Phase transition thresholds for some natural subclasses of the computable functions

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Abstract

In this paper we first survey recent advances on phase transition phenomena which are related to natural subclasses of the recursive functions. Special emphasis is put on descent recursive functions, witness bounding functions for well-partial orders and Ramsey functions. In the last section we prove in addition some results which show how the asymptotic of the standard Ramsey function is affected by phase transitions for associated parameterized Ramsey functions.

Keywords: Subrecursive hierarchies, phase transition, threshold functions, proof-theoretic ordinals, well-partial orderings, Ramsey theory, rapidly growing Ramsey functions.

1 Introduction

Phase transition is a type of behaviour wherein small changes of a parameter of a system cause dramatic shifts in some globally observed behaviour of the system, such shifts being usually marked by a sharp 'threshold point'. (An everyday life example of such thresholds are ice melting and water boiling temperatures.) This kind of phenomena nowadays occur throughout many mathematical and computational disciplines: statistical physics, evolutionary graph theory, percolation theory, computational complexity, artificial intelligence etc.

The last few years have seen an unexpected series of achievements that bring together independence results in logic, analytic combinatorics and Ramsey Theory. These achievements can be intuitively described as phase transitions from provability to unprovability of an assertion by varying a threshold parameter [23, 26]. Another face of this phenomenon is the transition from slow-growing to fast-growing computable functions [25, 28].

In this paper we survey recent advances on phase transition phenomena which are related to natural subclasses of the recursive functions.

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For the purpose of motivation let us assume that we have given some algorithm A which performs computations on a given set D of data. We assume that D is equipped with a norm function $N : D \to \mathbb{N}$ such that for every $k \in \mathbb{N}$ the set $\{d \in D : N(d) \le k\}$ is finite. Moreover let us assume that every computation tree for A is finitely branching. A transition in the computation tree is denoted by $\to$. Thus $d \to d'$ indicates that the the algorithms performs a calculation to obtain d' out of d. If A is terminating then every sequence $d_0 \to d_1 \to d_2 \ldots$ must terminate after finitely many steps and by our assumption on N and the branching we obtain by König's Lemma the following: Given $k \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that every sequence $d_0 \to d_1 \to d_2 \ldots$ with $N(d_0) \le k$ terminates after M steps. The minimal such m determines the computation lengths function $CompL_A$, i.e. $CompL_A(k)$ is the least m such that every sequence $d_0 \to d_1 \to d_2 \ldots$ with $N(d_0) \le k$ terminates after m steps.

It is now quite obvious to look for methods M for proving termination of A. Moreover in this respect it is also very natural to classify $CompL_A$ which can be considered as some measure for the computational complexity of A.

From the logical point of view it would be nice to see whether there are general principles yielding classifications for $CompL_A$ depending on the method M which is used for proving termination of A.

To study phase transition phenomena in this context we equip the problem under investigation with a control function $f : \mathbb{N} \to \mathbb{N}$ and we demand that $N(d_i) \le k + f(i)$ for any sequence of computations $d_0 \to d_1 \to d_2 \ldots \to d_i \ldots$. Hereby we assume that f is reasonably simple. Then classifying $CompL_A$ can be seen as a problem depending on parameters M and f. In particular when phase transitions for $CompL_A$ are studied the function f will play the role the order parameter plays in physics. The expectation is that for very slow growing functions f the function $CompL_A$ has moderate complexity but that the complexity of $CompL_A$ explodes as soon as f exceeds a certain threshold function. In analogy with physics it is natural to consider renormalization and universality in this context [27].

Investigations on this subject have given rise to rich and intriguing peaces of logic and mathematics where methods from Ramsey theory, analytic combinatorics and logic can be cross-fertilized.

In the following sections we consider different types of termination proof methods and consider the phase transition problem in each case separately. It is our aim to provide Rules of thumb so that it is possible to guess the phase transition thresholds a priori. In the final section we study phase transitions in Ramsey theory.

2 Phase transitions for ordinal sequences

In this and the following section we base our investigations on a principle suggested by Harvey Friedman. This turns out to be tailor made for our intended applications.

Obviously termination proofs can be carried out by using ordinals through mapping computation sequences into descending chains of ordinals. Typically such a mapping assigns an ordinal to a data element in an effective way so that resulting norms of ordinals from a descending ordinal sequence are also controlled by a function say $g : \mathbb{N} \to \mathbb{N}$.

After putting the problem into an abstract setting we arrive at Friedman's principle of combinatorial well-foundedness. For stating it let us fix a countable ordinal α and a norm function $N : \alpha \to \mathbb{N}$ such that for every $k \in \mathbb{N}$ the set $\{\beta < \alpha : N(\beta) \leq k\}$ is finite. Let
$\mathrm{CWF}(\alpha, g) = (\forall k)(\exists M)$
$(\forall \alpha_0, \ldots, \alpha_M < \alpha)\big[(\forall i \leq M)[N\alpha_i \leq k + g(i)] \to (\exists i < M)[\alpha_i < \alpha_{i+1}]\big]$.
The associatec complexity function is
$D(\alpha, g)(k) := \min\{M :$
$(\forall \alpha_0, \ldots, \alpha_M < \alpha)\big[(\forall i \leq M)[N\alpha_i \leq k + g(i)] \to (\exists i < M)[\alpha_i < \alpha_{i+1}]\big]\}$.

By Friedman's results is well known that proof-theoretic ordinals α and natural associated norm functions (e.g. given by a term length function) the function $D(\alpha, g)$ grows rapidly even for rather small values of α.

