# Rewriting theory for the Hydra battle and the extended Grzegorczyk hierarchy

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**Abstract**

A subrecursive rewriting framework for the classical Kirby and Paris hydra battle is introduced. The termination of a natural rewrite system $R_H$ for the Hydra battle is shown by using ordinals and additionally by proving an upper bound on the derivation lengths in terms of a fast growing function of ordinal index $\varepsilon_0$. It is shown that the $R_H$-derivation lengths cannot be bounded by a fast growing function of ordinal index less that $\varepsilon_0$, hence the termination of $R_H$ cannot be proved in first order Peano arithmetic. This yields that any natural *pointwise* termination ordering for the hydra battle rewrite system $R_H$ must have order type equal to the Howard Bachmann ordinal, as conjectured by E.A. Cichon. Rewrite systems for various levels of the extended Grzegorczyk hierarchy (up to ordinal level $\varepsilon_0$) are introduced and their derivation lengths are classified with appropriate functions from the fast growing hierarchy.

## 1 A subrecursive rewriting framework for the hydra battle

We begin with recalling the definition of the classical Kirby and Paris hydra battle [cf.[12]]. Since the geometrical formulation of this battle is well known we will not repeat it here. Instead we will directly deal with the underlying set of involved ordinal notations for the ordinals less than $\varepsilon_0$. The (commutative) natural sum of ordinals is denoted by #. See, for example, [14, 18] for a definition. Due to the Cantor normal form theorem every ordinal less than $\varepsilon_0$ can be

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represented in terms of $0, \omega$ and $\#$ (modulo permutative congruence in a unique way). A *hydra* $\alpha$ is an ordinal expression built up in terms of $0, \omega$ and $\#$. A *step* in the battle is described according to the following rules.
Rule 1. Assume that the hydra $\alpha$ has the form $\gamma[\![\omega^{\beta \# \omega^0}]\!]$, i.e. $\omega^{\beta \# \omega^0}$ appears as a subexpression in $\alpha$. If Hercules chops off the indicated *head* $\omega^0$ of $\alpha$ then the hydra $\alpha$ chooses a natural number $x$ and transforms itself into the hydra $\alpha' = \gamma[\![\omega^\beta \cdot x]\!]$. In this case the number of heads of $\alpha'$ will usually be larger than the number of heads of $\alpha$.
Rule 2. Assume that the hydra $\alpha$ has the form $\beta \# \omega^0$. If Hercules chops off the indicated *head* $\omega^0$ of $\alpha$ then the the hydra $\alpha$ transforms itself into the hydra $\beta$.

Using the fact that there does not exist a strictly descending chain of ordinals less than $\varepsilon_0$ it is easily seen that independently of Hercules' strategy the hydra $\alpha$ transforms itself into 0, i.e. a hydra without any head and Hercules always wins the battle (cf. [12]).

Now we consider a special type of the Hydra battle, its *miniaturization.* At the beginning the hydra $\alpha$ chooses a natural number $x$. In the first step of the battle the hydra can only reproduce $x$ many copies of expression of the form $\omega^\beta$ if rule 1 applies. In the second step the hydra can only reproduce $x+1$ many copies of expression of the form $\omega^\beta$ if rule 1 applies. So, in step $k$, the growth rate on reproductions is bounded by the parameter $x+k-1$. The hydra battle then yields a sequence of ordered pairs $\langle \alpha : x \rangle$, $\langle \alpha(x) : x+1 \rangle$, $\ldots, \langle \alpha(x)(x+1) \ldots (x+k) : x+k+1 \rangle$, where $\alpha(x)(x+1) \ldots (x+k)$ is the hydra after $k+1$ steps in the battle.

How many steps $k$ does Hercules need for winning the battle, i.e. how does the least $k$ so that $\alpha(x)(x+1) \ldots (x+k) = 0$ depend on $\alpha$ and $x$? It is well known that an appropriate bound on $k$ can be computed in terms of the Hardy hierarchy.
For $\alpha < \varepsilon_0$ and $x < \omega$ we define $\alpha[x]$ as follows.

1. $0[x] := 0$.
2. $(\beta + 1)[x] := \beta$.
3. If $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n + 1} > \alpha_1 \geq \ldots \geq \alpha_n$, then $\alpha[x] := \omega^{\alpha_1} + \cdots + \omega^{\alpha_n} \cdot x$.
4. If $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n} > \alpha_1 \geq \ldots \geq \alpha_n$ and $\alpha_n$ is a limit ordinal, then $\alpha[x] := \omega^{\alpha_1} + \cdots + \omega^{(\alpha_n)[x]}$.

The Hardy hierarchy $(\mathbf{H}_\alpha)_{\alpha < \varepsilon_0}$ of number-theoretic functions is defined as follows [cf.[16]].

1. $\mathbf{H}_0(x) := x$.
2. $\mathbf{H}_\alpha(x) := \mathbf{H}_{\alpha[x]}(x+1)$.

Then as it is well known the number $k$ of steps which Hercules does need using even the worst strategy for winning the restricted battle against the hydra $\alpha$, which has chosen $x$ at the beginning, is bounded in terms of $\mathbf{H}_\alpha(x)$ and this bound turns out to be essentially optimal. Since $(\mathbf{H}_\alpha)_{\alpha<\varepsilon_0}$ exhausts the provably total functions of Peano arithmetic, the $\Pi^0_2$-assertion that for each hydra $\alpha$ and each starting value $x$ there exists a $k$, so that Hercules wins in $k$ steps, is not provable in Peano arithmetic (cf. [12]).

For defining a rewrite system for the miniaturization of the Hydra battle we are going to mimic the definition of the Hardy functions. The appropriate framework for doing so is not immediately transparent. For example, if one uses a standard first order rewrite system [cf.[9]] then the formal term which corresponds to the involved ordinal operation on ordinals less than $\varepsilon_0$ has also to be considered as a possible number-theoretic argument of the term which represents the Hardy function. The resulting rewrite system [cf. the example in [9]] will then have a not absolutely transparent semantics.

The new idea in our approach is the introduction of typed (or higher order) *expressions* which are not considered as (first order) *terms* (of type zero) but which are used for defining terms. The rewriting itself applies only to *terms*. Once this decision is made the approach becomes transparent and it has its intended semantics. This approach is related to the notion of higher order rewriting which is investigated in [19, 20]. We now come to the formal development of the rewrite system for the hydra battle.

Let $X$ be a countable infinite set of variables. Let 0 be a nullary and $S$ be a unary function symbol. Let $H$ and $\omega$ be symbols.

**Definition 1.1** *Inductive definition of a set $E_H$ of* **expressions** *and a set $T_H$ of* **terms**.

1. *If $x \in X$ then $x \in T_H$.*
2. *$0 \in T_H$.*
3. *If $t \in T_H$ then $St \in T_H$.*
4. *If $t \in T_H$ and $\alpha \in E_H$ then $H_\alpha(t) \in T_H$.*
5. *If $t \in T_H$ then $\omega^0 \cdot t \in E_H$.*
6. *If $t_1, \ldots, t_n \in T_H$ and if $\alpha_1, \ldots, \alpha_n \in E_H$ then $\omega^{\alpha_1} \cdot t_1 + \cdots + \omega^{\alpha_n} \cdot t_n \in E_H$.*

**Definition 1.2** *Inductive definition of a set $C$ of* **contexts**.
*Let $*$ be a symbol.*

1. *If $\alpha, \beta \in E_H$ then $*, \alpha + *, \alpha + * + \beta, * + \beta \in C$.*
2. *If $\gamma \in C$, $t_0, \ldots, t_n \in T_H$ and if $\alpha_1, \ldots, \alpha_n \in E_H$, then $\omega^\gamma \cdot t_0 + \omega^{\alpha_1} \cdot t_1 + \cdots + \omega^{\alpha_n} \cdot t_n \in C$, $\omega^{\alpha_1} \cdot t_0 + \cdots + \omega^{\alpha_i} \cdot t_{i-1} + \omega^\gamma \cdot t_i + \omega^{\alpha_{i+1}} \cdot t_{i+1} \cdots + \omega^{\alpha_n} \cdot t_n \in C$ and $\omega^{\alpha_1} \cdot t_0 + \cdots + \omega^{\alpha_n} \cdot t_{n-1} + \omega^\gamma \cdot t_n \in C$.*

If $\gamma \in C$ and $\beta \in E_H$ then $\gamma[\![\beta]\!]$ denotes the result of replacing (the only occurrence of) $*$ in $\gamma$ by $\beta$; then $\gamma[\![\beta]\!] \in E_H$. In the sequel $\gamma$ ranges over contexts (and also over ordinal contexts which are defined similarly by replacing $+$ throughout $\#$).

