Phase transitions in logic and combinatorics*

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Abstract

A major theme in proof theory consists in classifying the proof strength of mathematical frameworks for reasoning about mathematics. The resulting phase transitions from provability to unprovability are interesting from the foundational as well as from the mathematical point of view. It is very surprising that during these investigations methods from analytic number theory, combinatorial probability theory, complex analysis and Ramsey theory enter the scene.

In this abstract we present the underlying research program. We try to explain our intuition about the nature of phase transition results under discussion. It is our objective to show that there are intriguing interrelations between "usual" mathematics and proof theory.

The abstract is intended to be accessible to a general mathematical audience. We therefore concentrate on basic examples and ideas without giving proofs.

1 Phase transitions and Gödel incompleteness

The phase transition phenomenon is familiar from statistical physics, [8], but also from percolation [17], random graphs [3, 19] and computational complexity [9]. Surprising interrelations between these fields have recently been discussed in [25].

In this abstract we treat phase transitions in the context of the Gödel incompleteness theorems [15] and resulting implications on the classical Ramsey function [16].

In simplified form Gödels incompleteness theorem states that for any reasonable system which extends the Peano axioms there exists an assertion A about non negative integers such that A is true but not provable in the system under consideration.

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Gödels example for such an assertion A is given by an assertion stating (after an appropriate arithmetization) its own unprovability in the system under consideration. This example has a bit of an exotic flavour.

Therefore logicians have been looking for instances of Gödels theorem where the assertion A is mathematically relevant and interesting. Breakthroughs in this respect have been achieved by G. Gentzen [13], J. Paris & L. Harrington [27] und H. Friedman [32]. Some of these have been popularized in G. Kolatas contribution 'Does Gödel's theorem matter to mathematics' in the science magazine [21] and we will take a new look at them from the phase transition perspective.

To fix the context let us explain briefly the axioms of first order Peano arithmetic PA. The axioms of PA have been chosen in a way that more or less every true assertion about the nonnegative integers should follow from these axioms. The language of PA contains besides the usual logical symbols (including $=$) a constant 0 for the number zero a function symbol S for the successor function and the function symbol $+, \cdot$ for addition and multiplication. In addition we include a relation symbol P for implicit quantification about arbitrary subsets of nonnegative integers. (The relation symbol P plays no role in the beginning and is only used to formulate the scheme of transfinite induction in section 3.) The mathematical axioms of PA describe elementary properties of $0, S, +, \cdot$. Most important is that PA contains the scheme of complete induction, i.e. for every formula φ in the language of PA an axiom $\varphi(0)\&(\forall x)[\varphi(x) \to \varphi(x{+}1)] \to (\forall x)\varphi(x)$.

We say that an assertion A follows from PA, or that PA proves A, if A holds in all models of PA. (This is completely analogous to usual mathematics. An assertion A follows from the axioms of group theory if A holds in all groups.) It is worth to note that a typical countable model of PA has the following form. There is an initial segment of individuals which is order isomorphic with the nonnegative integers. If there exists one nonstandard individual in the model then the end interval (formed solely from non standard individuals) of the model is order-isomorphic to $\mathbb{Q}\mathbb{Z}$ (ordered under the lexicographic product of the natural orderings involved).

The axiom system PA is relatively strong and it has been shown that large parts of countable mathematics can be developed in PA [33] (more precisely in its second order pendant).

To introduce the phase transition for instances of Gödel's theorem for PA let us assume for a moment that we have given an assertion A in the language of PA which depends on a parameter $r \in \mathbb{Q}$ (to which we can refer in PA without problems via coding) such that PA proves $A(r)$ for small r and such that PA does not prove $A(r)$ for large r. Further assume that A monotone with respect to these properties, i.e if $s < r$ and PA proves $A(r)$ then PA proves $A(s)$ as well. Analogously, if $s < r$ and PA does not prove $A(s)$ then PA does not prove $A(r)$. Classifying the phase transition then consists in computing the infimum of the set of all $r \in \mathbb{Q}$ such that PA proves $A(r)$. If this is not possible one tries to find numbers a, b such that $A(r)$ is provable for $r < a$ and unprovable for $r > b$ and such that the distance between a, b is as short as possible. We may then call $]a, b[$ the transition interval.