To fix the context let as consider the ordinals segment of ordinals below ε_0 and define a length norm function using Cantor normal forms as follows. $N(0) := 0$ and $N(\alpha) := n + N(\alpha_1) + \cdots + N(\alpha_n)$ if $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n} > \alpha_1 \geq \ldots \geq \alpha_n$. A corresponding sup norm function lh can be defined as follows $|0| := 0$ and $|\alpha| = \max\{m_1, \ldots, m_n, |\alpha_1|, \ldots, |\alpha_n|\}$ if $\alpha = \omega^{\alpha_1} \cdot m_1 + \cdots + \omega^{\alpha_n} \cdot m_n > \alpha_1 > \ldots > \alpha_n$.

Theorem 1 (Friedman). *Let $g(i) = i$ be the identity function and let N be the length or sup norm.*

1. *$D(\omega^d, g)$ is primitive recursive.*
2. *$D(\omega^\omega, g)$ is Ackermannian.*
3. *$D(\omega^{\omega^d}, g)$ is multiple recursive.*
4. *$D(\omega^{\omega^\omega}, g)$ is not multiple recursive.*
5. *If $\alpha < \varepsilon_0$ then $D(\alpha, g)$ is provably recursive in* PA.
6. *$D(\varepsilon_0, g)$ is not provably recursive in* PA.

So pushing ordinals beyond certain thresholds is reflected by (perhaps expected) phase transitions of the resulting complexity functions.

Another perhaps even more intriguing phase transition occurs when the ordinal notation under consideration is fixed but the control function g is varied.

Further let us agree on the following assignment of fundamental sequences. If $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n+1} > \alpha_1 \geq \ldots \geq \alpha_n + 1$ then $\alpha[x] := \omega^{\alpha_1} + \cdots + \omega^{\alpha_n} \cdot x$. If $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n} > \alpha_1 \geq \ldots \geq \alpha_n$ and α_n is a limit then $\alpha[x] := \omega^{\alpha_1} + \cdots + \omega^{\alpha_n[x]}$.

To be able to state the phase transition in terms of hierarchies of recursive functions let us recall the definition of the Hardy hierarchy.

$$\begin{array}{rcl} H_0(x) & := & x \\ H_{\alpha+1}(x) & := & H_\alpha(x+1) \\ H_\lambda(x) & := & H_{\lambda[x]}(x) \end{array}$$

For a given weakly increasing unbounded function $F : \mathbb{N} \to \mathbb{N}$ we define its inverse function F^{-1} as follows. $F^{-1}(x) := \min\{y : F(y) \geq x\}$. Typically such an inverse function F^{-1} grows rather slow when the original function F grows reasonably fast.

Theorem 2 (Weiermann [27]). *Let* $g_\alpha(i) := i^{\frac{1}{H_\alpha^{-1}(i)}}$ *and let* N *be the length or sup norm.*

1. *If* $\alpha < \omega^\omega$ *then* $D(\omega^\omega, g_\alpha)$ *is primitive recursive.*
2. $D(\omega^\omega, g_{\omega^\omega})$ *is Ackermannian.*

We are now going to formulate the general rationale behind this type of results. Of fundamental importance is here the study of count functions.

Definition 1.

$$c_\alpha(n) := |\{\beta < \alpha : N(\beta) \leq n\}|.$$

These count functions c_α come along with an intriguing mathematical theory which is based on generatingfunctionology [31]. To get good asymptotic bounds on $c_\alpha(n)$ for n large one applies Cauchy's integral formula to the generating function $C(z) := \sum_{i=0}^{\infty} c_\alpha(n) \cdot z^n$.

Theorem 3. *Let* N *be the length norm and let the count functions be defined with respect to this norm.*

1. $c_{\omega^d}(n) \sim \frac{n^d}{(d!)^2}$
2. $\log(c_{\omega^\omega}(n)) \sim \pi \cdot \sqrt{\frac{2n}{3}}$
3. $\log(c_{\omega^{\omega^\omega}}(n)) \sim \frac{\pi^2}{6} \frac{n}{\log(n)}$.

To formulate the phase transition principle let us further define $\alpha[\![x]\!] := \max\{\beta < \alpha : N(\beta) \leq N(\alpha) - 1 + x\}$.

Rule of thumb 1. *Let* $g_{\alpha,\beta}(x) := c^{-1}_{\alpha[\![H_\beta^{-1}(i)]\!]}(i)$*. If* $\beta \leq \alpha$ *then* $D(\alpha, g_{\alpha,\beta})$ *is primitive recursive in* H_β *and* H_β *is primitive recursive in* $D(\alpha, g_{\alpha,\beta})$

This would imply the following Rule of thumb.

Rule of thumb 2. *Let* $g_{\alpha,\beta}(x) := c^{-1}_{\alpha[\![H_\beta^{-1}(i)]\!]}(i)$*. Let* T *be a fragment of* PA *with proof-theoretic (*Π^0_2*-) ordinal* α *and let* N *be the length or sup norm.*

1. *If* $\beta < \alpha$ *then* $D(\alpha, g_{\alpha,\beta})$ *is provably recursive in* T
2. $D(\alpha, g_{\alpha,\beta})$ *is not provably recursive in* T*.*

These general rules apply to larger segments of ordinals and lead to the following applications (which already have been proved rigorously). Let $|x| := \log_2(x+1)$ be the binary length of x where $|0| := 0$.

Theorem 4 (Weiermann). *Let* $n \geq 1$*,* T *be* $I\Sigma_n$ *and* $\alpha := \omega_{n+1}$ *and let* N *be the length norm. Let* $g_{\alpha,\beta}(n) := |i| \cdot \sqrt[H_\beta^{-1}(i)]{\log_{n-1}(i)}$*.*

1. *If* $\beta < \alpha$ *then* $D(\alpha, g_{n,\beta})$ *is provably recursive in* T
2. $D(\alpha, g_{\alpha,\alpha})$ *is not provably recursive in* T*.*

In case of PA we obtain the following phase transition result.