**Definition 1.3** *We define a set $R_H$ of* **rewrite rules** *as follows.*

1. $H_{\omega^{\alpha_1}\cdot 0+\cdots+\omega^{\alpha_n}\cdot 0}(z) \to z$.
2. $H_{\gamma[\![\omega^{\alpha+\omega^0\cdot S0}\cdot Sx]\!]}(z) \to H_{\gamma[\![\omega^{\alpha+\omega^0\cdot S0}\cdot x+\omega^\alpha\cdot z]\!]}(Sz)$.
3. $H_{\gamma[\![\omega^0\cdot Sx]\!]}(z) \to H_{\gamma[\![\omega^0\cdot x]\!]}(Sz)$.

If $\sigma : X \to T_H$ is a substitution then $t\sigma$ denotes the result of replacing every occurrence of a variable $x$ in $t$ by $\sigma(x)$; then $t\sigma \in T_H$.

**Definition 1.4** *Definition of the* **rewrite relation** $\to_{R_H}$.
*Let $\to_{R_H}$ be the least binary relation on $T_H$ so that.*

1. *If $\sigma : X \to T_H$ and if $l \to r \in R_H$ then $l\sigma \to_{R_H} r\sigma$.*
2. *If $s \to_{R_H} t$ then $H_\alpha(s) \to_{R_H} H_\alpha(t)$.*
3. *If $s \to_{R_H} t$ then $H_{\gamma[\![\omega^\alpha\cdot s]\!]}(r) \to_{R_H} H_{\gamma[\![\omega^\alpha\cdot t]\!]}(r)$.*

**Definition 1.5** *Definition of a numeral* $\mathbf{n}$ *for $n < \omega$.*
*Let $\mathbf{0} := 0$ and $\mathbf{m+1} := S\mathbf{m}$.*

**Definition 1.6** *Definition of $\Phi : \varepsilon_0 \to E_H$.*

1. $\Phi(0) := 0$.
2. $\Phi(\omega^{\alpha_1}\cdot m_1 + \cdots + \omega^{\alpha_n}\cdot m_n) := \omega^{\Phi(\alpha_1)}\cdot \mathbf{m_1} + \cdots + \omega^{\Phi(\alpha_n)}\cdot \mathbf{m_n}$.

**Proposition 1.1** *For every $\alpha < \varepsilon_0$ and $n \in \omega$ there exists a reduction $H_{\Phi(\alpha)}(\mathbf{n}) \to_{R_H} \cdots \to_{R_H} S^{\mathbf{H}_\alpha(n)}0$ of length $\mathbf{H}_\alpha(n) - n$.*

**Proof.** The hydra $\alpha$ has to be in Cantor normal form with respect to $0, \omega$ and $+$. It has to mimic the definition of the fundamental sequences for the ordinals less than $\varepsilon_0$. Hercules has always to chop off the rightmost head. Then the battle corresponds the computations of the least $k$ so that $\alpha[n][n+1]\ldots[n+k] = 0$. This $k$ is equal to $\mathbf{H}_\alpha(n) - n$ by well known results. [cf., for example, [4] for a proof.] □

$R$ is therefore a natural formulation of a rewrite system for the hydra battle and the Hardy functions. Different strategies for Hercules correspond to a different number of steps which Hercules does need for a win. These differences are reflected in different numerals computed by the rewrite system $R_H$ which is not confluent.

For defining a system which computes uniquely determined normal forms one has to restrict the rewrite rules with respect to the following formal analogue of fundamental sequences.

**Definition 1.7** *Definition of $\alpha[x]$ for $\alpha \in E_H$ and $x \in X$.*

1. *If $\alpha = \beta + \omega^0 \cdot S0 + \omega^{\beta_1} \cdot 0 + \cdots + \omega^{\beta_l} \cdot 0$, then $\alpha[x]$ is defined and $\alpha[x] := \beta + \omega^0 \cdot 0 + \omega^{\beta_1} \cdot 0 + \cdots + \omega^{\beta_l} \cdot 0$.*

2. *If $\alpha = \omega^{\alpha_1} \cdot t_1 + \cdots + \omega^{\alpha_m} \cdot Sy + \omega^{\alpha_{m+1}} \cdot 0 + \cdots + \omega^{\alpha_n} \cdot 0$, and if $\alpha_m = \beta + \omega^0 \cdot S0 + \omega^{\beta_1} \cdot 0 + \cdots + \omega^{\beta_l} \cdot 0$, then $\alpha[x]$ is defined and $\alpha[x] := \omega^{\alpha_1} \cdot t_1 + \cdots + \omega^{\alpha_m} \cdot y + \omega^{\beta + \omega^0 \cdot 0 + \omega^{\beta_1} \cdot 0 + \cdots + \omega^{\beta_l} \cdot 0} \cdot x + \omega^{\alpha_{m+1}} \cdot 0 + \cdots + \omega^{\alpha_n} \cdot 0$.*

3. *If $\alpha = \omega^{\alpha_1} \cdot t_1 + \cdots + \omega^{\alpha_m} \cdot Sy + \omega^{\alpha_{m+1}} \cdot 0 + \cdots + \omega^{\alpha_n} \cdot 0$ and if $\alpha_m = \beta + \omega^{\beta_0} \cdot S0 + \omega^{\beta_1} \cdot 0 + \cdots + \omega^{\beta_l} \cdot 0$ and if $\beta_0$ has the form $\omega^{\delta_1} \cdot r_1 + \cdots + \omega^{\delta_i} \cdot Sr_i + \cdots + \omega^{\delta_k} \cdot r_k$ and if $\alpha_m[x]$ is defined, then $\alpha[x] := \omega^{\alpha_1} \cdot t_1 + \cdots + \omega^{\alpha_m} \cdot y + \omega^{\alpha_m[x]} + \omega^{\alpha_{m+1}} \cdot 0 + \cdots + \omega^{\alpha_n} \cdot 0$.*

Using this formal analogue of fundamental sequences one easily defines a rewrite system for the Hardy functions which computes unique normal forms consisting of numerals which correspond to the values of the Hardy functions.

Proposition 1.1 shows that the $R_H$-derivation lengths – this notion will be defined precisely in section 3 – can not be bounded in terms of a function $\mathbf{H}_\alpha$ for some fixed $\alpha < \varepsilon_0$. Hence the termination of $R_H$ can not be shown in Peano arithmetic. But perhaps $R_H$ is not terminating at all. For excluding this possibility we give in the next section a straightforward termination proof of $R_H$ using ordinals.

**Remark:** It is possible to include the usual notion of first order rewriting [cf.[9]] within our approach. Instead of using $S$ as the only non-constant standard function symbol one can generalize rule 3. in the definition of $T_H$ for including usual first order terms in $T_H$.

# 2 A "pointwise" termination ordering for $R_H$

This section is devoted for a termination proof of $R_H$ using a standard ordinal notation system for the Howard Bachmann ordinal. Familiarity with the theory of the unary ordinal function $\vartheta$ or the binary ordinal function $\overline{\theta}$ is here assumed. The theory of $\vartheta$ is developed in detail, for example, in [15]. The theory of $\overline{\theta}$ is developed, for example, in [18]. Readers who are not familiar with the involved ordinal-theoretic notions can skip this section at first reading. In section 3 an alternative termination proof for $R_H$ – which only involves ordinals less than or equal to $\varepsilon_0$ – will be given.

**Definition 2.1** *For closed $t \in T_H$ and closed $\alpha \in E_H$ we define $\psi t \in \Omega$ and $\Psi\alpha \in \varepsilon_{\Omega+1}$ as follows:*

1. $\psi 0 := 0$.
2. $\psi(St) := \psi t + 1$.
3. $\psi(H_\alpha(t)) := \vartheta(\Omega \cdot \Psi\alpha + \psi t)$.
4. $\Psi(\omega^0 \cdot t) := \psi t$.
5. $\Psi(\omega^{\alpha_1} \cdot t_1 + \cdots + \omega^{\alpha_n} \cdot t_n) := \Omega^{\Psi\alpha_1} \cdot (1 + \psi t_1) \# \cdots \# \Omega^{\Psi\alpha_n} \cdot (1 + \psi t_n)$.

