An extremely sharp phase transition threshold for the slow growing hierarchy

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Abstract. We investigate natural systems of fundamental sequences for ordinals below the Howard Bachmann ordinal and study growth rates of the resulting slow growing hierarchies. We consider a specific assignment of fundamental sequences which depends on a non negative real number $\varepsilon$. We show that the resulting slow growing hierarchy is eventually dominated by a fixed elementary recursive function if $\varepsilon$ is equal to zero. We show further that the resulting slow growing hierarchy exhausts the provably recursive functions of $ID_1$ if $\varepsilon$ is strictly greater than zero. Finally we show that the resulting fast growing hierarchies exhaust the provably recursive functions of $ID_1$ for all non negative values of $\varepsilon$.

Our result is somewhat surprising since usually the slow growing hierarchy along the Howard Bachmann ordinal exhausts precisely the provably recursive functions of $PA$. Note that the elementary functions are a very small subclass of the provably recursive functions of $PA$ and the provably recursive functions of $ID_1$ are a very small subclass of the provably recursive functions of $ID_1$. Thus the jump from $\varepsilon$ equal to zero to $\varepsilon$ greater than zero is one of the biggest jumps in growth rates for subrecursive hierarchies one might think of.

This article is part of our general research program on phase transitions in logic and combinatorics. Phase transition phenomena are ubiquitous in a wide variety of branches of mathematics and neighbouring sciences, in particular, physics (see, for example, [6]). An informal description of a ‘phase transition effect’ is the effect behaviour wherein ‘small’ changes in certain parameters of a system occasion dramatic shifts in some globally observed behaviour of the system, such shifts being marked by a ‘sharp threshold point’. An everyday life example of this is the change from one material state to a different one as temperature is increased, with the ‘threshold’ being given by melting/boiling point. Similar phenomena occur in mathematical and computational contexts like evolutionary

⋆ The author is a Heisenberg fellow of the DFG and has been supported in part by DFG grant We 2178 6/1
graph theory (see, e.g., [3, 10]), percolation theory (see, e.g., [9]), computational complexity theory and artificial intelligence (see, for example, [7, 11]).

The purpose of PTLC is to study Phase Transitions in Logic and Combinatorics. We are particularly interested in the transition from provability to unprovability of a given assertion by varying a threshold parameter. On the side of hierarchies of recursive functions this reduces to classifying the phase transition for the growth rates of the functions involved. In this article we are concerned with phase transitions for the slow growing hierarchy and we continue the investigations from [12–15].

From the pure logical side this article is motivated by the classical classification problem for the recursive functions and the resulting problem of comparing the slow and fast growing hierarchies. It has been claimed, for example in [3] p. 439 l.-5, that for sufficiently big prooftheoretic ordinals the slow and fast growing hierarchies will match up. The results of this paper may indicate that this claim might not be true in general.

To formulate the results precisely we introduce some notation. For an ordinal \( \alpha \) less than the Howard Bachmann ordinal let \( N\alpha \) be the number of symbols in \( \alpha \) which are different from 0 and \(+\). The idea is essentially that \( N\alpha \) is the number of edges in the tree which represents the term for \( \alpha \).

For a limit ordinal \( \lambda \) let \( \lambda[x] := \max\{\beta < \lambda : N\beta \leq N\lambda + x\} \). This assignment of fundamental sequences is natural and does not change, as we will show in the appendix, the growth rate of the induced fast growing hierarchy. But, as our first main theorem shows, the induced slow growing hierarchy (along the Howard Bachmann ordinal) consists of elementary functions only. This generalizes results from [4] where we showed that the resulting slow growing hierarchy along \( \Gamma_0 \) consists of elementary recursive functions only.

At first sight the resulting slow growing hierarchies seem always to collapse under this assignment of fundamental sequences and one may wonder how robust this phenomenon is. We prove therefore in a separate section a very surprising and extremely sharp phase transition threshold for the slow growing hierarchy. The upshot is that small changes prevent the hierarchies from collapsing. For a given real number \( \varepsilon \geq 0 \) let \( \lambda[x]_\varepsilon := \max\{\beta < \lambda : N\beta \leq (1 + \varepsilon) \cdot N\lambda + x\} \). Then, as we just said, for \( \varepsilon = 0 \) the resulting slow growing hierarchy is very slow growing but for any \( \varepsilon > 0 \) the resulting slow growing hierarchy becomes fast growing and matches up with the fast growing hierarchy at all \( \varepsilon \)-numbers below the Howard-Bachmann ordinal. We conjecture that within the phase transition, i.e. when in the definition of \( \lambda[x]_\varepsilon \) the number \( \varepsilon \) is a function of \( \lambda \) and \( x \), we may arrange other behaviours of the resulting slow growing hierarchy.

The paper is not fully self contained. The proof of the first main theorem requires basic familiarity with Buchholz style notation systems for the Howard Bachmann ordinal. (Knowledge of [4] is more then sufficient.) The proof of the second main result should be generally accessible (at least when one restricts the consideration to ordinals below \( \varepsilon_0 \).
Proof of the first main result

1.1 Basic concepts

We recall some basic definitions and facts from Buchholz papers on ordinal notations. Missing proofs can be found in [4].

**Definition 1** Inductive definition of a set of terms \( T \) and a set \( P \subseteq T \).

- \( 0 \in T \), \( a \in T \) & \( i \in \{0,1\} \Rightarrow D_i a \in P \),
- \( a_0, \ldots, a_n \in P \) and \( n \geq 1 \Rightarrow (a_0, \ldots, a_n) \in T \).

**Notations for Section 1:**
1. \( a, b, c, d, e \) range over \( T \).
2. If \( a \in P \), then we identify the one element sequence \( (a) \) with the term \( a \).
3. The empty sequence \( ( \) \) is identified with the term \( 0 \).
4. \( x, y, z, i, l, m, n \) range over non negative integers.

**Definition 2** Recursive definition of \( a \prec b \) for \( a, b \in T \).

\( a \prec b \) holds, iff one of the following cases holds:
1. \( a = 0 \) and \( b \neq 0 \),
2. \( a = D_0 a_0 \& b = D_1 b_0 \),
3. \( a = D_i a_0 \& b = D_i b_0 \& a_0 \prec b_0 \),
4. \( a = (a_0, \ldots, a_m) \& b = (b_0, \ldots, b_n) \& 1 \leq m + n \) and
   \( (m < n \& \forall i \leq n(a_i = b_i)) \) or \( (\exists k \leq \min\{m, n\}(\forall i < k(a_i = b_i) \& a_k \prec b_k)) \)

**Lemma 1** \((T, \prec)\) is a linear order.

\((T, \prec)\) is not a well-order. To see this, let \( a_0 := D_1 0 \) and \( a_{n+1} := D_0 a_n \). Then \( a_{i+1} \prec a_i \) for all \( i \). In the sequel we thin out \( T \) to a smaller set \( O T \) which is well-ordered by \( \prec \).

**Definition 3** Assume that \( M, N \subseteq T \).

\( M \preceq N : \iff \forall x \in M \exists y \in N(x \preceq y) \),
\( a \succeq N : \iff \{a\} \preceq N \),
\( M \prec a : \iff \forall x \in M(x \prec a) \).

\( (a_1, \ldots, a_m) + (b_1, \ldots, b_n) := (a_1, \ldots, a_i, b_1, \ldots, b_n) \)
where \( i \in \{1, \ldots, m\} \) is maximal with \( b_1 \leq a_i \).

**Definition 4** Recursive definition of \( K^* a \) and \( K a \) for \( a \in T \).

\( K^* 0 := \emptyset \), \( K 0 := \emptyset \),
\( K^*(a_0, \ldots, a_n) := \bigcup_{i \leq n} K^* a_i \), \( K(a_0, \ldots, a_n) := \bigcup_{i \leq n} K a_i \),
\( K^* D_1 a := K^* a \), \( K D_1 a := K a \),
\( K^* D_0 a := \{a\} \cup K^* a \), \( K D_0 a := \{D_0 a\} \).