Theorem 5 (Arai, Weiermann[1, 23]). *Let* $g_\beta(n) := |i| \cdot |i|_{H_\beta^{-1}(i)}$ *and let* N *be the length norm.*

1. *If* $\beta < \varepsilon_0$ *then* $D(\varepsilon_0, g_\beta)$ *is provably recursive in* PA
2. $D(\varepsilon_0, g_{\varepsilon_0})$ *is not provably recursive in* PA*.*

Remark: The phase transitions for CWF principles are in a sense continuous since they involve H_α^{-1} for varying α. In analogy with physics one might consider this as a second order phase transition.

3 Phase transitions for sequences in well partial orderings

A partial ordering $\langle X, \leq_X \rangle$ is a well-partial ordering iff for all functions $F : \mathbb{N} \to X$ there exist natural numbers i, j such that $i < j$ and $F(i) \leq_X F(j)$. A sequence $F : \mathbb{N} \to X$ is called bad if there do not exist natural numbers i, j such that $i < j$ and $F(i) \leq_X F(j)$. So a partial order is a well-partial order iff there does not exist an infinite bad sequence for it.

Obviously termination proofs can be carried out by using well partial orders through mapping computation sequences into bad sequences. Typically such a mapping assigns an initial sequence of data elements to an element of the well-partial order in an effective way so that resulting sequences of elements in the well-partial-order are again also controlled in norm by some function $g : X \to \mathbb{N}$.

After putting the problem into an abstract setting we arrive at Friedman's principle of combinatorial well-partial-orderedness. For stating it let us fix a well partial order $\langle X, \leq_X \rangle$ and a norm function $N : X \to \mathbb{N}$ such that for every $k \in \mathbb{N}$ the set $\{\beta \in X : N(\beta) \leq k\}$ is always finite. Let

CWP$(X, g) = (\forall k)(\exists M)$
$(\forall \alpha_0, \ldots, \alpha_M < \alpha)\big[(\forall i \leq M)[N\alpha_i \leq k + g(i)] \rightarrow (\exists i \leq M)(\exists j \leq M)[i < j \wedge \alpha_i \leq al_j\big]$. The associated complexity function is
$D(X, g)(k) := \min\{M :$
$(\forall \alpha_0, \ldots, \alpha_M < \alpha)\big[(\forall i \leq M)[N\alpha_i \leq k + g(i)] \rightarrow (\exists i \leq M)(\exists j \leq M)[i < j \wedge \alpha_i \leq \alpha_j\big]\}$.

By Friedman's results it is well known that for several natural well-partial oders and associated norm functions (e.g. given by a length function) the function $D(\alpha, g)$ grows rapidly.

Basic examples are provided by Dickson's Lemma and Higman's Lemma. Assume that $\langle Y, \leq_Y \rangle$ is a partial ordering. Then we can induce a partial ordering $\leq_Y^k$ on Y^k the set of k-tuples of elements in X as follows $\langle x_0, \ldots, x_{k-1} \rangle \leq_Y^k \langle y_0, \ldots, y_{k-1} \rangle$ if $x_i \leq_Y y_i$ for $i = 0, \ldots, k-1$. Moreover we can induce a partial ordering on $Y^\star$ the set of finite sequences of elements in X as follows $\langle x_0, \ldots, x_{k-1} \rangle \leq_Y^\star \langle y_0, \ldots, y_{l-1} \rangle$ if there exist $i_0, \ldots, i_{k-1}$ such that $0 \leq i_0 < i_1 < \ldots < i_{k-1}$ and such that $x_m \leq_Y y_{i_m}$ for $m = 0, \ldots, k-1$.

Theorem 6 (Dickson,Higman). *If $\langle Y, \leq_Y \rangle$ is a well partial ordering then so are $\langle Y^k, \leq_Y^k \rangle$ and $\langle Y^\star, \leq_Y^\star \rangle$*

If $Y \subseteq \mathbb{N}$ then for finite sequences with values in Y there are two obvious norm functions. Let $|\langle x_0, \ldots, x_{k-1} \rangle| := \max\{x_0, \ldots, x_{k-1}\}$ be the sup-norm and $N(\langle x_0, \ldots, x_{k-1} \rangle) := \sum_{l=0}^{k-1} x_l$ be the lengths norm. (In more general situations one defines the norms on sequences of course in terms of the norm given on the space Y.)

Theorem 7 (Friedman). *Let $d = \{0, \ldots, d-1\}$ and $\mathbb{N}$ be well-quasiordered by their natural orderings. Let $\mathbb{N}^d$ and $d^\star$ be ordered be ordered by the induced orderings. Let $g(i) = i$ be the identity function and N be either the sup norm or the length norm.*

1. *$D(\mathbb{N}^d, g)$ is primitive recursive.*
2. *$k \mapsto D(\mathbb{N}^k, g)(k)$ is Ackermannian.*
3. *$D(d^\star, g)$ is multiple recursive.*
4. *$k \mapsto D(k^\star, g)(k)$ is not multiple recursive.*

Theorem 8 (Weiermann [30]). *Let $k = \{0, \ldots, k-1\}$ be well-quasiordered by its natural ordering. Let $d^\star$ be ordered by the induced ordering. Let $g_r(i) = r \cdot |i|$ and N be the length norm.*

1. *If $r < 1$ then $k \mapsto D(k^\star, g_r)$ is multiple recursive.*
2. *If $r > 1$ then $k \mapsto D(k^\star, g)(k)$ is not multiple recursive.*

We conjecture that $k \mapsto D(k^\star, g_r)$ is multiple recursive for $r = 1$.

