CLASSIFYING THE PHASE TRANSITION THRESHOLD FOR REGRESSIVE RAMSEY FUNCTIONS

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ABSTRACT. We classify the sharp phase transition threshold from provability to unprovability in fragments of Peano Arithmetic for the Kanamori-McAloon principle for fixed dimension. For a non negative integer d let $\mathrm{I}\Sigma_d$ be the fragment of Peano arithmetic where the induction is restricted to formulas of alternating quantifier depths d (bounded quantifiers are not counted). We prove that the threshold for $\mathrm{I}\Sigma_d$-unprovable totality of f-regressive Ramsey numbers lies above all functions $n \mapsto \sqrt[g^{-1}(n)]{\log_{d-1}(n)}$ where g^{-1} is the functional inverse of an increasing function g which is primitive recursive in some fast growing function F_α from the Schwichtenberg-Wainer-hierarchy for some $\alpha < \omega_d$. Moreover we show that the threshold for $\mathrm{I}\Sigma_d$-provable totality of f-regressive Ramsey numbers lies below the function $n \mapsto \sqrt[F_{\omega_d}^{-1}(n)]{\log_{d-1}(n)}$.

1. Introduction and Motivation

The Peano Axioms (PA) for the natural numbers have been designed in a way such that every true statement about the natural numbers should follow from these axioms. It came therefore as a great surprise when Gödel showed in 1931 that there are true statements about the natural numbers which do not follow from these axioms. Gödel's original witnesses for the incompleteness of PA have a peculiar logical flavour and it has been suspected that incompleteness might be a purely logical phenomenon. Nevertheless logicians have searched, since Gödel's discovery, for mathematically interesting examples of incompleteness phenomena. A breakthrough has been obtained in 1977 by Paris and Harrington [16] who showed that a slight modification of the finite Ramsey theorem does not follow from PA. Further examples have later been given by Kirby and Paris [11], Pudlak [8], Friedman [19] and Kanamori-McAloon [9].

It is a natural mathematical problem to investigate how the incompleteness emerges in these examples. These investigations led to a recent research program on phase transitions for incompleteness results which surprisingly is connected to areas like analytic number theory, combinatorial probability and, of course, finite combinatorics [23, 24].

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The underlying idea can be briefly outlined as follows. Assume that we have a given assertion $A(F)$ which depends on a (weakly increasing) number-theoretic function $f : \mathbb{N} \to \mathbb{N}$. Let us assume that for a slow growth of f the assertion $A(f)$ does follow from PA and that for a fast-growing f the assertion $A(f)$ is not a consequence of PA. In analogy with random graph theory or statistical mechanics one is tempted to ask for a threshold function where the phase transition from provability to unprovability occurs.

In the examples considered so far a threshold classification has always been obtained by using a certain phase transition principle. The principle asserts that finite combinatorics places restrictions on the logical strength of a combinatorial principle. In cases where we have an assertion with a built-in condition which forces to exceed the bounds from finite combinatorics an independence emerges.

Let us explain this principle in case of the Paris-Harrington assertion. There the homogeneous set Y which is asserted to exist for a given partition exceeds in cardinality a given number and in addition has to satisfy the (so called largeness-) condition that the cardinality of Y is larger than the minimum of Y.

The largeness condition can be fulfilled easily by applying compactness to the infinite Ramsey theorem. Unexpectedly adding the largeness condition to the assertion of the finite Ramsey theorem leads to an independent statement for PA. One might argue that adding an artificial extra condition might produce incompleteness phenomena and logical tricks have been introduced thus somehow indirectly.

But finite combinatorics allows us to give an intrinsic explanation. By bounds from Erdös and Rado [5] it is known that adding a largeness condition of the form $card(Y) > \log^*(\min(Y))$ (where $\log^*$ is the functional inverse of the superexponential function) does not change the nature of the finite Ramsey theorem. If we choose a function like an iterated log function (which slightly exceeds $\log^*$) in the largeness condition which prevents the Erdös-Rado bounds then we come to an incompleteness phenomenon. Moreover, measuring the excess of the Erdös-Rado bounds in terms of hierarchies of recursive functions leads directly to a mathematical proof for the resulting incompleteness.

In this paper we classify the phase transition threshold for the Kanamori-McAloon assertion [9]. This assertion is considered by some authors to yield the most natural incompleteness result for PA. Our proof strategy is, as sketched above, to measure the excess of bounds from finite combinatorics in terms of hierarchies of recursive functions. Unfortunately, there is no underlying theory of Ramsey numbers available for this purpose and, thus, we develop this theory from scratch in the course of the proof of the main theorem.

The corresponding bootstrapping as well as the modified Erdös-Rado bounds seem to be of independent interest by themselves.

The final result will be somewhat surprising. It is well known that for a fixed dimension d the corresponding Kanamori-McAloon and Paris-Harrington statements are equivalent over a weak theory like $\mathrm{I}\Sigma_1$ or even $\mathrm{I}\Delta_0 + \exp$. Instead, the corresponding phase transitions turn out to be intrinsically different. Only in the limit, when unbounded dimension is assumed, the phase transition thresholds will become identical.

The paper is organized as follows. In section 2 we recall some relevant material from logic which is needed later on. In section 3 we show that finite combinatorics yields upper bounds on the strength of the Kanamori-McAloon principle in case of a slow growing parameter functions. This bound is obtained by adapting the classical Erdös and Rado [5] bound to the present situation.

In section 4 we analyze in detail the excess of our Erdös-Rado-style bounds in terms of hierarchies of recursive functions. For this purpose we develop a bit of hierarchy theory which is needed later on. To excess the Erdös-Rado-style bounds we prove lower bounds for certain Ramsey functions which are defined with respect to min-homogeneity. The proof is an adaptation of the Stepping up Lemma (see, for example, [7]). Surprisingly, we are forced to simultaneously consider Ramsey functions which are defined with respect to max-homogeneity as well. (These results will presumably have generalizations to other homogeneity constraints as well.) A main crux of the proof will be the proof of the Sparseness Lemma (Lemma 4.15, Section 4.3). Here we use a peculiar colour-diagonalization construction to excess the Erdös-Rado-style bounds.

This part of the proof is very different from the corresponding treatment of the Paris-Harrington assertion in [27], where the diagonalization proceeds along the cardinality of homogeneous sets). After passing this basic threshold we adapt the machinery from Kanamori and McAloon [9] to iterate the sparseness properties through the hierarchy of fast growing functions.

In the last section we sum up the main results of this paper. This paper is essentially self contained. Nevertheless, basic familiarity with the Kanamori and McAloon paper [9] and the Ketonen and Solovay paper [10] might be useful.

2. Background Notions and Results

We recall the definition of the Fast-Growing Hierarchy $(F_\alpha)_{\alpha<\varepsilon_0}$ (see [2, 18, 21]). If f is a function and $d \geq 0$ we denote by f^d the d-th iteration of f, with $f^0(x) := x$.

$$\begin{aligned} F_0(x) &:= x+1 \\ F_{\alpha+1}(x) &:= F_\alpha^x(x) \\ F_\lambda(x) &:= F_{\lambda[x]}(x) \end{aligned}$$

Here $\cdot\,[\cdot] : \varepsilon_0 \times \mathbb{N} \to \varepsilon_0$ is a fixed assignment of fundamental sequences to ordinals below ε_0, defined as follows. We assume a normal form condition, e.g. Cantor Normal Form. $(\gamma+\omega^\lambda)[x] := \gamma+\omega^{\lambda[x]}$, $(\gamma+\omega^{\beta+1})[x] := \gamma+\omega^\beta \cdot x$, $\varepsilon_0[x] := \omega_{x+1}$, where $\omega_0(x) := x$, $\omega_{d+1}(x) := \omega^{\omega_d(x)}$ and $\omega_d := \omega_d(1)$. For technical reasons we put $(\beta+1)[x] := \beta$ and $0[x] := 0$.

Let $\mathrm{I}\Sigma_d$ be the subsystem of PA obtained by replacing full first-order induction with induction restricted to $\mathrm{I}\Sigma_d$- formulas ($\mathrm{I}\Sigma_d$-induction). It is well-known that the provably total function of $\mathrm{I}\Sigma_d$ are characterized as the functions that are primitive recursive in some F_α for $\alpha < \omega_d$ (see, e.g. [2, 18, 22]). Also, the function F_{ω_d} eventually dominates every provably total function of $\mathrm{I}\Sigma_d$. We summarize these facts in the following Theorem.

Theorem 2.1. Let $d > 0$. Then

1. (1) $\mathrm{I}\Sigma_d \vdash (\forall x)(\exists y)[F_\alpha(x) = y]$ iff $\alpha < \omega_d$.
2. (2) Let f be a Σ_1-definable function. Then $\mathrm{I}\Sigma_d$ proves the totality f if and only if f is primitive recursive in F_α for some $\alpha < \omega_d$.
3. (3) F_{ω_d} eventually dominates all $\mathrm{I}\Sigma_d$-provably total functions.
4. (4) $\mathrm{PA} \vdash (\forall x)(\exists y)[F_\alpha(x) = y]$ iff $\alpha < \varepsilon_0$.