Later we give an example in which the phase transition can be classified exactly. Without going into details we would like to remark, that it is possible to find assertions A for which a classification of the transition interval depends on number-theoretic hypotheses like the *abc*-conjecture or the Riemann hypothesis. In this case there exists real numbers $a < a' < b' < b$, such that $]a, b[$ is the transition interval für A without assuming the number-theoretic hypothesis and such that, for example, under the Riemann hypothesis we can shorten the transition interval for A in a non trivial way to to $]a', b'[$. Examples for such assertions A can be can be constructed from independent assertions over combinatorial properties of $0-1$ sequences. These sequences can be coded via square free numbers and prime numbers for which their densities in short intervals depend non trivially on corresponding hypotheses.

In principle these results can be used to disprove number-theoretic hypotheses with logical methods. Perhaps it is possible to obtain a line of attack to the Cramer conjecture on the density of primes. This conjecture, which in contrast to the Riemann hypothesis is assumed to be false by some experts, states $\limsup_{n\to\infty} \frac{p_{n+1}-p_n}{(\log(p_n))^2} = 1$.

More generally it is possible to study the phase transition for assertions A which depend on a function parameter $F : \mathbb{N} \to \mathbb{N}$ (which is definable in PA). The underlying assumption is that for very slow growing F the assertion $A(F)$ is provable and that for faster growing F the assertion $A(F)$ becomes unprovable. The classification problem then consists in finding the threshold function for the provability of $A(F)$. As before we concentrate on examples where $A(F)$ is suitably monotone F. (It seems also to be interesting to study parameter functions F which are oscillating but we have not yet obtained any interesting results in this direction.)

The phase transitions discussed in this article are far away from the phase transitions considered in statistical mechanics. Nevertheless statistical mechanics serves as an interesting source of inspiration in our context. Well known phenomena in physics are universality and renormalization. For reasons which are not well understood yet it turns out in the examples which we investigated that the threshold functions for provability (unprovability) stem from a scale of very few basic functions. Moreover the phase transition results are usually stable under various forms of rescaling (renormalization). We have no general explanations for these effects. Maybe it is just the phenomenon that natural objects should have natural properties.

Another more common line of research in proof theory is ordinal analysis of systems extending the Peano axioms in a strong way. Here the idea is to associate a mathematical invariant, the proof-theoretic ordinal, to the theory in question. This can also be interpreted as a phase transition phenomenon. Ordinal analysis separates the order types of primitive recursive well-orderings for which the scheme of transfinite induction is provable from the order types of unprovable instances. A basic example will be discussed in section 3. We will not go further in this direction. Major contributions in this area have been achieved by [31, 5, 34, 18, 30, 1] and a comprehensive survey is, for example, provided in [29].

So far we have outlined our research theme on phase transitions for incompleteness phenomena. An alternative view of proof theory (going back to Kreisel) with concrete relevance to mathematics is to extract non trivial constructive information from given (maybe in-constructive) proofs of mathematical assertions. Recent success in this line of research has been obtained by U. Kohlenbach [20]. It might be interesting to investigate interrelation of this approach with the phase transition enterprise.

2 Phase transitions for Kruskals theorem

In this section we survey a somewhat spectacular phase transition result for the Peano axioms. To fix the context let us define a finite tree to be finite partial order $\langle B, \leq_B \rangle$, such that for every $b \in B$ the set $\{b' \in B : b' \leq_B b\}$ is linearly (i.e. totally) ordered trough $\leq_B$ and such that $\mathcal{B}$ contains a minimum, the root. In other words $\mathcal{B}$ is a non planar rooted tree. For two given vertices $b, b' \in B$ there exists an infimum, which we denote by $b \wedge_{\mathcal{B}} b'$. (If we go from b and b' to the root the infimum is the first vertex where the paths meet.) We say that a tree $\mathcal{B}$ is embeddable into a tree $\mathcal{B}'$ (this situation is denoted by $\mathcal{B} \trianglelefteq \mathcal{B}'$) if there exists an one to one mapping $h : B \to B'$ such that $h(b \wedge_{\mathcal{B}} b') = h(b) \wedge_{\mathcal{B}} h(b')$ for all $b, b' \in B$.

