Analytic combinatorics, proof-theoretic ordinals, and phase transitions for independence results ⋆

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Abstract

This paper is intended to give for a general mathematical audience (including non logicians) a survey about intriguing connections between analytic combinatorics and logic. We define the ordinals below ε_0 in non-logical terms and we survey a selection of recent results about the analytic combinatorics of these ordinals. Using a versatile and flexible (logarithmic) compression technique we give applications to phase transitions for independence results, Hilbert's basis theorem, local number theory, Ramsey theory, Hydra games, Goodstein sequences. We discuss briefly universality and renormalization issues in this context. Finally, we indicate how regularity properties of ordinal count functions can be used to prove logical limit laws.

Key words: Analytic combinatorics, proof-theoretic ordinals, ordinal analysis, independence results for systems of arithmetic, Tauberian theory, logical limit laws

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1 Introduction

It is a great honour for the author to contribute to Wolfram Pohlers 60-th birthday volume. The author learned the basics in ordinal analysis from Pohlers's lectures and seminars starting in Münster from 1986 onwards. These lectures and seminars started *medias in res* with the ordinal analysis of PA [56] and impredicative theories like ID_1 [54] and ID_ν [55]. These culminated finally in seminars on Jäger's and Pohlers's treatment of KPi [36].

The subject emerged quite quickly and led to substantial advance as e.g. documented by Rathjen's achievements on the ordinal analysis of KPM, $KP + (\Pi_3 - Ref)$ and $\Pi^1_2 - (CA)$ [58–61]. Independently Arai obtained similar achievements in the ordinal analysis of strong systems [3,4]. Methods from ordinal analysis turned out to be versatile and found applications in classifications of provably recursive functions [12,76], strong normalization proofs [75] and bounded arithmetic [8].

The author followed with great interest Pohlers's research over the years. In the beginning of the nineties Pohlers explored so called alternative interpretations. Here the idea is to replace in the development of strong ordinal notation systems the regular cardinals by their recursive or proof-theoretic analogues. An initial part of that research is documented in [56].

Moreover he worked on extending Beckmann's boundedness theorem to a more general setting [9].

In the last years he worked on simplifying and streamlining the local predicativity approach to proof theory. Over the years he was also very interested in the interfaces between advanced set theory (core model theory, fine structure theory) and the proof theory of strong subsystems of set theory. Here further achievements will be of great general interest.

Since Pohlers's 1986 lectures the author was fascinated by the internal beauty of the theory of ordinal notations [53] and their applications e.g. to well partial ordering theory [62,67], subrecursive hierarchies [77], term rewriting [78], provably recursive functions, strong normalization proofs (for fragments of Gödel's system T)[75], and combinatorial independence results [76]. In working on these applications he profited a lot from the know-how of the Schütte school, which was made available to him mainly by Pohlers (and Buchholz).

In this paper we take the opportunity to survey some recent progress on analytic combinatorics of ordinals, thereby indicating surprising connections between analytic number theory, the Peano axioms and ordinal notation systems for ε_0.

In general, analytic combinatorics aims at predicting precisely the asymptotic properties of structured combinatorial configurations and at quantifying effectively metric properties of large random structures. Accordingly, it is susceptible to many applications, within combinatorics itself, but also in statistical physics, computational biology, electric engineering and the analysis of algorithms.

The theory of notation systems for transfinite ordinals is devoted to developing systems of finitary term notations for infinite ordinals. These systems are used for classifying the provability strengths of axioms systems for the natural numbers, and in particular for scaling the provable instances of schemes for transfinite induction and hierarchies of provably recursive functions. Moreover, this theory finds applications in proving termination for algorithms and classifying the lengths of computations.

This paper is about bridging the gap between these different areas thereby showing on the one hand, that Cauchy's integral formula, generating functions and Tauberian theorems can be used to obtain intrinsic insights into classical ordinal notation systems.

On the other hand the problems arising in analytic combinatorics from ordinals and independence issues are well motivated and they provide challenges of varying degree of difficulty to be solved using analytic methods. The author expects here a long term synergy between two different communities.

This paper aims as a survey for a general readership and thus we include an attractive description of ε_0 in terms of Hardy's orders of infinity and in terms of Matula numbers, which until now found their main applications in chemistry. We hope that even experts in ordinal notations may obtain some additional insights into these (comparably small) ordinals.

A continuous impetus in the study of ordinal notations systems is provided by Feferman's natural well ordering challenge. The underlying problem has a conceptual nature and is not formulated as a concise mathematical problem. There is a common agreement that the ordinal notation system for the ordinals below ε_0 (after a suitable coding into the positive integers) constitutes a natural well-ordering. However, Kreisel devised a definition of a primitive recursive well-ordering of order type ω (using built in consistency assertions) such that PA does not prove the transfinite induction along ω. So the question is what distinguishes natural well-orderings for ε_0 from such "monsters". The feeling is that what distinguishes such orderings are certain intrinsic mathematical properties that are independent of their possible use in proof-theoretic work.

The author proposes a bottom up approach without claiming that final solutions will be ever obtained. By studying existing sets of notations one may single out a collection of intrinsic properties of ordinal notation systems. This might be used to provide an intriguing catalogue of properties for natural well orderings. Among these properties might be:

(1) Natural well orderings come with a short definition. (This is in contrast to Kreisel's example which provides a short ordering with a long definition.)
(2) They have intriguing algebraic properties (functoriality, end-extendibility, relativization,. . .) [20,26,37,32].
(3) Their well-foundedness can be proved constructively.
(4) They can be obtained from the analysis of finite proof figures (Takeuti's and Arai's approach) [5].
(5) They can be obtained from provability logic considerations (Beklemishev's approach) [10].
(6) They come with logical limit laws [79,82].
(7) They come with independence results [71].
(8) They come with a robust hierarchy of fast growing functions [7].
(9) They come with intriguing slow growing hierarchies [77].
(10) They have a smooth enumeration, i.e. their additive generating functions have coefficients which are in the Compton class RT_α (see, for example, [13] for a definition) and their multiplicative Dirichlet generating functions have coefficients which are regularly varying (as defined in [13]).
(11) Their statistical parameters of ordinal segments satisfy a Gaussian law. For example, for segments formed by additive principal ordinals exceeding ω^{ω^2} the number of summands of a random ordinal shall satisfy a central and local limit theorem.
(12) Their contour and profile processes are related to the Brownian excursion or similar stochastic processes [18,27].
(13) They are maximal linear extensions of natural well partial orders [67]. (They produce maximal possible closure ordinals [66,73].)
(14) They induce reduction orders stable under substitution and application of function symbols.

This list is not meant to be complete but it may form a basis of a general research program on proof-theoretic ordinals which may lead to some insight on Feferman's problem.

To sum up, in this article we aim at showing that ordinal analysis, which is Wolfram Pohlers's main area of research, is a vivid and very active area in research and that it has intriguing interrelations with cutting-edge mathematics.

2 Some attractive descriptions of ε_0

The ordinals below ε_0 are well established in logic but less so in other parts of mathematics. There are attractive descriptions of ε_0 without referring to ordinals at all, the only price to pay is that the well-foundedness proof does not come for free.

Let $\mathbb{E}$ be the least set of unary number-theoretic functions such that the constant zero function $x \mapsto 0$ is in $\mathbb{E}$ and such that with $\alpha, \beta \in \mathbb{E}$ the function $x \mapsto x^{\alpha(x)} + \beta(x)$ is in $\mathbb{E}$. For α, β define $\alpha \prec \beta$ iff there is an n_0 such that $\alpha(n) < \beta(n)$ for all $n \geq n_0$. Then $\langle \mathbb{E}, \prec \rangle$ is isomorphic to $\langle \{\alpha : \alpha < \varepsilon_0\}, < \rangle$ and we may identify each ordinal with its corresponding function and even with the (normal form) term denoting the function. Moreover we write $<$ instead of $\prec$ when there is no danger of confusion. In particular we have that $\langle \{\alpha : \alpha < \varepsilon_0\}, < \rangle$ is a linear order. The ordinal exponentiation function with respect to base ω is then $\alpha \mapsto (x \mapsto x^{\alpha(x)})$ and we may thus define ω^α as the function $x \mapsto x^{\alpha(x)}$. Moreover the natural sum of α and β is defined by $(\alpha \# \beta)(x) := \alpha(x) + \beta(x)$. Every not constant zero function $\alpha \in \mathbb{E}$ can be written uniquely as $\omega^{\alpha_1} \# \cdots \# \omega^{\alpha_n}$ with $\alpha_1 \geq \ldots \geq \alpha_n$. In this case we write $\alpha =_{NF} \omega^{\alpha_1} \# \cdots \# \omega^{\alpha_n}$ or even shorter $\alpha =_{NF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ Alternatively every not constant zero function $\alpha \in \mathbb{E}$ can be written uniquely as $\omega^{\alpha_1} \cdot m_1 \# \cdots \# \omega^{\alpha_n} \cdot m_n$ with $\alpha_1 > \ldots > \alpha_n$ and $m_1, \ldots, m_n > 0$. Here $\alpha \cdot m$ is defined as $x \mapsto \alpha(x) \cdot m$. In this case we write $\alpha =_{CNF} \omega^{\alpha_1} m_1 + \cdots + \omega^{\alpha_n} m_n$.

The crucial property of $\mathbb{E}$ is the following. For every function $F : \mathbb{N} \to \mathbb{E}$ there exists an n such that $F(n) \preceq F(n+1)$. This can be proved, for example, by an appeal to ordinals, to König's Lemma, or to Kruskal's theorem about the well-quasi orderedness of the embeddability relation on finite trees. (See, for example, [70] for more details.)

To each α in $\mathbb{E}$ which is not of the form 0 or $\beta + 1$ (α is then called a limit) we may associate a canonical sequence $\alpha[x]$ so that α is the supremum of the $\alpha[x]$. If α_n is a limit and if $\alpha =_{NF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ then let $\alpha[x] =_{NF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n[x]}$. If $\alpha =_{NF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n + 1}$ then let $\alpha[x] := \omega^{\alpha_1} + \cdots + \omega^{\alpha_n} \cdot x$.

Using recursion one can define the so called slow growing hierarchy on $\mathbb{E}$ as follows:

$$
\begin{aligned}
G_0(n) &:= 0, \\
G_{\alpha+1}(n) &:= G_\alpha(n) + 1, \\
G_\lambda(n) &:= G_{\lambda[n]}(n).
\end{aligned}
$$

Then G_α induces a canonical direct limit presentation of $\alpha \in \mathbb{E}$ in the sense of Girard [26] via the identity $G_\alpha = \alpha$.

An intriguing and attractive description of ε_0 can be obtained via so called Matula numbers [46] or equivalently by a Schütte-style coding [68]. Let $\mathbb{P}$ denote the prime numbers and let $(p_i)_{i\geq 1}$ be an increasing enumeration of $\mathbb{P}$. Let $ind(p_i) := i$ and let $|$ denote the divisibility relation on the integers. Further let (m,n) denote the greatest common divisor of m and n. Define by recursion (where p,q range over $\mathbb{P}$)

$$m \prec n :\iff (m \neq n) \ \& \ [m = 1 \vee n = 0 \vee (\forall p | \frac{m}{(m,n)} \exists q | \frac{n}{(m,n)}) ind(p) \prec ind(q)].$$

Then $\langle \{\mathbb{N} \setminus \{0\}, \prec \rangle$ is isomorphic to $\langle \{\alpha : \alpha < \varepsilon_0\}, < \rangle$. The corresponding isomorphism ord is given by $\mathrm{ord}(1) := 0$ and

$$\mathrm{ord}(p_{m_1} \cdot \ldots \cdot p_{m_k}) := \omega^{\mathrm{ord}(m_1)} \# \cdots \# \omega^{\mathrm{ord}(m_k)}.$$

This is well-defined due to the unique factorization theorem for positive integers into prime number products. Note that this isomorphism sends multiplicative indecomposable numbers into additive indecomposable ordinals. (An extension of this coding has been used by Schütte in [68] to code the ordinals below Γ_0 by positive integers.)