**Lemma 2.1** *If $s, t \in T_H$ are closed and if $s \to_{R_H} t$, then $\psi s > \psi t$.*

**Proof.** Let $\zeta < \Omega, \alpha < \varepsilon_{\Omega+1}$ and $\gamma$ be an ordinal context so that $\gamma[\![0]\!] < \varepsilon_{\Omega+1}$. $\xi < \eta < \Omega$ yields $\vartheta(\Omega \cdot \gamma[\![\Omega^\alpha \cdot \xi]\!] + \zeta) < \vartheta(\Omega \cdot \gamma[\![\Omega^\alpha \cdot \eta]\!] + \zeta)$ and $\vartheta(\Omega \cdot \alpha + \xi) < \vartheta(\Omega \cdot \alpha + \eta)$. Hence $\psi$ is a monotone interpretation and we are left in showing that the rewrite rules are reducing under $\psi$. We have $\vartheta(\Omega \cdot \gamma[\![\Omega^{\alpha+1} \cdot (\xi + 1))]\!] + \zeta) > \vartheta(\Omega \cdot \gamma[\![\Omega^{\alpha+1} \cdot \xi \# \Omega^\alpha \cdot \zeta)]\!] + \zeta)$, $\vartheta(\Omega \cdot \gamma[\![\xi + 1]\!] + \zeta) > \vartheta(\Omega \cdot \gamma[\![\xi]\!] + \zeta + 1)$ and $\vartheta(\Omega \cdot \alpha + \xi) > \xi$. □

**Corollary 2.1** *$R_H$ is terminating.*

So using the theory of ordinal notations we can easily prove the termination of $R_H$. It is not so obvious to use a strict segment of ordinals below the Howard Bachmann ordinal for a termination proof. [Although in the next section we will show that one can in fact use the well-foundedness of $\varepsilon_0$ for a more involved termination proof of $R_H$.] A (slight variant of a) problem posed by E.A. Cichon in LNCS 488 is as follows: Must any (pointwise) termination ordering for the hydra battle have order type equal to the Howard Bachmann ordinal? We have already seen that the Howard Bachmann ordinal is a convenient termination ordering for $R_H$. In the next section we are going to show that we have used the Howard Bachmann ordinal in a pointwise way. This means that one can use the pointwise collapsing operation (which corresponds to functions from the slow growing hierarchy) to collapse the involved ordinals less than the Howard Bachmann ordinal down to natural numbers for obtaining a majorant for the $R_H$-derivation lengths in terms of the pointwise collapsing hierarchy up to level equal to the Howard Bachmann ordinal. The latter hierarchy matches – by definition – up with the fast growing hierarchy of level up to $\varepsilon_0$.

If a strict segment of the Howard Bachmann ordinal would yield a “pointwise” termination ordering for $R_H$, then by the results of the next section the $R_H$ derivation lengths would be bounded in terms of a pointwise collapsing function of level less than the Howard Bachmann ordinal. This would imply that the $R_H$-derivation lengths would be bounded in terms of a Hardy function $\mathbf{H}_\alpha$ for some $\alpha < \varepsilon_0$. But as we have seen in Proposition 1.1 this is not the case. In this sense the answer to the modification of Cichon's question is affirmative.

More generally it can be shown that the termination of higher order rewrite systems for the hydra battle(s) $\mathcal{H}$ of Buchholz [cf.[2]] can be shown by pointwise termination orderings of order type equal to the proof-theoretic ordinal of $ID_{n+1}$ if only $n$ labels are involved resp. of order type equal to the proof-theoretic ordinal of $ID_{\prec^*}$ if $\omega$ labels are involved. A classification of the derivation lengths using the appropriate fast growing functions is also possible. Since these investigations turn out to be technically involved [cf. [23]] we have not included this material here.

All these results confirm the general principle that under some mild conditions – as suggested by Cichon in [8] – the derivation lengths of rewrite systems are non-trivially related to the order type of the (pointwise) termination ordering which is used in the termination proof via the slow growing hierarchy [cf.[22]].

# 3 A second termination proof for $R_H$ and a classification of the $R_H$-derivation lengths

**Definition 3.1** *Definition of $K(\alpha)$ for $\alpha \in E_H$.*

1. $K(\omega^0 \cdot t) := \{t\}$.
2. $K(\omega^{\alpha_1} \cdot t_1 + \cdots + \omega^{\alpha_n} \cdot t_n) := \{t_1, \ldots, t_n\} \cup K(\alpha_1) \cup \cdots \cup K(\alpha_n)$.

**Definition 3.2** *Definition of $c(\alpha)$ for $\alpha \in E_H$.*

1. $c(\omega^0 \cdot t) := 1$.
2. $c(\omega^{\alpha_1} \cdot t_1 + \cdots + \omega^{\alpha_n} \cdot t_n) := n + c(\alpha_1) + \cdots + c(\alpha_n)$.

**Proposition 3.1** *The cardinality of $K(\alpha)$ is equal to $c(\alpha)$.*

**Definition 3.3** *Recursive definition of $dp(t)$ for $t \in T_H$.*

1. $dp(0) := 0$.
2. $dp(x) := 0$.
3. $dp(St) := dp(t) + 1$.
4. $dp(H_\alpha(t)) := 1 + \max\{dp(t), c(\alpha), \max\{dp(s) : s \in K(\alpha)\}\}$.

**Definition 3.4** *Definition of $D_R : \omega \to \omega$.*
$D_R(m) := \max\{n : \exists \text{ closed } t_1, \ldots, t_n \in T_H t_1 \to_{R_H} \cdots \to_{R_H} t_n dp(t_1) \leq m\}$.

Following [4] we define the norm for the ordinals in question. The norm function counts the number-theoretic content of the ordinal under consideration and is the basic ingredient for defining subrecursive hierarchies a la [4].

**Definition 3.5** *Definition of $N\alpha$ for $\alpha \leq \varepsilon_0$.*

1. $N0 := 0$.

2. $N\alpha := m_1 + \cdots + m_n + N\alpha_1 + \cdots + N\alpha_n$ *if* $\alpha = \omega^{\alpha_1} \cdot m_1 + \cdots + \omega^{\alpha_n} \cdot m_n$ *where* $\alpha_1 > \ldots, \alpha_n$ *and* $m_1, \ldots, m_n \neq 0$.

3. $N\varepsilon_0 := 1$.

**Proposition 3.2** $N\gamma[\![\alpha]\!] := N\gamma[\![0]\!] + N\alpha$.

**Proof.** The proof is by induction on the built up of the context $\gamma[\![*]\!]$.
$N(*[\![\alpha]\!]) = N0 + N\alpha$. $N((\beta\#*)[\![\alpha]\!]) = N((*\#\beta)[\![\alpha]\!] = N\beta + N\alpha$. $N((\beta\#*\#\gamma)[\![\alpha]\!]) = N(\beta\#0\#\gamma) + N\alpha$. $N(\omega^{\gamma[\![\alpha]\!]} \cdot t_0 + \cdots + \omega^{\alpha_n} \cdot t_n) = N(\omega^{\gamma[\![0]\!]} \cdot t_0 + \cdots + \omega^{\alpha_n} \cdot t_n) + N\alpha$, etc. □

We denote the $n-$th iteration of a number-theoretic function $f : \omega \to \omega$ by $f^n$.

**Definition 3.6** *Definition of $\mathcal{F}_\alpha : \omega \to \omega$ for $\alpha \leq \varepsilon_0$.*

1. $\mathcal{F}_0(n) := 2^x$.

2. $\mathcal{F}_\alpha(n) := \max\{\mathcal{F}_\beta^{2\cdot(n+2)}(x) : \beta < \alpha N(\beta) \leq 3^{N\alpha+n+1}\}$.

This definition of the fast growing hierarchy is taken from [4] where it has been shown that this definition yields a hierarchy which is equivalent to the classical extended Grzegorczyk hierarchy.