Kruskals tree theorem [23] states that for every infinite sequence $(\mathcal{B}_i)_{i=0}^{\infty}$ of finite trees there exist nonnegative integers i, j such that $i < j$ and $\mathcal{B}_i \trianglelefteq \mathcal{B}_j$. It turns out that Kruskals theorem has first order consequences which are not provable in PA. Let us denote the cardinality of a finite tree, i.e. the number of its nodes, with $|\mathcal{B}|$. For a given function $F : \mathbb{N} \to \mathbb{N}$ let FKT(F) be the assertion: For every $K \in \mathbb{N}$ exists an $M \in \mathbb{N}$, such that for every finite sequence $(\mathcal{B}_i)_{i=0}^{M}$ of finite trees satisfying $(\forall i \leq M)[|\mathcal{B}_i| \leq K + F(i)]$ there exist indices $i, j \in \mathbb{N}$ with $i < j$ and $\mathcal{B}_i \trianglelefteq \mathcal{B}_j$. Using a compactness argument (the same as used in the proof of the Bolzano Weierstraß theorem) one can prove FKT(F) for any function F.

H. Friedman proved that PA does not prove FKT(id). (Here id denotes the identity function.) An elementary counting argument yields that for constant functions with value c the assertion FKT(F) follows from PA. For the threshold function F for FKT it therefore holds that $c \leq F(i) \leq i$ for all but finitely many $i \in \mathbb{N}$. (Here again we tacitly assume that F is weakly increasing.)

A significant improvement on the threshold has been obtained by J. Matoušek and M. Loebl. Let $|i|$ denote the binary lengths of i and put $F_\alpha(i) := \alpha \cdot |i|$. J. Matoušek and M. Loebl showed, that for $\alpha \leq \frac{1}{2}$ the assertion FKT(F_α) follows from PA but that for $\alpha \geq 4$ the assertion FKT(F_α) does not follow from PA.

It is an immediate question to ask for the threshold function resp the threshold value for α. The surprising answer runs as follows [35]. Let $T(z) := \sum_{n=0}^{\infty} t_n \cdot z^n$ be a power series such that $T(z) = z \cdot \exp(\sum_{i=1}^{\infty} \frac{T(z^i)}{i})$. Let ρ be the radius of convergence of T. Then we first have that $1 > \rho > 0$. The desired threshold value can be defined as follows: $c := -\frac{1}{\log_2(\rho)}$. Indeed, if $\alpha \leq c$ then PA does prove $\mathrm{FKT}(F_\alpha)$ and if $\alpha > c$ then PA proves $\mathrm{FKT}(F_\alpha)$.

3 Phase transitions for ε_0

In naive set theory ordinals are used to count into the transfinite. Ordinals begin with $0, 1, 2, 3, \ldots$. After infinitely many steps we reach the first limit point ω and we can continue the counting with $\omega + 1, \omega + 2, \omega + 3, \ldots$. In this way we obtain $\omega^2, \omega^3, \omega^4, \ldots$ and hence ω^ω. With more and more efforts we reach $\omega^{\omega^\omega}, \omega^{\omega^{\omega^\omega}}, \ldots$. Finally we obtain ε_0 as Limit of the sequence $\omega, \omega^\omega, \omega^{\omega^\omega} \ldots$.

This description is fine if one already has seen a thorough introduction into the theory of ordinals. Otherwise the counting process becomes more and more obscure the larger the ordinals in question become. Therefore it seems justified to provide an alternative description of the ordinals below ε_0 without referring to set theory. The desired description is possible by an appeal to Hardys order of infinity. The description is simple and accessible even to high school students.

Let $\mathcal{E}$ be the least set of functions $f : \mathbb{N} \to \mathbb{N}$, such that

1. $x \mapsto 0 \in \mathcal{E}$,
2. With $f, g \in \mathcal{E}$ the function $x \mapsto x^{f(x)} + g(x)$ is in $\mathcal{E}$ too.

For $f, g \in \mathcal{E}$ we define $f \prec g$ if there exists a $K \in \mathbb{N}$ such that $f(m) < g(m)$ for all $m \geq K$, i.e. if g eventually dominates f.