Theorem 9 (Weiermann [27]). *Let* $g_\alpha(i) := i^{\frac{1}{H_\alpha^{-1}(i)}}$ *and* N *be the sup or length norm.*

1. *If* $\alpha < \omega^\omega$ *then* $k \mapsto D(\mathbb{N}^k, g_\alpha)(k)$ *is primitive recursive.*
2. $k \mapsto D(\mathbb{N}^k, g_{\omega^\omega})(k)$ *is Ackermannian.*

To obtain farer reaching well-partial orders it is convenient to consider finite trees under homeomorphic embeddability. To stay within the realm of ε_0 it is convenient to restrict the consideration to the set $\mathcal{B}$ of binary trees. A convenient way to introduce $\mathcal{B}$ is as follows. Let 0 be a constant (a 0-ary function symbol) and let φ be a binary function symbol. Let $\mathcal{B}$ be the least set of terms such that

1. $0 \in \mathcal{B}$
2. If $\alpha, \beta \in \mathcal{B}$ then $\varphi(\alpha, \beta) \in \mathcal{B}$.

In the sequel we abbreviate $\varphi(\alpha, \beta)$ by $\varphi\alpha\beta$. The *homeomorphic embeddability relation* $\trianglelefteq$ is the least binary relation on $\mathcal{B}$ such that

1. If $\alpha = 0$ then $\alpha \trianglelefteq \beta$.
2. If $\alpha = \varphi\alpha_1\alpha_2$ and $\beta = \varphi\beta_1\beta_2$ and $\alpha \trianglelefteq \beta_1$ or $\alpha \trianglelefteq \beta_2$ then $\alpha \trianglelefteq \beta$.
3. If $\alpha = \varphi\alpha_1\alpha_2$ and $\beta = \varphi\beta_1\beta_2$ and $\alpha_1 \trianglelefteq \beta_1$ and $\alpha_2 \trianglelefteq \beta_2$ then $\alpha \trianglelefteq \beta$.

Theorem 10 (Higman,Kruskal [9]). $\langle \mathcal{B}, \trianglelefteq \rangle$ *is a well partial order.*

Theorem 11 (Friedman). *Let* $g(i) = i$. *Then the function* $D(\mathcal{B}, g)$ *is not provably recursive in* PA.

The associated phase transition result runs as follows. Let $N(0) := 0$ and $N(\varphi\alpha\beta) := 1 + (\alpha) + N(\beta)$.

Theorem 12 (Weiermann [30]). *Let* $g_r(i) = r \cdot |i|$ *and let* N *be a length norm.*

1. *If* $r \leq \frac{1}{2}$ *then* $k \mapsto D(\mathcal{B}, g_r)$ *is elementary recursive.*
2. *If* $r > \frac{1}{2}$ *then* $k \mapsto D(\mathcal{B}, g)(k)$ *is not provably recursive in* PA.

Extraction of a general pattern from Theorem 8 and Theorem 12 leads to the following rule.

Rule of thumb 3. *Assume that* $\langle Y, \leq_Y \rangle$ *is a normed well partial ordering of maximal order type* α *such that* $\log_2 |\{y \in Y : N(y) \leq n\}| \sim n \cdot c$ *for some* $c > 1$. *Let* $g_r(i) := r \cdot |i|$.

1. *If* $r \leq \frac{1}{\log_2(c)}$ *then* $D(Y, g_r)$ *is elementary recursive.*
2. *If* $r > \frac{1}{\log_2(c)}$ *then* $D(Y, g_r)$ *eventually dominates* H_β *for all* $\beta < \alpha$.

The phase transitions for CWP principles in case of Higman's Lemma or the Higman-Kruskal theorem. are in a sense discontinuous since they appear at a real number threshold. In analogy with physics one might consider this as a first order phase transition.

4 Phase transitions in Ramsey theory

In principle termination proofs can also be carried out by using Ramseyan theorems by providing appropriate partitions having only finite homogeneous sets but this connection seems us to be artificial at present. In this section we study therefore thresholds which are associated to Ramseyan statements as an investigation in its own right. We also indicate how classical open problems in Ramsey theory can be attacked via studying associated phase transitions.

4.1 Phase transitions for rapidly growing Ramsey functions

Let us recall the classical Kanamori-McAloon and the Paris-Harrington principles. If $X \subseteq \mathbb{N}, d \in \mathbb{N}$, let $[X]^d$ be the set of all subsets of X with d elements. As usual in Ramsey Theory, we identify a positive integer m with its set of predecessors $\{0, \dots, m-1\}$. If C is a colouring defined on $[X]^d$ (with values in $\mathbb{N}$) we write $C(x_1, \dots, x_d)$ for $C(\{x_1, \dots, x_d\})$ where $x_1 < \dots < x_d$. A subset H of X is called *homogeneous* or *monochromatic* for C if C is constant on $[H]^d$. We write

$$X \to (m)^d_k$$

if for all $C : [X]^d \to k$ there exists $H \subseteq X$ s.t. $card(H) = m$ and H is homogeneous for C. Ramsey [17] proved the following result, known as the Finite Ramsey Theorem.

$$(\forall d)(\forall k)(\forall m)(\exists \ell)[\ell \to (m)^d_k].$$

Let Let $R^d_k(m) := \min\{\ell : \ell \to (m)^d_k\}$. Erdös and Rado gave in [7] a primitive recursive upper bound on $R^d_k(m)$ as a function of d, k, m. The asymptotics of R^d_k is a main concern in Ramsey Theory [8] and we will come back to it later.