(5) Let f be a Σ_1-definable function. Then PA proves the totality f if and only if f is primitive recursive in F_α for some $\alpha < \varepsilon_0$.
(6) F_{ε_0} eventually dominates all $\mathrm{I}\Sigma_d$-provably total functions.

Let us formally introduce the Kanamori-McAloon and the Paris-Harrington principles. If $X \subseteq \mathbb{N}, d \in \mathbb{N}$, let $[X]^d$ be the set of all subsets of X with d elements. As usual in Ramsey Theory, we identify a positive integer m with its set of predecessors $\{0, \ldots, m-1\}$. If C is a colouring defined on $[X]^d$ we write $C(x_1, \ldots, x_d)$ for $C(\{x_1, \ldots, x_d\})$ where $x_1 < \cdots < x_d$. A subset H of X is called *homogeneous* or *monochromatic* for C if C is constant on $[H]^d$. We write

$$X \to (m)^d_k$$

if for all $C : [X]^d \to k$ there exists $H \subseteq X$ s.t. $card(H) = m$ and H is homogeneous for C. Ramsey [17] proved the following result, known as the Finite Ramsey Theorem.

$$(\forall d)(\forall k)(\forall m)(\exists \ell)[\ell \to (m)^d_k].$$

Erdös and Rado gave in [5] a primitive recursive upper bound for such an ℓ. This shows that the Finite Ramsey Theorem is provable in $\mathrm{I}\Sigma_1$. The asymptotics of Ramsey numbers is a main concern in Ramsey Theory [7].

The Paris-Harrington principle is a seemingly innocent variant of the Finite Ramsey Theorem. Let f be a number-theoretic function. A set X is called *f-relatively large* if $card(X) \geq f(\min X)$. If $f = id$, the identity function, we call such a set *relatively large* or just *large*. We write

$$X \to^*_f (m)^d_k$$

if for all $C : [X]^d \to k$ there exists $H \subseteq X$ s.t. $card(H) = m$, H is homogeneous for C and H is relatively f-large. The Paris-Harrington principle is just the Finite Ramsey Theorem with the extra condition that the homogeneous set is also relatively large.

$$(\mathrm{PH}) :\equiv (\forall d)(\forall k)(\forall m)(\exists \ell)[\ell \to^*_{id} (m)^d_k].$$

Let $f : \mathbb{N} \to \mathbb{N}$ be a number-theoretic function. A function $C : [X]^d \to \mathbb{N}$ is called *f-regressive* is for all $s \in [X]^d$ such that $f(\min(s)) > 0$ we have $C(s) < f(\min(s))$. When f is the identity function we just say that C is regressive. A set H is *min-homogeneous* for C if for all $s, t \in [H]^d$ with $\min(s) = \min(t)$ we have $C(s) = C(t)$. We write

$$X \to (m)^d_{f\text{-}reg}$$

if for all f-regressive $C : [X]^d \to \mathbb{N}$ there exists $H \subseteq X$ s.t. $card(H) = m$ and H is min-homogeneous for C. In [9] Kanamori and McAloon introduced the following statement and proved it for any choice of f.

$$(\mathrm{KM})_f :\equiv (\forall d)(\forall m)(\exists \ell)[\ell \to (m)^d_{f\text{-}reg}].$$

The main result of [9], proved by a model-theoretic argument, is that $(\mathrm{KM})_{id}$ is unprovable in PA. As a corollary one obtains the (provable in PA) equivalence of (KM) with (PH).

Weiermann considered the concept of f-largeness in [25] order to study the phase transition for (PH). He accordingly introduced the following parametrized Paris-Harrington principle

$$(\mathrm{PH})_f :\equiv (\forall d)(\forall k)(\forall m)(\exists \ell)[\ell \to^*_f (m)^d_k]$$

and characterized for which f the principle $(\mathrm{PH})_f$ remains unprovable in PA. We summarize his results in the following Theorem. Let $|\cdot|_d$ be the d-times iterated binary length function and $\log^*$ the inverse of the superexponential function:

$$|x| := \log_2(x+1), \quad |x|_{d+1} := ||x|_d|, \quad \text{and} \quad \log^* x := \min\{d : |x|_d \leq 2\}$$

For an unbounded function $f : \mathbb{N} \longrightarrow \mathbb{N}$ we denote by f^{-1} the inverse of f defined as follows. $f^{-1}(x) := \min\{y : f(y) > x\}$. Observe that we have $f^{-1}(x) \leq y$ if and only if $x < f(y)$.

Theorem 2.2 (Weiermann [25]). For $\alpha \leq \varepsilon_0$ let

$$f_\alpha(i) = |i|_{F_\alpha^{-1}(i)}.$$

Then

1. $\mathrm{I}\Sigma_1 \vdash (\mathrm{PH})_{\log^*}$.
2. For all $d \in \mathbb{N}$, $\mathrm{PA} \nvdash (\mathrm{PH})_{|\cdot|_d}$.
3. $\mathrm{PA} \vdash (\mathrm{PH})_{f_\alpha}$ if and only if $\alpha < \varepsilon_0$.

In his Ph.D. thesis [15], the second author showed that the same situation occurs in the case of (KM). That is, the phase transition threshold is the same when unbounded dimensions are considered. Lee also obtained partial results for (KM) with fixed dimensions with respect to fragments of PA.

Let us formally define the version of (PH) and (KM) for fixed dimensions, the latter being the main concern of the present paper:

$$\begin{aligned} (\mathrm{PH})^d_f \quad &:\equiv \quad (\forall k)(\forall m)(\exists \ell)[\ell \to^*_f (m)^d_k] \\ (\mathrm{KM})^d_f \quad &:\equiv \quad (\forall m)(\exists \ell)[\ell \to (m)^d_{f\text{-}reg}] \end{aligned}$$

When f is the identity function we drop the subscript. In our investigation we will study the growth rate of the functions that are naturally associated with Π^0_2 combinatorial principles. In particular, we define

$$R(\mu)^n_f(m) := \min\{\ell : \ell \to (m)^d_{f\text{-}reg}\}.$$

That is, $R(\mu)^n_f$ is the Skolem-function associated with $(\mathrm{KM})^n_f$.

In his Ph.D. thesis [15], the second author proved the following Theorem.

Theorem 2.3 (Lee, [15]). Let $d \geq 1$. Then

1. $\mathrm{I}\Sigma_1 \vdash (\mathrm{KM})^{d+1}_{\lfloor \log_d \rfloor}$.
2. $\mathrm{I}\Sigma_d \nvdash (\mathrm{KM})^{d+1}_{\lfloor \log_{d-2} \rfloor}$.

The case $d-1$ was left open. Lee formulated the following Conjecture.

Conjecture 2.4 (Lee, 2005). For all $n \geq 1$, for all $d \geq 1$

$$\mathrm{I}\Sigma_d \nvdash (\mathrm{KM})^{d+1}_{\lfloor \sqrt[n]{\log_{d-1}} \rfloor}.$$

We will prove a general Theorem that implies the truth of Conjecture 2.4 and closes the gap in Theorem 2.3.

Recall that from [9] we have the following.

Theorem 2.5. Let $d \geq 1$.

1. $\mathrm{I}\Sigma_1 \vdash (\mathrm{PH})^{d+1} \leftrightarrow (\mathrm{KM})^{d+1}$.
2. $\mathrm{I}\Sigma_d \vdash (\mathrm{KM})^d$.
3. $\mathrm{I}\Sigma_d \nvdash (\mathrm{KM})^{d+1}$.

The third author has characterized as follows the phase transition for $(\mathrm{PH})^d$:

Theorem 2.6 (Weiermann [27]). Let

$$f^d_\alpha(i) = \left\lfloor \frac{|i|_d}{F^{-1}_\alpha(i)} \right\rfloor.$$

Then

$$\mathrm{I}\Sigma_d \vdash (\mathrm{PH})^{d+1}_{f^d_\alpha} \quad \text{iff} \quad \alpha < \omega_d.$$

In this paper we classify the phase transition threshold for the $(\mathrm{KM})^d$ principles. Surprisingly (in view of Theorems 2.2 and 2.5 (i) above), for fixed dimensions, the phase transition of the Kanamori-McAloon principle turns out to be different from that of the Paris-Harrington principle. Let $\log_d$ be the d-th iterated logarithm in base 2. We stipulate $\log(0) := 0$. Let 2^c_d be the tower function in base 2 with height d and exponent c. $2_0(x) := x$ and we sometimes write $2_d(c)$ for 2^c_d. Our main result (Theorem 5.1 in section 5) is the following.