Let k_n be the constant function $x \mapsto n$ and let ω be the identity function $x \mapsto x$. Further define $+, \cdot$ on $\mathcal{E}$ via pointwise operation. We observe: $k_0 \prec k_1 \prec k_2 \prec \ldots \prec \omega \prec \omega + k_1 \prec \omega + k_2 \prec \ldots \prec \omega + \omega \prec \ldots \prec \omega + \omega + \omega \prec \ldots \prec \omega^{k_2} \prec \ldots \prec \omega^{k_3} \prec \ldots \prec \omega^\omega \prec \ldots \prec \omega^{\omega^\omega} \prec \ldots$ and one can verify that the order type of $\mathcal{E}$ with respect $\prec$ is precisely ε_0. Therefore we can identify ε_0, hence the segment of ordinals below ε_0 ordered in the natural way, with the order $\langle \mathcal{E}, \prec \rangle$.

Without problems one verifies that $\langle \mathcal{E}, \prec \rangle$ is a linear order. Moreover for a given $f \in \mathcal{E}$ with $f \neq k_0$ there exist uniquely determined $f_1, \ldots, f_n$ with $f = \omega^{f_1} + \ldots + \omega^{f_n}$ and $f_n \preceq \ldots \preceq f_1$. In this case we write $f =_{NF} \omega^{f_1} + \ldots + \omega^{f_n}$ and call this representation the normal form of f. Using this normal form representation theorem one can thus identify elements from $\mathcal{E}$ with their term representations.

The order $\langle \mathcal{E}, \prec \rangle$ is a well-order i.e. for every non empty set $X \subseteq \mathcal{E}$ there exists $f \in X$ such that $\neg g \prec f$ for all $g \in X$; equivalently we may state: For every function $F : \mathbb{N} \to \mathcal{E}$ there exists an $i \in \mathbb{N}$ such that $F(i) \preceq F(i + 1)$.

The ordinal ε_0 resp. the structure $\langle \mathcal{E}, \prec \rangle$, is the so called proof-theoretic ordinal (for further background information the reader may, for example, consult [31, 34, 29]) of the Peano axioms which has been determined by G. Gentzen [13]. Due to its fundamental importance of this result in proof theory we provide some more details.

Let

$$TI(\prec, P) := (\forall f(\forall g \prec f P(g)) \to P(f)) \to \forall f P(f).$$

After a suitable coding of the function terms for elements of $\mathcal{E}$ via nonnegative integers this assertion is a statement in the language of PA. G. Gentzen showed: The assertion $TI(\prec, P)$ is true but unprovable in PA. More precisely we define an arithmetization of $\mathcal{E}$ in the following way: Let p_i be the i-th prime for $i \geq 1$. Let $\mathbb{N}^+ := \mathbb{N} \setminus \{0\}$ and let $\lceil k_0 \rceil := 1$ und $\lceil f \rceil := p_{\lceil f_1 \rceil} \cdot \ldots \cdot p_{\lceil f_n \rceil}$, if $f =_{NF} \omega^{f_1} + \ldots + \omega^{f_n}$. Then the mapping $\lceil \cdot \rceil$ is a bijection between $\mathcal{E}$ and $\mathbb{N}^+$ and we may induce an ordering (which we again denote with $\prec$) on $\mathbb{N}^+$. It can easily be seen that the induced ordering can be defined by a formula in the language of PA.

The scheme $TI(\prec, P)$ can be written as follows:

$$TI(\prec, P) := (\forall n \in \mathbb{N}^+)[(\forall m \prec n P(m)) \to P(n))] \to \forall n \in \mathbb{N}^+ P(n).$$

The ordinal ε_0 is characteristic for an ordinal-theoretic phase transition for PA in so far as PA proves the transfinite induction for every initial segment of ε_0. More explicitly, PA proves for every $k \in \mathbb{N}^+$ the assertion
$\forall m \prec k[(\forall n \prec m P(n)) \to P(m)] \to \forall m \prec k P(m)$.