The Paris-Harrington principle is a seemingly innocent variant of the Finite Ramsey Theorem. Let f be a number-theoretic function. A set X is called *f-relatively large* if $card(X) \geq f(\min X)$. If $f = id$, the identity function, we call such a set *relatively large* or just *large*. We write

$$X \to^*_f (m)^d_k$$

if for all $C : [X]^d \to k$ there exists $H \subseteq X$ s.t. $card(H) = m$, H is homogeneous for C and H is relatively f-large. The Paris-Harrington principle is just the Finite Ramsey Theorem with the extra condition that the homogeneous set is also relatively large.

$$(\text{PH}) :\equiv (\forall d)(\forall k)(\forall m)(\exists \ell)[\ell \to^*_{id} (m)^d_k].$$

Paris and Harrington showed by model-theoretic methods that (PH) is true but unprovable in PA.

Let $R^d_k(f)(m) := \min\{\ell : \ell \to^*_{id} (m)^d_k\}$.

The following phase transition result has been obtained for the parameterized Ramsey functions $R^d_k(f)$. Let $|\cdot|_d$ be the d-times iterated binary length function and $\log^*$ the inverse of the superexponential function: Recall that

$$|x| := \log_2(x+1), \quad |x|_{d+1} := ||x|_d|, \quad \text{and} \quad \log^* x := \min\{d : |x|_d \leq 2\}$$

Recall that for a weakly increasing and unbounded function $f : \mathbb{N} \longrightarrow \mathbb{N}$ we we denote by f^{-1} the functional inverse of f.

Theorem 13 (Weiermann [26]). *For $\alpha \leq \varepsilon_0$ let*

$$f_\alpha(i) = |i|_{H_\alpha^{-1}(i)}.$$

Then

1. *The function $d, k, m \mapsto R^d_k(\log^*)(m)$ is primitive recursive*
2. *For any fixed positive integer the function $d, k, m \mapsto R^d_k(|\cdot|_q)(m)$ is not provably recursive in* PA.
3. *The function $d, k, m \mapsto R^d_k(f_\alpha)(m)$ is provably recursive in* PA *iff $\alpha < \varepsilon_0$.*

The phase transition in case of fixed dimension d can be characterized as follows.

Theorem 14 (Weiermann [29]). *Let*

$$f^d_\alpha(i) = \left\lfloor \frac{|i|_d}{H_\alpha^{-1}(i)} \right\rfloor.$$

Then for d fixed the function
$k, m \mapsto R^{d+1}_k(f^d_\alpha)(m)$ *is provably total in* $I\Sigma_d$ *iff* $\alpha < \omega_{d+1}$.

Now let us consider the Kanamori McAloon Ramseyan theorem.

Fix a number-theoretic function $f : \mathbb{N} \to \mathbb{N}$. A function $C : [X]^d \to \mathbb{N}$ is called *f-regressive* if for all $s \in [X]^d$ such that $f(\min(s)) > 0$ we have $C(s) < f(\min(s))$. When f is the identity function we just say that C is regressive. A set H is *min-homogeneous* for C if for all $s, t \in [H]^d$ with $\min(s) = \min(t)$ we have $C(s) = C(t)$. We write

$$X \to (m)^d_{f\text{-}reg}$$

if for all f-regressive $C : [X]^d \to \mathbb{N}$ there exists $H \subseteq X$ s.t. $card(H) = m$ and H is min-homogeneous for C. In [10] Kanamori and McAloon introduced the following statement and proved it for any choice of f.

$$(\mathrm{KM})_f :\equiv (\forall d)(\forall m)(\exists \ell)[\ell \to (m)^d_{f\text{-}reg}].$$

The main result of [10], proved by a model-theoretic argument, is that $(\mathrm{KM})_{id}$ is unprovable in PA. As a corollary one obtains the (provable in PA) equivalence of (KM) with (PH).

Let $R^d_{\min}(f)(m) := \min\{[\ell : \ell \to (m)^d_{f\text{-}reg}]\}$.

In his Ph.D. thesis [15], Lee showed that the situation of Theorem 13 occurs in the case of (KM). That is, the phase transition threshold is the same as the one for (PH) when unbounded dimensions are considered.

Theorem 15 (Lee [15]). *For $\alpha \leq \varepsilon_0$ let*

$$f_\alpha(i) = |i|_{H_\alpha^{-1}(i)}.$$

Then

1. *The function $d, k, m \mapsto R^d_{\min}(\log^*)(m)$ is primitive recursive*
2. *For any fixed positive integer the function $d, k, m \mapsto R^d_{\min}(|\cdot|_q)(m)$ is not provably recursive in* PA.
3. *The function $d, k, m \mapsto R^d_{\min}(f_\alpha)(m)$ is provably recursive in* PA *iff $\alpha < \varepsilon_0$.*

Carlucci, Lee and Weiermann obtained the following phase transition in case of fixed dimensions.

Theorem 16 (Carlucci,Lee,Weiermann[3]). *Let*

$$f^d_\alpha(i) = \lfloor \sqrt[H_\alpha^{-1}(i)]{\log_d(i)} \rfloor.$$

Then the function $d, m \mapsto R^{d+1}_{\min}(f_\alpha)(m)$ is provably recursive in $\mathrm{I}\Sigma_d$ *iff $\alpha < \omega_{d+1}$.*

Remarks:

1. The case $d = 1$ has been treated already by Kojman, Lee, Omri and Weiermann in [12] generalizing methods from Kojman and Shelah [13] and [5].
2. Related phase transition results can be shown for Friedman's Ramsey theorem and the canonical Ramsey theorem [Carlucci Weiermann, in preparation].