**Proposition 3.3** 1. $x < y \Rightarrow \mathcal{F}_\alpha(x) < \mathcal{F}_\alpha(y)$.

2. $\alpha < \beta N(\alpha) \leq 3^{N\beta+x+1} \Rightarrow \mathcal{F}_\alpha(x) < \mathcal{F}_\beta(x)$.

3. $F_{\alpha+1}(x) \geq F_\alpha^{2\cdot(x+2)}(x)$.

**Proof.** The proposition follows easily from the definition of $F_\alpha$. □

**Theorem 3.1** $D_R(m) \leq \mathcal{F}_{\varepsilon_0}(m)$.

**Proof.** For closed $t \in T_H$ and closed $\alpha \in E_H$ define $\mathcal{I}(t) \in \omega$ and $\mathcal{J}(\alpha) < \varepsilon_0$ as follows:

1. $\mathcal{I}(0) := 0$.
2. $\mathcal{I}(St) := \mathcal{I}(t) + 1$.
3. $\mathcal{I}(H_\alpha(t)) := \mathcal{F}_{\omega\cdot\mathcal{J}(\alpha)+\mathcal{I}(t)}(0)$.
4. $\mathcal{J}(\omega^0 \cdot t) := \omega^0 \cdot \mathcal{I}(t)$.
5. $\mathcal{J}(\omega^{\alpha_1} \cdot t_1 + \cdots + \omega^{\alpha_n} \cdot t_n) := \omega^{\mathcal{J}(\alpha_1)} \cdot (1 + \mathcal{I}(t_1)) \# \cdots \# \omega^{\mathcal{J}(\alpha_n)} \cdot (1 + \mathcal{I}(t_n))$.

**Claim:** $s \to_{R_H} t \Rightarrow \mathcal{I}(s) > \mathcal{I}(t)$.
*Proof* of the claim. If $m < n$, then $\omega\cdot\mathcal{J}(\beta)+m < \omega\cdot\mathcal{J}(\beta)+n$ and, by Proposition 3.2, $N(\omega\cdot\mathcal{J}(\beta)+m) < N(\omega\cdot\mathcal{J}(\beta)+n)$, hence $\mathcal{F}_{\omega\cdot\mathcal{J}(\beta)+m}(0) < \mathcal{F}_{\omega\cdot\mathcal{J}(\beta)+n}(0)$ by assertion 2 of Proposition 3.3. If $m < n$, then $\omega\cdot\mathcal{J}(\gamma)[\![\omega^{\mathcal{J}(\alpha)}\cdot m]\!]+l < \omega\cdot\mathcal{J}(\gamma)[\![\omega^{\mathcal{J}(\alpha)}\cdot n]\!]+l$ and, by Proposition 3.2, $N(\omega\cdot\mathcal{J}(\gamma)[\![\omega^{\mathcal{J}(\alpha)}\cdot m]\!]+l) < N(\omega\cdot\mathcal{J}(\gamma)[\![\omega^{\mathcal{J}(\alpha)}\cdot n]\!]+l)$, hence $\mathcal{F}_{\omega\cdot\mathcal{J}(\gamma)[\![\omega^{\mathcal{J}(\alpha)}\cdot m]\!]+l}(0) < \mathcal{F}_{\omega\cdot\mathcal{J}(\gamma)[\![\omega^{\mathcal{J}(\alpha)}\cdot n]\!]+l}(0)$ by assertion 2 of Proposition 3.3. This discussion shows that $\mathcal{I}$ is a monotone interpretation. So we are left in showing that the rules are reducing under $\mathcal{I}$.
We have $\omega\cdot\mathcal{J}(\gamma)[\![\omega^{\mathcal{J}(\alpha)+1}\cdot(m+1)]\!]+n > \omega\cdot\mathcal{J}(\gamma)[\![\omega^{\mathcal{J}(\alpha)+1}\cdot m\#\omega^{\mathcal{J}(\alpha)}\cdot n]\!]+n$ and, by Proposition 3.2,
$3^{N(\omega\cdot\mathcal{J}(\gamma)[\![\omega^{\mathcal{J}(\alpha)+1}\cdot(m+1)]\!]+n)+1} > N(\omega\cdot\mathcal{J}(\gamma)[\![\omega^{\mathcal{J}(\alpha)+1}\cdot m\#\omega^{\mathcal{J}(\alpha)}\cdot n]\!]+n)$, hence $\mathcal{F}_{\omega\cdot\mathcal{J}(\gamma)[\![\omega^{J(\alpha)+1}\cdot(m+1)]\!]+n}(0) > \mathcal{F}_{\omega\cdot\mathcal{J}(\gamma)[\![\omega^{\mathcal{J}(\alpha)+1}\cdot m\#\omega^{\mathcal{J}(\alpha)}\cdot n]\!]+n}(0)$ by assertion 2 of Proposition 3.3.
Furthermore we have
$\omega\cdot\mathcal{J}(\gamma)[\![m+1]\!]+n > \omega\cdot\mathcal{J}(\gamma)[\![m]\!]+n+1$ and, by Proposition 3.2,
$3^{N(\omega\cdot\mathcal{J}(\gamma)[\![m+1]\!]+n)+1} > N(\omega\cdot\mathcal{J}(\gamma)[\![m]\!]+n+1)$, hence
$\mathcal{F}_{\omega\cdot\mathcal{J}(\gamma)[\![m+1]\!]+n}(0) > \mathcal{F}_{\omega\cdot\mathcal{J}(\gamma)[\![m]\!]+n+1}(0)$ by assertion 2 of Proposition 3.3. We also have $\mathcal{F}_{\omega\cdot 0+n}(0) > n$. This proves the claim.
$\mathcal{F}_{\omega_{3\cdot dp(t)}+1}(dp(t)) > \mathcal{I}(t)$ follows by a straightforward induction on $dp(t)$. Finally we have $\mathcal{F}_{\varepsilon_0}(dp(t)) > \mathcal{F}_{\omega_{3\cdot dp(t)}+1}(dp(t))$. Putting things together, the theorem follows. □

**Comment:** The pointwise collapsing operation $k \mapsto C_k(\alpha)$ is defined for ordinals $\alpha$ less than the Howard Bachmann ordinal as follows:

1. $C_k(0) := 0$.
2. $C_k(\omega^\alpha+\beta) := (k+2)^{C_k(\alpha)} + C_k(\beta)$.
3. $C_k(\Omega) := \omega$.
4. $C_k(\vartheta\alpha) := \mathcal{F}_{C_k(\alpha)}(k)$.

The pointwise ordering $<_k$ for the ordinals less than the Howard Bachmann ordinal is defined by $\alpha <_k \beta :\iff \alpha < \beta C_k(\alpha) < C_k(\beta)$. The proof just given yields that we have used the Howard ordinal in the termination proof given in section 2 in a pointwise way with parameter equal to 0. More general investigations on using pointwise termination orderings for termination proofs are carried out in [22]. In these investigations it turned out that for proving the desired results it is very convenient to use the Buchholz, Cichon and Weiermann 1994 approach to fast growing hierarchies.

**Remark.** If one replaces in the definition of the rewrite system $R_H$ the clause
$H_{\gamma[\![\omega^{\alpha+\omega^0\cdot S0}\cdot Sx]\!]}(z) \to H_{\gamma[\![\omega^{\alpha+\omega^0\cdot S0}\cdot x+\omega^\alpha\cdot z]\!]}(Sz)$ by its slow growing variant
$H_{\gamma[\![\omega^{\alpha+\omega^0\cdot S0}\cdot Sx]\!]}(z) \to S(H_{\gamma[\![\omega^{\alpha+\omega^0\cdot S0}\cdot x+\omega^\alpha\cdot z]\!]}(z))$ then the corresponding derivation lengths are bounded in terms of the function $\mathcal{F}_2$, hence by a primitive recursive function. The simple proof of this fact is left to the reader.

# 4 Subrecursive rewriting theory for the extended Grzegorczyk hierarchy

In this section we describe how the extended Grzegorczyk hierarchy can be modeled within the framework of subrecursive rewriting theory. We define a rewrite system $R_F$ for the Grzegorczyk hierarchy and we introduce subsystems for various levels of this hierarchy. We classify the resulting derivation lengths in terms of the $F$ hierarchy thereby showing that termination of some rewrite systems cannot be proved in the corresponding fragments of first order Peano arithmetic.