Via a compactness argument $TI(\prec, P)$ yields for every function $F : \mathbb{N} \to \mathbb{N}$ the truth of the following assertion FWO(F):
$(\forall K)(\exists M)(\forall m_1, \ldots, m_n \in \mathbb{N}^+)\big[(\forall i \leq M)\lceil m_i \rceil \leq K + F(i) \to (\exists i < M) m_i \prec m_{i+1}\big]$. A deep result which essentially goes back to H. Friedman [32, 12] states that for $F(i) = 2^i$ the assertion FWO(F) is unprovable in PA.

Again it is very natural to ask for a classification of the resulting phase transition for FWO(F). During working on the solution it became apparent that one is more or less forced to deal with problems from analytic number theory.

Especially it seems to be necessary to obtain bounds on the following count function

$$C_m(n) := \#\{l \prec m : \lceil l \rceil \leq n\}.$$

Let $o_1 := 3 = \lceil \omega \rceil$ and $o_{k+1} := p_{o_k}$ so that the order-type of o_k is given by an exponential tower of ωs of hight k. In particular we have $5 = \lceil \omega^\omega \rceil$. Using analytic combinatorics for Dirichlet generating series (which reflect the multiplicative nature of the counting) one can obtain the following rough estimates. (Further background material can be found, for example, in in [11, 22, 26, 28, 36, 38].)

1. $\ln(C_5(n)) \sim \pi\sqrt{\frac{2\ln(n)}{3}\ln(2)}$
2. $\ln(C_{o_{k+2}}(n) = \Theta(\frac{\ln(n)}{\underbrace{\ln(\ldots(\ln}_{k\text{-times}}(\ln(n)))\ldots)})$ falls $k \geq 1$.

As a problem of interest in its own right one might be interested in the asymptotic of $n \mapsto \ln(C_{o_{k+2}}(n))$. There are indications for the following estimate: $\ln(C_{o_{k+2}}(n)) \sim \frac{\pi^2}{6\ln(2)} \frac{n}{\underbrace{\ln(\ldots(\ln}_{k \text{ times}}(n))\ldots)}$, where $k \geq 1$.

Let $|i|$ denote as before the binary length of i. Let $|i|_1 := |i|$ and $|i|_{k+1} := ||i|_d|$ and finally $\log^*(i) := \min\{d : |i|_d \leq 2\}$. A combination of logical methods with the number-theoretic estimates yields the following result over the phase transition for FWO. If $F(i) = 2^{|i|\cdot|i|_d}$ then the assertion FWO(F) is unprovable in PA. If $F(i) = 2^{|i|\cdot\log^*(i)}$ then PA proves the assertion FWO(F).

From the viewpoint of analytic number theory the last phase transition result refers to a multiplicative norm on ordinals. It is a natural question to investigate phase transitions in the additive setting. To this end we define a norm $N : \varepsilon_0 \to \mathbb{N}$ as follows. $N(k_0) := 0$ and $Nf := n + Nf_1 + \ldots + Nf_n$, if $f =_{NF} \omega^{f_1} + \ldots + \omega^{f_n}$. Via the natural isomorphis between ε_0 and $\mathbb{N}^+$ the norm can be extended to $\mathbb{N}^+$.

As before a compactness argument applied to $TI(\prec, P)$ yields for every function $F : \mathbb{N} \to \mathbb{N}$ the truth of the following assertion FWON(F): $(\forall K)(\exists M)(\forall m_1, \ldots, m_n \in \mathbb{N}^+)\big[(\forall i \leq M)Nm_i \leq K + F(i) \to (\exists i < M)m_i \prec m_{i+1}\big]$. H. Friedman [32, 12] showed that for $F(i) = i$ the assertion FWON(F) is unprovable in PA.

Again it is very natural to ask for a classification of the resulting phase transition for FWON(F). Additive analytic number theory can then be applied to the following count function

$$c_m(n) := \#\{l \prec m : Nl \leq n\}.$$

Using the generating function technology one obtains the following estimates. (Further background material can be found, for example, in auf [11, 22, 28].)

1. $\ln(c_5(n)) \sim \pi\sqrt{\frac{2\ln(n)}{3}}$
2. $\ln(c_{o_{k+2}}(n) \sim \frac{\pi^2}{6}(\frac{n}{\underbrace{\ln(\ldots(\ln}_{k \text{ times}}(n))\ldots)})$ if $k \geq 1$.