**Lemma 2** \( K^* c \prec a \Rightarrow K c \prec D_0 a \).
Lemma 4 1. $\text{tp}(a) = \Omega \wedge c \in T_0 \Rightarrow a\{c\} \subseteq a$.
2. $\text{tp}(a) = \Omega \wedge c, d \in T_0 \wedge c < d \Rightarrow a\{c\} \prec a\{d\}$.
3. $\text{tp}(a) = \Omega \wedge c \in T_0 \Rightarrow K^*a\{c\} \subseteq K^*a\{0\} \cup K^*c$.
4. $a \in \text{OT}, \text{tp}(a) = \Omega \wedge c \in \text{OT}_0 \Rightarrow a\{c\} \in \text{OT}$.

Definition 8 Recursive definition of $a\{c\} \in T$ for $c \in T_0$ and $a \in T$ with $\text{tp}(a) = \Omega$.

$\Omega[c] := c$.

$(D_1a)\{c\} := D_1a\{c\}$.

$a = (a_0, \ldots, a_n) \Rightarrow a\{c\} := (a_0, \ldots, a_{n-1}) + a_n\{c\}$.

Theorem 1 $(\text{OT}, \prec)$ is a well-order. The order type of $(\text{OT}_0, \prec)$ is equal to the Howard Bachmann ordinal.

Definition 6 $b \triangleleft a : \iff b < a \wedge \forall d \in \text{OT} \ (b \leq d \leq a \Rightarrow K^*b \leq K^*d \cup K^*c)$.

Lemma 3 1. $a \prec D_1a$.
2. $a, b \in \text{OT}_0 \wedge a < b \Rightarrow K^*a \leq K^*b$.
3. $b \triangleleft a \wedge K^*a < b \wedge b, a \in \text{OT} \Rightarrow K^*b < b$.
4. $b \triangleleft a \Rightarrow d + b \triangleleft d + a \wedge D_1b \triangleleft D_1a \ (i \in \{0, 1\})$.

Proof. We prove assertion 3. Assume that $b \leq K^*b$. Choose a subterm $d$ of $b$ with $b \leq K^*d$ such that the length of $d$ is minimal possible. Then $d = D_0c$ with $K^*c < b \leq e \leq a$, since $K^*b \leq K^*c \cup K^*a < a$. Then we obtain $K^*b \leq K^*c \cup K^*e < b$. Contradiction.

Definition 7 Definition of $\text{tp}(a) \in \{0, 1, \omega, \Omega\}$ for $a \in T$.

$\text{tp}(0) := 0$.

$\text{tp}(1) := 1$.

$\text{tp}(\Omega) := \Omega$.

$\text{tp}(a) = 1 \Rightarrow \text{tp}(D_1a) := \omega$.

$\text{tp}(a) = \omega \Rightarrow \text{tp}(D_1a) := \omega$.

$\text{tp}(a) = \Omega \Rightarrow \text{tp}(D_0a) := \omega$.

$\text{tp}(a) := \text{tp}(a)$.

$\text{tp}(\{a_0, \ldots, a_n\}) := \text{tp}(a_n)$.
1.2 Refined concepts

Here we collect some technical material which is needing during the proof of the first main result.

**Definition 9** Recursive definition of $Na$ for $a \in T$.

$N0 := 0$.

$Na := Na0 + \ldots + Na_n$.

$NDa := 1 + Na$.

**Definition 10** Definition of $a[x]$ and $a[x]$ for $a \in OT \setminus \{0\}$.

$a[x] := \max\{b \in OT : b < a & Nb \leq Na + x\}.$

$a[x] := \max\{b \in OT : b < a & Nb \leq x\}.$

**Lemma 5** 1. $(a0, \ldots, an) [x] = (a0, \ldots, an−1) + a_n [x]$. 2. $a = (a0, \ldots, an), b = (a1, \ldots, an), x \geq Na0 \Rightarrow a[x] = a0 + b[x − Na0].$ 3. $a = (a0, \ldots, an), x < Na0 \Rightarrow a[x] = a0 [x].$ 4. $a[0] = 0.$ 5. $(D1a)[x] = D1a[x].$ 6. $x > 0 \Rightarrow (D1a)[x] = D1a[x − 1].$

**Definition 11** Recursive definition of $Ga(x)$ for $a \in OT0$.

$Ga0(x) := 0.$

$Ga_{a+1}(x) := Ga(x) + 1.$

$Ga(a[x]) := Ga[a](x)$ if $tp(a) = \omega$.

**Lemma 6** Let $a, b \in OT0$.

1. $a = (a0, \ldots, an) \Rightarrow Ga(a) = Ga0(a) + \ldots + Ga_n(a).$ 2. $a = D0(b + 1) \Rightarrow Ga(a) = GD0b(x) + Ga[x+1](x).$ 3. $a â‰‡ b & Na \leq x + 1 \Rightarrow Ga(x) \leq Ga(x).$

Assertion 2 motivates the definition of the assignment $\llbracket \cdot \rrbracket$. Moreover it gives a first indication why the resulting slow growing hierarchy will collapse since the second term $Ga[x+1](x)$ refers to $a[x + 1]$ which is in general very small when compared to $a$. In the sequel we verify that this phenomenon also holds true in more complicated situations.

**Definition 12** Definition of $Ta(x)$ for $a \in T$.

$T0(a) := 1.$

$Ta(x) := Ta[a](x) + 2.$

**Remark.** $Ta(x)$ is defined by recursion on the cardinality of the set $\{b < a : Nb \leq x\}$. The asymptotic of $Ta$ is very interesting from the analytic number theory point of view. We conjecture that sharp bounds on $Ta$ will prove useful to obtain good upper bounds on $Ga$ but in this article we will just prove that each $Ga$ is elementary for $a \in OT$. 
Lemma 7  1. $a \leq b \& Na \leq x \Rightarrow T_a(x) \leq T_b(x)$.
2. $a \leq b \Rightarrow T_a(x) \leq T_b(x)$.

Proof. 1. By induction on the cardinality of the set $\{c < b : Nc \leq x\}$. Assume that $T_b(x) = T_b[x]](x) + 2$ and $a \neq b$. Then $a \leq b[[x]]$ and the induction hypothesis yields $T_a(x) \leq T_b[x]](x)$ and the assertion follows.
2. If $a = 0$ then the assertion is clear. Assume that $T_a(x) = T_a[x]](x) + 2$ and $T_b(x) = T_b[[x]](x) + 2$. Then $a[[x]] \leq b[[x]]$ and the assertion follows from 1.

Definition 13 Recursive definition of $C_x(a, g)$ for $a \in OT$ and $g, x < \omega$.
1. $C_x(0, g) := 0$.
2. $C_x((a_0, \ldots, a_n), g) := C_x(a_0, g) + \cdots + C_x(a_n, g)$.
3. $C_x(D_\alpha a, g) := g \cdot G_{D_\alpha a}(x)$.
4. $C_x(D_\beta a, g) := g^{2^\tau_{D_\beta a}(x+1)} \cdot (C_x(a, g) + 1)$.

Lemma 8 If $a \in OT_\alpha$ then $C_x(a, g) = g \cdot G_a(x)$.

Proof by induction on $Na$.
1. $a = 0$. Then the assertion is obvious.
2. $a = (a_0, \ldots, a_n)$. Then the induction hypothesis yields $C_x(a, g) = C_x(a_0, g) + \cdots + C_x(a_n, g) = g \cdot G_{a_0}(x) + \cdots + g \cdot G_{a_n}(x) = g \cdot G_{a_0, \ldots, a_n}(x)$.
3. $a = D_\beta b$. Then $C_x(a, g) = g \cdot G_a(x)$.

The following Lemma is a crucial tool in proving the hierarchy collapse.