For $\alpha < \varepsilon_0$ we define a number-theoretic function $\mathbf{F}_\alpha$ as follows.

1. $\mathbf{F}_0(x) := x + 1$.
2. $\mathbf{F}_{\alpha+1}(x) := \mathbf{F}_\alpha^{x+1}(x)$.
3. $\mathbf{F}_\lambda(x) := \mathbf{F}_{\lambda[x]}(x)$ if $\lambda$ is a limit ordinal.

We are going to define a natural rewriting system for $(\mathbf{F}_\alpha)_{\alpha<\varepsilon_0}$. Let $X$ be a countable infinite set of variables. Let 0 be a nullary and $S$ be a unary function symbol. Let $F$, $I$ and $\omega$ be symbols.

**Definition 4.1** *Inductive definition of a set $E_F$ of* **expressions** *and a set $T_F$ of* **terms**.

1. *If $x \in X$ then $x \in T_F$.*
2. *$0 \in T_F$.*
3. *If $t \in T_F$ then $St \in T_F$.*
4. *If $t \in T_F$ and $\alpha \in E_F$ then $F_\alpha(t) \in T_H$.*
5. *If $t, s \in T_F$ and $\alpha \in E_F$ then $I(F_\alpha)(s,t) \in T_F$.*
6. *If $t \in T_F$ then $\omega^0 \cdot t \in E_F$.*
7. *If $t_1, \ldots, t_n \in T_F$ and if $\alpha_1, \ldots, \alpha_n \in E_F$ then $\omega^{\alpha_1} \cdot t_1 + \cdots + \omega^{\alpha_n} \cdot t_n \in E_F$.*

The corresponding set of contexts is defined similarly as for $T_H$ and $E_H$.

**Definition 4.2** *We define a set $R_F$ of* **rewriting rules** *as follows.*

1. *$F_{\omega^{\alpha_1}\cdot 0+\cdots+\omega^{\alpha_n}\cdot 0}(z) \to Sz$.*
2. *$F_{\beta[\![\omega^{\alpha+\omega^0\cdot S0}\cdot Sx]\!]}(y) \to F_{\beta[\![\omega^{\alpha+\omega^0\cdot S0}\cdot x+\omega^\alpha\cdot y]\!]}(y)$.*
3. *$F_{\beta[\![\omega^0\cdot Sx]\!]}(y) \to I(F_{\beta[\![\omega^0\cdot x]\!]})(Sy, y)$.*

4. $I(F_\alpha)(0,y) \to y$.

5. $I(F_\alpha)(Sy,z) \to F_\alpha(I(F_\alpha)(y,z))$.

**Definition 4.3** *Inductive definition of the* **rewrite relation** $\to_{R_F}$.

1. *If* $\sigma : X \to T_F$ *and if* $l \to r \in R_H$ *then* $l\sigma \to_{R_F} r\sigma$.

2. *If* $s \to_{R_F} t$ *then* $F_\alpha(s) \to_{R_F} F_\alpha(t)$.

3. *If* $s \to_{R_F} t$ *then* $F_{\gamma[\![\omega^\alpha \cdot s]\!]}(r) \to_{R_F} F_{\gamma[\![\omega^\alpha \cdot t]\!]}(r)$.

4. *If* $s \to_{R_F} t$ *then* $I(F_\alpha)(s,u) \to_{R_F} I(F_\alpha)(t,u)$ *and* $I(F_\alpha)(u,s) \to_{R_F} I(F_\alpha)(u,t)$.

5. *If* $s \to_{R_F} t$ *then* $I(F_{\gamma[\![\omega^\alpha \cdot s]\!]})(u,v) \to_{R_F} I(F_{\gamma[\![\omega^\alpha \cdot t]\!]})(u,v)$.

It is easy to see that the $R_F$ derivation lengths cannot be bounded in terms of a single function $\mathbf{H}_\alpha$ for some $\alpha < \varepsilon_0$. Hence the termination of $R_F$ cannot be proved in Peano arithmetic.

**Remark.** For obtaining a confluent version of $R_F$ one has to restrict the rules similar as in the case for $R_H$.

**Theorem 4.1** $R_F$ *is terminating.*

**Proof.** The following proof gives a pointwise termination proof using the Howard Bachmann ordinal. An alternative proof only involving only ordinals less than or equal to $\varepsilon_0$ is given in the proof of theorem 4.2. For closed $t \in T_F$ and closed $\alpha \in E_F$ we define $\psi(t) < \Omega$ and $\Psi(\alpha) < \varepsilon_{\Omega+1}$ as follows.

1. $\psi(0) := 0$.

2. $\psi(St) := \psi(t) + 1$.

3. $\psi(F_\alpha(t)) := \vartheta(\Omega^2 \cdot \Psi(\alpha) + \psi(t))$.

4. $\psi(I(F_\alpha)(s,t) := \vartheta(\Omega^2 \cdot \Psi(\alpha) + \Omega \cdot \psi(s) + \psi(t))$.

5. $\Psi(\omega^0 \cdot t) := \psi(t)$.

6. $\Psi(\omega^{\alpha_1} \cdot t_1 + \cdots + \omega^{\alpha_n} \cdot t_n) := \Omega^{\Psi(\alpha_1)} \cdot (1 + \psi(t_1)) \# \cdots \# \Omega^{\Psi(\alpha_n)} \cdot (1 + \psi(t_n))$.

Then $s \to_{R_F} t$ implies $\psi(s) > \psi(t)$ for closed $s, t \in T_F$. □

**Definition 4.4** *1.* $K_F(\omega^0 \cdot t) := \{t\}$.

2. $K_F(\omega^{\alpha_1} \cdot t_1 + \cdots + \omega^{\alpha_n} \cdot t_n) := \{t_1, \ldots, t_n\} \cup K_F(\alpha_1) \cup \ldots \cup K_F(\alpha_n)$.

**Definition 4.5** *1.* $c_F(\omega^0 \cdot t) := 1$.

2. $c_F(\omega^{\alpha_1} \cdot t_1 + \cdots + \omega^{\alpha_n} \cdot t_n) := n + c_F(\alpha_1) + \ldots + c_F(\alpha_n)$.

**Definition 4.6** *Definition of $dp_F(t)$ for $t \in T$.*

1. $dp_F(0) := 0$.
2. $dp_F(x) := 0$.
3. $dp_F(St) := dp_F(t) + 1$.
4. $dp_F(F_\alpha(t)) := 1 + \max\{dp_F(t), c_F(\alpha), \max\{dp_F(s) : s \in K_F\alpha\}\}$.

In the sequel we drop the superscript $F$ in $K_F, c_F$ and $dp_F$.

**Definition 4.7** 1. *$\omega^0 \cdot t$ occurs directly in $\omega^0 \cdot t$ with height 0.*

2. *$\omega^0 \cdot t$ occurs directly in $\omega^{\alpha_1} \cdot t_1 + \cdots + \omega^{\alpha_n} \cdot t_n$ with height $m+1$ if it occurs directly for some $i \le n$ in $\alpha_i$ with height $m$.*

**Definition 4.8** 1. $E_F^d := \{\alpha \in E_F : ht(\alpha) \le d\}$

2. $T_F^d := \{t \in T_F :$ *If $\alpha$ occurs in $t$ then* $\alpha \in E_F^d\}$
3. $E_F^{d,\omega} := \{\alpha \in E_F^d :$ *If $\omega^0 \cdot t$ occurs directly in $\alpha$ with height $d$ then $t$ is a numeral*$\}$.
4. $T_F^{d,\omega} := \{s \in T_F^n :$ *If $\alpha$ occurs in $s$ then* $\alpha \in E_F^{d,\omega}\}$.
5. $E_F^{d,e} := \{\alpha \in E_F^{d,e} :$ *If $\omega^0 \cdot t$ occurs directly in $\alpha$ with height $d$ then* $t \in \{0, S0, \ldots, \mathbf{e}\}\}$.
6. $T_F^{d,e} := \{s \in T_F^{n,\omega} :$ *If $\alpha$ occurs in $s$ then* $s \in E_F^{d,e}\}$.