Theorem. Let

$$f^d_\alpha(i) = \lfloor \sqrt[F^{-1}_\alpha(i)]{\log_d(i)} \rfloor.$$

Then

$$\mathrm{I}\Sigma_d \vdash (\mathrm{KM})^{d+1}_{f^{d-1}_\alpha} \quad \text{iff} \quad \alpha < \omega_d.$$

The case $d = 1$ has been proved by Kojman, Lee, Omri and Weiermann in [12] generalizing methods from Kojman and Shelah [13] and [4]. Our proof does *not* follow these lines.

3. Provability (Upper Bounds)

In this section we show the provability part of our main result. Essentially, the bound for standard Ramsey functions from Erdös-Rado's [5] is adapted to the case of regressive functions.

Definition 3.1. Let $C\colon [\ell]^d \to k$ be a coloring. Call a set H *s-homogeneous* for C if for any s-element set $U \subseteq H$ and for any $(d-s)$-element sets $V, W \subseteq H$ such that $\max U < \min\{\min V, \min W\}$, we have

$$C(U \cup V) = C(U \cup W).$$

$(d-1)$-homogeneous sets are called *end-homogeneous*.

Note that 0-homogeneous sets are homogeneous and 1-homogeneous sets are min-homogeneous. Let

$$X \to_s \langle m \rangle^d_k$$

denote that given any coloring $C\colon [X]^d \to k$, there is H s-homogeneous for C such that $card(H) \geq m$. The following lemma shows a connection between s-homogeneity and homogeneity.

Lemma 3.2. Let $s \leq d$ and assume

(1) $\ell \to_s \langle p \rangle^d_k$,
(2) $p - d + s \to (m - d + s)^s_k$.

Then we have

$$\ell \to (m)^d_k.$$

Proof. Let $C\colon [\ell]^d \to k$ be given. Then assumption 1 implies that there is $H \subseteq \ell$ such that $|H| = p$ and H is s-homogeneous for C. Let $z_1 < \cdots < z_{d-s}$ be the last $d - s$ elements of H. Set $H_0 := H \setminus \{z_1, \ldots, z_{d-s}\}$. Then $card(H_0) = p - d + s$. Define $D\colon [H_0]^s \to k$ by

$$D(x_1, \ldots, x_s) := C(x_1, \ldots, x_s, z_1, \ldots, z_{d-s}).$$

By assumption 2 there is Y_0 such that $Y_0 \subseteq H_0$, $card(Y_0) = m - d + s$, and homogeneous for D. Hence $D \restriction [Y_0]^s = e$ for some $e < k$. Set $Y := Y_0 \cup \{z_1, \ldots, z_{d-s}\}$. Then $card(Y) = m$ and Y is homogeneous for C. Indeed, we have for any sequence $x_1 < \cdots < x_d$ from Y

$$C(x_1, \ldots, x_d) = C(x_1, \ldots, x_s, z_1, \ldots, z_{d-s}) = D(x_1, \ldots, x_s) = e.$$

The proof is complete. $\square$

Given d, s such that $s \leq d$ define $R^s_\mu(d, \cdot, \cdot)\colon \mathbb{N}^2 \to \mathbb{N}$ by

$$R^s_\mu(d, k, m) := \min\{\ell\colon \ell \to_s \langle m\rangle^d_k\}.$$

Then

- $R^0_\mu(1, k, m - d + 1) = k \cdot (m - d) + 1$,
- $R^d_\mu(d, k, m) = R^s_\mu(d, 1, m) = m$,
- $R^s_\mu(d, k, d) = d$,
- $R^s_\mu(d, k, m) \leq R^{s-1}_\mu(d, k, m)$ for any $s > 0$.

R^s_μ are called *Ramsey functions*. Set

$$R(d, k, m) := R^0_\mu(d, k, m) \quad \text{and} \quad R_\mu(d, k, m) := R^1_\mu(d, k, m).$$

Then $R(\mu)^d_{f_k}(m) = R^1_\mu(d, k, m)$ where f_k is the constant function with value k. Define a binary operation $*$ by putting, for positive natural numbers x and y,

$$x * y := x^y.$$

Further, we put for $p \geq 3$

$$x_1 * x_2 * \cdots * x_p := x_1 * (x_2 * (\cdots * (x_{p-1} * x_p) \cdots))$$

Erdös and Rado [5] gave an upper bound for $R(d, k, m)$: Given d, k, m such that $k \geq 2$ and $m \geq d \geq 2$, we have

$$R(d, k, m) \leq k * (k^{d-1}) * (k^{d-2}) * \cdots * (k^2) * (k \cdot (m - d) + 1).$$

Theorem 3.3 ($\mathrm{I}\Sigma_1$). Let $2 \leq d \leq m$, $0 < s \leq d$, and $2 \leq k$.

$$R^s_\mu(d, k, m) \leq k * (k^{d-1}) * (k^{d-2}) * \cdots * (k^{s+1}) * (m - d + s) * s.$$

In particular, $R_\mu(2, k, m) \leq k^{m-1}$.

Proof. The proof construction below is motivated by Erdös and Rado [5]. We shall work with s-homogeneity instead of homogeneity.

Let X be a finite set. In the following construction we assume that $card(X)$ is large enough. How large it should be will be determined after the construction has been defined. Throughout this proof the letter Y denotes subsets of X such that $card(Y) = d - 2$.

Let $C\colon [X]^d \to k$ be given and $x_1 < \ldots < x_{d-1}$ the first $d - 1$ elements of X. Given $x \in X \setminus \{x_1, \ldots, x_{d-1}\}$ put

$$C_{d-1}(x) := C(x_1, \ldots, x_{d-1}, x).$$

Then $\mathrm{Im}(C_{d-1}) \subseteq k$, and there is $X_d \subseteq X \setminus \{x_1, \dots, x_{d-1}\}$ such that C_{d-1} is constant on X_d and

$$card(X_d) \geq k^{-1} \cdot (card(X) - d + 1).$$

Let $x_d := \min X_d$ and given $x \in X_d \setminus \{x_d\}$ put

$$C_d(x) := \prod \{C(Y \cup \{x_d, x\}) \colon Y \subseteq \{x_1, \dots, x_{d-1}\}\}.$$

Then $\mathrm{Im}(C_d) \subseteq c * \binom{d-1}{d-2}$, and there is $X_{d+1} \subseteq X_d \setminus \{x_d\}$ such that C_d is constant on X_{d+1} and

$$card(X_{d+1}) \geq k^{-\binom{d-1}{d-2}} \cdot (card(X_d) - 1)\,.$$

Generally, let $p \geq d$, and suppose that $x_1, \dots, x_{p-1}$ and X_d, $X_{d+1}, \dots, X_p$ have been defined, and that $X_p \neq \varnothing$. Then let $x_p := \min X_p$ and for $x \in X_p \setminus \{x_p\}$ put

$$C_p(x) := \prod \{C(Y \cup \{x_p, x\}) \colon Y \subseteq \{x_1, \dots, x_{p-1}\}\}.$$

Then $\mathrm{Im}(C_p) \subseteq k * \binom{p-1}{d-2}$, and there is $X_{p+1} \subseteq X_p \setminus \{x_p\}$ such that C_p is constant on X_{p+1} and

$$card(X_{p+1}) \geq k^{-\binom{p-1}{d-2}} \cdot (card(X_p) - 1)\,.$$

Now put

$$\ell := 1 + R^s_\mu(d-1, k, m-1).$$

Then $\ell \geq m \geq d$. If $card(X)$ is sufficiently large, then $X_p \neq \varnothing$, for all p such that $d \leq p \leq \ell$, so that $x_1, \dots, x_\ell$ exist. Note also that $x_1 < \cdots < x_\ell$. For $1 \leq \rho_1 < \cdots < \rho_{d-1} < \ell$ put

$$D(\rho_1, \dots, \rho_{d-1}) := C(x_{\rho_1}, \dots, x_{\rho_{d-1}}, x_\ell).$$

By definition of ℓ there is $Z \subseteq \{1, \dots, \ell - 1\}$ such that Z is s-homogeneous for D and $card(Z) = m - 1$. Finally, we put

$$X' := \{x_\rho \colon \rho \in Z\} \cup \{x_\ell\}.$$

We claim that X' is min-homogeneous for C. Let

$$H := \{x_{\rho_1}, \dots, x_{\rho_d}\} \quad \text{and} \quad H' = \{x_{\eta_1}, \dots, x_{\eta_d}\}$$

be two subsets of X' such that $\rho_1 = \eta_1, \dots, \rho_s = \eta_s$ and

$$1 \leq \rho_1 < \cdots < \rho_d \leq \ell, \quad 1 \leq \eta_1 < \cdots < \eta_d \leq \ell.$$

Since x_{ρ_d}, $x_\ell \in X_{\rho_d}$, we have $C_{\rho_{d-1}}(x_{\rho_d}) = C_{\rho_{d-1}}(x_\ell)$ and hence

$$C(x_{\rho_1}, \dots, x_{\rho_{d-1}}, x_{\rho_d}) = C(x_{\rho_1}, \dots, x_{\rho_{d-1}}, x_\ell).$$

Similarly, we show that

$$C(x_{\eta_1}, \dots, x_{\eta_{d-1}}, x_{\eta_d}) = C(x_{\eta_1}, \dots, x_{\eta_{ddn-1}}, x_\ell).$$

In addition, since $\{x_{\rho_1}, \dots, x_{\rho_{d-1}}\} \cup \{x_{\eta_1}, \dots, x_{\eta_{d-1}}\} \subseteq X'$, we have

$$D(\rho_1, \dots, \rho_{d-1}) = D(\eta_1, \dots, \eta_{d-1}),$$

i.e.,

$$C(x_{\rho_1}, \dots, x_{\rho_{d-1}}, x_\ell) = C(x_{\eta_1}, \dots, x_{\eta_{d-1}}, x_\ell).$$

This means that $C(H) = C(H')$. So X' is min-homogeneous for C.

