Phase transitions for monotone increasing sequences, the Erdös-Szekeres theorem and the Dilworth theorem

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Abstract

Motivated by the classical Ramsey for pairs problem in reverse mathematics we investigate the recursion-theoretic complexity of certain assertions which are related to the Erd"os-Szekeres-theorem. We show that resulting density principles give rise to Ackermannian growth. We then parameterize these assertions with respect to a number-theoretic function f and investigate for which functions f Ackermannian growth is still preserved. We show that this is the case for $f(i) = sqrt[d]i$ but not for $f(i) = \log(i)$.

1 Introduction

It is well known that every infinite sequence of natural numbers contains an infinite subsequence which is weakly monotonic increasing. It is quite natural to ask, which strength can be generated from this principle (which we call MISP) and for this purpose we consider a miniaturization of MISP in terms of densities. This density approach Paris's original independence result for PA in terms of Ramsey-densities. As usual the density statement for MISP follows from an application of König's Lemma to MISP.

We call a set of natural numbers X 0-dense if $|X| \geq \min X$ and X $n+1$ dense if for all regressive $F : X \to \mathbb{N}$ there exists a $Y \subseteq X$ such that $F \restriction Y$ is weakly monotonic decreasing. Then for every natural number n and natural number a there exists a natural number $b := \mathrm{MISP}(n, a)$ such that the interval $[a, b]$ is n-dense.

In a first step we will show that the function $n \mapsto \mathrm{MISP}(n, n)$ is n-dense.

In a second step we consider phase transitions related to MISP. This contributes to a general research program of the second author about phase transitions in logic and combinatorics (See, for example, [4, 5, 6, 7, 8, 9, **?**] for more information.)

Given a number-theoretic function f we call a set of natural numbers X 0-f-dense if $|X| \geq f(\min X)$ and X $n+1$-f-dense if for all regressive $F : X \to \mathbb{N}$ there exists a $Y \subseteq X$ such that $F \restriction Y$ is weakly monotonic decreasing and such that Y is n-f-dense. Then for any fixed f and every natural number n and natural number a there exists a natural number $b := \mathrm{MISP}_f(n,a)$ such that the interval $[a,b]$ is n-f-dense.

It is easy to see that for a constant function f the function $n \mapsto \mathrm{MISP}_f(n,n)$ is elementary recursive. Moreover as we have announced above the function $n \mapsto \mathrm{MISP}_f(n,n)$ is Ackermannian for $f(i) = i$. So inbetween constant functions and the identity function there will be a threshold region for f where the function $n \mapsto \mathrm{MISP}_f(n,n)$ switches from being primitive recursive to being Ackermannian. We show that for $f(i) = \log(i)$ the function $n \mapsto \mathrm{MISP}_f(n,n)$ remains elementary recursive whereas for every fixed d and $f(i) = \sqrt[d]{(i)}$ the function $n \mapsto \mathrm{MISP}_f(n,n)$ becomes Ackermannian.

Our results are intended to contribute partly to the RT^2_2 problem in reverse mathematics. It is obvious that RT^2_2 yields MISP and the related MISP-, Erdös-Szekeres- or CAC-principles. Moreover RT^2_2 also yields the infinitary Erdös-Moser principle EM stating that every complete infinite directed graph has an infinite transitive subgraph. Now EM and CAC are particularly interesting for studying RT^2_2 since EM + CAC prove RT^2_2. Therefore classifying the strength of EM and CAC may yield progress in classifying the strength of RT^2_2. It is somewhat surprising that even MISP generates all primitive recursive functions with its miniaturization. But this should not be seen as an indication that RT^2_2 proves the totality of the Ackermann function.

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