There are several natural norm functions on $\mathbb{E}$ so that the number of elements in $\mathbb{E}$ of bounded norm is always finite. Our favourite choice is

$$N0 := 0 \text{ and } N\alpha = n + N\alpha_1 + \cdots + N\alpha_n$$

if $\alpha =_{NF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$. This function is best motivated by associating finite rooted non-planar trees to elements in ε_0. To 0 associate the tree $t(0)$ consisting of one node. If $\alpha =_{CNF} \omega^{\alpha_1} m_1 + \cdots + \omega^{\alpha_n} m_n$ then we may assume that trees $t(\alpha_1), \ldots, t(\alpha_n)$ are associated to $\alpha_1, \ldots, \alpha_n$. Then $t(\alpha)$ is the tree built up from a new root node which is connected to the root nodes of $t(\alpha_1), \ldots, t(\alpha_n)$. Now the number of nodes is a canonical complexity measure for a given finite tree and $N(\alpha)$ is nothing but the number of nodes in $t(\alpha)$ minus one.

Another norm function is provided by the inverse function ord^{-1} (which we will denote by S in the sequel) of ord. This function has been used by Schütte in [68] for coding ordinals by natural numbers. S is intriguing since it is multiplicative in the sense of analytic number theory. The norm function N is additive in the sense of number theory when considered as defined on the Matula numbers since $N(S\alpha \cdot S\beta) = N(S(\alpha)) + N(S(\beta))$. Using these properties one easily sees that $\mathbb{E}$ and each segment of $\mathbb{E}$ determined by an ordinal of the form ω^α can be considered as an additive (with respect to N) or multiplicative (with respect to S) number system in the sense of Knopfmacher [41]. This gives interest in the study of the following count functions

$$c_\beta(n) := c_\beta^N(n) := \#\{\alpha < \beta : N\alpha \leq n\},$$

and

$$c^S_\beta(n) := \#\{\alpha < \beta : S(\alpha) \leq n\},$$

In the sequel we will survey how information on the asymptotic of these functions yield applications in logic.

3 Some intriguing results about the ordinals not exceeding ω^ω

We start with some remarkable results about the ordinals below ω^ω. These ordinals can be identified with the polynomials in $\mathbb{N}[X]$ ordered by eventual domination. For $\beta \leq \omega^\omega$ let the local count function for β be defined as follows.

$$c^{=}_\beta(n) := \#\{\alpha < \beta : N\alpha = n\}.$$

The goal is to find an expression for c_β or at least an asymptotic expression. In case of $\beta = \omega^\omega$ this turns out to be more complicated then expected at first sight. For real numbers a_n, b_n let $a_n \sim b_n$ be an abbreviation for $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. Hardy and Ramanujan proved in [31] that

$$c^{=}_{\omega^\omega}(n) \sim \frac{1}{4\sqrt{3}n} \exp(\pi\sqrt{\frac{2}{3}n}). \tag{1}$$

In fact they proved a corresponding result for integer partitions. Let $p(n)$ be the number of partitions of n, i.e. the number of sequences $\langle i_1, \ldots, i_k\rangle$ such that $i_1 + \cdots + i_k = n$ and $i_1 \geq \ldots \geq i_k \geq 1$. Then an integer partition $\langle i_1, \ldots, i_k\rangle$ of n corresponds exactly to an ordinal $\omega^{i_1-1} + \cdots + \omega^{i_k-1}$ of norm n and vice versa, hence $p(n) = c^{=}_{\omega^\omega}(n)$.

To obtain their result Hardy and Ramanujan considered the corresponding generating function $P(z) := \sum_{i=0}^\infty p(n)z^n$. Then for complex numbers z with $\mid z \mid< 1$ one obtains $P(z) = \prod_{i=1}^\infty \frac{1}{1-z^i}$. Now they evaluated $p(n) = \frac{1}{2\pi i} \oint \frac{P(\zeta)}{\zeta^{n+1}} d\zeta$ (as a result of Cauchy's integral formula where integration is over a suitable circle around the origin) with the use of the circle method and applying results from the theory of modular functions. In [57] Rademacher was able to obtain an exact expression for $c^{=}_{\omega^\omega}(n)$ but the term is too complicated to be included here. Still it seems very intriguing to see how Cauchy's integral formula shows up in the context of ordinals.

For the sake of uniformity it turns out to be convenient to work with the global count function c_β from the previous section.

For each $k < \omega$ we have that

$$c_{\omega^k}(n) \sim \frac{1}{k!k!} n^k \tag{2}$$

by an old result credited in [13] to Schur. An exact formula for $c_{\omega^k}(n)$ is contained in [1] but it is too complicated to be included here.

Since

$$c_{\omega^\alpha+\beta}(n) = \begin{cases} c_{\omega^\alpha}(n) + c_\beta(n - N\omega^\alpha) & \text{if } n \geq N \\ c_{\omega^\alpha}(n) & \text{if } n < N \end{cases}$$

we obtain a complete picture of these count functions.

A main application of these asymptotic results is proving independence results for fragments of arithmetic.

Recently Friedman challenged to continue this line of research as a high point of proof theory. Friedman states [22]: "This [i.e. independence results] is an obvious high point of proof theory that provides clear applications that everyone in logic and many people outside logic appreciate. It is obvious to me that this will be a major topic in the coming century."

In this paper we follow Friedman's theme and calibrate the strength of various independence results.

With PA we denote the Peano axioms stating the usual properties of 0, successor, addition and multiplication. Moreover PA contains a scheme of complete induction for all formulas in the language of arithmetic.

Let $\mathrm{I}\Sigma_k$ be the subset of the axioms of of PA where the induction schema is restricted to Σ_k formulas, i.e. formulas beginning with an alternating sequence of k quantifiers where the first is existential and with a kernel consisting of a formula where each quantifier is bounded. Let $\omega_0(\alpha) := \alpha$ and $\omega_{n+1}(\alpha) := \omega^{\omega_n(\alpha)}$. Moreover let $\omega_k := \omega_k(1)$. Then the proof-theoretic ordinal of PA is ε_0 and the prooftheoretic ordinal of $\mathrm{I}\Sigma_k$ is ω_{k+1}. These ordinals are the smallest upper bounds for the order types of provable well orderings of the theories in question.

The well orderedness of $\alpha \in \mathbb{E}$ can be expressed by the second order assertion $WO(\alpha, f)$ which stands for

$$(\forall F : \omega \to \alpha)(\exists n)[F(n) \leq F(n+1)].$$

For this assertion there exists a canonical miniaturization in the following way, when an underlying norm function is assumed. (The norm function has to satisfy that the number of α of norm bounded by a fixed natural number is always finite.) Thus given a norm function no and an arbitrary number-theoretic function f let $\mathrm{SWO}(\alpha, no, f)$, the slowly well orderedness of α with respect to no and f, be the assertion

$$(\forall K)(\exists M)(\forall \alpha_1, \ldots, \alpha_M < \alpha)[\forall i \leq M(no(\alpha_i) \leq K + f(i)) \to \exists i < n(\alpha_i \leq \alpha_{i+1})].$$

Then for any f the assertion $\mathrm{SWO}(\alpha, no, f)$ is true by König's Lemma since $WO(\alpha, f)$ is true. Indeed, assume that the assertion is false. Then consider the set of all strictly decreasing sequences $\langle \alpha_1, \ldots, \alpha_M \rangle$ such that $no(\alpha_i) \leq K + f(i))$ for $i \leq M$. This is then an infinite (by assumption) but (due to the norm property) finitely branching tree. But every path would produce an infinite strictly descending chain of ordinals, contradicting real life.

According to [71] we have the following general independence result by Friedman. (Recall that $N0 := 0$ and $N(\alpha) := n + N(\alpha_1) + \cdots + N(\alpha_n)$ if $\alpha =_{NF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$.)

Theorem 1 *Let T be* PA *or one of its fragments* $\mathrm{I}\Sigma_n$. *Let $\alpha \leq \varepsilon_0$ be the proof-theoretic ordinal of T. Let f be the identity function. Then $T \nvdash \mathrm{SWO}(\alpha, N, f)$*

It is quite natural to ask for phase transition classification from provable to unprovable instances of $\mathrm{SWO}(\alpha, no, f)$ depending on variations of f (for various fixed norms no). Let A denote the Ackermann function defined as usual by

$$\begin{aligned} A(0, n) &:= n + 1, \\ A(m + 1, 0) &:= A(m, 1), \\ A(m + 1, n + 1) &:= A(m, A(m + 1, n)). \end{aligned}$$

Then for each m the function $n \mapsto A(m, n)$ is primitive recursive but the function $m, n \mapsto A(m, n)$ is not. With A^{-1} we denote the inverse function of $d \mapsto A(d, d)$ and with A_k we denote the inverse function of the k-th branch of A, thus

$$\begin{aligned} A^{-1}(n) &:= \min\{d : n < A(d, d)\}, \\ A_k^{-1}(n) &:= \min\{d : n < A(k, d)\}. \end{aligned}$$

Moreover let (the threshold functions be defined by)

$$a(n) := \sqrt[A^{-1}(n)]{n}, \tag{3}$$
$$a_k(n) := \sqrt[A_k^{-1}(n)]{n}. \tag{4}$$

Then from [76] and [5] one obtains the following refinement of Theorem 1.

Theorem 2 *Let g be a number-theoretic function whose graph is Σ_1-definable in* $\mathrm{I}\Sigma_1$.

(1) If $g(n) \geq a(n)$ for all but finitely many n then $\mathrm{I}\Sigma_1 \nvdash \mathrm{SWO}(\omega^\omega, N, g)$.
(2) If $\mathrm{I}\Sigma \vdash (\exists x_0)(\forall x \geq x_0)[g(x) \leq a_k(x)]$ *for some k then* $\mathrm{I}\Sigma_1 \vdash \mathrm{SWO}(\omega^\omega, N, g)$.

The basic idea behind the proof of the independence result of Theorem 2 is to obtain from a (with respect to id) long slowly descending sequence of ordinals α_i a more slowly descending sequence of the form

$$\omega^l \cdot \alpha_{|i|} + c(i) \tag{5}$$

for some appropriate l where $c(i)$ enumerates the ordinals below ω^l which have norms bounded by the threshold function. That enough of these ordinals are available is guaranteed by the underlying analytic combinatorics. The provability part is carried out by a straightfoward counting argument. These two arguments are quite powerful and general and can be applied in various situations where it is possible to split descending ordinal sequences into two rather independent processes. In this paper we consider various variations on this theme.

Another classical paper by Hardy and Ramanujan [30] deals with a multiplicative coding of ω^ω. Recall that p_i denotes the i-th prime where $p_1 := 2$. Let $HR(0) := 1$ and $HR(\alpha) := p_1^{m_1} \cdot \ldots \cdot p_n^{m_n}$ if $\omega^\omega > \alpha =_{NF} \omega^{m_1} + \cdots + \omega^{m_n}$. For $\beta \leq \omega^\omega$ let

$$c_\beta^{HR}(n) := \#\{\alpha < \beta : hr(\alpha) \leq n\}.$$

Then, according to Hardy and Ramanujan, we obtain

$$\log(c_{\omega^\omega}^{HR}(n)) \sim \pi \frac{2}{\sqrt{3}} \sqrt{\frac{\log(n)}{\log(\log(n))}}. \tag{6}$$

Note the analogy with the classical Hardy Ramunujan result $\log(c_{\omega^\omega}(n)) \sim \pi\sqrt{\frac{2}{3}n}$ which can be proved much simpler than (1). The multiplicative asymptotic results roughly from the additive asymptotic by replacing n through $\frac{\log(n)}{\log(\log(n))}$. We will see in the sequel that this phenomenon shows up more often and we will indicate that in several examples a uniform treatment can be provided by using Tauberian theorems for Laplace transforms.