We now return to the question how large $card(X)$ should be in order to ensure that the construction above can be carried through.

Set

$$\begin{aligned} t_d &:= k^{-1} \cdot (card(X) - d + 1), \\ t_{p+1} &:= k^{-\binom{p-1}{d-2}} \cdot (t_p - 1) \qquad (d \leq p < \ell). \end{aligned}$$

Then we require that $t_\ell > 0$, where

$$\begin{aligned} t_\ell &= k^{-\binom{\ell-2}{d-2}} \cdot \Big(k^{-\binom{\ell-3}{d-2}} \cdot \Big(\cdots \Big(k^{-\binom{d-1}{d-2}} \cdot \big(t_d - 1\big)\Big) \cdots \Big) - 1\Big) \\ &= k^{-\binom{\ell-2}{d-2} - \cdots - \binom{d-1}{d-2}} \cdot t_d - k^{-\binom{\ell-2}{d-2} - \cdots - \binom{d-1}{d-2}} - \cdots - k^{-\binom{\ell-2}{d-2} - \binom{\ell-3}{d-2}} - k^{-\binom{\ell-2}{d-2}}. \end{aligned}$$

Since $k = k^{\binom{d-2}{d-2}}$ a sufficient condition on $card(X)$ is then

$$card(X) - d + 1 > k^{\binom{\ell-3}{d-2} + \cdots + \binom{d-2}{d-2}} + k^{\binom{\ell-4}{d-2} + \cdots + \binom{d-2}{d-2}} + \cdots + k^{\binom{d-2}{d-2}}.$$

A possible value is

$$card(X) = d + \sum_{p=d-1}^{\ell-2} k^{\binom{p}{d-1}},$$

so that

$$\begin{aligned} R^s_\mu(d,k,m) &\leq d + \sum_{p=d-1}^{\ell-2} k^{\binom{p}{d-1}} \quad \leq \quad d + \sum_{p=d-1}^{\ell-2} k^{p^{d-1}} \\ &\leq d + \sum_{p=d-1}^{\ell-2} \left(k^{(p+1)^{d-1}} - k^{p^{d-1}}\right) \\ &= d + k^{(\ell-1)^{d-1}} - k^{(d-1)^{d-1}} \\ &\leq k^{(\ell-1)^{d-1}} \\ &= k^{R_\mu(d-1,k,m-1)^{d-1}}. \end{aligned}$$

Hence

$$R^s_\mu(d,k,m) * d \quad \leq \quad (k^d) * R^s_\mu(d-1,k,m-1) * (d-1).$$

After $(d-s)$ times iterated applications of the inequality we get

$$\begin{aligned} R^s_\mu(d,k,m) * d &\leq (k^d) * (k^{d-1}) * \cdots * (k^{s+1}) * R^s_\mu(s,k,m-d+s) * s \\ &= (k^d) * (k^{d-1}) * \cdots * (k^{s+1}) * (m-d+s) * s. \end{aligned}$$

This completes the proof. □

Remark 3.4. Lemma 26.4 in [3] gives a slight sharper estimate for $s = d-1$:

$$R^{d-1}_\mu(d,k,m) \leq d + \sum_{i=d-1}^{m-2} k^{\binom{i}{d-1}}$$

Corollary 3.5. Let $2 \leq d \leq m$ and $2 \leq k$.

$$R_\mu(d,k,m) \leq k * (k^{d-1}) * (k^{d-2}) * \cdots * (k^2) * (m-d+1).$$

Now we come back to f-regressiveness and prove the key upper bound of the present section.

Lemma 3.6. Given $d \geq 1$ and $\alpha \leq \varepsilon_0$ set $f_\alpha^d(i) := \lfloor \sqrt[F_\alpha^{-1}(i)]{\log_d(i)} \rfloor$. Then

$$R(\mu)^{d+1}_{f_\alpha^{d-1}}(m) \leq 2_{d-1}^{F_\alpha(q)^{m+p}}$$

for some p, $q \in \mathbb{N}$ depending (primitive-recursively) on d and α.

Proof. Given d and α note first that there are two natural numbers p and q such that $d < p < q$ and for all m

$$\ell := 2_{d-1}^{F_\alpha(q)^{m+d}+1} + F_\alpha(q) \leq 2_{d-1}^{F_\alpha(q)^{m+p}} =: N.$$

Let $C\colon [N]^{d+1} \to \mathbb{N}$ be any f_α^d-regressive function and

$$D\colon [F_\alpha(q), \ell]^{d+1} \to \mathbb{N}$$

be defined from C by restriction. Then for any $y \in [F_\alpha(q), \ell]$, we have

$$\begin{aligned} \sqrt[F_\alpha^{-1}(y)]{\log_{d-1}(y)} &\leq \sqrt[F_\alpha^{-1}(F_\alpha(q))]{\log_{d-1}(2_{d-1}(F_\alpha(q)^{m+p}))} \\ &= \sqrt[q]{F_\alpha(q)^{m+p}}. \end{aligned}$$

Hence

$$\mathrm{Im}(D) \subseteq \lfloor F_\alpha(q)^{(m+p)/q} \rfloor + 1.$$

Put now $k := \lfloor F_\alpha(q)^{(m+p)/q} \rfloor + 1$. Then

$$(k) * (k^d) * \cdots * (k^2) * (m-d) < 2_{d-1}(F_\alpha(q)^{m+d} + 1)$$

if q is sufficiently larger than p. By Theorem 3.3 there is an $H \subseteq N$ min-homogeneous for D, hence for C, such that $card(H) \geq m$. □

Theorem 3.7. Let $d \geq 1$.

(1) $(\mathrm{KM})_{\log^*}$ is provable in $\mathrm{I}\Sigma_1$.
(2) $(\mathrm{KM})^{d+1}_{\lfloor \log_d \rfloor}$ is provable in $\mathrm{I}\Sigma_1$.
(3) $(\mathrm{KM})^{d+1}_{\lfloor \sqrt[F_\alpha^{-1}(\cdot)]{\log_{d-1}(\cdot)} \rfloor}$ is provable in $\mathrm{I}\Sigma_d$ if $\alpha < \omega_d$.

Proof. (1) Let d, $m \geq 1$ be given. Note first that there is x larger than d and m such that for $k := x + m$ and $\ell := x + 2_d^{x+m}$

$$k * (k^{d-1}) * (k^{d-2}) * \cdots * (k^2) * (k \dot{-} d + 1) < 2_d^{x+m}.$$

and

$$\log^* \ell \leq k$$

We claim that $R(\mu)^d_{\log^*}(m) \leq \ell$. Let $C\colon [\ell]^d \to \mathbb{N}$ be $\log^*$-regressive and $C'\colon [x, \ell]^d \to \mathbb{N}$ be defined from C by restriction. By Theorem 3.3 we can find an $H \subseteq \ell$ min-homogeneous for C', hence for C, such that $card(H) \geq m$.

(2) Let d, $m \geq 1$ be given. Note first that there is x larger than d and m such that for $k := 2x + m$ and $\ell := x + 2_d^{x+m}$

$$k * (k^d) * (k^{d-1}) * \cdots * (k^2) * (m \dot{-} d) < 2_d^{x+m}.$$

and

$$\lfloor \log_d(\ell) \rfloor \leq k$$

We claim that $R(\mu)^{d+1}_{\log_d}(m) \leq \ell$. Let $C\colon [\ell]^{d+1} \to \mathbb{N}$ be $\log_d$-regressive and $C'\colon [x, \ell]^{d+1} \to \mathbb{N}$ be defined from C by restriction. By Theorem 3.3

we can find an $H \subseteq \ell$ min-homogeneous for C', hence for C, such that $card(H) \geq m$.