A combination of logical methods with the number-theoretic estimates yields the following result over the phase transition for FWO. If $F(i) = |i| \cdot |i|_d$ then the assertion FWON(F) is unprovable in PA. If $F(i) = |i| \cdot \log^*(i)$ then PA proves the assertion FWON(F).

The optimal phase transition in this case has been obtained by Arai[2].

At the end of this section we would like to mention the following problem which appears from our investigations and which we expect to have an interesting solution. It is known that the contour process for planar trees leads to a Brownian excursion [14]. We would like to know what process is related to the contour process for $\mathcal{E}$. Due to the normal form condition for term representations which says that the exponents are weakly decreasing it seems clear that the process does not have the Markov property. Thus it seems to be very complex. Another problem in this context would be the investigation of limit shapes for typical elements in $\mathcal{E}$.

4 Intermezzo

At this point we can not resist including some supplementary results which arose from studying the count functions for elements of ε_0. We just include them for their beauty.

By inspection of Burris's book [6] it becomes apparent that ε_0 shows all basic features of an additive number system. Therefore it seemed highly plausible that ε_0 comes equipped with logical limit laws and in joint work with Alan Woods it was possible to show that the ordinals in $[\omega^\omega, \varepsilon_0[$ are characterized by a zero one law.

Let φ be a sentence in the language of linear oders. Given $m \in \mathbb{N}^+$ we write $m \models \varphi$ if φ becomes true when the quantifiers occuring in φ range over $\{l : l \prec m\}$ and the order relation symbol is interpreted as $\prec$. Similarly we write $\mathbb{N}^+ \models \varphi$ if φ becomes true when the quantifiers occuring in φ range over $\mathbb{N}^+$. Given $m \in \mathbb{N}^+$ such that $5 \preceq m$ it is natural to investigate the probability that $m \models \varphi$. It will be either zero or one (for natural choices of the model)! More precisely we have the following results where probability is measured via associated asymptotic densities.

1. $\lim_{n\to\infty} \frac{\#\{l \prec n : l \models \varphi \& \lceil l \rceil \le n\}}{\#\{l \prec n : \lceil l \rceil \le n\}} \in \{0, 1\}$.

2. $\lim_{n\to\infty} \frac{\#\{l \prec n : l \models \varphi \& Nl \le n\}}{\#\{l \prec n : Nl = n\}} \in \{0, 1\}$.

This zero one law does not hold if we reach ε_0. The limits $\lim_{n\to\infty} \frac{\#\{l \in \mathbb{N}^+ n : l \models \varphi \& \lceil l \rceil \le n\}}{\#\{l \prec n : \lceil l \rceil \le n\}}$ and $\lim_{n\to\infty} \frac{\#\{l \in \mathbb{N}+ : l \models \varphi \& Nl \le n\}}{\#\{l \prec n : Nl \le n\}}$ still exist but for certain choices of φ are elements of $]0, 1[$.

5 Phase transitions in Ramsey theory

To discuss Ramsey theory we use the following standard terminology [16]. For $m \in \mathbb{N}$ let $[m] := \{1, \dots, m\}$ and four a set Y let $[Y]^d$ be the set of d-element subsets of Y. In addition we write $[m]^d$ for $[[m]]^d$. The infinitary Ramsey theorem states that for all $d, c \in \mathbb{N}$ and for every partition $P : \mathbb{N}^d \to [c]$ there exists an infinite set $Y \subseteq \mathbb{N}$ such that $P \restriction [Y]^d$ is constant.

An application of compactness yields the finite Ramsey theorem which states that for all $d, c, m \in \mathbb{N}$ there exists a minimal nonnegative integer $R =: R^d_c(m)$ such that for all partitions $P : [R]^d \to [c]$ there exists a subset $Y \subseteq [R]$ with $|Y| \ge m$ and such that $P \restriction [Y]^d$ is constant. The asymptotic of R^d_c is in many cases not known. It is known that for R^3_3 we have the following lower and upper bounds

$$2^{m^2(\log(m))^2 \cdot const} \le R^3_3(m) \le 2^{2^{const \cdot m}}. \tag{1}$$

It is a classical Erdös problem (USD 500 have been offered as far as we know) whether R^3_3 has a double exponential lower bound.

