cdot m_n$.

Thus $a[i := \alpha]$ is the result of replacing in the complete base $i$ representation of $a$ the base $i$ by $\alpha$. The following is the classical result on Goodstein sequences by Kirby and Paris [3] and its refinment by Hodgson and Kent [4]. It follows immediately from Theorem 6 hence we omit its proof.

**Definition 6 (Goodstein sequences).** $m_{f,0} := m$ and $m_{f,i+1} := m_{f,i}[1 + f(i) := 1 + f(i+1)] - 1$.

**Theorem 5 (Kirby and Paris, Hodgson and Kent).** *1. If $f(i) = i + 1$ then*

$$\mathrm{PA} \nvdash \forall m \exists i m_{f,i} = 0.$$

*2. If $h \in \mathbb{N}$ and $f(i) = |i|_h$ then*

$$\mathrm{PA} \nvdash \forall m \exists i m_{f,i} = 0.$$

**Theorem 6 (The phase transition for Goodstein sequences).** *1. Let $f(i) = |i|_{H_{\varepsilon_0}^{-1}(i)}$. Then*

$$\mathrm{PA} \nvdash \forall m \exists i m_{f,i} = 0.$$

*2. Let $\alpha < \varepsilon_0$ and $f_\alpha(i) = |i|_{H_\alpha^{-1}(i)}$. Then*

$$\mathrm{PA} \vdash \forall m \exists i m_{f,i} = 0.$$

*Proof.* We prove assertion 1. Assertion 2 follows by the same pattern and Theorem 3 Assume otherwise. Let

$$e(m) := 2_{m+1+m} + 2_{m+1}.$$

Then

$$\mathrm{PA} \vdash \forall m \exists i e(m)_{f,i} = 0.$$

Let $m$ be given. Assume that $m \geq 2$. Let $\alpha := \omega_{m+1+m} + \omega_{m+1}$ be the result of replacing 2 in the complete base 2 representation of $e(m)$ by $\omega$ and let $m' := e(m)$. Then

$$m'_{f,0} = m' = G_{1+f(0)}(\alpha).$$

Further

$$m'_{f,1} = P_{1+f(1)} G_{1+f(1)}(\alpha) = G_{1+f(1)} P_{1+f(1)} \alpha.$$

and

$$m'_{f,2} = P_{1+f(2)} G_{1+f(2)}(P_{1+f(1)}\alpha) = G_{1+f(2)} P_{1+f(2)} P_{1+f(1)} \alpha.$$

Thus by iteration we obtain for $i \leq H_{\omega_m}(1)$

$$m'_{f,i+1} = G_{1+f(i+1)} P_{1+f(i+1)} \ldots P_{1+f(1)} \alpha.$$

Therefore, by lemma 4

$$\mu i(m'_{f,i+1} = 0) = \mu i(P_{1+f(i)} \ldots P_{1+f(1)} \alpha = 0) \geq H_{\omega_m}(1).$$

Since $m \mapsto H_{\omega_m}(1)$ is not provably recursive in PA we obtain a contradiction. $\square$

There are versions of the results obtained so far which apply to the fragments $I\Sigma_h$ of PA in which the induction schema is restricted to $\Sigma_h$ formulas.

**Lemma 8.** $\omega_h(k) > \alpha \implies (\omega_h(k) \cdot \alpha)[x] = \omega_h(k) \cdot \alpha[x]$

Using this lemma one can show using the techniques from above the following theorem.

**Theorem 7.** *Let* $f^h_\alpha(i) := \sqrt[H_\alpha^{-1}(i)]{|i|_{h-1}}$.

1. *If* $\alpha = \omega_{h+1}$ *then* $I\Sigma_h \nvdash \forall K \exists M Q_M^{f^h_\alpha} \dots Q_1^{f^h_\alpha} \omega_h(K) = 0$.
2. *If* $\alpha < \omega_{h+1}$ *then* $I\Sigma_h \vdash \forall K \exists M Q_M^{f^h_\alpha} \dots Q_1^{f^h_\alpha} \omega_h(K) = 0$.

## References


[1] T. Arai: On the slowly well orderedness of $\varepsilon_0$, Mathematical Logic Quarterly 48 (2002), 125-130.

[2] Eugeniusz Adam Cichon. *A short proof of two recently discovered independence results using recursion theoretic methods.* Proceedings of the AMS 87 (1983) 704-706.

[3] Laurie Kirby and Jeff Paris: *Accessible independence results for Peano Arithmetic.* Bull. London Math. Soc. 14 (1982), 285-293.

[4] Clement F. Kent and Bernard R. Hodgson: *Extensions of arithmetic for proving termination of computations.* The Journal of Symbolic Logic 54 (1989), pages 779–794.

[5] Andreas Weiermann: *What makes a (point-wise) subrecursive hierarchy slow growing?* In *Sets and proofs (Leeds, 1997)*, pages 403–423. Cambridge Univ. Press, Cambridge, 1999.

[6] Andreas Weiermann: An application of graphical enumeration to PA. The Journal of Symbolic Logic 68 (2003), 5-16.

[7] A. Weiermann. A classification of rapidly growing Ramsey functions. *Proc. Am. Math. Soc.*, 132(2):553–561, 2004.