(3) F_α is provably recursive in $\mathrm{I}\Sigma_d$ for $\alpha < \omega_d$. Then the assertion follows from Lemma 3.6 □

4. Unprovability (Lower Bounds)

In this section we present the unprovability part of the phase transition for the Kanamori-McAloon principle with fixed dimension. The key arguments in subsection 4.4 are a non-trivial adaptation of Kanamori-McAloon's [9], Section 3. Before being able to apply those arguments we need to develop, by bootstrapping, some relevant bounds for the parametrized Kanamori-McAloon principle. This is done in subsection 4.3 by adapting the idea of the Stepping up Lemma in [7]. We begin with the base case $d = 1$ which is helpful for a better understanding of the coming general cases. The following subsection 4.1, covering the base case $d = 1$ of our main result, is already done in [12, 15].

4.1. **Ackermannian Ramsey functions.** Throughout this subsection m denotes a *fixed* positive natural number. Set

$$h_\omega(i) := \lfloor \sqrt[F_\omega^{-1}(i)]{i} \rfloor \quad \text{and} \quad h_m(i) := \lfloor \sqrt[m]{i} \rfloor .$$

Define a sequence of strictly increasing functions $f_{m,n}$ for as follows:

$$f_{m,n}(i) := \begin{cases} i+1 & \text{if } n = 0, \\ f_{m,n-1}^{(\lfloor \sqrt[m]{i} \rfloor)}(i) & \text{otherwise.} \end{cases}$$

Note that $f_{m,n}$ are strictly increasing.

Lemma 4.1. $R(\mu)^2_{h_m}(R(2, c, i+3)) \geq f_{m,c}(i)$ for all c and i.

Proof. Let $k := R(2, c, i+3)$ and define a function $C_m \colon [R(\mu)^2_{h_m}(k)]^2 \to \mathbb{N}$ as follows:

$$C_m(x, y) := \begin{cases} 0 & \text{if } f_{m,c}(x) \leq y, \\ \ell & \text{otherwise,} \end{cases}$$

where the number ℓ is defined by

$$f_{m,p}^{(\ell)}(x) \leq y < f_{m,p}^{(\ell+1)}(x)$$

where $p < c$ is the maximum such that $f_{m,p}(x) \leq y$. Note that C_m is h_m-regressive since $f_{m,p}^{(\lfloor \sqrt[m]{x} \rfloor)}(x) = f_{m,p+1}(x)$. Let H be a k-element subset of $R(\mu)^2_{h_m}(k)$ which is min-homogeneous for C_m. Define a c-coloring $D_m \colon [H]^2 \to c$ by

$$D_m(x, y) := \begin{cases} 0 & \text{if } f_{m,c}(x) \leq y, \\ p & \text{otherwise,} \end{cases}$$

where p is as above. Then there is a $(i+3)$-element set $X \subseteq H$ homogeneous for D_m. Let $x < y < z$ be the last three elements of X. Then $i \leq x$. Hence, it suffices to show that $f_{m,c}(x) \leq y$ since $f_{m,c}$ is an increasing function.

Assume $f_{m,c}(x) > y$. Then $f_{m,c}(y) \geq f_{m,c}(x) > z$ by the min-homogeneity. Let $C_m(x, y) = C_m(x, z) = \ell$ and $D_m(x, y) = D_m(x, z) = D_m(y, z) = p$. Then

$$f_{m,p}^{(\ell)}(x) \leq y < z < f_{m,p}^{(\ell+1)}(x).$$

By applying $f_{m,p}$ we get the contradiction that $z < f_{m,p}^{(\ell+1)}(x) \leq f_{m,p}(y) \leq z$. □

We are going to show that R_{h_m} is not primitive recursive. This will be done by comparing the functions $f_{m,n}$ with the Ackermann function.

Lemma 4.2. Let $i \geq 4^m$ and $\ell \geq 0$.

(1) $(2i+2)^m < f_{m,\ell+2m^2}(i)$ and $f_{m,\ell+2m^2}((2i+2)^m) < f^{(2)}_{m,\ell+2m^2}(i)$.

(2) $F_n(i) < f^{(2)}_{m,n+2m^2}(i)$.

Proof. (1) By induction on k it is easy to show that $f_{m,k}(i) > (\lfloor \sqrt[m]{i} \rfloor)^k$ for any $i > 0$. Hence for $i \geq 4^m$

$$f_{m,2m^2}(i) > (\lfloor \sqrt[m]{i} \rfloor)^{2m^2} \geq (\lfloor \sqrt[m]{i} \rfloor)^{m^2} \cdot 2^{m^2+m} \geq (\sqrt[m]{i+1}\,)^{m^2} \cdot 2^m = (2i+2)^m$$

since $2 \cdot \lfloor \sqrt[m]{i} \rfloor \geq \sqrt[m]{i+1}$. The second claim follows from the first one.

(2) By induction on n we show the claim. If $n = 0$ it is obvious. Suppose the claim is true for n. Let $i \geq 4^m$ be given. Then by induction hypothesis we have $F_n(i) \leq f^{(2)}_{m,n+2m^2}(i)$. Hence

$$F_{n+1}(i) \leq F_n^{(i+1)}(i) \leq f^{(2i+2)}_{m,n+2m^2}(i) \leq f_{m,n+2m^2+1}((2i+2)^m) < f^{(2)}_{m,n+2m^2+1}(i).$$

The induction is now complete. □

Corollary 4.3. $F_n(i) \leq f_{m,n+2m^2+1}(i)$ for any $i \geq 4^m$.

Theorem 4.4. $R(\mu)^2_{h_m}$ and $R(\mu)^2_{h_\omega}$ are not primitive recursive.

Proof. Lemma 4.1 and Corollary 4.3 imply that $R(\mu)^2_{h_m}$ is not primitive recursive. For the second assertion we claim that

$$N(i) := R(\mu)^2_{h_\omega}(R(2, i+2i^2+1, 4^i+3)) > F_\omega(i)$$

for all i. Assume to the contrary that $N(i) \leq A(i)$ for some i. Then for any $\ell \leq N(i)$ we have $A^{-1}(\ell) \leq i$, hence $\sqrt[i]{\ell} \leq \sqrt[A^{-1}(\ell)]{\ell}$. Hence

$$\begin{array}{rcl} R(\mu)^2_{h_\omega}(R(2, i+2i^2+1, 4^i+3)) & \geq & R(\mu)^2_{h_i}(R(2, i+2i^2+1, 4^i+3)) \\ & \geq & f_{i,i+2i^2+1}(4^i) \\ & > & F_\omega(i) \end{array}$$

by Lemma 4.1 and Corollary 4.3. Contradiction! □

Now we are ready to begin with the general cases.

4.2. **Fast-growing hierarchies.** We introduce some variants of the Schwichtenberg-Wainer hierarchy and prove that they are still fast-growing, meaning they match up with the Schwichtenberg-Wainer hierarchy.

Definition 4.5. Let $d > 0, c \geq 2$ be natural numbers. Let ϵ be a real number such that $0 < \epsilon \leq 1$.

$$\begin{array}{rcl} B_{\epsilon,c,d,0}(x) & := & 2_d^{\lfloor \log_d(x) \rfloor^c} \\ B_{\epsilon,c,d,\alpha+1}(x) & := & B_{\epsilon,c,d,\alpha}^{\lfloor \epsilon \cdot \sqrt[c]{\log_d(x)} \rfloor}(x) \\ B_{\epsilon,c,d,\lambda(x)} & := & B_{\epsilon,c,d,\lambda[\lfloor \epsilon \cdot \sqrt[c]{\log_d(x)} \rfloor]}(x) \end{array}$$

In the following we abbreviate $B_{\epsilon,c,d,\alpha}$ by B_α when c, d, ϵ are fixed.

Lemma 4.6. Let c, d, ϵ as above.

(1) $B_{i+1}(2_d^{\lfloor \epsilon^{-1}\cdot(x+1)\rfloor^c}) \geq 2_d^{\lfloor \epsilon^{-1}\cdot(F_i(x)+1)\rfloor^c}$ for all $i \in \omega$ and $x > 0$.
(2) $B_\alpha(2_d^{\lfloor \epsilon^{-1}\cdot(x+1)\rfloor^c}) \geq 2_d^{\lfloor \epsilon^{-1}\cdot(F_\alpha(x)+1)\rfloor^c}$ for all $\alpha \geq \omega$ and $x > 0$.