The bound provided by (6) is a weak asymptotic result since it provides only information on $\log(c_{\omega^\omega}^{HR}(n))$. To the author's best knowledge the following impressive result of Richmond [63] is the best known currently available refinement.

$$\begin{aligned}
\log(c_{\omega^\omega}^{HR}(n)) = {} & (2\pi/\sqrt{3})(\log x/\log\log x)^{1/2} \\
& \times \{1 - (2\log\pi + 12B_1/\pi^2 - 2)/(2\log\log x) \\
& - (\log 3 - \log\log\log x)/(2\log\log x) \\
& + O((\log\log\log x)/(\log\log x))^2\},
\end{aligned}$$

where $B_1 = -\int_0^\infty \log(1-e^{-y})\log y\, dy$.

In case of $\beta < \omega^\omega$ a strong asymptotic can be obtained for $c_\beta^{HR}(n)$ by Karamata's Tauberian theorem [80]. Indeed, for any k we obtain

$$c_{\omega^k}^{HR}(n) \sim \frac{1}{(k!)^2}\Big(\frac{\log(n)}{\log(\log(n))}\Big)^k. \tag{7}$$

Note the analogy between (7) and $c_{\omega^k}(n) \sim \frac{1}{(k!)^2} n^k$, the transfer from the additive to the multiplicative setting is again provided by substituting n through $\frac{\log(n)}{\log(\log(n))}$.

This suggest some deeper correspondence between these results and perhaps one uniform proof. Such a proof is indeed possible by using Karamata's Tauberian theorem once applied to power series and once applied to Dirichlet series. (This technique is explained in detail in [30].)

By adapting the proof of Theorem 2 we arrive at the following result.

Theorem 3 *Let g be a number-theoretic function whose graph is Σ_1 definable in* $\mathrm{I}\Sigma_1$.

(1) If $g(n) \geq 2^{a(n)}$ for all but finitely many n then $\mathrm{I}\Sigma_1 \nvdash \mathrm{SWO}(\omega^\omega, HR, g)$.
(2) If $\mathrm{I}\Sigma \vdash (\exists x_0)(\forall x \geq x_0)[g(x) \leq 2^{a_k(x)}]$ *for some k then* $\mathrm{I}\Sigma_1 \vdash \mathrm{SWO}(\omega^\omega, HR, g)$.

Note that the phase transition result for $\mathrm{SWO}(\omega^\omega)$ with respect to a multiplicative norm is obtained by replacing the threshold functions a, a_k from Theorem 2 through $2^a, 2^{a_k}$. Of course the exponential function provides a natural homomorphism between addition and multiplication. But now we will show that the phase transition for $\mathrm{SWO}(\omega^\omega)$ shares features of a more general universality phenomenon. It turns out that different multiplicative norms lead to the same phase transitions for $\mathrm{SWO}(\omega^\omega)$. The specific choice of the norm effects the microscopic asymptotic for the count functions but not the macroscopic behaviour of the resulting independence result. (There are some vague similarities to universality phenomena in statistical physics where phase transition also obey universality laws. There the phase transition depends also only on few parameters like dimension and symmetries but for example not on the speciffic matter under consideration.)

Let us thus consider the Schütte or Matula coding S from the previous section. Recall that $S(0) = 1$ and $S(\alpha) = p_{S(\alpha_1)} \cdot \ldots \cdot p_{S(\alpha_n)}$ if $\omega^\omega > \alpha =_{NF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$. Moreover recall that

$$c_\beta^S(n) = \#\{\alpha < \beta : S(\alpha) \leq n\}.$$

The exponential Tauberian theorem of Hardy and Ramanujan [30] and some additional calculations yield [79]

$$\log(c^S_{\omega^\omega}(n)) \sim \pi\sqrt{\frac{2}{3\log(2)}}\sqrt{\log(n)}. \tag{8}$$

Again the similarity between (8) and $\log(c_{\omega^\omega}(n)) \sim \pi\sqrt{\frac{2}{3}}\sqrt{n}$ is striking. Moreover both results can be obtained from the exponential Tauberian theorem of Hardy and Ramanujan for Laplace transfoms, once applied to power series and once applied to Dirichlet series. (The factor $\log(2)$ appears since the primes start with 2.) Moreover Karamata's theorem yields

$$c^S_{\omega^k}(n) \sim \frac{1}{k!(\log(p_{2^0}) \cdot \ldots \cdot \log(p_{2^k})}(\log(n))^k. \tag{9}$$

The result follows also directly from Corollary 2.49 of Burris's book [13]. Again note the similarity of (9) with (2).

Theorem 4 *Let g be a number-theoretic function whose graph is Σ_1 definable in $\mathrm{I}\Sigma_1$.*

(1) If $g(n) \geq 2^{a(n)}$ for all but finitely many n then $\mathrm{I}\Sigma_1 \nvdash \mathrm{SWO}(S, \omega^\omega, g)$.
(2) If $\mathrm{I}\Sigma \vdash (\exists x_0)(\forall x \geq x_0)[g(x) \leq 2^{a_k(x)}]$ for some k then $\mathrm{I}\Sigma_1 \vdash \mathrm{SWO}(S, \omega^\omega, g)$.

Finally we would like introduce some further well investigated additive norms on ω^ω. Let $P0 := 0$ and $P\alpha := p_{\alpha_1} + \ldots + p_{\alpha_n}$ if $\omega^\omega > \alpha =_{NF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$. Let

$$c^P_\beta(n)) := \#\{\alpha < \beta : P\alpha \leq n\}.$$

Then [30] yields the following result (concerning partitions into primes)

$$\log(c^P_{\omega^\omega}(n)) \sim \pi\frac{2}{\sqrt{3}}\sqrt{\frac{n}{\log n}}. \tag{10}$$

Again, note that (10) results roughly speaking from (1) through replacing n by $\frac{n}{\log n}$ (which is roughly the number of primes below n).

One might conjecture that all additive norms have a common feature with (1) but that is not the case. A natural exception is provided by the classical Mahler partitions. Let us define the Mahler norm as follows.

$$\begin{aligned} M(0) &:= 0, \\ M(\alpha) &:= 2^{M(\alpha_1)} + \ldots + 2^{M(\alpha_n)} \end{aligned}$$

if $\omega^\omega > \alpha =_{NF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$. Let $c^M_\beta(n)) := \#\{\alpha < \beta : M\alpha \leq n\}$ Then $\log(c^M_{\omega^\omega}(n)) \sim \frac{\log^2(n)}{2\log 2}$. (In this case even a good strong asymptotic is available [15].)

As before Karamata's theorem yields strong asymptotics for the count functions with index ω^k. Surprisingly we obtain the same phase transitions for $\text{SWO}(\omega^\omega, M, f)$ as in Theorem 2.

To sum up: In all cases where the underlying norm is additive we obtained the same phase transition as in Theorem (2). In all situations where the underlying norm is multiplicative we obtained the same phase transition as in Theorem (3). Moreover in all cases the underlying analytic combinatorics can be treated via Karamata's Tauberian theorem which therefore seems to be intrinsically related to the Ackermann function.

Since Theorems 2 and 3 do not depend on the specific fine structure of the underlying norms we would like to say that these theorems share features of a general universality.

We close this section by an application to Hilbert's basis theorem. It is well known that the ordinal complexity of Hilbert's theorem is measured by ω^ω [69]. This reflects recursion theoretically in the classification of the lengths of effectively given ascending chains of ideals in terms of the Ackermann function. A corresponding result has been credited (through folklore) to Harvey Friedman. By a direct calculation Moreno Socias arrived independently at comparable results [47]. We found Socia's exposition very useful for our computations since it provides very explicit bounds. We give now a refinement of these results.

Let A be Ackermann's function and let $A_d(n) = A(d, m)$. Let the corresponding inverse functions a and a_k be defined as in (3) and (4).

Theorem 5 *Let K be a field and $n \in \mathbb{N}$. Let $R := K[X_1, \ldots, X_{n+n+1}]$. Then for $d < n$ and $M := A(n-1, d) - 1$ there is a sequence $(m_i)_{i+1}^M$ of polynomials in R with $m_i \notin (m_1, \ldots, m_{i-1})$ and $deg(m_i) \leq \sqrt[a(i)]{i}$ for $1 \leq i \leq M$.*

PROOF. We give a rough outline of the argument. The basic theme is a modification of [76] with an additional ingredient of [5]. By elementary arguments there is roughly a chain (with respect to the antilexicgraphic ordering taken from Moreno Socias paper [47]) of length n^{d+1} of different normed monomials in $\{q \in K[X_1, \ldots, X_n] : deg(q) \leq d\}$

According to a result by Moreno Socias [47] we choose a sequence $(p_i)_{i=1}^M$ of polynomials in $K[X_{n+1}, \ldots, X_{n+n}]$ such that $deg(p_i) \leq d + i$ and $p_i \notin (p_1, \ldots, p_{i-1})$ for $1 \leq i \leq M$. Then for $1 \leq i \leq M$ we have $a(n) \leq A(\cdot, d)^{-1}(i) + 1 \leq n$. Let $M_i := \{q \in K[X_1, \ldots, X_n] : q$ normed monomial and $deg(q) \leq \sqrt[a(i)]{2^{|i|}}\}$. Then $\#M_i \geq \sqrt[n]{2^{|i|}} \geq 2^{|i|-1}$ for i large enough. Choose a constant C large enough for the following. For small i let $m_i := X_{2n+1}^{d+C-i}$. For i large enough put $m_i := p_{|i|} \cdot enum_{M_i}(2^{|i|-1} - i)$ where M_i is ordered with the antilexicographic ordering $\prec$ and $enum_{M_i}(k)$ is the k-th member of M_i with respect to $\prec$. Then $m_i \notin (m_1, \ldots, m_{i-1})$ for $1 \leq i \leq M$. Moreover $deg(m_i) \leq C + d + |i| + \sqrt[n]{2^{|i|}} \leq d + |i| + \sqrt[a(i)]{2^{|i|}}$ for $1 \leq i \leq M$. $\square$

Theorem 6 *Let K be a field and let $n \in \mathbb{N}$. Let $R := K[X_1, \ldots, X_n]$. Choose $d \in \mathbb{N}$. Assume that a sequence of polynomials $(m_i)_{i+1}^{M}$ satisfies $deg(m_i) \leq d + \sqrt[a_k(i)]{i}$ and $m_i \notin (m_1, \ldots, m_{i-1})$ for $1 \leq i \leq M$. Then $M \leq A(k+1, n+d) \cdot 2$.*

PROOF. Let $N := 2 \cdot A(k+1, n+d)$. We claim that $M \leq N$. Assume otherwise. We have $deg(m_i) \leq d + \sqrt[A_k^{-1}(\frac{N}{2})]{N}$ for $\frac{N}{2} \leq i \leq N$. Then $\frac{N}{2} < (d + n^{\sqrt[a_k(\frac{N}{2})]{N}}) \leq \frac{N}{2}$. Contradiction. $\square$

4 Applications to local number theory

In this section we establish (to our best knowledge for the first time) a connection between independence results for $I\Sigma_1$ and local results in (prime) number theory. We conjecture that this will be a starting point for a plethora of further interrelations between logic and number theory and we hope that these investigations will improve the understanding of central unknown conjectures in pure mathematics. The underlying theme is to translate local asymptotical results for certain classes of natural numbers in short intervals to global results about large intervals for natural numbers.