In this context it is quite natural to ask whether it is possible to regain the strength of the infinitary Ramsey theorem from suitable iterations of the finite

Ramsey theorem, whether the infinitary Ramsey theorem leads to independence results and whether these logical investigations might be helpful in the study of Erdös problems.

To this end we consider the following variant of the finite Ramsey theorem. Let PH(F) stand for the following assertion: For all d, c, m exists a nonnegative integer $R =: R_c^d(F)(m)$ such that for all partitions $P : [R]^d \to [c]$ there exists a subset $Y \subseteq [R]$ with $P \restriction [Y]^d$ is constant and $|Y| \geq \max\{m, F(\min(Y)\}$. An application of compactness to the infinitary Ramsey theorem yields that PH(F) is true for all parameter functions F.

J. Paris and L. Harrington [27] proved that PH(id) is unprovable in PA. Using estimates of Erdös and Rado for R_c^d [10] one easily verifies that for constant functions F the theory PA proves the assertion PH(F).

An exact description of the resulting phase transition would exceed the scope of this abstract and therefore we present a rough approximation.

It has been shown (see, for example, [37]) that for $F(i) := \log^*(i)$ the assertion PH(F) is provable in PA but that for every fixed $d \in \mathbb{N}$ and the function $F(i) := |i|_d$ the assertion PH(F) is unprovable in PA. Corresponding assertion hold also for the fragments of PA in which the induction scheme is suitably restricted. Then classification of the resulting phase transitions is based on the probabilistic method [39].

Without going too much into detail we would like to remark that with methods from non standard models one can show that suitable iterations of the Paris Harrington principle reach the proof strength of the infinitary Ramsey theorem as far as provably computable functions are concerned [4].

Analogous properties and phase transitions can be obtained for the canonical Ramsey theorem, the Ramsey theorem for regressive partitions and further variants [7].

Finally let us reconsider the phase transition for the function $R_3^3(F)$. It is our hope to obtain advance in Ramsey theory from investigations on this function. First one can show that for any $\varepsilon > 0$ and the function $F(i) = \varepsilon \cdot \log_2(i)$ the induced Ramsey function $R_3^3(F)$ is not bounded by a primitive recursive function. Thus in this case the growth of $R_3^3(F)$ is gigantic. Second one can show that there exists a constant C such that for $F(i) := \frac{1}{C} \cdot \log_2(\log_2(i))$ one has $R_3^3(F)(m) \leq 2^{2^{C \cdot m}}$. Thus by varying the function parameter from double logarithmic to logarithmic there appears an extreme phase transition. Moreover, if it is possible to pin down this transition precisely that one obtains advance on the asymptotic of R_3^3.

To approximate the phase transition one shows thirdly that for $F(i) := (\log_2(i))^{\frac{1}{2}+\varepsilon}$ there exists a triple exponential lower bound for $R_3^3(F)$. Finally one verifies for $F(i) := (\log_2(i))^{\frac{1}{3}+\varepsilon}$ the following implication:

If the estimate $R_3^3(F)(m) \geq 2^{m^3}$ holds for all but finitely many $m \in \mathbb{N}$ then also $R_3^3(m) \geq 2^{m^3}$ holds for all but finitely many $m \in \mathbb{N}$.

The conclusion would improve the lower bound from (1) and would therefore give real advance in Ramsey theory. Similar implications from phase transitions can be obtained for the Ramsey function R_2^2.

6 The underlying principle for unprovability phase transitions

So far we have seen phase transition results for well quasi orders, well orders and Ramsey properties. Despite their differences our proofs for all of these classifications share a common feature. The truth of the assertions in question is proved from certain infinitary principles using compactness. In proving lower bounds for the unprovability thresholds one employs bounds from elementary combinatorics. But if one exceeds the bounds dictated by finite combinatorics then one is lead naturally to true but unprovable (in PA) assertions. This principle (which is formulated in the last sentence) proved to be very useful. It provides immediate conjectures on provability thresholds. Moreover in classifying resulting thresholds we have not encountered a situation in which the principle did not apply.

It would be interesting to explore whether this principle can be proved in some abstract framework.

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