Lemma 9 (Mainlemma) If $a, b \in OT$, $K^b \prec a$, $N_b \leq x + 1$ and $g = G_{(D_\alpha a)[x+1]}(x)$ then

$$C_x(b, g) \leq g^{2^\tau_b(x+1)+1}$$

Proof by induction on $N_b$.
1. $b = 0$. Then the assertion is obvious.
2. $b = (b_0, \ldots, b_n)$. Then $n + 1 \leq x + 1$, $N_b \leq x + 1$ and $K^b b_i \leq K^b b \prec a$ for $i = 0, \ldots, n$. Thus $C_x(b, g) = C_x(b_0, g) + \cdots + C_x(b_n, g) = g^{2^\tau_{b_0}(x+1)+1} + \cdots + g^{2^\tau_{b_n}(x+1)+1} \leq (x + 1) \cdot g^{2^\tau_{b}(x+1)+1} \leq g^{2^\tau_{b}(x+1)+1}$, since $x + 1 \leq g$ and $T_b(x + 1) < T_b(x + 1)$ by Lemma 7.
3. $b = D_\beta c$. Then $C_x(b, g) = g \cdot G_{D_c c}(x)$. $K^b b \prec a$ yields $b \prec D_\alpha a$. Thus $N_b \leq x + 1$ yields $b \preceq (D_\alpha a)[x+1]$ and $G_b(x) \leq G_{(D_\alpha a)[x+1]}(x)$ hence $C_x(b, g) \leq g^2$.
4. $b = D_\gamma c$. Then the induction hypothesis yields $C_x(c, g) \leq g^{2^\tau_c(x+1)+1}$ hence

$$C_x(b, g) = g^{2^\tau_{D_\gamma c}(x+1)} \cdot (C_x(c, g) + 1) \leq g^{2^\tau_{D_\gamma c}(x+1)} \cdot (g^{2^\tau_c(x+1)+1} + 1) \leq g^{2^\tau_{D_\gamma c}(x+1)+1}$$

since $T_c(x + 1) + 1 < T_{D_\gamma c}(x + 1)$ because $c \prec D_\gamma c$ and $N_c \leq x$. 
1.3 Collapsing ordinals with countable cofinalities

Lemma 10 If \( x < Na \) then \( K^*a[x] \preceq K^*a \).

Proof by induction on \( Na \) using Lemma 5.

1. \( a = 0 \). Then the assertion is obvious.
2. \( a = (a_0, \ldots, a_n) \). Let \( b := (a_1, \ldots, a_n) \).
3. \( x < Na \). Then \( a[x] = a_0[x] \) and the induction hypothesis yields \( K^*a[x] = K^*a_0[x] \preceq K^*a \) by assertion 2 of Lemma 3.
4. \( x = Na \). Then \( a[x] = a_0 \) and \( K^*a_0 \subseteq K^*a \).
5. \( x > Na \). Then \( a[x] = a_0 + b[x - Na_0] \), \( x < Na \) yields \( x - Na_0 < Nb \) and the induction hypothesis yields \( K^*b[x - Na_0] \preceq K^*b \). Thus \( K^*a[x] = K^*a_0 \cup K^*b[x - Na_0] \preceq K^*a_0 \cup K^*b = K^*a \).

Lemma 11 Assume that \( a, b \in OT \), \( \text{tp}(b) = \omega \) and \( b[x] \preceq a \preceq b \) then \( K^*b[x] \preceq K^*a \).

Proof by induction on \( Nb \).

1. \( b = (b_0, \ldots, b_n) \). Then we have \( b[x] = (b_0, \ldots, b_{n-1}) + b_n[x] \) by assertion 1 of Lemma 5. \( b[x] \preceq a \preceq b \) yields \( a = (b_0, \ldots, b_{n-1}) + c \) for some \( c \) with \( b_n[x] \preceq c \preceq b_n \). The induction hypothesis yields \( K^*b_n[x] \preceq K^*c \) hence \( K^*b[x] \preceq K^*a \).
2. \( b = D_1c \). Then \( b[x] \preceq a \preceq b \prec \Omega \) and \( K^*b[x] \preceq K^*a \).
3. \( b = D_1c \).
   1. \( \text{tp}(c) = \omega \). Then \( b[x] = D_1c[x] \preceq a \preceq D_1c \). Thus \( a = D_1d + e \) for some \( d \) with \( c[x] \preceq d \preceq c \). The induction hypothesis yields \( K^*c[x] \preceq K^*d \) hence \( K^*b[x] \preceq K^*a \).
   2. \( c = d + 1 \). Then \( (D_1c)[x] = D_1d \cdot y + (D_1d)[z] \) with \( z < ND_1d \) and \( y > 0 \).
   3. \( a = (D_1d)[z] \). Hence \( a \preceq D_1c \) yields \( a = D_1d + e \) for some \( e \prec D_1c \) hence \( K^*b[x] \preceq K^*a \).

Lemma 12 Assume that \( b \in OT \) and \( \text{tp}(b) = \omega \).

1. \( b[x] \prec_c b \).
2. \( K^*b[x] \prec b[x] \).
3. \( D_0b[x] \in OT \).

Proof. Assertions 2 and 3 follow from assertion 1. Assertion 1 itself follows from Lemma 11.

Lemma 13 Assume \( D_0b \in OT \). If \( x < \omega \) and \( \text{tp}(b) = \omega \) then \( D_0b[x] \in OT \) and \( (D_0b)[x] = D_0b[x] \).
Proof. Lemma 12 yields $K^*b[x] < b[x]$, hence $D_0b[x] \in OT$. By definition we have $D_0b[x] \leq (D_0b)[x]$. Assume now that $(D_0b)[x] = D_0c + d$ for some $d < D_0(c + 1)$.

If $d \neq 0$ then $D_0(c + 1)$ would be a better choice for $(D_0b)[x]$ than $D_0c + d$. Hence $d = 0$. We have $N(D_0b)[x] = 1 + Nb + x = 1 + Nc$, thus $Nc = Nb + x$. $c \prec b$ yields $c \preceq b[x]$ hence $D_0c \leq D_0b[x]$. Therefore $D_0b[x] = (D_0b)[x]$.

Lemma 14 Assume that $a, b \in OT$, $tp(b) = \omega$, $K^*b \prec a$ and $g = G(D_0a)[x+1](x)$. Then $C_x(b[x], g) \leq C_x(b, g)$.

Proof by induction on $b$.
1. $b = (b_0, \ldots, b_n)$ with $tp(b_n) = \omega$.

Then $K^*b_n \prec a$ and the induction hypothesis yields $C_x(x, g) \leq C_x(b_n, g)$. Then $C_x(b[x], g) = C_x(b_0, g) + \cdots + C_x(b_n, g) \leq C_x(b_0, g) + \cdots + C_x(b_n, g) = C_x(b, g)$.
2. $b = D_0c$.

Then $b[x] \prec \Omega$ and Lemma 8 yields $C_x(b[x], g) = g \cdot G(b[x])(x) = g \cdot G(b) = C_x(b, g)$.
3. $b = D_1c$ where $tp(c) = \omega$.

We have $K^*c \subseteq K^*b \prec a$ and the induction hypothesis yields $C_x(c[x], g) \leq C_x(c, g)$. Therefore Lemma 7 yields

$C_x(b[x], g) = C_x(D_1c[x], g)$
$= g^{2^{TD_1c[z]} \cdot (C_x(c[x], g) + 1)}$
$\leq g^{2^{TD_1c[z]} \cdot (C_x(c, g) + 1)}$
$= C_x(b, g)$.

4. $b = D_1c$ where $c = d + 1$.

In this critical case we have $b[x] = D_1d + (D_1c)[x + 1] = D_1c + y + (D_1c)[z]$ where $z < ND_1c$ and $y > 0$. Lemma 10 yields $K^*(D_1c)[z] \preceq K^*D_1c \preceq K^*b \prec a$. Hence Lemma 9 yields

$C_x(b[x], g) = C_x(D_1d, g) + C_x((D_1c)[x + 1], g)$
$\leq g^{2^{TD_1c[z]} \cdot (C_x(d, g) + 1) + g^{2^{TD_1c[z]} \cdot (C_x(c, g) + 1)}}$
$= C_x(b, g)$

since $TD_1c(x + 1) > TD_1c(x + 1) + 1$ and $TD_1c(x + 1) \geq TD_1d(x + 1)$.

1.4 Collapsing ordinals with uncountable cofinalities

Lemma 15 $tp(a) = \Omega \& a\{0\} \prec b \prec a \Rightarrow Na\{0\} < Nb$.

Proof by induction on $Na$.
1. $a = \Omega$. Then $a\{0\} = 0$ and the assertion is obvious.
2. $a = (a_0, \ldots, a_n)$. $a \{0\} \prec b \prec a$ yields $b = (a_0, \ldots, a_{n-1}, c)$ for some $c$ with $a_n \{0\} \prec c \prec a_n$. The induction hypothesis yields $Na_n \{0\} < Nc$ hence $Na \{0\} < Nb$.