Our investigations are further prompted in part by remarks of Harvey Friedman posted to the newsgroup foundations of mathematics [22]:

"Anand [Pillay] frequently says that it is important for logicians to incorporate mathematics outside logic in their work - and particularly cutting edge mathematics, and not just classical mathematics.

I certainly agree with part of this remark. But whether classical mathematics or cutting edge mathematics is appropriate depends on what you are trying to accomplish.

If one is trying to make direct applications and connections, then what Anand says makes some real sense. However, if one is trying to join the issue of new axioms for mathematics, then one should - in fact one must (in my opinion) start at the most fundamental level possible and build things up from there. Preferably build things up starting at the high school level. This is what I am doing. Later, with the benefit of experience, one can perhaps profitably venture into more modern contexts."

We think that this section provides a contribution to these challenges. The new idea is to use famous unproven hypotheses to prove Π^0_2-independence results e.g. for $\mathrm{I}\Sigma_1$. This will provide a definition of a number-theoretic function, say g, which has Ackermannian growth. If for some unexpected reason it is possible to show that g can be bounded by a primitive recursive function, then the underlying hypothesis would be falsified.

Let us explain briefly the underlying methodology which can be used for providing several results of such a type. We start with the standard miniaturization of Higman's lemma for $0-1$-words, which is known to be independent of $\mathrm{I}\Sigma_1$. Using compression, the $0-1$-words occuring in long bad sequences can be chosen of short length. Now we replace $0-1$-words by squarefree numbers, since their local density is affected by the ABC-conjecture. In the resulting independence result for squarefree numbers we demand that the i-th squarefree number has to be contained in an appropriately short interval I_i of length depending on i. If these intervals are very short then the existence of squarefree numbers in these intervals requires, for example, a non trivial application of the ABC-conjecture. If we further code squarefree numbers s by the s-th prime p_s, the existence of such primes in related short intervals requires a non trivial application of the Riemann hypothesis.

Using this methodology is is in principle possible to transfer all sorts of number-theoretic hypothesis into independence results. Presently we are not able to claim that this approach is useful to get some breakthrough but we hope that this connection to pure mathematics is intriguing.

In a first step we carry out the methodology in detail for the ABC conjecture which is famous due to its consequences on theorems by Baker, Roth, Bombieri, Wiles and Faltings. (See, for example, [29] for further details.) In a second step we treat the Riemann hypothesis [64] which is problem number 8 in Hilbert's 1900 list of open problems and which is one of the Problems of the Millenium [14]. We start with some basic definitions. For $n \in \mathbb{N}$ let $|n|$ be the binary length of n, i.e. the least integer k such that $\log_2(n+1) \leq k$. Let Z_i be the set of finite sequences σ over $\{0,1,2\}$ such that σ contains at most i occurrences of 2. For $\sigma = \langle s_1, \ldots, s_m \rangle, \tau = \langle t_1, \ldots, t_n \rangle \in Z_1$ we write $\sigma \trianglelefteq \tau$ iff there are indices $i_1, \ldots, i_m$ such that $i_1 < \ldots < i_m$ and $s_l \leq t_{i_l}$ for $1 \leq l \leq m$. Then $\trianglelefteq$ is a well partial ordering, i.e. for every map $f : \mathbb{N} \to Z_1$ exist natural numbers i, j such that $f(i) \trianglelefteq f(j)$. For $\sigma = \langle s_1, \ldots, s_m \rangle$ let $N\sigma = s_1 + 1 + \cdots + s_m + 1$. By weakening, Theorem 2 yields the following result.

Theorem 7 $\mathrm{I}\Sigma_1 \nvdash (\forall K)(\exists M)(\forall \alpha_1, \ldots, \alpha_M < \omega^\omega)\Big[(\forall i \leq M)(N\alpha_i \leq K + \sqrt{i}) \to (\exists j, k)(1 \leq j < k \leq M \ \& \ \alpha_j \leq \alpha_k)\Big]$.

Via the translation $f = \alpha^k \cdot m_k + \cdots + \alpha^0 \cdot m_0 \mapsto 0^{m_k} 1 0^{m_{k-1}} \ldots 1 0^{m_0} \in Z_0$ one easily proves the following result.

Theorem 8 $\mathrm{I}\Sigma_1 \nvdash (\forall K)(\exists M)(\forall \sigma_1, \ldots, \sigma_M \in Z_0)\Big[(\forall i \leq M)(N\sigma_i \leq K + \sqrt{i}) \rightarrow (\exists j,k)(1 \leq j < k \leq M \;\&\; \sigma_i \trianglelefteq \sigma_j)\Big]$.

By standard analytic combinatorics there exists a constant $c > 1$ such that $\#\{\sigma \in Z_0 : |\sigma| \leq k\} \geq c^k$ for all but finitely many k. Thus using the logarithmic compression technique [i.e. transforming bad sequences into slowed down bad sequencess following the advice provided by (5)] one obtains the following lemma.

Lemma 9 $\mathrm{I}\Sigma_1 \nvdash (\forall K)(\exists M)(\forall \sigma_1, \ldots, \sigma_M \in Z_1)\Big[(\forall i \leq M)(N\sigma_i \leq K + 2|i|) \rightarrow (\exists j,k)(1 \leq j < k \leq M \;\&\; \sigma_i \trianglelefteq \sigma_j)\Big]$.

Recall that $\mathbb{P}$ denotes the set of prime numbers and that p_i denotes the i-th prime. In the sequel p ranges over $\mathbb{P}$. For $x \in \mathbb{N}^+$ let $\nu_p(x) = n$ if $x = p^n s$ for some s such that p does not divide s. ($\nu_p(x)$ is the usual p-adic valuation of x.) Let $Q_2 = \{x \in \mathbb{N}^+ : \nu_p(x) \leq 1\}$ be the set of squarefree numbers and $Q_2^\star = \{x \in \mathbb{N}^+ : (\exists p \in \mathbb{P})(\exists y \in Q_2) x = p \cdot y\}$. Put $s \mid^\star t$ iff there is a strictly monotonic function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $\nu_{p_i}(s) \leq \nu_{p_{f(i)}}(t)$. Note that if f is the identity function in this definition then s divides t. For $s = \prod_{j=1}^n p_i^{\nu_i}$ with $\nu_m > 0$ and $t = \prod_{j=1}^n p_j^{\mu_j}$ $\mu_n > 0$ let $s \star t := \prod_{i=1}^m p_i^{\nu_i} \cdot \prod_{j=1}^n p_{m+j}^{\mu_j}$ be the concatenation of s and t.

Theorem 10 *Let $a = 25$ and $b = 5$.*

(1) $\mathrm{I}\Sigma_1 \nvdash (\forall K)(\exists M)(\forall s_1, \ldots, s_M \in Q_2^\star)(\forall t_1, \ldots t_M \in Q_2)$
$\Big[(\forall i \leq M)(\{t_i, s_i + t_i\} \subseteq [0, K] \cup [2^{a|i|}, 2^{a|i|} + 2^{b|i|} + K]) \rightarrow$
$(\exists j,k)(1 \leq j < k \leq M \;\&\; s_j \star 4 \star t_j \mid^\star s_k \star 4 \star t_k)\Big]$.

(2) Assume that the ABC conjecture is true.
Then $\mathrm{I}\Sigma_1 \nvdash (\forall K)(\exists M)(\forall s_1, \ldots, s_M \in Q_2^\star)(\forall t_1, \ldots t_M \in Q_2)$
$\Big[(\forall i \leq M)(\{t_i, s_i + t_i\} \subseteq [0, K] \cup [2^{a|i|}, 2^{a|i|} + 2^{|i|} + K]) \rightarrow$
$(\exists j,k)(1 \leq j < k \leq M \;\&\; s_j \star 4 \star t_j \mid^\star s_k \star 4 \star t_k)\Big]$.

PROOF. We show the second assertion. The proof of the first is almost identical. Let $\delta := \frac{1}{100}$. Choose $\epsilon := \frac{1}{6}$. Then according to Granville [28] we obtain from the ABC conjecture that

$$[x, x + x^\epsilon] \cap Q_2$$

contains at least $x^{\frac{\epsilon}{2} - \frac{1}{200}}$ elements for large enough x. (For the proof of the first assertion one may employ the bound

$$[x, x + x^{\frac{1}{5}} \cdot \log(x)] \cap Q_2 \neq \emptyset$$

which is proved in [21] without assuming any unproved conjecture.) Let

$$M_i = Q_2 \cap [2^{a|i|}, 2^{a|i|} + 2^{|i|}(1-\delta)]$$

and for $j < \#M_i$ let $\mathrm{enum}_{M_i}(j)$ be the j-th element in M_i.

There is a constant D which depends on ϵ and δ only such that $\#M_i \geq 2^{|i|-1}$ for $i \geq D$. Moreover we may assume that for $i \geq D$ we have $2^{(K+||i||)^2} \leq \delta \cdot 2^{|i|} + 2^{K^4}$ for all K.

Assume now that K is given. According to Lemma 9 choose $\sigma_1, \ldots, \sigma_{M-1}$ in Z_1 such that the sequence is bad with respect to $\trianglelefteq$, i.e. there are no i, j such that $i < j < M$ and $\sigma_i \trianglelefteq \sigma_j$, such that $N\sigma_i \leq K + 2|i|$ for $1 \leq i \leq M-1$, and that M, when chosen minimal possible, then as a function depending on argument K eventually dominates every provably recursive function of $\mathrm{I}\Sigma_1$. Moreover assume that the last entry in all σ_i is not 0, and that 2 occurs exactly once in every σ_i.

Assume that $\sigma_i = \langle s_{i1}, \ldots, s_{ik_i}\rangle$. For $1 \leq i \leq D$ put $s_i := p_1^{s_{11}} \cdot \ldots \cdot p_{k_1}^{s_{1k_1}}$ and $t_i := p_1 \cdot \ldots \cdot p_{D+1-i}$. Then $s_i + t_i \leq 2^{D^2} + 2^{(K+1)^4}$ for $i \leq D$. For $i > D$ put $s_i := p_1^{s_{|i|1}} \cdot \ldots \cdot p_{k_{|i|}}^{s_{|i|k_{|i|}}}$ and $t_i := \mathrm{enum}_{M_i}(2^{|i|} - i - 2^{|i|-1})$. This is possible since $\#M_i \geq 2^{|i|-1}$ and $2^{|i|} - i \geq 2^{|i|-1}$. Then $s_i \leq 2^{(K+||i||)^2} \leq \delta \cdot 2^{|i|} + 2^{K^4}$ and $t_i \in [2^{a|i|}, 2^{a|i|} + (1-\delta)2^{|i|}]$ thus $s_i + t_i \in [2^{a|i|}, 2^{a|i|} + 2^{|i|} + 2^{K^4}]$ for $i \geq D$. Moreover we see that $s_i \star 4 \star t_i \,|^\star\, s_j \star 4 \star t_j$ does not hold for $i < j$. For otherwise assume $s_i \star 4 \star t_i \,|^\star\, s_j \star 4 \star t_j$. Then $s_i \,|^\star\, s_j$ and $t_i \,|^\star\, t_j$. If $|i| < |j|$ then this conflicts with σ_i being bad and if $|i| = |j|$ then $t_i > t_j$ hence not $t_i \,|^\star\, t_j$. $\square$

The proof of the second assertion of Theorem 10 clearly does not exploit the full strength of Granville's result. However, we do not see whether it is possible to prove the second assertion of Theorem 1 without any yet unproved consequence of the ABC conjecture.

Theorem 11 *Let* $a = \frac{24}{13}, b = \frac{15}{13}, c = \frac{25}{24}$.