**Definition 4.9** *Let ${R_F}^d, {R_F}^{d,\omega}, {R_F}^{d,e}$ be the rewrite system $R_F$ restricted to $T_F^d$, $T_F^{d,\omega}$, $T_F^{d,e}$.*

**Definition 4.10** *Definition of $\omega_n(\alpha)$.*

1. $\omega_0(\alpha) := \alpha$.
2. $\omega_{n+1}(\alpha) := \omega^{\omega_n(\alpha)}$.

**Definition 4.11** *Recursive definition of $K'(\alpha)$ for $\alpha < \varepsilon_0$.*

1. $K'(\omega^0 \cdot m) := \{m\}$.
2. $K'(\omega^{\alpha_1} \cdot m_1 + \cdots + \omega^{\alpha_n} \cdot m_n) := \{m_1, \ldots, m_n\} \cup K'(\alpha_1) \cup \ldots \cup K'(\alpha_n)$.

We put $maxcoeff(\alpha) := \max K'(\alpha)$.

**Theorem 4.2** *Let $d, e < \omega$.*

1. $D_{R_F}(m) \le \mathcal{F}_{\varepsilon_0}(m)$.

2. $D_{R_F{}^d}(m) \leq \mathcal{F}_{\omega_d(\omega)+1}(m)$.

3. $D_{R_F{}^{d,\omega}}(m) \leq \mathcal{F}_{\omega_d(\omega)}(m)$.

4. $D_{R_F{}^{d,e}}(m) \leq \mathcal{F}_{\omega_d(e+1)+1}(m)$.

**Proof.** For closed $t \in T_F$ and closed $\alpha \in E_F$ we define $\mathcal{I}(t) \in \omega$ and $\mathcal{J}(\alpha) < \varepsilon_0$ as follows.

1. $\mathcal{I}(0) := 0$.
2. $\mathcal{I}(S(t)) := \mathcal{I}(t) + 1$.
3. $\mathcal{I}(F_\alpha(t)) := \mathcal{F}_{\mathcal{J}(\alpha)}(\mathcal{I}(t))$.
4. $\mathcal{I}(I(F_\alpha)(s,t)) := \mathcal{F}_{\mathcal{J}(\alpha)}^{\mathcal{I}(s)+1}(\mathcal{I}(t))$.
5. $\mathcal{J}(\omega^0 \cdot t) := \mathcal{I}(t)$.
6. $\mathcal{J}(\omega^{\alpha_1} \cdot t_1 + \dots + \omega^{\alpha_n} \cdot t_n) := \omega^{\mathcal{J}(\alpha_1)} \cdot (1+\mathcal{I}(t_1)) + \dots + \omega^{\mathcal{J}(\alpha_n)} \cdot (1+\mathcal{I}(t_n))$.

Assume $k < l$. Then $k+1 < l+1$, $\mathcal{F}_\alpha(k) < \mathcal{F}_\alpha(l)$, $F_{\gamma[\![\omega^\alpha \cdot k]\!]}(m) < F_{\gamma[\![\omega^\alpha \cdot l]\!]}(m)$, $\mathcal{F}_\alpha^{m+1}(k) < \mathcal{F}_\alpha^{m+1}(l)$, $\mathcal{F}_\alpha^{k+1}(m) < \mathcal{F}_\alpha^{l+1}(m)$ and $\mathcal{F}_{\gamma[\![\omega^\alpha \cdot k]\!]}^{m+1}(n) > \mathcal{F}_{\gamma[\![\omega^\alpha \cdot l]\!]}^{m+1}(n)$. Hence $\mathcal{I}$ yields a monotone interpretation for $R_F$.
We show that the rules are reducing under $\mathcal{I}$.
$\mathcal{F}_\alpha(m) > m$.
$\mathcal{F}_{\gamma[\![m+1]\!]}(n) \geq \mathcal{F}_{\gamma[\![m]\!]}^{2\cdot(n+2)}(n) > \mathcal{F}_{\gamma[\![m]\!]}^{n+2}(n)$,
$\mathcal{F}_\alpha^{2\cdot(m+1+2)}(n) > \mathcal{F}_\alpha(F_\alpha^{2\cdot(m+2)}(n))$.
$\mathcal{F}_{\gamma[\![\omega^{\alpha+1}\cdot(m+1)]\!]}(n) > \mathcal{F}_{\gamma[\![\omega^{\alpha+1}\cdot m \# \omega^\alpha \cdot n]\!]}(n)$.
Therefore, $t_1 \to_{R_F} \cdots \to_{R_F} t_n$ implies $\mathcal{I}(t_1) > \ldots > \mathcal{I}(t_n)$, hence $n \leq \mathcal{I}(t_1)$. The rest of the proof consists in proving upper bounds on $\mathcal{I}(t)$ in terms of $\mathcal{F}_\alpha(dp(t)$ for appropriate $\alpha$. These calculations are done in the following lemmata.

**Proposition 4.1** $\alpha \in E_F \Rightarrow N(\mathcal{J}(\alpha)) \leq c(\alpha) \cdot maxcoeff(\mathcal{J}(\alpha))$

**Proof.** By induction on $c(\alpha)$. □

**Proposition 4.2** $\alpha \in E_F \Rightarrow ht(\alpha) \leq c(\alpha)$.

**Proof.** By induction on $c(\alpha)$. □

**Proposition 4.3** $\alpha \in E_F \Rightarrow \mathcal{J}(\alpha) < \omega_{ht(\alpha)+1}(1)$

**Proof.** By induction on $c(\alpha)$. □

**Lemma 4.1** $s \in T_F \Rightarrow I(s) < \mathcal{F}_{\omega_{dp(s)+1}}^{2\cdot(dp(s)+1)}(0)$.

**Proof.** By induction on $dp(s)$.
If $s = 0$, then the assertion is true.
If $s = S(t)$, then the assertion follows easily from the induction hypothesis.
Assume that $s = F_\alpha(t)$. The induction hypothesis yields $\mathcal{I}(t), \mathcal{I}(u) < \mathcal{F}^{2\cdot dp(s)}_{\omega_{dp(s)}}(0)$ and $maxcoeff(\mathcal{J}(\alpha)) \leq \mathcal{F}^{2\cdot dp(s)}_{\omega_{dp(s)}}(0)$.
Thus, $N(\mathcal{J}(\alpha)) \leq c(\alpha) \cdot \mathcal{F}^{2\cdot dp(s)}_{\omega_{dp(s)}}(0) < 3^{N(\omega_{ht(\alpha)+1}(1)) + \mathcal{F}^{2\cdot dp(s)}_{\omega_{dp(s)}}(0)}$. Hence, $\mathcal{I}(s) = \mathcal{F}_{\mathcal{J}(\alpha)}(\mathcal{I}(t)) < F_{\mathcal{J}(\alpha)}(\mathcal{F}^{2\cdot dp(s)}_{\omega_{dp(s)}}(0)) < \mathcal{F}_{\omega_{ht(\alpha)+1}(1)}(\mathcal{F}^{2\cdot dp(s)}_{\omega_{dp(s)}}(0)) < \mathcal{F}^{2\cdot dp(s)+1}_{\omega_{dp(s)}+1}(0)$.
Finally, assume that $s = I(F_\alpha)(t, u)$. The induction hypothesis yields $\mathcal{I}(t) < \mathcal{F}^{2\cdot dp(s)}_{\omega_{dp(s)}}(0)$ and $maxcoeff(\mathcal{J}(\alpha)) \leq \mathcal{F}^{2\cdot dp(s)}_{\omega_{dp(s)}}(0)$.
Thus, $N(\mathcal{J}(\alpha) + 1) \leq c(\alpha) \cdot \mathcal{F}^{2\cdot dp(s)}_{\omega_{dp(s)}}(0) + 1 < 3^{N(\omega_{ht(\alpha)+1}(1)) + \mathcal{F}^{2\cdot dp(s)}_{\omega_{dp(s)}}(0)}$. Hence, $\mathcal{I}(s) = \mathcal{F}^{\mathcal{I}(t)+1}_{\mathcal{J}(\alpha)}(\mathcal{I}(u)) < \mathcal{F}_{\mathcal{J}(\alpha)+1}(\mathcal{I}(u) + 1 + \mathcal{I}(t)) < \mathcal{F}_{\mathcal{J}(\alpha)+1}(\mathcal{F}^{2\cdot dp(s)}_{\omega_{dp(s)}(1)}(0) \cdot 2 + 1) < \mathcal{F}_{\omega_{ht(\alpha)+1}(1)}(\mathcal{F}^{2\cdot dp(s)}_{\omega_{dp(s)}(1)}(0) \cdot 2 + 2) < \mathcal{F}^{2\cdot dp(s)+1}_{\omega_{dp(s)}+1}(0)$. □