3. $a = D_1 c$. $a \{0\} = D_1 c \{0\} \prec b \prec D_1 c$ yields $b = D_1 d + e$ for some $e < D_1 (d+1)$ and $c \{0\} \leq d < c$. The induction hypothesis yields $Nc \{0\} \leq Nd$. If $e \neq 0$ then $Nb \geq Nd + Ne + 1 > Na$. If $e = 0$ then $c \{0\} < d < c$. The induction hypothesis yields $Nc \{0\} < Nd$ hence the assertion.

**Lemma 16** Assume that $D_0 a, b, c \in OT$, $tp(a) = \Omega$, $b \preceq c \preceq a$, $K^* b \prec a$ and $Nb \leq Nc + x$. Then $b \preceq c \{D_0 a[x + 1]\}$.

*Proof.* by induction on $Nb$.

1. $b = 0$. Then the assertion is obvious.

2. $b = (b_0, \ldots, b_m)$. Assume that $c = (c_0, \ldots, c_n)$.

2.1. $b_0 = c_0, \ldots, b_m = c_m$ and $m < n$. Then $b \preceq (c_0, \ldots, c_m) \preceq c \{D_0 a[x + 1]\}$.

2.2. $\exists i \leq \min \{m, n\}[b_0 = c_0, \ldots, b_{i-1} = c_{i-1}, b_i < c_i]$. If $i < n$ then $b \prec (c_0, \ldots, c_i) \preceq c \{D_0 a[x + 1]\}$.

Assume $i = n$ and $b_0, \ldots, b_m \prec c_n$.

2.2.1. $m = n$. $Nb \leq Nc + x$ yields $Nb_n \leq Nc_n + x$. The induction hypothesis yields $b_n \preceq c_n \{D_0 a[x + 1]\}$ hence $b \preceq c \{D_0 a[x + 1]\}$.

2.2.2. $m > n$. $Nb \leq Nc + x$ yields $Nb_0, \ldots, Nb_m \leq Nc_n + x - 1$. The induction hypothesis yields for $x > 0$ that $b_n, \ldots, b_m \preceq c_n \{D_0 a[x]\} \prec c_n \{D_0 a[x + 1]\}$.

For $x = 0$ we obtain $b_0, \ldots, b_m \preceq c_n \{0\} = c_n \{D_0 a[x]\}$ by Lemma 15. Since $c_0 \{D_0 a[x + 1]\} \in P$ we obtain $(b_0, \ldots, b_m) \prec c_0 \{D_0 a[x + 1]\}$ hence $b \prec c \{D_0 a[x + 1]\}$ holds for $x \geq 0$.

3. $b = D_0 c$. $K^* b \prec a$ yields $K^* c \cup \{c\} \prec a$ hence $b \prec D_0 a$, thus $b \preceq (D_0 a)[Nb]$.

3.1. $c = \Omega$. Then $Nc = 1$ and $b \preceq (D_0 a)[x + 1] = c \{D_0 a[x + 1]\}$.

3.2. $\Omega \prec c$. Then $b \preceq \Omega \preceq c \{0\} \preceq c \{D_0 a[x + 1]\}$.

4. $b = D_1 d$.

4.1. $c = (c_0, \ldots, c_n)$ with $n \geq 1$.

Then $b = D_1 d \preceq c_0 \preceq c \{0\} \preceq c \{D_0 a[x + 1]\}$.

4.2. $c = D_1 e$. $Nb \leq Nc + x$ yields $Nd \leq Ne + x$. The induction hypothesis yields $d \preceq c \{D_0 a[x + 1]\}$ since $e \prec D_1 e \preceq a$. Thus $b = D_1 d \preceq D_1 c \{D_0 a[x + 1]\} = c \{D_0 a[x + 1]\}$.

**Corollary 1** Assume that $tp(a) = \Omega$, $b \preceq a$, $K^* b \prec a$ and $Nb \leq Na + x$. Then $b \preceq a \{D_0 a[x + 1]\}$.

*Proof.* Put $c = a$ in Lemma 16.

**Lemma 17** Assume that $D_0 a \in OT$ and $tp(a) = \Omega$. Let $z < ND_0 a$. Then $(D_0 a)[z] = D_0 a \{0\}$ if $z = ND_0 a \{0\}$ and $(D_0 a)[z] = (D_0 a \{0\})[z]$ else.

*Proof.* It suffices to show $(D_0 a)[z] \preceq D_0 a \{0\}$. Assume for a contradiction that $D_0 a \{0\} \prec (D_0 a)[z] \prec D_0 a$. Then $(D_0 a)[z] = D_0 b + d$ for some $b$ with $a \{0\} \preceq b \prec a$. Lemma 15 yields $Nb \geq Na \{0\}$. If $d \neq 0$ then $N(D_0 b + d) = \geq N(D_0 a \{0\} + 1 =$
ND_0a. This contradicts \( z < ND_0a \). Hence \( d = 0 \) and \( a\{0\} \prec b \prec a \). Lemma 15 yields \( Nb > Na\{0\} \) hence \( ND_0b \geq ND_0a \). This contradicts \( z < ND_0a \).

**Lemma 18** Assume that \( D_0a \in OT \) and \( tp(a) = \Omega \). Let \( z < N(D_0a) \). Let \( d_0a(0, z) := (D_0a(0))\{z\} \) and \( d_0a(y + 1, z) := D_0a(d_0a(y, z)) \). Then \( d_0a(y, z) \prec d_0a(y + 1, z) \) and \( d_0a(y, z) \in OT \). Moreover \( (D_0a)[x] = d_0a(y, z') \) where \( y \) and \( z' \) are chosen such that \( (Na + 1) \cdot y + z' = x \) and \( z' < Na + 1 \).

**Proof.** By induction on \( y \) we show \( d_0a(y, z) < d_0a(y + 1, z) \).

Assume first that \( y = 0 \).

Then \( d_0a(0, z) = (D_0a(0))\{z\} \) and \( d_0a(1, z) = D_0a((D_0a(0))\{z\}) \). If \( z = 0 \) then \( d_0a(0, z) < d_0a(1, z) \) is obvious. Assume that \( z \neq 0 \). Lemma 10 yields \( K^*(D_0a(0))\{z\} \leq K^*(D_0a(0)) \leq K^*\{a\{0\} \cup \{a\{0\} \prec a(D_0a(0))\{z\}) \}. \) Lemma 2 yields \( K(D_0a(0))\{z\} < D_0a((D_0a(0))\{z\}) \) hence \( (D_0a(0))\{z\} < D_0a((D_0a(0))\{z\}) \).

Now assume that \( y = y' + 1 \).

The induction hypothesis yields \( d_0a(y', z) < d_0a(y' + 1, z) \) hence \( a\{d_0a(y', z)\} < a\{d_0a(y' + 1, z)\} \) thus \( d_0a(y, z) < d_0a(y + 1, z) \).

By induction on \( y \) we show \( d_0a(y, z) \in OT \).

Assume \( y = 0 \).

Then \( d_0a(y, z) = (D_0a(0))\{z\} \in OT \).

Assume \( y = y' + 1 \).

Then \( d_0a(y, z) := D_0a\{d_0a(y', z)\} \). The induction hypothesis yields \( d_0a(y', z) \in OT \) hence \( a\{d_0a(y', z)\} \in OT \).

We have to show \( K^*\{a\{d_0a(y', z)\} \prec a\{d_0a(y', z)\} \} \) and compute \( K^*\{a\{d_0a(y', z)\} \leq K^*\{a\{0\} \cup K^*\{d_0a(y', z)\} \). Lemma 4 yields \( a\{0\} <_0 a \) hence \( K^*\{a\{0\} \prec a\{0\} \) since \( K^*\{a\{0\} \prec a\{0\} \).