(1) $\mathrm{I}\Sigma_1 \nvdash (\forall K)(\exists M)(\forall s_1, \ldots, s_M \in Q_2^\star)(\forall t_1, \ldots t_M \in Q_2)$
$\Big[(\forall i \leq M)(\{p_{t_i}, s_i + p_{t_i}\} \subseteq [0, K] \cup [2^{a|i|}, 2^{a|i|} + 2^{b|i|} + K]) \rightarrow$
$(\exists j, k)(1 \leq j < k \leq M \;\&\; s_j \star 4 \star t_j \,|^\star\, s_k \star 4 \star t_k)\Big]$.

(2) Assume that the Riemann hypothesis is true.
Then $\mathrm{I}\Sigma_1 \nvdash (\forall K)(\exists M)(\forall s_1, \ldots, s_M \in Q_2^\star)(\forall t_1, \ldots t_M \in Q_2)$
$\Big[(\forall i \leq M)(\{p_{t_i}, s_i + p_{t_i}\} \subseteq [0, K] \cup [2^{a|i|}, 2^{a|i|} + 2^{c|i|} + K]) \rightarrow$
$(\exists j, k)(1 \leq j < k \leq M \;\&\; s_j \star 4 \star t_j \,|^\star\, s_k \star 4 \star t_k)\Big]$.

PROOF. We show the second assertion. The proof of the first is almost identical.

Let $Q_2(x) := \#\{y \in Q_2 : y \le x\}$. From [65] we know that

$$\mid Q_2(x) - \frac{6}{\pi^2}x \mid \quad \le O(x^{\frac{1}{2}} \exp(C2^{-\frac{8}{5}} \log(x)^{\frac{3}{5}} \log(\log(x))^{-\frac{1}{5}}))$$

for large x and a certain constant C not depending on x. Thus for any $\epsilon > \frac{1}{2}$ the interval

$$[x, x + x^{\epsilon}] \cap Q_2$$

contains at least $\frac{1}{2}x^{\epsilon}$ elements for large enough x. According to [17] we obtain from the Riemann hypothesis that

$$\#([x, x + x^{\frac{1}{2}} \log(x)] \cap \mathbb{P}) \sim x^{\frac{1}{2}}.$$

(For the proof of the first assertion one may instead employ the bound

$$\#[x, x + x^{\frac{7}{12}} \cdot \log(x)] \cap \mathbb{P} \sim x^{\frac{7}{12}}$$

which is proved in [33] without assuming any unproved conjecture.) Choose $\delta > 0$ so small that $c(1 - 2\delta) > 1$. Then there is a constant D depending only on δ such that

$$\#([2^{a|i|}, 2^{a|i|} + 2^{c|i|}(1 - \delta)] \cap \mathbb{P}) \ge 2^{c(1-\delta)|i|} + 1.$$

$$\#\{p \in \mathbb{P} : p \le 2^{a|i|}\} \ge 2^{a(1-\delta)|i|}$$
$$\#([r, r + 2^{c(1-\delta)|i|}] \cap Q_2) \ge 2^{c(1-2\delta)|i|} \tag{11}$$

for $i \ge D$ and $r \ge 2^{a(1-\delta)|i|}$. Moreover we may assume that for $i \ge D$ we have $2^{(K+||i||)^2} \le \delta \cdot 2^{|i|} + 2^{K^4}$.

Assume that K is given. According to Lemma 1 choose $\sigma_1, \ldots, \sigma_{M-1}$ in Z_1 such that the sequence is bad with respect to $\trianglelefteq$, such that $N\sigma_i \le K + 2|i|$ for $1 \le i \le M - 1$, and that M, when chosen minimal possible, then as a function of argument K eventually dominates every provably recursive function of $\mathrm{I}\Sigma_1$. Moreover assume that the last entry in all σ_i is not 0, that 2 occurs exactly once in every σ_i.

Assume that $\sigma = \langle s_{i1}, \ldots, s_{ik_i} \rangle$. For $1 \le i \le D$ put $s_i := p_1^{s_{11}} \cdot p_{k_1}^{s_{1k_1}}$ and $t_i := p_1 \cdot \ldots \cdot p_{D+1-i}$. Then $s_i + t_i \le 2^{D^2} + 2^{(K+1)^2}$ for $i \le D$.

Assume now that $i > D$. Choose r minimal such that $p_r \ge 2^{a|i|}$. Then $r - 1 \ge 2^{a(1-\delta)|i|}$. We then have

$$p_{r+j} \in [2^{a|i|}, 2^{a|i|} + 2^{c|i|}]$$

by (11) for $j \in [0, 2^{c(1-\delta)|i|}]$. Moreover

$$\#([r, r + 2^{c(1-\delta)|i|}] \cap Q_2) \ge 2^{c(1-2\delta)|i|} \ge 2^{|i|}.$$

Let $M_i := \{p_t : t \in [r, r + 2^{b(1-\delta)|i|}] \cap Q_2\}$. Let $t_i := \text{enum}_{M_i}(2^{|i|} - i)$. This is possible since $\#M_i \geq 2^{|i|-1}$ and $2^{|i|} - i \geq 2^{|i|-1}$. As before let $s_i := p_1^{s_{|i|1}} \cdot \ldots \cdot p_{k_{|i|}}^{s_{|i|k_{|i|}}}$ for $i > D$.

Then $s_i \leq 2^{(K+||i||)^2} \leq \delta \cdot 2^{|i|} + 2^{K^4}$ and $p_{t_i} \in [2^{a|i|}, 2^{a|i|} + (1-\delta)2^{c|i|}]$ thus $s_i + p_{t_i} \in [2^{a|i|}, 2^{a|i|} + 2^{c|i|} + 2^{K^4}]$ for $i \geq D$. Moreover we see that $s_i \star 4 \star t_i \mid^\star s_j \star 4 \star t_j$ does not hold for $i < j$. For otherwise assume $s_i \star 4 \star t_i \mid^\star s_j \star 4 \star t_j$. Then $s_i \mid^\star s_j$ and $t_i \mid^\star t_j$. If $|i| < |j|$ then this conflicts with σ_i being bad and if $|i| = |j|$ then $t_i > t_j$ hence not $t_i \mid^\star t_j$. Contradiction. $\square$

Remarks: 1. A further fine tuning of Theorem 1 and Theorem 2 is clearly possible. For expository reasons we decided to work with bounds which work smoothly.
2. We assume that there will be further applications of the ABC conjecture, the Riemann hypothesis and further related hypotheses (like the Cramer conjecture on the distribution of primes in short intervals) to independence results.

5 Enumeration theory for the ordinal segment $]\omega^\omega, \omega^{\omega^\omega}]$

We consider the corresponding count functions for the ordinals below ω^{ω^ω}. Since ω^{ω^ω} is the proof-theoretic ordinal of $\text{I}\Sigma_2$ we obtain corresponding independence results for $\text{I}\Sigma_2$. It turns out that most independence results for $\text{I}\Sigma_2$ can be treated uniformly with Kohlbecker's Tauberian theorem [42].

However, for Mahler type codings and prime number exponential codings Parameswaran's Tauberian theorem [50] seems more appropriate and the resulting phase transition results are different from the ones which rely on Kohlbecker's theorem. Thus the universality of the phase transition for ω^{ω^ω} is not as strong as in the ω^ω case.

Recall that $N\alpha$ is the number of occurrences of ω in α and that $c_\beta(n) := \#\{\alpha < \beta : N\alpha \leq n\}$. Kohlbecker's Tauberian theorem applied to (2) yields

$$\log(c_{\omega^{\omega^k}}(n)) \sim C \cdot n^{\frac{k}{k+1}} \tag{12}$$

for a certain explicitly calculable constant C. Let $a_n = \Theta(b_n)$ denote that there exist constants $0 < r, s$ such that $s \cdot b_n < a_n < r \cdot b_n$ for all n. By some elementary calculations we showed in [76]

$$\log(c_{\omega^{\omega^\omega}}(n)) = \Theta(\frac{n}{\log(n)}).$$

A more detailed calculation yields Yamashita's result [83,25]

$$\log(c_{\omega^{\omega^\omega}}(n)) \sim \zeta(2)\frac{n}{\log(n)}. \tag{13}$$

Now we are going to define the scales for measuring the phase transitions for $\mathrm{I}\Sigma_2$. For $\alpha < \varepsilon_0$ let let H_α be defined as follows:

$$\begin{aligned} H_0(n) &:= n, \\ H_{\beta+1}(n) &:= H_\beta(n+1), \\ H_\lambda(n) &:= H_{\lambda[n]}(n). \end{aligned}$$

(Refer to, e.g. [7] for more details concerning these functions.) Let $|n|$ be the binary length of n. Let $B(n) := H^{-1}_{\omega^{\omega^\omega}}(n)$ and let $B_k(n) := H^{-1}_{\omega^{\omega^k}}(n)$. The relevant scale functions are as follows:

$$b(n) := \sqrt[B(n)]{|n|}, \tag{14}$$
$$b_k(n) := \sqrt[B_k(n)]{|n|}. \tag{15}$$

By applying the logarithmic compression technique [confer (5)] and (12) we obtain from Theorem 1 the following result in the additive situation.

Theorem 12 *Let g be a number-theoretic function whose graph is Σ_1 definable in $\mathrm{I}\Sigma_2$.*

(1) If $g(n) \geq b(n)$ for all but finitely many n then $\mathrm{I}\Sigma_2 \nvdash \mathrm{SWO}(\omega^{\omega^\omega}, N, g)$.
(2) If $\mathrm{I}\Sigma \vdash (\exists x_0)(\forall x \geq x_0) g(x) \leq b_k(x)$ for some k then $\mathrm{I}\Sigma_2 \vdash \mathrm{SWO}(\omega^{\omega^\omega}, N, g)$.

Now we consider the multiplicative situation. Thus let us recall that the Schütte Matula coding has defined as follows:

$$S(0) := 1 \text{ and } S(\alpha) := p_{S(\alpha_1)} \cdot \ldots \cdot p_{S(\alpha_n)}$$

if $\alpha =_{NF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$.

Again Kohlbecker's theorem applied to (7) yields

$$\log(c^S_{\omega^{\omega^k}}(n)) \sim C \cdot (\log(n))^{\frac{k}{k+1}}$$

for a certain explicitly caculable constant C.

Moreover in case of ω^{ω^ω} using elementary calculation with Dirichlet series we arrived at

$$\log(c^S_{\omega^{\omega^\omega}}(n)) = \Theta(\frac{\log(n)}{\log\log(n)}).$$

Korevaar in his book on Tauberian theory [40] remarked (referring to these counting problems) that appropriate Tauberian theory for dealing with iterated exponentials is provided by results due to Geluk, de Haan and Stadtmüller [25].

Based on this advice we therefore expect that by applying such results one should be able to prove the following analogue of Yamashita's result (13), just by replacing n through $\log(n)$ and taking a correction factor $\log(2)$ into account:

$$\log(c^S_{\omega^{\omega^\omega}}(n)) \sim \frac{\zeta(2)}{\log(2)}\frac{\log(n)}{\log\log(n)}.$$

Following our philosophy one might expect that the phase transition in the multiplicative situation results from the additive situation by applying the exponential function to the threshold functions. Indeed, let b and b_k be our scales as defined in (14) and (15).

Theorem 13 *Let g be a number-theoretic function whose graph is Σ_1 definable in $\mathrm{I}\Sigma_2$.*

(1) If $g(n) \geq 2^{b(n)}$ for all but finitely many n then $\mathrm{I}\Sigma_2 \nvdash \mathrm{SWO}(\omega^{\omega^\omega}, S, g)$.
(2) If $\mathrm{I}\Sigma \vdash (\exists x_0)(\forall x \geq x_0) g(x) \leq b_k(x)$ for some k then $\mathrm{I}\Sigma_2 \vdash \mathrm{SWO}(\omega^{\omega^\omega}, S, g)$.