**Corollary 4.1** *$s \in T_F \Rightarrow \mathcal{I}(s) < \mathcal{F}_{\varepsilon_0}(dp(s))$. Hence $D_{R_F}(m) \leq \mathcal{F}_{\varepsilon_0}(m)$.*

**Proof.** $\mathcal{F}_{\varepsilon_0}(dp(s)) > \mathcal{F}_{\omega_{dp(s)+1}(1)}(dp(s)) > \mathcal{I}(s)$. □

**Proposition 4.4** *$\alpha \in E^d_F \Rightarrow \mathcal{J}(\alpha) < \omega_d(1 + \max\{dp(s) : s \in K(\alpha)\})$.*

**Proof.** By induction on $d$. □

**Lemma 4.2** *$s \in T^d_F \Rightarrow \mathcal{I}(s) < \mathcal{F}^{2\cdot(dp(s)+1)}_{\omega_d(\omega)}(0)$.*

**Proof.** By induction on on $dp(s)$.
If $s = 0$ or $s = S(t)$, then the assertion follows easily.
Assume that $s = F_\alpha(t)$. The induction hypothesis yields $\mathcal{I}(t) < \mathcal{F}^{2\cdot dp(s)}_{\omega_d(\omega)}(0)$ and $maxcoeff(\mathcal{J}(\alpha)) \leq \mathcal{F}^{2\cdot dp(s)}_{\omega_d(\omega)}(0)$.
Thus, $N(\mathcal{J}(\alpha)) \leq c(\alpha) \cdot \mathcal{F}^{2\cdot dp(s)}_{\omega_d(\omega)}(0) < 3^{N(\omega_{ht(\alpha)+1}(1)) + \mathcal{F}^{2\cdot dp(s)}_{\omega_d(\omega)}(0)}$. Hence, $\mathcal{I}(s) = \mathcal{F}_{\mathcal{J}(\alpha)}(\mathcal{I}(t)) < \mathcal{F}_{\mathcal{J}(\alpha)}(\mathcal{F}^{2\cdot dp(s)}_{\omega_d(\omega)}(0)) < \mathcal{F}_{\omega_d(\omega)}(\mathcal{F}^{2\cdot dp(s)}_{\omega_d(dp(s))}(0)) < \mathcal{F}^{2\cdot dp(s)+1}_{\omega_d(\omega)}(0)$.
Finally, assume that $s = I(F_\alpha)(t, u)$. The induction hypothesis yields $\mathcal{I}(t), I(u) < \mathcal{F}^{2\cdot dp(s)}_{\omega_d(\omega)}(0)$ and $maxcoeff(\mathcal{J}(\alpha)) \leq \mathcal{F}^{2\cdot dp(s)}_{\omega_d(\omega)}(0)$.
Thus, $N(\mathcal{J}(\alpha) + 1) \leq c(\alpha) \cdot \mathcal{F}^{2\cdot dp(s)}_{\omega_d(\omega)}(0) + 1 < 3^{N(\omega_{ht(\alpha)+1}(1)) + \mathcal{F}^{2\cdot dp(s)}_{\omega_d(\omega)}(0)}$. Hence, $\mathcal{I}(s) = \mathcal{F}^{\mathcal{I}(t)+1}_{\mathcal{J}(\alpha)}(\mathcal{I}(u)) < \mathcal{F}_{\mathcal{J}(\alpha)+1}(\mathcal{I}(u)+1+\mathcal{I}(t)) < \mathcal{F}_{\mathcal{J}(\alpha)+1}(\mathcal{F}^{2\cdot dp(s)}_{\omega_d(\omega)}(0) \cdot 2 + 1) < \mathcal{F}^{2\cdot dp(s)+1}_{\omega_d(\omega)}(0)$.

**Corollary 4.2** *$s \in T^d_F \Rightarrow \mathcal{I}(s) < \mathcal{F}_{\omega_d(\omega)+1}(dp(s))$. Hence $D_{R_F}(m) \leq \mathcal{F}_{\omega_d(\omega)+1}(m)$.*

**Proof.** $\mathcal{F}_{\omega_d(\omega)+1}(dp(s)) \geq \mathcal{F}^{2\cdot(dp(s)+1)}_{\omega_d(\omega)}(dp(s)) > \mathcal{I}(s)$. □

**Lemma 4.3** *$s \in T^{d,\omega}_F \Rightarrow \mathcal{I}(s) < \mathcal{F}^{2\cdot(dp(s)+1)}_{\omega_d(dp(s)+1)}(0)$.*

**Proof.** By induction on $dp(s)$.
If $s = 0$ or $s = S(t)$ then the assertion follows easily.
Assume that $s = F_\alpha(t)$. The induction hypothesis yields $\mathcal{I}(t) < \mathcal{F}^{2\cdot dp(s)}_{\omega_d(dp(s))}(0)$ and $maxcoeff(\mathcal{J}(\alpha)) \leq \mathcal{F}^{2\cdot dp(s)}_{\omega_d(dp(s))}(0)$.
Thus, $N(\mathcal{J}(\alpha)) \leq c(\alpha)\cdot\mathcal{F}^{2\cdot dp(s)}_{\omega_d(dp(s))}(0) < 3^{N(\omega_{ht(\alpha)+1}(1))+\mathcal{F}^{2\cdot dp(s)}_{\omega_d(dp(s))}(0)}$. Hence, $\mathcal{I}(s) = \mathcal{F}_{\mathcal{J}(\alpha)}(\mathcal{I}(t)) < \mathcal{F}_{\mathcal{J}(\alpha)}(\mathcal{F}^{2\cdot dp(s)}_{\omega_d(dp(s))}(0)) < \mathcal{F}_{\omega_d(dp(s)}(\mathcal{F}^{2\cdot dp(s)}_{\omega_d(dp(s))}(0)) < \mathcal{F}^{2\cdot dp(s)+1}_{\omega_d(dp(s)+1)}(0)$.
Finally, assume that $s = I(F_\alpha)(t, u)$. The induction hypothesis yields $\mathcal{I}(t), \mathcal{I}(u) < \mathcal{F}^{2\cdot dp(s)}_{\omega_d(dp(s))}(0)$ and $maxcoeff(\mathcal{J}(\alpha)) \leq \mathcal{F}^{2\cdot dp(s)}_{\omega_d(dp(s))}(0)$.
Thus, $N(\mathcal{J}(\alpha)+1) \leq c(\alpha)\cdot\mathcal{F}^{2\cdot dp(s)}_{\omega_d(dp(s))}(0)+1 < 3^{N(\omega_{ht(\alpha)+1}(1))+\mathcal{F}^{2\cdot dp(s)}_{\omega_d(dp(s))}(0)}$. Hence, $\mathcal{I}(s) = \mathcal{F}^{\mathcal{I}(t)+1}_{\mathcal{J}(\alpha)}(\mathcal{I}(u)) < \mathcal{F}_{\mathcal{J}(\alpha)+1}(\mathcal{I}(u) + 1 + \mathcal{I}(t)) < \mathcal{F}_{\mathcal{J}(\alpha)+1}(\mathcal{F}^{2\cdot dp(s)}_{\omega_d(dp(s))}(0) \cdot 2 + 1) < \mathcal{F}^{2\cdot dp(s)+1}_{\omega_d(dp(s))}(0)$. □

**Corollary 4.3** *$s \in T^{d,\omega}_F \Rightarrow \mathcal{I}(s) < \mathcal{F}_{\omega_d(\omega)}(dp(s))$. Hence $D_{R_F}(m) \leq \mathcal{F}_{\omega_d(\omega)}(m)$.*