We first consider the case \( y' = 0 \). If \( z = 0 \) then \( d_0a(y', z) = 0 \) hence \( K^*\{d_0a(y', z)\} = K^*\{a\{0\} \prec a\{0\} \). If \( z > 0 \) then \( (D_0a(0))\{z\} \not\leq 0 \) and Lemma 10 yields \( K^*\{d_0a(y', z)\} \leq K^*\{d_0a(0) \cup \{a\{0\} \prec a\{0\} \) since \( K^*\{a\{0\} \prec a\{0\} \) as we have already shown that \( \{d_0a(y' - 1, z)\} \prec d_0a(y', z) \). The induction hypothesis yields \( d_0a(y', z) \in OT \) hence \( K^*\{d_0a(y' - 1, z)\} \prec a\{d_0a(y' - 1, z)\} \) hence \( K^*\{d_0a(y', z)\} = K^*\{d_0a(d_0a(y' - 1, z)\} \leq K^*\{a\{0\} \cup K^*\{d_0a(y' - 1, z)\} \cup \{d_0a(y' - 1, z)\} \leq \{d_0a(y' - 1, z)\} \prec d_0a(y', z) \).

Now we prove \( (D_0a)[x] = d_0a(y, z') \) by induction on \( x \) where \( (Na + 1) \cdot y + z' = x \) and \( z' < Na + 1 \). If \( x < N(D_0a) \) then the assertion follows from Lemma 17. Now assume that \( x \geq N(D_0a) \). The choice of \( y \) and \( z' \) and the definition of \( D_0a[x] \) yield \( (D_0a)[x] \geq d_0a(y, z') \) since \( Nd_0a(y, z') = x \).

Now assume that \( (D_0a)[x] = D_0b + c \) with \( b \prec a \) and \( c \prec D_0(b + 1) \). Then \( c = 0 \) since otherwise \( D_0(b + 1) \) would be a better choice than \( D_0b + c \) for \( (D_0a)[x] \).

We have \( b \preceq a, \), \( tp(a) = \Omega, \) \( K^*b \prec b \) and \( Nb \leq x = Na + 1 + x - Na - 1 \). The induction hypothesis yields \( (D_0a)[x - Na - 1] = d_0a(y, z) \). Lemma 1 yields \( b \preceq a((D_0a)[x - Na]) \) hence \( D_0b \triangleq D_0a((D_0a)[x - Na]) = d_0a(y, z) \in OT \).
Lemma 18 yields

**Proof.**

1. **Corollary 2** Assume that \( \text{tp}(a) = \Omega \). Then \( D_{0}(D_{0}[a][x]) \in \Omega \text{ and } (D_{0}[a])[x] = D_{0}(D_{0}[a][x + 1]) \). 

2. **Recursive definition of a nominal form** \( C_{x}(a, g) \) for \( a \in \Omega \text{ and } g < \omega \).

   1. \( C_{x}(\Omega, g) := * \).
   2. \( C_{x}((a_{0}, \ldots, a_{n-1}, a_{n}), g) := C_{x}(a_{0}, g) + \cdots + C_{x}(a_{n-1}, g) + C_{x}(a_{n}, g) \).
   3. \( C_{x}(D_{1}, g) := g^{2^{|D_{1}|g^{x+1}}} \cdot (C_{x}(a, g) + 1) \).

   If \( C \) is a nominal form then \( C[\star := c] \) denotes the result of replacing every occurrence of \( * \) in \( C \) by \( c \).

3. **Lemma 19** If \( a \in \Omega \text{ and } \text{tp}(a) = \Omega \text{ and } c \in OT_{0} \) then \( C_{x}(a[c], g) \leq C_{x}(a, g)[\star := g \cdot C_{c}(x)] \).

   **Proof by induction on \( Na \).**
   1. \( a = \Omega \). Then Lemma 8 yields \( C_{x}(a[c], g) = C_{x}(c, g) = g \cdot C_{c}(x) = \star[\star := g \cdot C_{c}(x)] \).
   2. \( a = (a_{0}, \ldots, a_{n}) \). Then the induction hypothesis yields \( C_{x}(a[c], g) = C_{x}(a_{0}, g) + \cdots + C_{x}(a_{n-1}, g) + C_{x}(a_{n}[c], g) \leq C_{x}(a_{0}, g) + \cdots + C_{x}(a_{n-1}, g) + C_{x}(a_{n}, g)[\star := g \cdot C_{c}(x)] \).
   3. \( a = D_{1}b \).

   Then assertion 2 of Lemma 7 and the induction hypothesis yields
   
   \[
   C_{x}(a[c], g) = C_{x}(D_{1}(b[c]), g) = g^{2^{|D_{1}|g^{x+1}}} \cdot (C_{x}(b[c], g) + 1) \leq g^{2^{|D_{1}|g^{x+1}}} \cdot (C_{x}(b, g)[\star := g \cdot C_{c}(x)] + 1) = C_{x}(a, g)[\star := g \cdot C_{c}(x)].
   \]

4. **Lemma 20** If \( a \in \Omega \text{ and } \text{tp}(a) = \Omega \) then \( C_{x}(a, g)[\star := g^{2}] \leq C_{x}(a, g) \).

   **Proof by induction on \( Na \).**
   1. \( a = \Omega \). Then \( C_{x}(a, g)[\star := g^{2}] = g^{2} \) and \( C_{x}(D_{1}0, g) = g^{2^{|D_{1}|0^{x+1}}} \cdot (0 + 1) \geq g^{2} \).
   2. \( a = (a_{0}, \ldots, a_{n}) \).

   Then the induction hypothesis yields \( C_{x}(a, g)[\star := g^{2}] = C_{x}(a_{0}, g) + \cdots + C_{x}(a_{n-1}, g) + C_{x}(a_{n}, g)[\star := g^{2}] \leq C_{x}(a_{0}, g) + \cdots + C_{x}(a_{n-1}, g) + C_{x}(a_{n}, g) = C_{x}(a, g) \).
   3. \( a = D_{1}b \).

   Then the induction hypothesis yields
   
   \[
   C_{x}(a, g)[\star := g^{2}] = g^{2^{|D_{1}|g^{x+1}}} \cdot (C_{x}(b, g)[\star := g^{2}] + 1) \leq g^{2^{|D_{1}|g^{x+1}}} \cdot (C_{x}(b, g) + 1) = C_{x}(a, g).
   \]
1.5 Putting things together

**Theorem 2** Let $D_{0}a \in OT_{0}$ and $g := G_{(D_{0}a)[x+1]}(x)$. Let $D_{0}b \in OT$ and assume that $K^{*}D_{0}b \prec a$. Then

$$G_{D_{0}b}(x) \leq 1 + C_{x}(b, g).$$

**Proof** by induction on $D_{0}b$.

1. $G_{D_{0}a}(x) = 1 \leq 1 + C_{x}(0, g)$.

2. $b = c + 1$. Then $G_{D_{0}b}(x) = G_{D_{0}c+(D_{0}b)[x+1]}(x) = G_{D_{0}c}(x) + G_{(D_{0}b)[x+1]}(x) \leq 1 + C_{x}(c, g) + g = 1 + C_{x}(c + 1, g)$ since $C_{x}(1, g) = g$ and $G_{(D_{0}b)[x+1]}(x) \leq G_{(D_{0}a)[x+1]}(x)$ by assertion 3 of Lemma 6.

3. $tp(b) = \omega$.

Then the induction hypothesis, Lemma 13 and Lemma 14 yield

$$G_{D_{0}b}(x) = G_{(D_{0}b)[x]}(x) = G_{D_{0}(b|x)}(x) \leq 1 + C_{x}(b|x, g) \leq 1 + C_{x}(b, g).$$

4. $tp(b) = \Omega$.

Then the induction hypothesis, Lemma 2, Lemma 19 and Lemma 20 yield

$$G_{D_{0}b}(x) = G_{D_{0}(b|[x+1])}(x)$$

$$\leq 1 + C_{x}(b|[x+1], g)$$

$$\leq 1 + C_{x}(b, g)[\ast := g \cdot G_{(D_{0}b)[x+1]}(x)](x)$$

$$\leq 1 + C_{x}(b, g)[\ast := g^{2}]$$

$$\leq 1 + C_{x}(b, g).$$

**Lemma 21** Let $U_{x} := \{a \in T : Na \leq x\}$ and $\#U_{x}$ be the cardinality of $U_{x}$. Then $\#U_{x} \leq 4^{4^{x}}$.