6 Enumeration theory for the ordinals in $]\omega_d, \omega_{d+1}]$

For the rest of this section fix $d \geq 3$. With elementary calculations similar to those in [76] one verifies rather easily that

$$\log(c_{\omega_d(k)}(n)) = \Theta(\frac{n}{\sqrt[k]{\log_{d-2}(n)}}). \tag{16}$$

This is already sufficient for our intended applications on phase transitions. Nevertheless, as the author recently learned, one can sharpen the result as follows. (See, e.g. Petrogradsky [52] for a proof.)

$$\log(c_{\omega_d(k)}(n)) \sim C \cdot \frac{n}{\sqrt[k]{\log_{d-2}(n)}}$$

for a certain explicitly calculable constant C (for which a formula is provided in [52]).

Let $B^d(n) := H^{-1}_{\omega_{d+1}}(n)$ and $B^d_k(n) := H^{-1}_{\omega_d(k)}(n)$. The relevant scale functions in this section are defined as follows:

$$c^d(n) := \sqrt[B^d(n)]{|n|_{d-1}},$$
$$c_k^d(n) := \sqrt[B_k(n)]{|n|_{d-1}}.$$

Using (16) and by applying the logarithmic compression technique to the results of Theorem 1 we obtain the following phase transition result.

Theorem 14 *Let g be a number-theoretic function whose graph is Σ_1 definable in $\mathrm{I}\Sigma_d$.*

(1) If $g(n) \geq c^d(n)$ for almost all n then $\mathrm{I}\Sigma_d \nvdash \mathrm{SWO}(N, \omega_{d+1}, g)$.
(2) If for some k we have $\mathrm{I}\Sigma \vdash (\exists x_0)(\forall x \geq x_0) g(x) \leq c_k^d(x)$ then $\mathrm{I}\Sigma_d \vdash \mathrm{SWO}(N, \omega_{d+1}, g)$.

By considering the Schütte Matula norm S we obtain a corresponding phase transition in the multiplicative setting.

With elementary calculations for Dirichlet series we verified that

$$\log(c^S_{\omega_d(k)}(n)) = \Theta(\frac{\log(n)}{\sqrt[k]{\log_{d-2}(\log(n)}})).$$

Moreover we conjecture that by applying results from Geluk de Haan and Stadtmüller [25] one should obtain following asymptotic equation:

$$\log(c^S_{\omega_d(k)}(n)) \sim C \cdot \frac{\log(n)}{\sqrt[k]{\log_{d-2}(\log(n)}})$$

for a certain explicitly calculable constant C. As by now expected the phase transition for $\mathrm{I}\Sigma_d$ in the multiplicative setting results again from the additive setting by applying the exponential function. The result is as follows.

Theorem 15 *Let g be a number-theoretic function whose graph is Σ_1 definable in $\mathrm{I}\Sigma_d$.*

(1) If $g(n) \geq 2^{c^d(n)}$ for all but finitely many n then $\mathrm{I}\Sigma_d \nvdash \mathrm{SWO}(\omega_{d+1}, S, g)$.
(2) If $\mathrm{I}\Sigma_d \vdash (\exists x_0)(\forall x \geq x_0)[g(x) \leq 2^{c_k^d(x)}]$ for some k then $\mathrm{I}\Sigma_d \vdash \mathrm{SWO}(\omega_{d+1}, S, g)$.

The corresponding phase transitions for the Mahler norm and the exponential coding norm are more involved. Let us just mention that the phase transitions differ from the ones found in Theorems 14 and 15. For example, the commonly used exponential coding, which can be seen as a canonical extension of the Hardy Ramanujan coding for ω^ω to all ordinals below ε_0 can be defined by

$$H0 := 1 \text{ and } H\alpha := p_1^{H\alpha_1} \cdot \ldots \cdot p_n^{H\alpha_n}$$

if $\alpha =_{NF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$. Then, for $\beta \leq \omega^\omega$ let

$$c^H_\beta(n) := \#\{\alpha < \beta : H\alpha \leq n\}.$$

By elementary calculations and applying Parameswaran's Tauberian theorem from [50] we obtained:

$$\log_d(c^M_{\omega_{k+1}(d)}(n)) = \Theta(\log_{k+1}(n)\Big(\frac{\log_{k+1}(n)}{\log_{k+2}(n)}\Big)^d).$$

We intend to cover the resulting phase transitions in a separate paper.

7 Analytic combinatorics of ε_0 and finite trees

Using Otter's result [49] (– a short exposition of this result can be found e.g. in exercise 4 on p. 395 in Knuth's textbook [39] –) we have for a suitable constant C

$$c_{\varepsilon_0}(n) \sim C\frac{\alpha^n}{n^{\frac{3}{2}}}$$

where α is Otter's tree constant. This follows from the standard identification of rooted trees with ordinals below ε_0.

Therefore this section contains just a survey about recent results concerning the embeddability relation on the set of finite trees. Recall (e.g. from Knuth's textbook [39] section 2.3 p. 305) that a finite rooted tree T (with outdegree bounded by a natural number l) is a nonvoid set of nodes such that there is one distinguished node, root(T), called the root of T and the remaining nodes are partitioned into $m \geq 0$ ($l \geq m \geq 0$) disjoint sets $T_1, \ldots, T_m$, and each of these sets is a finite rooted tree (with outdegree bounded by l). The trees $T_1, \ldots, T_m$ are called the immediate subtrees of T. The cardinality of T is denoted by $|T|$. We say that a finite rooted tree S is embeddable into a finite rooted tree T, $S \trianglelefteq T$, if either S is embeddable into an immediate subtree of T or if there exist listings $(S^1_i)_{i\leq m}$, $(T_j)_{j\leq n}$ of the (distinct) immediate subtrees of S and T and natural numbers $j_1 < \ldots < j_m \leq n$ such that S_k is embeddable into T_{j_k} for $1 \leq k \leq m$. Then $\trianglelefteq$ is transitive and $S \trianglelefteq T$ yields $|S| \leq |T|$. (Note that this definition of $\trianglelefteq$ is equivalent to the notion of homeomorphic embedding used in [70].)

Kruskal's famous tree theorem is as follows.

Theorem 16 *For any ω-sequence $(T_i)_{i<\omega}$ of finite rooted trees there exist natural numbers i and j such that $i < j$ and $T_i \trianglelefteq T_j$.*

Using König's Lemma one easily proves the following lemma.

Lemma 17 *Let f be a number-theoretic function. For any K there is an N such that for all sequences $(T_i)_{i \le N}$ of finite rooted trees with $|T_i| \le K + f(i)$ for $1 \le i \le N$ there exist natural numbers i and j such that $1 \le i < j \le N$ and $T_i \trianglelefteq T_j$.*

Assume that the set of finite rooted trees is coded as usual primitive recursively into the set of natural numbers. For a binary function f let $\mathrm{SWQ}(f)$, the slowly well quasi orderedness of the finite trees with respect to f, be the following statement (formula) about finite rooted trees:

$$\forall K \exists M \forall T_1, \ldots, T_M \Big((\forall i \le M)|T_i| \le K + f(i) \implies \exists i,j[i < j \,\&\, T_i \trianglelefteq T_j] \Big).$$

Then Friedman's celebrated miniaturization result is as follows.

Theorem 18 (cf. [69,71]) *Let $f(i) := i$. Then $\mathrm{PA} \nvdash \mathrm{SWQ}(f)$. (In fact we even have $\mathrm{ATR}_0 \nvdash \mathrm{SWQ}(f)$.)*

This result has later been sharpended considerably by Loebl and Matoušek as follows.

Theorem 19 (cf. [45]) *Let $f(i) := 4 \cdot \log(i)$. Then $\mathrm{PA} \nvdash \mathrm{SWQ}(f)$.*

This result is rather sharp since Loebl and Matoušek obtained the following lower bound.

Theorem 20 (cf. [45]) *Let $f(i) := \frac{1}{2} \cdot \log(i)$. Then $\mathrm{PRA} \vdash \mathrm{SWQ}(f)$.*

Using the compression method we obtained the following refinement in [76].

Theorem 21 *Let $c := \frac{\log(2)}{\log(\alpha)}$ where α is Otter's tree constant. Let r be a primitive recursive real number and let f_r be defined by $f_r(i) := r \cdot \log(i)$. Assume that f is Σ_1-definable in* PA

(1) If $r > c$ and $f(i) \ge f_r(i)$ for all but finitely many i then $\mathrm{PA} \nvdash \mathrm{SWQ}(f)$. (In fact we have $\mathrm{ACA}_0 + (\Pi^1_2 - BI) \nvdash \mathrm{SWQ}(f)$. [62])
(2) If $r \le c$ $\mathrm{PRA} \vdash (\exists n_0)(\forall n \ge n_0)[f(n) \le f_r(n)]$ then $\mathrm{PRA} \vdash \mathrm{SWQ}(f)$.

Using the saddle point method for large exponents [24] (the author acknowledges here gratefully valuable help by Flajolet) we have been able to extend this result to an unprovability result for a upper and lower bounds for threshold function in terms of $f_c^{c_i}(i) := c \cdot \log(i) + c_i \log(\log(i))$ for some real numbers $c_1 > c_2 > 0$. A suitable choice for c_2 is $c \cdot \frac{3}{2}$, since then the $\log\log$ term and the constant c_2 reflect the $\sqrt{n^3}$ term in the tree count function. A precise threshold determination, i.e. closing the gap between c_1 and c_2 is still open.

It seems to be of interest whether one can "compute" a Poincare series in iterated logarithms for the threshold function classifying the miniaturization of Kruskal's theorem. We believe that this problem is very difficult.

Very similar as Otter's constant is related to the threshold function for Kruskal's theorem the real number 5.6465442616239497... is related to ATR_0 when one considers Γ_0 as represented by + and the binary Veblen function. (For this work the author acknowledges valuable help by H. Prodinger.) Later it turned out that such a result follows also in part from [43]. The calculation of some characteristic real numbers for stronger theories has been carried out by G. Lee in his PhD thesis [44].

8 Applications to Ramsey theory

There are two famous independence results for PA which are related to Ramsey theory: the Paris Harrington theorem and the Kanamori McAloon theorem. It turns out that these come along with intriguing phase transition phenomena which are related to ordinals.

Let us first consider the Paris Harrington theorem. To state it concisely we need some notation. For a given set X of natural numbers let $[X]^d$ be the set of sequences $\langle x_1, \ldots, x_n\rangle$ such that $x_1, \ldots, x_n \in X$ and $x_1 < \ldots < x_n$. Thus $[X]^d$ is (modulo some reinterpretation) the set of d-element subsets of X. Moreover let us agree that we consider a natural number as the set of its predecessors. Thus a positive integer c is also a typical c-element set. For a given positive integer d and a number-theoretic function f let $\mathrm{PH}^d(f)$ be the assertion

$(\forall c, m)(\exists R)(\forall F : [R]^d \to c)(\exists Y \subseteq R)$
$[f \restriction [Y]^d = \textit{constant}\ \&\ \textit{card}(Y) \geq \max\{m, f(\min(Y))\}]$.

Let $\mathrm{PH}(f)$ be the assertion $(\forall d)PH^{d+1}(f)$. According to Paris Harrington we have the following theorem.

Theorem 22 *Let id denote the identity function and* const *denote a constant function.*

(1) $\mathrm{PA} \nvdash \mathrm{PH}(f)$.
(2) $\mathrm{I}\Sigma_d \nvdash \mathrm{PH}^{d+1}(id)$.
(3) $\mathrm{I}\Sigma_1 \vdash (\forall d)\mathrm{PH}^{d+1}(\mathrm{const})$.