4.2 A phase transition for R^3_3

Let us recall that for given positive integers d and c the Ramsey function R^d_c is defined as follows: $R^d_c(m)$ is the least number R such that for every function $F : [R]^d \to c$ there exists a set $Y \subseteq [R]$ such that $F \restriction [Y]^d$ has a constant value and $card(Y) \geq m$. It is well known that there exists constants c_1, c_2, c_3, c_4 such that for all but finitely many m

$$2^{c_1 \cdot m^2} \leq R^3_2(m) \leq 2^{2^{c_2 \cdot m}} \tag{1}$$

and

$$2^{c_3 \cdot m^2 (\log(m))^2} \leq R^3_3(m) \leq 2^{2^{c_4 \cdot m}}. \tag{2}$$

For $c \geq 4$ it is known that there exists a double exponential lower bound for the function R^3_c. The asymptotics of R^3_2 and R^3_3 are not known. It is a longstanding open problem to prove or disprove that R^3_3 has a double exponential lower bound. (As far as we know the Erdös award offered for solving this problem is USD 500.)

For attacking this problem (and related problems) we propose to investigate the phase transition problem for the associated Paris Harrington function $R^3_3(f)$ (resp. other Paris Harrington functions in question). We show that a classification for the phase transition for $R^3_3(f)$ will yield advance on the asymptotic of R^3_3.

We now study the grwoth rate behaviour of the function $R^3_3(f)$ from the previous section when f varies from very slow growing functions f to slow growing functions f.

Theorem 17. *Let $f(i) = \frac{1}{c_4}||i||$. Then $R^3_3(f)(m) \leq 2^{2^{c_4 \cdot m}}$ for all but finitely many m.*

Proof. By (2) there is a number K such that for $R^3_3(m) \leq 2^{2^{c_4 \cdot m}}$ for all $m \geq K$. We show that $R^3_3(f)(m) \leq R^3_3(m) =: R$. Let $F : [R]^3 \to 3$ be given. Then there exists $Y \subseteq R$ such that $F \upharpoonright [Y]^3$ has constant value and $card(Y) \geq m$. We claim that even $card(Y) \geq f(\min(Y))$ is true. Indeed $f(\min(Y)) \leq f(R) \leq f(2^{2^{c_4 \cdot m}}) = m \leq card(Y)$. □

Theorem 18. *Let $\alpha > 0$ and $f(i) = |i|^\alpha$. If $R^3_3(m) \leq 2^{m^{\frac{1}{\alpha}}}$ for infinitely many m then $R^3_3(f)(m) \leq 2^{m^{\frac{1}{\alpha}}}$ for infinitely many m.*

Proof. Pick an m such that $R^3_3(m) \leq 2^{m^{\frac{1}{\alpha}}}$. We show that $R^3_3(f)(m) \leq R^3_3(m) =:$ R. Let $F : [R]^3 \to 3$ be given. Then there exists $Y \subseteq [R]$ such that $F \upharpoonright [Y]^3$ has constant value and $card(Y) \geq m$. We claim that even $card(Y) \geq f(\min(Y))$ is true. Indeed $f(\min(Y)) \leq f(R) \leq f(2^{m^{\frac{1}{\alpha}}}) = m \leq card(Y)$. □

Corollary 1. *Let $\alpha > 0$ and $f(i) = |i|^\alpha$. If $R^3_3(f)(m) > 2^{m^{\frac{1}{\alpha}}}$ for all but finitely many m then $R^3_3(m) > 2^{m^{\frac{1}{\alpha}}}$ holds for all but finitely many m.*

Proof. If $R^3_3(f)(m) > 2^{m^{\frac{1}{\alpha}}}$ for all but finitely many m then there are not infinitely many m such that $R^3_3(f)(m) \leq 2^{m^{\frac{1}{\alpha}}}$ hence there are not infinitely many m such that $R^3_3(m) \leq 2^{m^{\frac{1}{\alpha}}}$ and hence $R^3_3(m) > 2^{m^{\frac{1}{\alpha}}}$ holds for all but finitely many m. □

Theorem 19. *Let $\varepsilon > 0$. If $f(i) = \varepsilon \cdot |i|$ then $R^3_3(f)$ eventually dominates every primitive recursive function.*

Theorem 20. *Let $\varepsilon > 0$, $\alpha := \frac{1}{2} + \varepsilon$ and $f(i) = |i|^\alpha$. Then $R^3_3(f)(m) \geq 2^{2^{(1+\varepsilon)^{m/2}}}$ for all but finitely many m.*

Proof. Choose $\delta > 0$ sufficiently small so that

$$(2-\delta)(\frac{1}{2}+\varepsilon) > 1 + \frac{3}{2}\varepsilon. \tag{3}$$

Choose K_0 such that $R^3_2(m) \geq 2^{m^{2-\delta}}+1$ for $m \geq K_0$. We show that the asserted inequality holds for $m \geq 2^{K_0^{1+\frac{3}{2}\varepsilon}}+1$. Define

$$\begin{aligned} v_0 &:= 1, \\ v_1 &:= R^3_2(m) - 1, \\ v_{i+1} &:= R^3_2(f(v_i)) - 1 \text{ for } i \geq 1, \\ v &:= v_{m'-1}, \end{aligned}$$

where m' is the least integer not greater then $\frac{m}{2}$.

Choose $G_1 : [v_0, v_1[^3 \to 2$ such that for all Y with $G_0 \restriction [Y]^3$ having constant value we have $card(Y) < m$. Choose $G_{i+1} : [v_i, v_{i+1}[^3 \to 2$ such that for all Y with $G_0 \restriction [Y]^3$ having constant value we have $card(Y) < f(v_i)$.