Proof. (1) We claim that $B_0^m(x) = 2_d^{\lfloor \log_d(x)\rfloor^{c^m}}$ for $m > 0$. Proof by induction on m. The base case holds trivially. For the induction step we calculate:

$$\begin{aligned}
B_0^{m+1}(x) &= B_0(B_0^m(x)) \\
&= 2_d^{\lfloor \log_d(B_0^m(x))\rfloor^c} \\
&= 2_d^{\lfloor \log_d(2_d^{\lfloor \log_d(x)\rfloor^{c^m}})\rfloor^c} \\
&= 2_d^{\lfloor \lfloor \log_d(x)\rfloor^{c^m}\rfloor^c} \\
&= 2_d^{\lfloor \log_d(x)\rfloor^{c^{m+1}}}
\end{aligned}$$

We now claim that $B_{i+1}(2_d^{\lfloor \epsilon^{-1}\cdot(x+1)\rfloor^c}) \geq 2_d^{\lfloor \epsilon^{-1}\cdot(F_i(x)+1)\rfloor^c}$ Proof by induction on i. For $i = 0$ we obtain

$$\begin{aligned}
B_1(2_d^{\lfloor \epsilon^{-1}\cdot(x+1)\rfloor^c}) &= B_0^{\lfloor \epsilon\cdot \sqrt[c]{\log_d(2_d^{\lfloor \epsilon^{-1}\cdot(x+1)\rfloor^c})}\rfloor}(2_d^{\lfloor \epsilon^{-1}\cdot(x+1)\rfloor^c}) \\
&= B_0^{\lfloor \epsilon\cdot\lfloor \epsilon^{-1}\cdot(x+1)\rfloor\rfloor}(2_d^{\lfloor \epsilon^{-1}\cdot(x+1)\rfloor^c}) \\
&\geq B_0^{x+1}(2_d^{\lfloor \epsilon^{-1}\cdot(x+1)\rfloor^c}) \\
&= 2_d^{\lfloor \log_d(2_d^{\lfloor \epsilon^{-1}\cdot(x+1)\rfloor^c})\rfloor^{c^{x+1}}} \\
&= 2_d^{\lfloor \lfloor \epsilon^{-1}\cdot(x+1)\rfloor^c\rfloor^{c^{x+1}}} \\
&= 2_d^{\lfloor \epsilon^{-1}\cdot(x+1)\rfloor^{c^{x+2}}} \\
&\geq 2_d^{\lfloor \epsilon^{-1}\cdot(F_0(x)+1)\rfloor^c}
\end{aligned}$$

since $x > 0$ and $c > 1$. For the induction step we compute

$$\begin{aligned}
B_{i+1}(2_d^{\lfloor \epsilon^{-1}\cdot(x+1)\rfloor^c}) &= B_i^{\lfloor \epsilon\cdot \sqrt[c]{\log_d(2_d^{\lfloor \epsilon^{-1}\cdot(x+1)\rfloor^c})}\rfloor}(2_d^{\lfloor \epsilon^{-1}\cdot(x+1)\rfloor^c}) \\
&\geq B_i^{x+1}(2_d^{\lfloor \epsilon^{-1}\cdot(x+1)\rfloor^c}) \\
&\geq B_i^{x}(2_d^{\lfloor \epsilon^{-1}\cdot(F_{i-1}(x)+1)\rfloor^c}) \\
&\geq B_i^{x-1}(2_d^{\lfloor \epsilon^{-1}\cdot(F_{i-1}^2(x)+1)\rfloor^c}) \\
&\geq \cdots \\
&\geq 2_d^{\lfloor \epsilon^{-1}\cdot(F_{i-1}^{x+1}(x)+1)\rfloor^c} \\
&\geq 2_d^{\lfloor \epsilon^{-1}\cdot(F_i(x)+1)\rfloor^c}
\end{aligned}$$

(2) We prove the claim by induction on $\alpha \geq \omega$. Let $\alpha = \omega$. We obtain

$$\begin{aligned} B_\omega(2_d^{\lfloor \epsilon^{-1}\cdot(x+1) \rfloor^c}) &= B_{x+1}(2_d^{\lfloor \epsilon^{-1}\cdot(x+1) \rfloor^c}) \\ &\geq 2_d^{\lfloor \epsilon^{-1}\cdot(F_x(x)+1) \rfloor^c} \\ &= 2_d^{\lfloor \epsilon^{-1}\cdot(F_\omega(x)+1) \rfloor^c} \end{aligned}$$

For the successor case $\alpha+1$ we compute

$$\begin{aligned} B_{\alpha+1}(2_d^{\lfloor \epsilon^{-1}\cdot(x+1) \rfloor^c}) &= B_\alpha^{\lfloor \epsilon\cdot \sqrt[c]{\log_d(2_d^{\lfloor \epsilon^{-1}\cdot(x+1) \rfloor^c})} \rfloor}(2_d^{\lfloor \epsilon^{-1}\cdot(x+1) \rfloor^c}) \\ &= B_\alpha^{x+1}(2_d^{\lfloor \epsilon^{-1}\cdot(x+1) \rfloor^c}) \\ &= B_\alpha^{x}(B_\alpha(2_d^{\lfloor \epsilon^{-1}\cdot(x+1) \rfloor^c})) \\ &\geq B_\alpha^{x}(2_d^{\lfloor \epsilon^{-1}\cdot(F_\alpha(x)+1) \rfloor^c}) \\ &\geq \cdots \\ &\geq 2_d^{\lfloor \epsilon^{-1}\cdot(F_\alpha^{x+1}(x)+1) \rfloor^c} \\ &\geq 2_d^{\lfloor \epsilon^{-1}\cdot(F_{\alpha+1}(x)+1) \rfloor^c} \end{aligned}$$

If λ is a limit we obtain

$$\begin{aligned} B_\lambda(2_d^{\lfloor \epsilon^{-1}\cdot(x+1) \rfloor^c}) &= B_{\lambda[\lfloor \epsilon\cdot \sqrt[c]{\log_d(2_d^{\lfloor \epsilon^{-1}\cdot(x+1) \rfloor^c})} \rfloor]}(2_d^{\lfloor \epsilon^{-1}\cdot(x+1) \rfloor^c}) \\ &\geq B_{d,\lambda[x+1]}(2_d^{\lfloor \epsilon^{-1}\cdot(x+1) \rfloor^c}) \\ &\geq 2_d^{\lfloor \epsilon^{-1}\cdot(F_{\lambda[x+1]}(x)+1) \rfloor^c} \\ &\geq 2_d^{\lfloor \epsilon^{-1}\cdot(F_\lambda(x)+1) \rfloor^c} \end{aligned}$$

□

Theorem 4.7. Let $d > 0, c > 1$ be natural numbers. Let $0 < \epsilon \leq 1$.

(1) $B_{\epsilon,c,d,\omega}$ is Ackermannian.
(2) $B_{\epsilon,c,d,\alpha}$ is provably total in $\mathrm{I}\Sigma_n$ iff $\alpha < \omega_n$.
(3) $(B_{\epsilon,c,d,\alpha})_{\alpha<\varepsilon_0}$ is fast-growing.

Proof. Obvious by Lemma 4.6. □

4.3. **Bootstrapping.** In this section we show how one can use $(\mathrm{KM})^{d+1}_{\sqrt[c]{\log_{d-1}}}$ to obtain min-homogeneous sets whose elements are "spread apart" with respect to the function $2_{d-1}(\log_{d-1}(x)^c)$ (i.e. $B_{\epsilon,c,d-1,0}$). This fact will be used next (Proposition 4.21) to show that one can similarly obtain from the same assumption even sparser sets (essentially sets whose elements are $F_{\omega^c_{d-1}}$ "spread apart").

For the sake of clarity we work out the proofs of the main results of the present section for the base cases $d = 2$ and $d = 4$ in detail in section 4.3.1 before generalizing them in section 4.3.2. We hope that this will improve the readability of the arguments.

Definition 4.8. We say that a set X if f*-sparse* iff for all $a, b \in X$ we have $f(a) \leq b$. We say that two elements a, b of a set X are n*-apart* iff there exist $e_1, \ldots, e_n$ from

X such that $a < e_1 < \cdots < e_n < b$. We say that a set is (f,n)-*sparse* iff for all $a,b \in X$ such that a and b are n-apart we have $f(a) \leq b$.

Definition 4.9. Let X be a set of cardinality $> m \cdot k$. We define X/m as the set $\{x_0, x_m, x_{2m}, \dots, x_{k\cdot m}\}$, where x_i is the $(i+1)$-th smallest element of X.

Thus, if a set X is (f,m)-sparse of cardinality $> k \cdot m$ we have that X/m is f-sparse and has cardinality $> k$.

4.3.1. *$B_{\epsilon,2,1,0}$-sparse min-homogeneous sets - Base Cases.* Given $P : [\ell]^d \to \mathbb{N}$ we call $X \subseteq \ell$ max-homogeneous for P if for all $U, V \in [X]^d$ with $\max(U) = \max(V)$ we have $P(U) = P(V)$.

Let $\mathrm{MIN}^d_k(m) := R_\mu(d,k,m)$, i.e., the least natural number ℓ such that for all partitions $P : [\ell]^d \to k$ there is a min-homogeneous $Y \subseteq \ell$ such that $card(Y) \geq m$. Let $\mathrm{MAX}^d_k(m)$ be the least natural number ℓ such that for all partitions $P : [\ell]^d \to k$ there is a max-homogeneous $Y \subseteq \ell$ such that $card(Y) \geq m$.