It is quite natural to ask for what functions f the assertion $\mathrm{PH}(f)$ is unprovable in PA. The Erdös Rado theorem yields that $\mathrm{PH}(f)$ is provable in PA if f is the inverse of the superexponential function. In [81] we showed that $\mathrm{PH}(f)$

becomes unprovable in PA if f grows a little faster than the inverse of the superexponential function.

For a proof we introduced a specific master coloring of ordinals which we define in the sequel. If $\alpha =_{CNF} \omega^{\alpha_1} \cdot n_1 + \cdots + \omega^{\alpha_t} \cdot n_t$ we write $S_i(\alpha) := \omega^{\alpha_i} \cdot n_i$, $E_i(\alpha) := \alpha_i$, $K_i(\alpha) := n_i$ for $i \leq t$. For $i > t$ we put $S_i(\alpha) := E_i(\alpha) := K_i(\alpha) := 0$. Let $\mathrm{mc}(0) := 0$, and $\mathrm{mc}(\alpha) := \max\{n_1, \ldots, n_t, \mathrm{mc}(\alpha_1), \ldots, \mathrm{mc}(\alpha_t)\}$ and put $p(\alpha) := \mathrm{mc}(\alpha) + h(\alpha)$.

To define the critical coloring, which has only relatively small homogeneous sets, we use an appropriate coloring of ordinals.

For $\alpha > \beta > \gamma$ set $d(\alpha, \beta) := \min\{i : S_i(\alpha) > S_i(\beta)\}$, $K(\alpha, \beta) := K_{d(\alpha,\beta)}(\alpha)$, $E(\alpha, \beta) = E_{d(\alpha,\beta)}(\alpha)$.

Definition 23 *If $\beta_n > \beta_1 > \beta_2$ then*
$\chi(\beta_n, \beta_1, \beta_2) := \nearrow$ *if* $d(\beta_1, \beta_2) < d(\beta_n, \beta_1)$,
$\chi(\beta_n, \beta_1, \beta_2) := \uparrow$ *if* $d(\beta_n, \beta_1) \leq d(\beta_1, \beta_2)$ & $K(\beta_1, \beta_2) < K(\beta_n, \beta_1)$,
$\chi(\beta_n, \beta_1, \beta_2) := \downarrow$ *otherwise.*

Lemma 24 *Let $\beta_n > \ldots > \beta_m$ with $m \geq 2$ and assume that $c \in \{\nearrow, \uparrow, \downarrow\}$ satisfies*

$$\{\chi(\beta_i, \beta_{i+1}, \beta_{i+2}) : i \leq m-2\} = \{c\}.$$

(1) If $c = \uparrow$ then $m \leq m(\beta_n) < p(\beta_n)$.
(2) If $c = \nearrow$ then $E(\beta_n, \beta_1) < \ldots < E(\beta_{m-1}, \beta_m)$.
(3) If $c = \downarrow$ then $E(\beta_n, \beta_1) > \ldots > E(\beta_{m-1}, \beta_m)$.

We write $k^{(s)} := k + 3 + 3^2 + \cdots + 3^s$.

Definition 25 *Definition of $\mathbb{C}^k_s$ and $\chi^k_s : [\omega^k_s]^{s+1} \to \mathbb{C}^k_s$.*

(1) $BbbC^k_1 := \{0, \ldots, k-1\}$, $\mathbb{C}^k_{s+1} := \mathbb{C}^k_s \cup \{\nearrow, \uparrow, \downarrow\}^s$. Note that card$(\mathbb{C}^k_s) = k^{(s)}$.
(2) If $\alpha = \omega^{k-1} \cdot m_0 + \cdots + \omega^0 \cdot m_{k-1} > \omega^{k-1} \cdot n_0 + \cdots + \omega^0 \cdot n_{k-1} = \beta$ where $m_i, n_i \geq 0$ for $0 \leq i \leq k-1$ then $\chi^k_1(\alpha, \beta) := \min\{i : n_i < m_i\}$. Note that $\chi^k_1(\alpha, \beta) = 1 + d(\alpha, \beta)$.
(3) Assume that $s \geq 1$, $\omega^k_{s+1} > \beta_n > \ldots > \beta_{s+1}$, $\delta_i := E(\beta_i, \beta_{i+1})$ and $c_i := \chi(\beta_i, \beta_{i+1}, \beta_{i+2})$.
If $c_0 = \ldots = c_{s-1} = \downarrow$ then $\chi^k_{s+1}(\beta_n, \ldots, \beta_{s+1}) := \chi^k_s(\delta_n, \ldots, \delta_s)$.
If $c_0 = \ldots = c_{s-1} = \nearrow$ then $\chi^k_{s+1}(\beta_n, \ldots, \beta_{s+1}) := \chi^k_s(\delta_s, \ldots, \delta_n)$.
In all other cases put $\chi^k_{s+1}(\beta_n, \ldots, \beta_{s+1}) := (c_0, \ldots, c_{s-1})$.

The intrinsic complexity of the coloring χ^k_{s+1} is measured in the following lemma.

Lemma 26 *Let* $1 \leq s, k$ *&* $s < m$ *&* $c \in \mathbb{C}_s^k$ *&* $\omega_s^k > \beta_n > \ldots > \beta_m$*. If* $\chi_s^k(\beta_i, \ldots, \beta_{i+s}) = c$ *for all* $i \leq m - s$ *then* $m < p(\beta_n)$*.*

By applying the compression method to χ_s^k we arrive at the following phase transition result for the Paris Harrington Ramsey numbers.

Theorem 27 *Assume that* f *is* Σ_1*-definable in* PA

(1) Let $f(i) \geq |i|_{H_{\varepsilon_0}^{-1}(i)}$*. Then* $\mathrm{PA} \nvdash \mathrm{PH}(f)$*.*
(2) If for some $\alpha < \varepsilon_0$ $\mathrm{PA} \vdash (\exists n_0)(\forall n \geq n_0)[f(n) \leq |n|_{H_\alpha^{-1}(n)}]$ *then* $\mathrm{PA} \vdash \mathrm{PH}(f)$*.*

The first assertion follows easily from Lemma 26 (full details can be found in [81]) and the second by the Erdös Rado theorem [19].

It is now an obvious question whether our master ordinal coloring also provides sharp thresholds for the phase transition of $\mathrm{PH}^{d+1}(f)$ in $\mathrm{I}\Sigma_d$. Unfortunately this is not the case. This resembles a phenomenon in Ramsey theory where the best lower bounds for the Ramsey function is obtained by the probabilistic method. By modifying the definition of our ordinal coloring χ_2 by using a probabilistic method device we have been able to prove the following result (during a corresponding NWO-funded workshop on Ramsey theory in Utrecht).

Theorem 28 *Let* d *be a positive integer. Assume that* f *is* Σ_1*-definable in* $\mathrm{I}\Sigma_d$

(1) Let $f(i) \geq \frac{|i|_d}{H_{\omega_{d+1}}^{-1}(i)}$*. Then* $\mathrm{I}\Sigma_d \nvdash \mathrm{PH}^{d+1}(f)$*.*
(2) If for some $\alpha < \omega_{d+1}$ $\mathrm{PA} \vdash (\exists n_0)(\forall n \geq n_0)[f(n) \leq \frac{|n|_d}{H_\alpha^{-1}(n)}]$ *then* $\mathrm{I}\Sigma_d \vdash \mathrm{PH}^d(f)$*.*

Now let us discuss the Kanamori McAloon result. Again we need some terminology. Given a number-theoretic function f, a set of positive integers X we call a coloring $F : [X]^d \to \mathbb{N}$ f-regressive iff $F(x_1, \ldots, x_n) < f(x_1)$ whenever $F(x_1) > 0$. We call a set Y of natural numbers F-min homogeneous if for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in Y$ with $x_1 < \ldots < x_n, y_1 < \ldots < y_n$ and $x_1 = y_1$ we have $F(x_1, \ldots, x_n) = F(y_1, \ldots, y_n)$. Let $\mathrm{KM}^d(f)$ be the assertion

$(\forall m)(\exists R)(\forall F : [R]^d \to \mathbb{N})[F$ f-regressive $\to (\exists Y \subseteq R)[Y$ is F-homogeneous$]]$.

Moreover let $\mathrm{KM}(f)$ be the assertion $(\forall d)\mathrm{KM}^{d+1}(f)$.

According to Kanamori McAloon [38] we have the following theorem.

Theorem 29 *Let id denote the identity function and* const *denote a constant function.*

(1) PA $\nvdash$ KM(id).
(2) I$\Sigma_d \nvdash$ KM$^{d+1}(id)$.
(3) I$\Sigma_1 \vdash (\forall d)$KM^{d+1}(const).
(4) I$\Sigma_1 \vdash (\forall d)[KM^{d+1}(id) \leftrightarrow$ PH$^{d+1}(id)$.

Thus it is not too surprising that one can obtain the following theorem (which in fact follows from Theorem 27 and methods from [38]). (Full details can be found in Lee's PhD thesis [44].)

Theorem 30 *Assume that f is Σ_1-definable in* PA

(1) Let $f(i) \geq |i|_{H_{\varepsilon_0}^{-1}(i)}$. Then PA $\nvdash$ KM(f).
(2) If for some $\alpha < \varepsilon_0$ PA $\vdash (\exists n_0)(\forall n \geq n_0)[f(n) \leq |n|_{H_\alpha^{-1}(n)}]$ *then* PA $\vdash$ KM(f).

Since I$\Sigma_1 \vdash (\forall d)[PH^{d+1} \leftrightarrow$ KM$^{d+1}(id)]$ it is now an obvious question whether the sharp thresholds for the phase transition of PH$^{d+1}(f)$ in IΣ_d also provides the sharp thresholds for KM$^{d+1}(f)$. Surprisingly this is not the case, although the full classification has not been achieved in the time of writing. Lee obtained in his PhD thesis [44] the following results.

Theorem 31 (Lee) *Let d be a positive integer. Assume that f is Σ_1-definable in* IΣ_d

(1) Let $f(i) \geq \sqrt[H_{\omega_2}^{-1}(i)]{(i)}$. Then I$\Sigma_1 \nvdash$ KM$^2(f)$.
(2) Let $f(i) \geq 2 \cdot |i|_{d-1}$. Then I$\Sigma_d \nvdash$ KM$^{d+1}(f)$.
(3) If for some $\alpha < \omega_{d+1}$ PA $\vdash (\exists n_0)(\forall n \geq n_0)[f(n) \leq \sqrt[H_\alpha^{-1}(n)]{|i|_{d-1}}]$ *then* I$\Sigma^d \vdash$ PH$^d(f)$.

The author conjectures that the following refinement of assertion 2 of Theorem 31 holds. If $f(i) \geq \sqrt[H_{\omega_{d+1}}^{-1}(i)]{|i|_{d-1}}$ holds for a Σ_1 definable function in IΣ_d then I$\Sigma_d \nvdash$ KM$^{d+1}(f)$. Moreover he conjecures that for a proof the ordinal coloring used above in the treatment of PH(f) can be employed (without using the probabilistic method).

We close this section by stating an independence result in terms of the maximal coefficient norm. It turns out that we arrive exactly at the same threshold function as for the Kanamori McAloon theorem. Interestingly these threshold functions also show up while classifying hydras and Goodstein sequences in the following section.

Theorem 32 *Assume that f is Σ_1-definable in* PA

(1) Let $f(i) \geq |i|_{H_{\varepsilon_0}^{-1}(i)}$. Then PA $\nvdash$ SWO$(\varepsilon_0, \mathrm{mc}(\cdot), f)$.
(2) If for some $\alpha < \varepsilon_0$ PA $\vdash (\exists n_0)(\forall n \geq n_0)[f(n) \leq |n|_{H_\alpha^{-1}(n)}]$ *then* PA $\vdash$

SWO($\varepsilon_0, \mathrm{mc}(\cdot), f$)

A similar result holds for the fragments $\mathrm{I}\Sigma_d$ of PA.