**Proof**. By induction on $x$. Obviously $\#U_{0} = 1$ and $\#U_{x+1}$ is less than or equal to one plus the cardinality of the Cartesian product $\{0, 1, 2\} \times U_{x} \times \cdots U_{x} \times U_{x}$ with $x + 2$ factors. For, if $a \in T$ then $a$ is either of the form $(a_{0}, \ldots, a_{n})$ with $n \leq x$ and $a_{i} \in T_{x}$ or $a$ is of the form $D_{0}b$ or $D_{1}b$ for $b \in T_{x}$. Hence, arguing by induction, $\#U_{x+1} \leq 3 \cdot (\#U_{x})^{2} \leq 3 \cdot (4^{4^{x}})^{2} \leq 4^{4^{x+1}}$.

**Theorem 3** Let $p(x) := 4^{4^{x}}$. If $a \in OT_{0}$ and $Na \leq x$ then $G_{a}(x) \leq p(p(T_{a}(x+1)))$.

**Proof**. By induction on $Na$.

1. $a = 0$. Then the assertion is obvious.

2. $a = (a_{0}, \ldots, a_{n})$.

Then the induction hypothesis yields $G_{a}(x) = G_{a_{0}}(x) + \cdots + G_{a_{n}}(x) \leq p^{2}(T_{a_{0}}(x+1)) + \cdots + p^{2}(T_{a_{n}}(x+1)) \leq p^{2}(T_{a}(x+1))$.

2. $a = D_{0}b$. Let $g := (D_{0}a)[x+1]$. Then the induction hypothesis and Lemma 9 yield

$$G_{a}(x) \leq 1 + C_{x}(b, g)$$
Corollary 3 Let $a_0(x) := x$ and $a_{n+1}(x) := 4^4_n(x)$. If $a \in \Omega_T$ and $Na \leq x$ then $G_a(x) \leq a_0(x)$.

Proof. By Lemma 21 and Theorem 3.

We find it an interesting open question to decide whether $G_a$ can be majorized eventually by a double (or triple) exponential function. Another interesting open question is whether our first main result extends to the proof-theoretic ordinal of the theory $ID_{<\omega}$. Then the usual first subrecursively inaccessible ordinal would not be subrecursively inaccessible for the assignment of fundamental sequences considered in this section. The corresponding phase transition would then even sharper than the one obtained in this paper.

2 The second main result

2.1 Preliminaries

We collect here some folklore material which proves useful in proofs later on. In this section we denote ordinals by small Greek letters. The idea is to indicate that the results of this chapter are independent of the notation system OT to a large extent. In particular, for $\varepsilon > 0$ the resulting slow growing hierarchy, when restricted to the segment of ordinals below $\varepsilon_0$ will exactly exhaust all provably recursive functions of $PA$. Therefore this section can be read by readers without knowledge of higher ordinal notations.

Definition 15 For a given real number $\varepsilon \geq 0$ let $\lambda[\varepsilon] := \max\{\beta < \lambda : N\beta \leq (1 + \varepsilon) \cdot N\lambda + x\}$. Any such system will be called a norm based assignment of fundamental sequences. If $\varepsilon = 0$ we call $\cdot[\cdot]$ the standard norm based assignment.

Definition 16 Let $\cdot[\cdot]$ be an assignment of fundamental sequences. With respect $\cdot[\cdot]$ we define certain ordinal relations as follows:

1. $\alpha \succ x \beta$ : $\iff (\exists n > 0)(\exists \gamma_0, \ldots, \gamma_n)[\alpha = \gamma_0 \land \beta = \gamma_n \land (\forall i < n)[\gamma_{i+1} = \gamma_i[x]]].$
2. $\alpha \preceq x \beta$ : $\iff \alpha \succ x \beta \lor \alpha = \beta.$
3. $\beta \succeq x m$ : $\iff (\exists a)[\beta \succeq x a \land Na \geq m].$

The following lemma provides some useful properties for investigating the growth rate of pointwise hierarchies.
Lemma 22 Let \([\cdot]\) be a norm based assignment of fundamental sequences and let the slow growing hierarchy \((G_\alpha)\) be defined with respect to \([\cdot]\). Then we have:

1. \(\alpha \geq x \beta \Rightarrow G_\alpha(x) \geq G_\beta(x)\).
2. \(G_\beta(x) \geq N\beta\).
3. \(\beta \succeq x m \Rightarrow G_\beta(x) \geq m\).

Proof. Straightforward.

Lemma 23 Assume that \([\cdot]\) is a norm based assignment. Let \(\lambda \in \text{Lim}\). Then \(N\lambda + x \leq N\lambda[x]\). If further \(\lambda[x] + 1 < \alpha < \lambda\) then \(\lambda[x] + 1 \leq \alpha[0]\). Contradiction.

Corollary 4 Assume that \([\cdot]\) is a norm based assignment. Let \(\succ y\) be defined with respect to \([\cdot]\).

1. Let \(\lambda \in \text{Lim}\). Then \(\lambda[x + 1] \geq y \lambda[x] + 1\).
2. Assume that \(\alpha \succ x \beta\) and \(\delta = \omega^\gamma + \alpha\) where \(\alpha < \omega^\gamma + 1\). Then \(\delta \succ x \omega^\gamma + \beta\).
3. Assume that \(\alpha \succ x\). Then \(\omega^\alpha \succ x \omega^\beta\).

Proof. Assertion 1) follows from Lemma 23. The assertions 2) are 3) are proved by induction on \(\beta\) with the use of 1).

The following lemma shows that monotonicity for the indices of the assignment of fundamental sequences yields the expected monotonicity for the induced assignments and pointwise hierarchies.

Lemma 24 Let \(\varepsilon, \varepsilon'\) be real numbers with \(1 \leq \varepsilon \leq \varepsilon'\).

Let \((G_\alpha)\) be defined with respect to \([\cdot]_\varepsilon\) and let \((G'_\alpha)\) be defined with respect to \([\cdot]_{\varepsilon'}\). Let \(\geq y\) be defined with respect to \([\cdot]\). Then the following holds:

1. If \(\lambda \in \text{Lim}\) then \(\lambda[x + 1]_{\varepsilon} \geq y \lambda[x]_{\varepsilon}\).
2. \(G_\alpha(x) \leq G'_\alpha(x)\) for any \(\alpha \in \text{OT}\) and \(x < \omega\).

Proof. Straightforward.

We are going to show that for \(\varepsilon > 0\) the pointwise hierarchies consists in fact of fast growing functions. For this purpose we recall some basic facts from hierarchy theory.

Definition 17 (The Hardy-Hierarchy) With regard to the a given norm based system \([\cdot]\) of fundamental sequences we define recursively numbertheoretic functions \(H_\alpha\) as follows.

1. \(H_0(x) := x\).
2. \(H_{\alpha+1}(x) := H_\alpha(x + 1)\).
3. $H_{\lambda}(x) := H_{\lambda[z]}(x)$ if $\lambda$ is a limit.

**Lemma 25** Let $\succeq_x$ be defined with respect to a given norm based assignment $\cdot[\cdot]$. Then $\alpha \succeq_x \beta$ yields $H_\alpha(y) \geq H_\beta(y)$ for all $y \geq x$. Furthermore each function $H_\alpha$ is strictly monotonic increasing.

**Lemma 26** Let $\varepsilon, \varepsilon'$ be real numbers with $1 \leq \varepsilon \leq \varepsilon'$. Let $(H_\alpha)$ be defined with respect to $\cdot[\cdot]_\varepsilon$ and let $(H_\alpha')$ be defined with respect to $\cdot[\cdot]_{\varepsilon'}$. Let $\succeq_y$ be defined with respect to $\cdot[\cdot]_1$. Then $H_\alpha(x) \leq H'_\alpha(x)$ for any $\alpha$ and $x < \omega$.

**Proof.** Straightforward.

**Lemma 27** Let $(H_\alpha)$ be defined with respect to a norm based assignment Then $(H_\alpha)$ is a fast growing hierarchy.

Proof. This is postponed into the appendix. Using techniques from [14] the proof is straightforward.

### 2.2 Putting things together

If $f$ is an operation on natural numbers we write $f(m/n)$ for $f(l)$ where $l$ is the largest integer less than or equal to $m/n := m \cdot n^{-1}$. In the sequel we show the fast growingness of $(G_\alpha)$ when defined with respect to $\cdot[\cdot]_\varepsilon$ for $\varepsilon > 0$ by a straightforward but tedious calculation.