**Proof.** $\mathcal{F}_{\omega_d(\omega)}(dp(s)) > \mathcal{F}_{\omega_d(dp(s)+1)}(dp(s)) > \mathcal{I}(s)$. □

**Proposition 4.5** *$\alpha \in E^{d,e}_F \Rightarrow \mathcal{J}(\alpha) < \omega_d(e + 1)$.*

**Proof.** By induction on $d$. □

**Lemma 4.4** *$s \in T^{d,e}_F \Rightarrow \mathcal{I}(s) < \mathcal{F}^{2\cdot(dp(s)+1)}_{\omega_d(e+1)}(0)$.*

**Proof.** By induction on $dp(s)$.
If $s = 0$ or $s = S(t)$, then the assertion follows easily.
Assume that $s = F_\alpha(t)$. The induction hypothesis yields $\mathcal{I}(t) < \mathcal{F}^{2\cdot dp(s)}_{\omega_d(e+1)}(0)$ and $maxcoeff(\mathcal{J}(\alpha)) \leq \mathcal{F}^{2\cdot dp(s)}_{\omega_d(e+1)}(0)$.
Thus, $N(\mathcal{J}(\alpha)) \leq c(\alpha) \cdot \mathcal{F}^{2\cdot dp(s)}_{\omega_d(e+1)}(0) < 3^{N(\omega_d(e+1))+\mathcal{F}^{2\cdot dp(s)}_{\omega_d(e+1)}(0)}$. Hence, $\mathcal{I}(s) = \mathcal{F}_{\mathcal{J}(\alpha)}(\mathcal{I}(t)) < \mathcal{F}_{\mathcal{J}(\alpha)}(\mathcal{F}^{2\cdot dp(s)}_{\omega_d(e+1)}(0)) < \mathcal{F}_{\omega_d(e+1)}(\mathcal{F}^{2\cdot dp(s)}_{\omega_d(e+1)}(0)) < \mathcal{F}^{2\cdot dp(s)+1}_{\omega_d(e+1)}(0)$.
Finally, assume that $s = I(F_\alpha)(t, u)$. The induction hypothesis yields $\mathcal{I}(t), \mathcal{I}(u) < \mathcal{F}^{2\cdot dp(s)}_{\omega_d(e+1)}(0)$ and $maxcoeff(\mathcal{J}(\alpha)) \leq \mathcal{F}^{2\cdot dp(s)}_{\omega_d(e+1)}(0)$.
Thus, $N(\mathcal{J}(\alpha) + 1) \leq c(\alpha) \cdot \mathcal{F}^{2\cdot dp(s)}_{\omega_d(e+1)}(0) + 1 < 3^{N(\omega_{ht(\alpha)+1}(1))+\mathcal{F}^{2\cdot dp(s)}_{\omega_d(e+1)}(0)}$. Hence, $\mathcal{I}(s) = \mathcal{F}^{\mathcal{I}(t)+1}_{\mathcal{J}(\alpha)}(\mathcal{I}(u)) < F_{\mathcal{J}(\alpha)+1}(\mathcal{I}(u)+1+\mathcal{I}(t)) < \mathcal{F}_{\mathcal{J}(\alpha)+1}(\mathcal{F}^{2\cdot dp(s)}_{\omega_d(e+1)}(0)\cdot 2+1) < \mathcal{F}^{2\cdot dp(s)+1}_{\omega_d(e+1)}(0)$. □

**Corollary 4.4** *$s \in T^{d,e}_F \Rightarrow \mathcal{I}(s) < \mathcal{F}_{\omega_d(e+1)+1}(dp(s))$. Hence $D_{R_{F^{d,e}}}(m) \leq \mathcal{F}_{\omega_d(e+1)+1}(m)$.*

**Proof.** $\mathcal{F}_{\omega_d(e+1)+1}(dp(s)) \geq \mathcal{F}^{2\cdot(dp(s)+1)}_{\omega_d(e+1)}(dp(s)) > \mathcal{I}(s)$. □

We end with the following table of results for the rewrite systems for the Grzegorczyk hierarchy.

**Definition 4.12** *Recursive definition of* $\Omega_n(\alpha)$.

*Let* $\Omega_0(\alpha) := \alpha$ *and* $\Omega_{n+1}(\alpha) := \Omega^{\Omega_n(\alpha)}$.

| **A list of rewrite systems and their derivation lengths.** | | |
|---|---|---|
| rewrite system | pointwise term. ordering | bound on der. lengths |
| ${R_F}^{0,e}$ | $\vartheta(\Omega \cdot e)$; *mpo* | $\mathcal{F}_{e+2} \in$ prim rec |
| ${R_F}^{0,\omega}$ | $\vartheta(\Omega \cdot \omega)$; *sup mpo* | $\mathcal{F}_{\omega}$ |
| ${R_F}^{0}$ | $\vartheta(\Omega^2) = \Gamma_0$ | $\mathcal{F}_{\omega+1}$ |
| ${R_F}^{1,e}$ | $\vartheta(\Omega^e)$; *lpo* | $\mathcal{F}_{\omega^{e+1}+1} \in$ multiply rec |
| ${R_F}^{1,\omega}$ | $\vartheta(\Omega^\omega) =$ s.Veblen n.; $v - lpo$ | $\mathcal{F}_{\omega^\omega}$ |
| ${R_F}^{1}$ | $\vartheta(\Omega^\Omega) =$ big Veblen number | $\mathcal{F}_{\omega^\omega+1}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| ${R_F}^{d,e}$ | $\vartheta(\Omega_d(e+1))$ | $\mathcal{F}_{\omega_d(e+1)+1}$ |
| ${R_F}^{d,\omega}$ | $\vartheta(\Omega_d(\omega))$ | $\mathcal{F}_{\omega_d(\omega)}$ |
| ${R_F}^{d}$ | $\vartheta(\Omega_d(\Omega))$ | $\mathcal{F}_{\omega_d(\omega)+1}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $R_F$ | $\vartheta(\varepsilon_{\Omega+1}) =$ H.B. ordinal | $\mathcal{F}_{\varepsilon_0}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |

**4.7 Definition.** For $t \in T_F$ we define its derivation lengths function $D_{R_F}(t)$ as follows. Assume that $FV(t) = \{x_1, \ldots, x_n\}$. $D_{R_F}(t)(m_1, \ldots, m_n)$ is the maximal possible length of an $\rightarrow_{R_F}$ derivation starting from $t[x_1 := \mathbf{m_1}, ..., x_n := \mathbf{m_n}]$.

**Explanation and remarks:** In the first line of the table *mpo* denotes the multiset path ordering over a signature consisting of finitely many but appropriately many varyadic function symbols. The order types of these orderings are bounded by ordinals less than the first primitive recursively closed ordinal. In general derivation lengths resulting from termination proofs with such a multiset ordering are bounded by a primitive recursive function [cf., for example, [10, 3] for a proof]. In line four *lpo* denotes the lexicographic path ordering over a signature consisting of appropriately many function symbols. In general derivation lengths resulting from termination proofs with such a lexicographic ordering are bounded by a multiply recursive function [cf., for example, [21, 3] for a proof]. In line five $v - lpo$ denotes the lexicographic path ordering over a signature consisting of one varyadic function symbol. The order type of this ordering is equal to the small Veblen ordinal $\vartheta(\Omega^\omega)$. The termination of ${R_F}^{d,e}$

can be shown for each $e \in \omega$ in the fragment $PA^- + (\Pi^0_d) - Ind$ of $PA$. The termination of ${R_F}^{d,\omega}$ can not be shown in the fragment $PA^- + (\Pi^0_d) - Ind$ of $PA$. The derivation lengths of terms in ${R_F}^0$ are elementary recursive in $\mathcal{F}_\omega$, i.e. elementary recursive in the Ackermann function. The derivation lengths of terms in ${R_F}^1$ are elementary recursive in $\mathcal{F}_{\omega^\omega}$. The derivation lengths of terms in ${R_F}^d$ are elementary recursive in $\mathcal{F}_{\omega_d}$. The termination of ${R_F}^0$ resp. ${R_F}^1$ cannot been shown in $PA^-$ plus pointwise transfinite induction along initial segments of $\Gamma_0$ resp in $PA^-$ plus pointwise transfinite induction along initial segments of the big Veblen number $\vartheta\Omega^\Omega$ [cf. [1, 17]].

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