9 Goodstein sequences and Hydra games

The independence of the termination of the Hydra game from PA follows immediately from the standard classification of the provably recursive functions of PA (See, for example, [12] for a smooth presentation). The parametrized Hydra game is formalized in mathematical terms as follows. Let $Q^f_x(\alpha)$ be the ordinal predecessor of α with respect to x and step growth rate bound f. Thus $Q^f_x 0 := 0$, $Q^f_x(\alpha+1) = \alpha$, and $Q^f_x\lambda := \lambda[f(x)]$. Let HYDRA$(\alpha, f)$, the statement that the corresponding hydra game terminates, stand for

$$(\forall K)(\exists M)[Q^f_M \ldots Q^f_1 \omega_K = 0].$$

Then HYDRA(α, f) is true for any $\alpha \leq \varepsilon_0$ and any number-theoretic function, since ε_0 resp. $\mathbb{E}$ is well-founded.

Theorem 33 (Classification of Hydra games) *Assume that f is Σ_1-definable in* PA

(1) Let $f(i) \geq |i| \cdot |i|_{H^{-1}_{\varepsilon_0}(i)}$. Then $\mathrm{PA} \nvdash \mathrm{HYDRA}(\varepsilon_0, f)$.
(2) If for some $\alpha < \varepsilon_0$ $\mathrm{PA} \vdash (\exists n_0)(\forall n \geq n_0)[f(n) \leq |n| \cdot |n|_{H^{-1}_\alpha(n)}]$ *then* $\mathrm{PA} \vdash \mathrm{HYDRA}(\varepsilon_0, f)$

Corresponding results for the fragments $\mathrm{I}\Sigma_k$ follow by the same method.

As a side product, assertion 2 of this theorem classifies Girard's notion of pointwiseness in a more or less satisfactory way in as far as we can measure precisely the excess of a non-pointwise descent which leads beyond (or does not lead beyond) the slow growing hierarchy.

Definition 34 (Goodstein sequences)

$$m_{f,0} := m \textit{ and } m_{f,i+1} := m_{f,i}[1 + f(i) := 1 + f(i+1)] - 1.$$

Let GOODSTEIN(f) be the assertion

$$(\forall m)(\exists i)[m_{f,i} = 0].$$

By now a classic is Cichon's exposition of the independence of GOODSTEIN(id).

Theorem 35 ([16]) $\mathrm{PA} \nvdash \mathrm{GOODSTEIN}(id)$.

By adapting Cichon's proof we obtain the following threshold classification. This also includes an improvement of results by Hodgson and Kent [34]. (Corresponding results for the fragments $\mathrm{I}\Sigma_k$ follow by the same method.)

Theorem 36 (Classification of Goodstein sequences) *Assume that f is Σ_1-definable in* PA.

(1) *Let $f(i) \geq |i|_{H_{\varepsilon_0}^{-1}(i)}$. Then* $\mathrm{PA} \nvdash \mathrm{GOODSTEIN}(f)$.
(2) *If for some $\alpha < \varepsilon_0$* $\mathrm{PA} \vdash (\exists n_0)(\forall n \geq n_0)[f(n) \leq |n|_{H_\alpha^{-1}(n)}]$ *then* $\mathrm{PA} \vdash \mathrm{GOODSTEIN}(f)$

In view of the preceding results it seems quite natural to ask wether the norm function N (or other norms) is intrinsically related to a Hydra game or a Goodstein sequence principle. In the remaining part of this section we show that this is indeed the case.

Definition 37 *Definition of $N_i a$ for $i \geq 2$.*

(1) *If $a < i$ then $N_i a = a$.*
(2) *If $a = i^{a_1} \cdot m_1 + \cdots + i^{a_n} \cdot m_n$ where $a_1 > \ldots > a_n$ and $i > m_j > 0$ then $N_i a := m_1 \cdot (1 + N_i a_1) + \cdots + m_n \cdot (1 + N_i a_n)$.*

Thus $N_i a$ counts the number of occurrences of i in the complete base i-representation of a, when multiplicities are taken into account.

Definition 38 *Definition of the modified Goodstein style sequences $m_{k,i}$ for $k \geq 2$.*

(1) $m_{k,0} := m$,
(2) $m_{k,i+1} := \max\{n < m_{k,i}[k+i := k+i+1] : N_{k+i+1} n \leq k+i\}$,
(3) $m_i := m_{2+N_2 m,i}$.

Theorem 39 *(1)* $\mathrm{PA} \nvdash (\forall m, k)(\exists i)[m_{k,i} = 0]$.
(2) $\mathrm{PA} \nvdash (\forall m)(\exists i)[m_i = 0]$.

Instead of proving this directly we show that the bounds obtained in [76] can be used to get even an optimal classification of these Goodstein-style sequences.

Lemma 40 (Lifting Lemma) $m < n \;\&\; i \geq 2 \implies m[i := \omega] < n[i := \omega]$

Lemma 41 (Collapsing Lemma) *(1)* $\alpha < \beta \;\&\; \mathrm{mc}(\alpha) < i \implies \alpha[\omega := i] < \beta[\omega := i]$
(2) $N\alpha \leq \mathrm{mc}(\alpha)$.

Lemma 42 *If $i < j \leq k$ then*

$$m[j := \omega][\![i]\!][\omega := k] = \max\{n < m[j := k] : N_k n \leq i\}.$$

Definition 43 *(1)* $m_0^{x,f} := m$,

(2) $m_{i+1}^{x,f} := \max\{n < m[x+1+f(i) := x+1+f(i+1)] : N_{x+1+f(i+1)}n \leq x+f(i)\}$.

Corollary 44 $m_{i+1}^{x,f} = m_i^{x,f}[x+1+f(i)][\![x+f(i)]\!][\omega := x+1+f(i+1)]$

Lemma 45 *Let* $\lambda_0^{x,f} := m[x+1+f(0) := \omega]$ *and* $\lambda_{i+1}^{x,f} := \lambda_i^{x,f}[\![x+f(i)]\!]$. *Then* $\lambda_i^{x,f} = m_i^{x,f}[x+f(i) := \omega]$

Let GOODSTEIN(N, f) be the assertion

$$(\forall m, x)(\exists i)[m_i^{x,f} = 0].$$

Then GOODSTEIN(N, f) is true.

Theorem 46 *Assume that* f *is* Σ_1*-definable in* PA

(1) *Let* $f(i) \geq |i| \cdot |i|_{H_{\varepsilon_0}^{-1}(i)}$. *Then* PA $\nvdash$ GOODSTEIN(N, f).

(2) *If for some* $\alpha < \varepsilon_0$ PA $\vdash (\exists n_0)(\forall n \geq n_0)[f(n) \leq |n| \cdot |n|_{H_\alpha^{-1}(n)}]$ *then* PA $\vdash$ GOODSTEIN(N, f).

Remark: Similar results as those proved in this section for ε_0 hold for Beklemishev's worm principle [11]. This follows by an translation of these worms into ε_0. (This has been observed by G. Lee and the author and independently by L Carlucci.)

10 A discussion of universality and renormalization issues

The classification of the phase transitions in various examples shares features of universality and renormalization phenomena in statistical physics. There the phase transition depends often on very few parameters like dimensions and symmetries but not on the specific matter under consideration. This phenomenon is called universality. Let $\mathbb{U}$ be the least set of unary functions $f : \mathbb{N} \to \mathbb{R}$ such that

(1) every constant function $x \mapsto r$ is in $\mathbb{U}$ where r is a recursive real,
(2) the identity function is in $\mathbb{U}$,
(3) for every $\alpha \leq \varepsilon_0$ the function H_α^{-1} is in $\mathbb{U}$
(4) with f, g in $\mathbb{U}$ also $f \cdot g$, $\frac{f}{g}$, $|f|_g$, $\sqrt[f]{g}$, and 2^f are in $\mathbb{U}$.

From the examples of this paper it appears that the threshold functions for independence results can be classified with functions stemming from $\mathbb{U}$, which may thus be considered to be a universality scale. As we have seen in the section on ω^ω there is a strong universality (in the examples considered in this paper) for the thresholds concerning inpendence results for $\mathrm{I}\Sigma_1$. Differences only depend on whether the underlying norm is additive or multiplicative. For the stronger fragments $\mathrm{I}\Sigma_d$ the universality becomes a bit weaker but it is still present, as reflected by the transition from the additive to the multiplicative situation via the exponential function.

Renormalization is used in physics to study the phase transition under rescaling and thus, e.g., to obtain information on critical exponents. If we investigate a parametrized assertion $A(f)$ with respect to independence we might be interested in rescaled variants $A(g)$ where e.g. $g(i) = f(\frac{i}{2})$. (For example, in case of the principle $\mathrm{SWO}(\alpha, N, g)$ this amounts in demanding that the growth rate condition reads as $(\forall i < \frac{M}{2})N(\alpha_{2\cdot i} \leq f(i)$ so that ordinals are rescaled into groups of pairs.) It turns out that the phase transitions considered in this paper are stable under linear renormalization, thus provable assertions remain provable in the system under consideration and unprovable system remain unprovable. Most of the considered examples are even stable under more general scalings, e.g. under $x \mapsto x^{\frac{1}{d}}$.

11 Limit laws for ordinals

Assume that we have given a sentence φ from the language of linear orders. One might be interested in the probability that φ holds on the set of predecessors of a randomly chosen ordinal $\alpha < \beta$. To introduce a concise definition of probability let us consider the following notion of asymptotic density which is similar to a notion from random graph theory. We write $\alpha \models \varphi$ if φ is true in the structure $\langle \{\gamma < \alpha\}, \in \rangle$ where $\in$ is the interpretation of the relation symbol for the less than relation. The additive density, $\delta_\varphi(\alpha)$, of φ on the predecessors of α is the following limit, if it exists.

$$\delta_\varphi(\beta) = \lim_{n\to\infty} \frac{\#\{\alpha < \beta : N\alpha \leq n \ \& \ \alpha \models \varphi\}}{\#\{\alpha < \beta : N\alpha \leq n\}}.$$

The multiplicative density, $\delta^S_\varphi(\alpha)$, of φ on the predecessors of α is the following limit, if it exists.

$$\delta^S_\varphi(\beta) = \lim_{n\to\infty} \frac{\#\{\alpha < \beta : S(\alpha) \leq n \ \& \ \alpha \models \varphi\}}{\#\{\alpha < \beta : S(\alpha) \leq n\}}.$$

For each $\alpha < \varepsilon_0$ we have proved in [79] that

$$c_\alpha(n) \sim c_\alpha(n+1)$$

and

$$c^S_\alpha(n) \sim c^S_\alpha(n+1).$$

These equations are Compton style conditions which are valuable in proving zero-one laws [13]. The following is a special result of a general theory developed by Alan R. Woods and the author.

Theorem 47 *Let φ be a first order sentence in the language of linear orders.*

(1) If $\alpha < \varepsilon_0$ is an additive principal number or has the form $\omega^\omega \cdot (1+\gamma)$ then $\delta_\varphi(\alpha) \in \{0,1\}$. If $\alpha = \varepsilon_0$ then $\delta_\varphi(\alpha)$ exists but need not be in $\{0,1\}$.
(2) If $\alpha < \varepsilon_0$ is an additive principal number or has the form $\omega^\omega \cdot (1+\gamma)$ then $\delta^S_\varphi(\alpha) \in \{0,1\}$.

Thus ε_0 is the minimal additive principle number such that a first order additive (multiplicative) zero-one law fails. For any of the term complexity functions which we considered one may ask wheter limit laws for ordinals hold. The author conjectures that at least a Cesaro limit law shall hold. Moreover he conjectures that any natural coding of the ordinals below ε_0 should come with corresponding logical limit laws.

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