**Theorem 4** Assume $k \geq 4$. Let $\lceil \cdot \rceil := \lceil \cdot \rceil_{1/k}$ and let $\exists_0 \succeq_0$ and $(H_\alpha)$ be defined with respect to $\cdot[\cdot]$. If $\gamma = NF \delta + \omega^\varepsilon \cdot k^k$ then $\gamma \succeq_0 H_\alpha(N\delta/k)$.

**Proof.** By induction on $\alpha$. In the following calculations we frequently make use of assertions 1),2) and 3) of Corollary 4. We may assume that $\alpha > 0$.

**Case 1.** $\alpha = \beta + 1$.

Then $\delta + \omega^\varepsilon \cdot k^k \succeq_0 \delta + \omega^\varepsilon \cdot (k^k - 1) + \omega^\varepsilon$ for some $\gamma < \omega^\beta + 1$ with $N\gamma \geq (N\beta + 1) \cdot k^{k-1} + 1$. Let $\xi_0 := \delta + \omega^\varepsilon \cdot (k^k - 1) + \omega^\varepsilon$. If $\gamma < \omega$ then $\xi_0 \geq_0 \delta + \omega^\varepsilon \cdot (k^k - 1) + \omega^\varepsilon + (1 + N\beta) \cdot k^{k-1} + 1 := \xi_1$. If $\gamma \geq \omega$ then $\xi_0 \geq_0 \delta + \omega^\varepsilon \cdot (k^k - 1) + \omega^\varepsilon \cdot \omega^\varepsilon \succeq_0 \xi_1$. In both cases $\xi_0 \succeq_0 \xi_1$. We have $\xi_1 \geq_0 \delta + \omega^\varepsilon \cdot (k^k - 1) + \omega^\varepsilon + (1 + N\beta) k^{k-1} \cdot 2 =: \xi_2$ since $N\xi_1 \cdot 1/k \geq (3 + N\beta) \cdot (k^k - 1)/k + (2 + N\beta + 1 + N\beta \cdot k^{k-1}) \cdot k \geq (3 + N\beta) \cdot k^{k-1} + N\beta \cdot k^{k-2} \geq N\omega^\varepsilon + (1 + N\beta) \cdot k^{k-1}$. Similarly we obtain

$$\xi_2 = \delta + \omega^\varepsilon \cdot (k^k - 1) + \omega^\varepsilon + (1 + N\beta) k^{k-1} \cdot 2 \succeq_0 \delta + \omega^\varepsilon \cdot (k^k - 1) + \omega^\varepsilon + (1 + N\beta) k^{k-1} + \omega^\varepsilon + (1 + N\beta) k^{k-1} \cdot 2.$$
Further since $N_{\xi_3} (2 + N_{\beta}) \cdot k^k$ hence the induction hypothesis yields

$$\sum_{i=1}^{\infty} \frac{1}{(k + (2 + N_{\alpha}) \cdot k^k - 1)} \geq (2 + N_{\beta}) \cdot k^k$$

$$\xi_3 = \delta + \omega^{\alpha + 1} \cdot (k^k - 1) + \omega^{\alpha + (1 + N_{\beta}) \cdot k^k - 1} + \omega^{\alpha + (1 + N_{\beta}) \cdot k^k - 1} + \ldots$$

$$\geq 0 \delta + \omega^{\alpha + 1} \cdot (k^k - 1) + \omega^{\alpha + (1 + N_{\beta}) \cdot k^k - 1} + \omega^{\alpha + (1 + N_{\beta}) \cdot k^k - 1} + \ldots$$

$$\equiv 0 H_{\beta}((N_{\delta} \cdot (k + 1)) / k) = H_{\alpha}(N_{\delta} / k) .$$

Case 2. $\alpha \in \text{Lim}$. We have

$$\delta + \omega^{\alpha} \cdot k^k \geq 0 \delta + \omega^{\alpha} \cdot (k^k - 1) + \omega^{\alpha \cdot \{N_{\delta} \cdot (k + 2 + N_{\alpha}) \cdot k^k - 1\}}$$

$$\geq 0 \delta + \omega^{\alpha} \cdot (k^k - 1) + \omega^{\alpha \cdot \{N_{\delta} \cdot (k + 1 + N_{\alpha}) \cdot k^k - 1\} + 1}$$

$$\equiv 0 \delta + \omega^{\alpha} \cdot (k^k - 1) + \omega^{\alpha \cdot \{N_{\delta} \cdot (k + 1 + N_{\alpha}) \cdot k^k - 1\} + 1}$$

$$\equiv 0 \delta + \omega^{\alpha} \cdot (k^k - 1) + \omega^{\alpha \cdot \{N_{\delta} \cdot (k + 1 + N_{\alpha}) \cdot k^k - 1\} + 2}$$

since $N_{\eta_1} \cdot 1 / k \geq N_{\delta} / k + (2 + N_{\alpha}) \cdot k^k + (1 + N_{\alpha}) \cdot k^k - 2 \geq N_{\eta_2}$. Further

$$\eta_1 \geq 0 \delta + \omega^{\alpha} \cdot (k^k - 1) + \omega^{\alpha \cdot \{N_{\delta} \cdot (k + 1 + N_{\alpha}) \cdot k^k - 1\} + 1}$$

$$\geq 0 \delta + \omega^{\alpha} \cdot (k^k - 1) + \omega^{\alpha \cdot \{N_{\delta} \cdot (k + 1 + N_{\alpha}) \cdot k^k - 1\} + 2}$$

$$\geq 0 \ldots$$

$$\geq 0 \delta + \omega^{\alpha} \cdot (k^k - 1) + \omega^{\alpha \cdot \{N_{\delta} \cdot (k + 1 + N_{\alpha}) \cdot k^k - 1\} + \omega^{\alpha \cdot \{N_{\delta} \cdot (k + 1 + N_{\alpha}) \cdot k^k - 1\} + \omega^{\alpha \cdot \{N_{\delta} \cdot (k + 1 + N_{\alpha}) \cdot k^k - 1\}}$$

Let $\eta_2 := \omega^{\alpha \cdot \{N_{\delta} \cdot (k + 1 + N_{\alpha}) \cdot k^k - 1\} + \ldots + \omega^{\alpha \cdot \{N_{\delta} \cdot (k + 1 + N_{\alpha}) \cdot k^k - 1\}}$. Then

$$\eta_2 = \delta + \omega^{\alpha} \cdot (k^k - 1) + \omega^{\eta_2 + N_{\delta} \cdot k^k - 1}$$

$$\geq 0 \delta + \omega^{\alpha} \cdot (k^k - 1) + \omega^{\eta_2 + N_{\delta} \cdot k^k - 1} + \omega^{\eta_2 + N_{\delta} \cdot k^k - 1}$$

$$\geq 0 \delta + \omega^{\alpha} \cdot (k^k - 1) + \omega^{\eta_2 + N_{\delta} \cdot k^k - 1} + \omega^{\eta_2 + N_{\delta} \cdot k^k - 1}$$
\[\geq 0 \delta + \omega^\alpha \cdot (k^k - 1) + \omega^\eta_2 + N\delta / k \cdot k^{k-1} - 1\\ + \omega^\alpha \cdot [N\delta / k + (1 + N\alpha) \cdot k^{k-1} - 1] + \omega^\alpha \cdot [N\delta / k + (1 + N\alpha) \cdot k^{k-1} - 2] + \ldots + \omega^\alpha \cdot [N\delta / k + (1 + N\alpha) \cdot k^{k-1} - (N\delta / k)] + \ldots + N\delta / k \cdot k^{k-1} - 1\\ \geq 0 \delta + \omega^\alpha \cdot (k^k - 1) + \ldots + \omega^\alpha \cdot [N\delta / k] + 1 := \eta_3\]

The induction hypothesis yields

\[\eta_3 = \ldots + \omega^\alpha \cdot [N\delta / k] + 1\]

\[\geq 0 \omega^\alpha \cdot [N\delta / k] \cdot k^k\]

\[\geq 0 H_\alpha \cdot [N\delta / k] (N\delta / k) = H_\alpha (k)\]

since \(N\eta_3 \geq N\delta / k \cdot (k^{k-1} + 1/2 \cdot (N\alpha + 1)^2 \cdot (k^{k-1})^2 \geq (2 + N\alpha(k + 1) / k + N\delta / k)^k)\).