Define $G : [v_0, v[^3 \to 3$ as follows

$$G(x,y,z) := \begin{cases} G_i(x,y,z) & \text{if } v_i \leq x < y < z \leq v_{i+1} \text{ for some } i \\ 3 & \text{otherwise} \end{cases} \tag{4}$$

We claim that $v < R^3_3(f)(m)$. The counter example partition is provided by G. Assume that $Y \subseteq v$ and that $G \restriction [Y]^3$ has constant value. We have to show that $card(Y) < \max\{m, f(\min(Y)\} =: m''$. We may assume that $card(Y) \geq 3$.

Case 1: $Y \subseteq [v_0, v_1[$. Then $G_0 \restriction [Y]^3 = G \restriction [Y]^3$ has constant value. Hence $card(Y) < m \leq m''$.

Case 2: $Y \subseteq [v_i, v_{i+1}[$ for some i with $0 < i < m' - 1$. Then $G_i \restriction [Y]^3 = G \restriction [Y]^3$ has constant value. Hence $card(Y) < f(v_i)) \leq f(\min(Y)) \leq m''$.

Case 3: For all $i < m' - 1$ the set Y is not contained in $[v_i, v_{i+1}[$. Then $G \restriction [Y]^3$ has constant value 3. Moreover for all $i < m' - 1$ we have

$$card(Y \cap [v_i, v_{i+1}[) \leq 2. \tag{5}$$

Indeed if we would find three elements x, y, z in some $Y \cap [v_i, v_{i+1}[$ then $G(x,y,z) \neq 3$. By (5) we obtain that $card(Y) \leq (m'-1) \cdot 2 < m \leq m''$.

We now claim that

$$v_i \geq 2^{K_0^{(1+\frac{3}{2}\varepsilon)^i}} \tag{6}$$

and

$$f(v_i) \geq K_0 \tag{7}$$

for $i \geq 1$. Proof of the claim by induction on i. Let us check the case $i = 1$. Then $v_1 = R^3_2(m) - 1 \geq m \geq 2^{K_0^{(1+\frac{3}{2})^1}}$ and $f(v_1) = |v_1|^{\frac{1}{2}+\varepsilon} \geq |2^{m^{2-\delta}}|^{\frac{1}{2}+\varepsilon} \geq$

$m^{1+\frac{3}{2}\varepsilon} \geq K_0$. Now assume that the claim holds for i. Since $f(v_i) \geq K_0$ we can apply the asymptotic for R_2^3 to $f(v_i)$. We thus obtain

$$\begin{aligned} v_{i+1} &= R_2^3(f(v_i)) - 1 \\ &\geq 2^{(f(v_i))^{2-\delta}} \\ &\geq 2^{(K_0^{1+\frac{3}{2}\varepsilon)^i})^{\alpha\cdot(2-\delta)}} \\ &\geq 2^{K_0^{(1+\frac{3}{2}\cdot\varepsilon)^{i+1}}}. \end{aligned}$$

This moreover implies $f(v_{i+1}) \geq K_0$.

Summing up we have shown $R_3^3(f)(m) > 2^{K_0^{(1+\frac{3}{2}\cdot\varepsilon)^{m'-1}}}$. Thus for all but finitely many m we obtain $R_3^3(f)(m) > 2^{K_0^{(1+\varepsilon)^{m/2}}}$.

□

Similarly one shows the following result indicating the relevance of the threshold at $\frac{1}{2}$.

Theorem 21. *Let* $\delta \geq 2$, $\gamma > 0$. *and* $\varepsilon > 0$. *Put* $\alpha := \frac{1}{\delta} + \varepsilon$ *and* $f(i) = |i|^\alpha$. *Assume that* $R_2^3(n) \geq 2^{n^\delta\cdot\gamma}$ *for all but finitely many* n. *Then* $R_3^3(f)(m) \geq 2^{2^{(1+\varepsilon)^m)}}$ *for all but finitely many* m.

Remark: Related phase transitions can be proved for all functions R_3^d for $d \geq 3$.

4.3 A phase transition for R_2^2

Finally we study the Ramsey function for pairs. Here we study a phase transition in terms of densities a concept which goes back to J. Paris [11]. Let f be a number theoretic function. We call a finite set X of natural numbers 0-dense(f,k,l) iff $card(X) \geq \max\{3, f(\min(X))\}$. We call X $n+1$-dense$(f$,$k,l)$ iff for any $F : [X]^k \to l$ there exists a $Y \subseteq X$ such that $F \restriction [Y]^2$ is constant and Y is n-dense$(f$,$k,l)$.

Recall that $R_2^2(k)$ is the least m such that for every $F : [m]^2 \to 2$ there exists a monochromatic $Y \subseteq m$ such that $card(Y) \geq k$. Then, by Erdös's probabilistic method, we know the classical lower bound $R_2^2(k) \geq 2^{\frac{k}{2}}$ for all k [8]. Elementary combinatorics yields further the well known upper bound $R_2^2(k) \leq 2^{2\cdot k}$ for all k [8]. It is open (an Erdös USD 100 problem) whether the limit $\lim_{k\to\infty}(R_2^2(k))^{\frac{1}{k}}$ exists. (Determing the value is an Erdös USD 250 problem.)

Let $\log_4$ denote the logarithm with respect to base 4.