Let $k \geq 2$ and $m \geq 1$. Given an integer $a < k^m$ let $a = k^{m-1} \cdot a(m-1) + \cdots + k^0 \cdot a(0)$ be in the unique representation with $a(m-1), \dots, a(0) \in \{0, \dots, k-1\}$. Then $D^{(k,m)} : [k^m]^2 \to m$ is defined by

$$D^{(k,m)}(a,b) := \max\{j : a(j) \neq b(j)\}.$$

Lemma 4.10. Let $k \geq 2$ and $m \geq 1$.

(1) $\mathrm{MIN}^2_{k\cdot m}(m+2) > k^m$.
(2) $\mathrm{MAX}^2_{k\cdot m}(m+2) > k^m$.

Proof. Let us show the first item. Define $R_1 : [k^m]^2 \to k \cdot m$ as follows.

$$R_1(a,b) := k \cdot D(a,b) + b(D(a,b)),$$

where $D := D^{(k,m)}$. Assume $Y = \{a_0, \dots, a_\ell\}$ with $a_0 < \dots < a_\ell$ is min-homogeneous for R_1. We claim $\ell \leq m$. Let $c_i := D(a_i, a_{i+1})$, $i < \ell$. Since $m > c_0$ it is sufficient to show $c_{i+1} < c_i$ for every $i < \ell - 1$.

Fix $i < \ell - 1$. We have $D(a_i, a_{i+1}) = D(a_i, a_{i+2})$ since $R_1(a_i, a_{i+1}) = R_1(a_i, a_{i+2})$ by min-homogeneity. Hence for any $j > D(a_i, a_{i+1})$ we have $a_i(j) = a_{i+1}(j) = a_{i+2}(j)$ which means $c_i \geq c_{i+1}$. Moreover, $R_1(a_i, a_{i+1}) = R_1(a_i, a_{i+2})$ further yields $a_{i+1}(D(a_i, a_{i+1})) = a_{i+2}(D(a_i, a_{i+2}))$, hence $c_i = c_{i+1}$ cannot be true, since $a_{i+1}(D(a_{i+1}, a_{i+2})) \neq a_{i+2}(D(a_{i+1}, a_{i+2}))$.

For the proof of the second item define $R'_1 : [k^m]^2 \to k \cdot m$ as follows.

$$R'_1(a,b) := k \cdot D(a,b) + a(D(a,b)),$$

where $D := D^{(k,m)}$. Assume $Y = \{a_0, \dots, a_\ell\}$ with $a_0 < \dots < a_\ell$ is max-homogeneous for R'_1. We claim $\ell \leq m$. Let $c_i := D(a_i, a_{i+1})$, $i < \ell$. Since $m > c_{\ell-1}$ it is sufficient to show $c_{i+1} > c_i$ for every $i < \ell - 1$.

Fix $i < \ell - 1$. We have $D(a_i, a_{i+2}) = D(a_{i+1}, a_{i+2})$ since $R'_1(a_i, a_{i+2}) = R'_1(a_{i+1}, a_{i+2})$ by max-homogeneity. Hence for any $j > D(a_{i+1}, a_{i+2})$ we have $a_i(j) = a_{i+1}(j) = a_{i+2}(j)$ which means $c_i \leq c_{i+1}$. Moreover, $R'_1(a_i, a_{i+2}) = R'_1(a_{i+1}, a_{i+2})$ further yields $a_i(D(a_i, a_{i+2})) = a_{i+1}(D(a_{i+1}, a_{i+2}))$, hence $c_i = c_{i+1}$ cannot be true, since $a_i(D(a_i, a_{i+1})) \neq a_{i+1}(D(a_i, a_{i+1}))$. $\square$

Lemma 4.11. Let $k, m \geq 2$.

(1) $\mathrm{MIN}^3_{2k\cdot m}(2m+4) > 2^{k^m}$.
(2) $\mathrm{MAX}^3_{2k\cdot m}(2m+4) > 2^{k^m}$.

Proof. (1) Let $k, m \geq 2$ be positive integers and put $e := k^m$. Let R_1 and R_1' be the partitions from Lemma 4.10. Define $R_2 : [2^e]^3 \to 2k \cdot m$ as follows:

$$R_2(u, v, w) := \begin{cases} R_1(D(u, v), D(v, w)) & \text{if } D(u, v) < D(v, w), \\ k \cdot m + R_1'(D(v, w), D(u, v)) & \text{if } D(u, v) > D(v, w), \end{cases}$$

where $D := D^{(2,e)}$. The case $D(u, v) = D(v, w)$ does not occur since we developed u, v, w with respect to base 2. Let $Y \subseteq 2^e$ be min-homogeneous for R_2. We claim $card(Y) < 2m + 4$.

Assume $card(Y) \geq 2m + 4$. Let $\{u_0, \ldots, u_{2m+3}\} \subseteq Y$ be min-homogeneous for R_2. We shall provide a contradiction. Let $d_i := D(u_i, u_{i+1})$ for $i < 2m + 3$.

Case 1: Assume there is some r such that $d_r < \ldots < d_{r+m+1}$. We claim that $Y' := \{d_r, \ldots, d_{r+m+1}\}$ is min-homogeneous for R_1 which would contradict Lemma 4.10.

Note that for all i, j with $r \leq i < j \leq r + m + 2$ we have

$$D(u_i, u_j) = \max\{D(u_i, u_{i+1}), \ldots, D(u_{j-1}, u_j)\}.$$

We have therefore for $r \leq i < j \leq r + m + 1$

$$R_1(d_i, d_j) = R_1(D(u_i, u_{i+1}), D(u_{i+1}, u_{j+1})) = R_2(u_i, u_{i+1}, u_{j+1}).$$

By min-homogeneity of Y we obtain similarly

$$R_2(u_i, u_{i+1}, u_{j+1}) = R_2(u_i, u_{i+1}, u_{p+1}) = R_1(d_i, d_p)$$

for all i, j, p such that $r \leq i < j < p \leq r + m + 1$.

Case 2: Assume there is some r such that $d_r > \ldots > d_{r+m+1}$. We claim that $Y' := \{d_{r+m+1}, \ldots, d_r\}$ is max-homogeneous for R_1' which would contradict Lemma 4.10.

Assume $r \leq i < j < p \leq r + m + 1$, hence $u_i < u_j < u_p$ and $d_p < d_j < d_i$. Note that we also have $d_j = D(u_j, u_p)$ and $d_i = D(u_i, u_p)$. Hence

$$k \cdot m + R_1'(d_p, d_j) = k \cdot m + R_1'(D(u_p, u_{p+1}), D(u_j, u_p)) = R_2(u_j, u_p, u_{p+1}).$$

By min-homogeneity we obtain

$$\begin{aligned} k \cdot m + R_1'(d_p, d_i) &= k \cdot m + R_1'(D(u_p, u_{p+1}), D(u_i, u_p)) \\ &= R_2(u_i, u_p, u_{p+1}) \\ &= R_2(u_i, u_j, u_{j+1}) \\ &= k \cdot m + R_1'(d_j, d_i). \end{aligned}$$

Case 3: There is a local maximum of the form $d_i < d_{i+1} > d_{i+2}$. Note then that $D(u_i, u_{i+2}) = d_{i+1}$. Hence we obtain the following contradiction using the min-homogeneity: $k \cdot m > R_1(d_i, d_{i+1}) = R_2(u_i, u_{i+1}, u_{i+2}) = R_2(u_i, u_{i+2}, u_{i+3}) = k \cdot m + R_1'(d_{i+2}, d_{i+1}) \geq k \cdot m$.

Case 4: Cases 1 to 3 do not hold. Then there must be two local minima. But then inbetween we have a local maximum and we are back in Case 3.

(2) Similar to the first claim. Define R_2' just by interchanging R_1 and R_1' and argue as above interchanging the role of min-homogeneous and max-homogeneous sets. $\square$

Lemma 4.12. Let $k, m \geq 2$.

(1) $\mathrm{MIN}^4_{4k \cdot m}(2(2m + 4) + 2) > 2^{2^{k^m}}$.
(2) $\mathrm{MAX}^4_{4k \cdot m}(2(2m + 4) + 2) > 2^{2^{k^m}}$.

Proof. (1) Let $k, m \geq 2$ be positive integers and put $\ell := 2^{k^m}$. Let R_2 and R_2' be the partitions from the Lemma 4.11. Let $D := D^{(2,\ell)}$. Then define $R_3 : [2^\ell]^4 \to 4k \cdot m$ as follows:

$$R_3(u, v, w, x) :=$$

$$\begin{cases} R_2(D(u,v), D(v,w), D(w,x)) & \text{if } D(u,v) < D(v,w) < D(w,x) \\ 2k \cdot m + R_2'(D(w,x), D(v,w), D(u,v)) & \text{if } D(u,v) > D(v,w) > D(w,x) \\ 0 & \text{if } D(u,v) < D(v,w) > D(w,x) \\ 2k \cdot m & \text{if } D(u,v) > D(v,w) < D(w,x) \end{cases}$$

The cases $D(u,v) = D(v,w)$ or $D(v,w) = D(w,x)$ don't occur since we developed u, v, w, x with respect to base 2.