**Lemma 28** \(\omega^{\alpha + 1} \cdot k^k + \omega^{\alpha + \omega} \geq k^k + \omega^{\alpha + x} \cdot k^k\)

**Proof.** We obtain

\[\omega^{\alpha + 1} \cdot k^k + \omega^{\alpha + \omega} \geq k^k + \omega^{\alpha + x + k^{k-1}} - 2\]

\[\geq 0 \omega^{\alpha + 1} \cdot k^k + \ldots + \omega^{\alpha + x + k^{k-1}}\]

\[\geq 0 \omega^{\alpha + 1} \cdot k^k + k^k + \omega^{\alpha + x + k^{k-1}} + \ldots + \omega^{\alpha + x} \cdot k^k\]

**Theorem 5** Assume \(k \geq 4\). Let \(\lceil \cdot \rceil := \lceil \cdot \rceil / k\). Assume that \((G_\alpha)\) is defined with respect to \(\lceil \cdot \rceil\) and that \((H_\alpha)\) is defined with respect to the standard norm based assignment. Then \(G_{\omega^{\alpha + 1} \cdot k^k + \omega^{\alpha + \omega}} (x) \geq H_\alpha (x)\).

**Proof.** This follows from assertion 2 of Lemma 22 and Lemma 28

**Corollary 5** Let \(\varepsilon > 0\) and assume that the hierarchy \((G_\alpha)\) is defined with respect to \(\lceil \cdot \rceil_\varepsilon\). Then \((G_\alpha)\) is fast growing.

**Proof.** This follows from Theorem 5

**Appendix**

We stick to the notational conventions of Section 1. In this appendix we first describe the standard system of fundamental sequences in terms of the norms
function and show that it gives rise to a normed Bachmann system. Second, we define the standard Hardy hierarchy \((H^*_n)\) along OT and compare it with \((H_n)\). For an intermediate calculation we introduce in addition a fast growing (as shown in [1]) hierarchy \((A_n)\) (which looks slow growing at first sight).

**Definition 18** For \(a \in \text{OT} \) with \(tp(a) = \omega \) and \(x < \omega \) we define a non negative integer \(p(a + x)\) as follows.

1. \(a = (a_0, \ldots, a_{n-1}, a_n) \Rightarrow p(a + x) := Na_0 + \ldots + Na_{n-1} + p(a_n + x).\)
2. \(a = D_i(b + 1) \Rightarrow p(a + x) := (Nb + 1) \cdot (x + 1).\)
3. \(a = D_i b \& \, tp(b) = \omega \Rightarrow p(a + x) := 1 + p(b + x).\)
4. \(a = D_0 b \& \, tp(b) = \Omega \Rightarrow p(a + x) := (Nb + 1) \cdot (x + 1).\)

**Definition 19** Definition of \(a\{x\}\) for \(a \in \text{OT}_0 \) with \(tp(a) = \omega \) and \(x < \omega \).

\[a\{x\} := \max\{b \in \text{OT} : b < a \& \, Nb \leq p(a + x)\}\]

**Lemma 29** The structure \(\langle \text{OT}_0, \cdot\{\cdot\}, N \rangle\) is a normed Bachmann system.

Proof. This follows from Theorem 5 of [5].

**Lemma 30** Characterization of \(a\{x\}\) for \(a \in \text{OT}_0 \) with \(tp(a) = \omega \) and \(x < \omega \).

1. \(a = (a_0, \ldots, a_{n-1}, a_n) \Rightarrow a\{x\} = (a_0, \ldots, a_{n-1}) + a_n\{x\}\).
2. \(a = D_0 b + 1 \Rightarrow a\{x\} = D_0 b \cdot (x + 1)\).
3. \(a = D_0 b \& \, tp(b) = \omega \Rightarrow a\{x\} = D_0 b\{x\}\).
4. \(a = D_0 b \& \, tp(b) = \Omega \Rightarrow a\{x\} = D_0 b x \text{ where } b_0 := b\{0\} \text{ and } b_{y+1} := b[D_0 b_y]\).

Lemma 30 shows that \(\cdot\{\cdot\}\) coincides with Buchholz usual definition of fundamental sequences for the limits below the Howard Bachmann ordinal.

In the sequel \(a, b, c, d\) range over \(\text{OT}_0\).

**Lemma 31**

1. \(N(a\{0\}) = N(a) + 1\).
2. \(N(\omega \cdot a\{0\}) \leq N(\omega \cdot a) + 1\).
3. \(N(a\{x\}) \leq Na \cdot (x + 1)\).
4. \(N(\omega^a \cdot a) \leq Na \cdot (a + 1)\).

**Definition 20**

1. \(a\)
   - \(H^*_a(x) := x\),
   - \(H^*_a(x + 1) := H^*_a(x + 1)\),
   - \(H^*_a(x) := H^*_a\{x\}(x) \text{ if } tp(a) = \omega\).
2. \(a\)
   - \(A_0(x) := x\),
   - \(A_a(x) := \max\{A_b(x) + 1 : b < a \& \, Nb \leq Na + x\}\).

**Lemma 32**

1. \(NF(a, b) \Rightarrow H_{a+b}(x) = H_a(H_b(x))\).
2. \(H_{\omega^10}(x) \geq 10^x\).
3. \( H_{\omega^2+\omega+\omega+\omega+(k+1)}(x) \geq H_{\omega^2+\omega+\omega+(k+1)}(10 \cdot x) \).

4. \( a < b \& Na \leq Nb + x \Rightarrow H_a(x) < H_b(x) \).

5. \( A_a(x) \leq H_a(x) \leq H_a^*(x) \).

**Definition 21**

1. \( a \triangleright b : \iff a \geq b \& Na \geq Nb + x \).
2. \( a \triangleright_* b : \iff (\exists a_0, \ldots, a_n)[a_0 = a \& a_n = b \& (\forall i < n)[a_i \triangleright a_{i+1}]] \).
3. \( a \triangleright k : \iff (\exists b)[a \triangleright b \& Nb \geq k] \).

**Lemma 33**

1. \( a \triangleright_* b \Rightarrow A_a(x) \geq A_b(x) \).
2. \( A_a(x) \geq Na \).
3. \( a \triangleright k \Rightarrow A_a(x) \geq k \).
4. \( \omega \cdot a \geq_{10} N \cdot (10 \cdot x - 1) \).

**Lemma 34**

1. \( x \geq 2 \Rightarrow \omega^2 \cdot a + \omega \cdot a + \omega \cdot k + \omega \geq x H_a^*(k + x) \).
2. \( x \geq 2 \Rightarrow H_{\omega^2+\omega+\omega+\omega+k}(x) \geq H_a^*(k + x) \).
3. \( c = \omega^{\omega+d} \Rightarrow (\forall a < c)(\exists b < c)(\forall x)[H_a^*(x) \leq H_b(x)] \)

**Proof of the first assertion by induction on \( a \).**

1. \( a = 0 \).
   \[
   \omega \cdot k + \omega \geq_{10} \omega \cdot k + 10 \cdot x + 1
   \geq_{10} \omega \cdot k + x = H_0^*(k + x).
   \]

2. \( a = b + 1 \). Then the induction hypothesis yields
   \[
   \omega^2 \cdot a + \omega \cdot a + \omega \cdot k + \omega
   \geq_{10} \omega^2 \cdot b + \omega \cdot b + \omega \cdot (k + 1) + \omega
   \geq_{10} H_a^*(k + x + 1) = H_a^*(k + x) + 1
   \]

3. \( \text{tp}(a) = \omega \). Then the induction hypothesis yields
   \[
   \omega^2 \cdot a + \omega \cdot a + \omega \cdot k + \omega
   \geq_{10} \omega^2 \cdot a + Na \cdot (10 \cdot x - 1) + \omega \cdot k + \omega
   \geq_{10} H_{a\{x\}}^*(k + x) = H_a^*(k + x).
   \]

The second assertion follows from assertion 3 of the Lemma 33 and assertion 5 of Lemma 32 and the last assertion follows from the second assertion.

**References**

13. A. Weiermann: $\Gamma_0$ may be minimal subrecursively inaccessible. MLQ (2001), no. 3, 397–408.