Theorem 22. *Let* $f(i) := \lceil \log_4(\log_4(i - \frac{1}{2}))\rceil$. *Assume that*

$$\rho := \lim_{k\to\infty}(R_2^2(k))^{\frac{1}{k}}$$

exists. Let

$$\mu := \inf\{r \in \mathbb{R} : (\exists K)(\forall m \geq K)\, [4^{4^m}, 4^{4^m} + \lfloor (r)^{\lfloor (r)^m \rfloor} \rfloor[\text{ is } 2-dense(f,2,2)\}.$$

Then $\rho = \mu$.

Proof. It is easy to see that $[4^{4^m}, 4^{4^m} + 4^{4^m}[$ is $2 - dense(f, 2, 2)\}$ using the well known upper bound on R^2_2. Thus $\mu \leq 4$ and $\rho \leq 4$. In the sequel we assume $\rho < 4$. In the case $\rho = 4$ we have $R^2_2(m) < 4^{4^m}$ for almost all m and the following argument shows that $\mu \leq \rho$.

To prove $\rho \geq \mu$ let $\varepsilon > 0$. We may assume $\rho + \varepsilon \leq 4$. Then for almost all m we have

$$R^2_2(m) < \lfloor(\rho + \varepsilon)^m\rfloor. \tag{8}$$

In particular for allmost all m we have

$$R^2_2(\lfloor(\rho + \varepsilon)^m\rfloor) < \lfloor(\rho + \varepsilon)^{\lfloor(\rho+\varepsilon)^m\rfloor}\rfloor. \tag{9}$$

For m large enough so that (8) and (9) hold let $I_m := [4^{4^m}, 4^{4^m} + \lfloor(\rho + \varepsilon)^{\lfloor(\rho+\varepsilon)^m\rfloor}\rfloor[$. We claim that I_m is $2 - dense(f, 2, 2)$. Let $P : [I_m]^2 \to 2$ be any partition. Then there exists by (9) a $Y \subset I_m$ such that $P \restriction [Y]^2$ is constant and such that $card(Y) \geq \lfloor(\rho+\varepsilon)^m\rfloor$. We claim that Y is $1-dense(f, 2, 2)$. For proving this let $Q : [Y]^2 \to 2$ be any partition. Then there exists by (8) a $Z \subseteq Y$ such that $Q \restriction [Z]^2$ is constant and $card(Z) \geq m$. We claim that Z is $0-dense(f, 2, 2)$. Indeed, $f(\min(Z) \leq f(4^{4^m} + \lfloor(\rho + \varepsilon)^{\lfloor(\rho+\varepsilon)^m\rfloor}\rfloor) \leq f(4^{4^m} \cdot 2) \leq m \leq card(Z)$. Thus $\rho + \varepsilon \geq \mu$ for any $\varepsilon > 0$, hence $\rho \geq \mu$.

To prove $\mu \geq \rho$ let again $\varepsilon > 0$. Then for almost all m we have

$$R^2_2(m) > \lfloor(\rho - \varepsilon)^m\rfloor. \tag{10}$$

In particular for allmost all m we have

$$R^2_2(\lfloor(\rho - \varepsilon)^m\rfloor) > \lfloor(\rho - \varepsilon)^{\lfloor(\rho-\varepsilon)^m\rfloor}\rfloor. \tag{11}$$

Let for large enough m $J_m := [4^{4^m}, 4^{4^m} + \lfloor(\rho - \varepsilon)^{\lfloor(\rho-\varepsilon)^m\rfloor}\rfloor[$. We claim that J_m is not $2 - dense(f, 2, 2)$. Indeed, by (11) there exists a partition $P : [J_m]^2 \to 2$ such that $card(Y) < \lfloor(\rho + \varepsilon)^m\rfloor$ for all $Y \subseteq J_m$ such that $P \restriction [Y]^2$ is constant. Pick any such Y. We claim that Y is not $1 - dense(f, 2, 2)$. Indeed by (11) there exists a partition $Q : [Y]^2 \to 2$ such that $card(Z) < m$ for all $Z \subseteq Y$ with $Q \restriction [Z]^2$ is constant. We claim that any such Z is not $0 - dense(f, 2, 2)$. Indeed, $f(\min(Z) \geq f(4^{4^m}) \geq m > card(Z)$. Thus $\rho - \varepsilon < \mu$ for any $\varepsilon > 0$, hence $\rho \leq \mu$.

□

Similar proofs yield the following results.

Theorem 23. *Let* $f(i) := \lceil\log_4(\log_4(i - \frac{1}{2}))\rceil$. *Assume that*

$$\rho := \lim_{k\to\infty} (R^2_2(k))^{\frac{1}{k}}$$

exists. Let

$$\mu := \sup\{r \in \mathbb{R} : (\exists K)(\forall m \geq K)[4^{4^m}, 4^{4^m} + \lfloor(r)^{\lfloor(r)^m\rfloor}\rfloor[\text{ is not } 2{-}dense(f, 2, 2)\}.$$

Then $\rho = \mu$.

Theorem 24. *Let* $f(i) := \lceil \log_4(\log_4(i - \frac{1}{2})) \rceil$*. Assume that*

$$\begin{aligned} \mu &= \sup\{r \in \mathbb{R} : (\exists K)(\forall m \geq K)\ [4^{4^m}, 4^{4^m} + \lfloor (r)^{\lfloor (r)^m \rfloor} \rfloor[\textit{ is not } 2-dense(f,2,2)\} \\ &= \inf\{r \in \mathbb{R} : (\exists K)(\forall m \geq K)\ [4^{4^m}, 4^{4^m} + \lfloor (r)^{\lfloor (r)^m \rfloor} \rfloor[\textit{ is } 2-dense(f,2,2)\}. \end{aligned}$$

Then

$$\rho := \lim_{k \to \infty} (R_2^2(k))^{\frac{1}{k}}$$

exists and $\rho = \mu$.

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