Let $Y \subseteq 2^\ell$ be min-homogeneous for R_3. We claim $card(Y) \leq 2(2m+4)+1$. Let $Y = \{u_0, \ldots, u_h\}$ be min-homogeneous for R_3, where $h := 2(2m+4)+1$. Put $d_i := D(u_i, u_{i+1})$ and $g := 2m+3$.

Case 1: Assume that there is some r such that $d_r < \ldots < d_{r+g}$. We claim that $Y' := \{d_r, \ldots, d_{r+g}\}$ is min-homogeneous for R_2 which would contradict Lemma 4.11.

Note again that for $r \leq i < j \leq r+g+1$ we have

$$D(u_i, u_j) = \max\{D(u_i, u_{i+1}), \ldots, D(u_{j-1}, u_j)\} = D(u_{j-1}, u_j).$$

Therefore for $r \leq i < p < q \leq r+g$

$$\begin{aligned} R_2(d_i, d_p, d_q) &= R_2(D(u_i, u_{i+1}), D(u_{i+1}, u_{p+1}), D(u_{p+1}, u_{q+1})) \\ &= R_3(u_i, u_{i+1}, u_{p+1}, u_{q+1}). \end{aligned}$$

By the same pattern we obtain for $r \leq i < u < v \leq r+g$

$$\begin{aligned} R_2(d_i, d_u, d_v) &= R_2(D(u_i, u_{i+1}), D(u_{i+1}, u_{u+1}), D(u_{u+1}, u_{v+1})) \\ &= R_3(u_i, u_{i+1}, u_{u+1}, u_{v+1}). \end{aligned}$$

By min-homogeneity of Y for R_3 we obtain then $R_2(d_i, d_p, d_q) = R_2(d_i, d_u, d_v)$. Thus Y' is min-homogeneous for R_2.

Case 2: Assume that there is some r such that $d_r > \ldots > d_{r+g}$. We claim that $Y' := \{d_{r+g}, \ldots, d_r\}$ is max-homogeneous for R_2' which would contradict Lemma 4.11.

Then for $r \leq i < p < q \leq r+g$

$$\begin{aligned} 2k \cdot m + R_2'(d_q, d_p, d_i) &= 2k \cdot m + R_2'(D(u_{p+1}, u_{q+1}), D(u_{i+1}, u_{p+1}), D(u_i, u_{i+1})) \\ &= R_3(u_i, u_{i+1}, u_{p+1}, u_{q+1}). \end{aligned}$$

By the same pattern we obtain for $r \leq i < u < v \leq r+g$

$$\begin{aligned} 2k \cdot m + R_2'(d_v, d_u, d_i) &= 2k \cdot m + R_2'(D(u_{u+1}, u_{v+1}), D(u_{i+1}, u_{u+1}), D(u_i, u_{i+1})) \\ &= R_3(u_i, u_{i+1}, u_{u+1}, u_{v+1}). \end{aligned}$$

By min-homogeneity of Y for R_3 we obtain then $R_2'(d_q, d_p, d_i) = R_2'(d_v, d_u, d_i)$. Thus Y' is max-homogeneous for R_2'.

Case 3: There is a local maximum of the form $d_i < d_{i+1} > d_{i+2}$. Then we obtain the following contradiction using the min-homogeneity

$$\begin{aligned} 0 &= R_3(u_i, u_{i+1}, u_{i+2}, u_{i+3}) \\ &= R_3(u_i, u_{i+2}, u_{i+3}, u_{i+4}) \\ &\geq 2k \cdot m \end{aligned}$$

since $D(u_i, u_{i+2}) = d_{i+1} > d_{i+2}$.

Case 4: Cases 1 to 3. do not hold. Then there must be two local minima. But then inbetween we have a local maximum and we are back in Case 3.

(2) Similar to the first claim. Define R'_3 just by interchanging R_2 and R'_2 and argue interchanging the role of min-homogeneous and max-homogeneous sets. $\square$

We now show how one can obtain sparse min-homogeneous sets for certain functions of dimension 3 from the bounds from Lemma 4.11. It will be clear that the same can be done for functions of dimension 4 using the bounds from Lemma 4.12. In section 4.3.2 we will lift the bounds and the sparseness results to the general case.

Lemma 4.13. Let $f(i) := \lfloor \sqrt{\log(i)} \rfloor$. Let $\ell := 2^{(16\cdot 17+1)^2}$. Then there exists an f-regressive partition $P : [\mathbb{N}]^3 \to \mathbb{N}$ such that if Y is min-homogeneous for P and of cardinality not below $3\ell - 1$, then we have $2^{(\log(a))^2} \leq b$ for all $a, b \in \bar{Y}/4$, where

$$\bar{Y} := Y \setminus (\{\text{the first } \ell \text{ elements of } Y\} \cup \{\text{the last } \ell - 2 \text{ elements of } Y\}).$$

Proof. Let $u_0 := 0$, $u_1 = \ell$ and $u_{i+1} := \mathrm{MIN}^3_{f(u_i)-1}(\ell+1) - 1$ for $i > 0$. Notice that $u_i < u_{i+1}$. This is because $u_i \geq 2^{(16\cdot 17+1)^2}$ implies by Lemma 4.11, letting $m = 8$,

$$\begin{aligned} u_{i+1} &= \mathrm{MIN}^3_{f(u_i)-1}(\ell+1) - 1 \\ &\geq \mathrm{MIN}^3_{f(u_i)-1}(20) - 1 \\ &\geq 2^{\lfloor \frac{f(u_i)-1}{16} \rfloor^8} \\ &> 2^{f(u_i)^4} \\ &= 2^{\log(u_i)^2} \\ &\geq u_i \end{aligned}$$

Let $G_0 : [u_1]^3 \to 1$ be the constant function with the value 0 and for $i > 0$ choose $G_i : [u_{i+1}]^3 \to f(u_i) - 1$ such that every G_i-min-homogeneous set $Y \subseteq u_{i+1}$ satisfies $card(Y) < \ell + 1$. Let $P : [\mathbb{N}]^3 \to \mathbb{N}$ be defined as follows:

$$P(x_0, x_1, x_2) := \begin{cases} G_i(x_0, x_1, x_2) + 1 & \text{if } u_i \leq x_0 < x_1 < x_2 < u_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then P is f-regressive by the choice of the G_i. Assume that $Y \subseteq \mathbb{N}$ is min-homogeneous for P and $card(Y) \geq 3\ell - 1$ and $\bar{Y}$ is as described, i.e., $card(\bar{Y}) \geq \ell + 1$. If $\bar{Y} \subset [u_i, u_{i+1}[$ then $\bar{Y}$ is G_i-min-homogeneous hence $card(\bar{Y}) \leq \ell$ which is excluded. Hence each interval $[u_i, u_{i+1}[$ contains at most two elements from Y since we have omitted the last $\ell - 2$ elements from Y.

If a, b are in $\bar{Y}/4$. Then there are $e_1, e_2, e_3 \in \bar{Y}$ such that $a < e_1 < e_2 < e_3 < b$, and so there exists an $i \geq 1$ such that $a \leq u_i < u_{i+1} \leq b$. Hence $b \geq u_{i+1} \geq 2^{f(u_i)^4} \geq 2^{\log(a)^2}$ as above by Lemma 4.11. $\square$

We just want to remark that $2^{(16\cdot 17+1)^2}$ is not the smallest number which satisfies Lemma 4.13.

4.3.2. *$B_{\epsilon,c,d,0}$-sparse min-homogeneous sets - Generalization.* We now show how the above results Lemma 4.12 and Lemma 4.13 can be generalized to arbitrary dimension. Let g_d be defined inductively as follows. $g_0(x) := x$, $g_{d+1}(x) := 2 \cdot g_d(x) + 2$. Thus

$$g_d(x) := \underbrace{2(\dots(2(2}_{d}x+2)+2)\dots)+2,$$

i.e. d iterations of the function $x \mapsto 2x+2$.

Lemma 4.14. Let $d \geq 1$ and $k, m \geq 2$.

(1) $\mathrm{MIN}^{d+1}_{2^{d-1}k\cdot m}(g_{d-2}(2m+4)) > 2_{d-1}(k^m)$.
(2) $\mathrm{MAX}^{d+1}_{2^{d-1}k\cdot m}(g_{d-2}(2m+4)) > 2_{d-1}(k^m)$.

Proof Sketch. By a simultaneous induction on $d \geq 1$. The base cases for $d \leq 2$ are proved in Lemma 4.10 and Lemma 4.11. Let now $d \geq 2$. The proof is essentially the same as the previous ones.

Let $R_d : [2_{d-1}(k^m)]^{d+1} \to 2^{d-1}k \cdot m$ (or $R'_d : [2_{d-1}(k^m)]^{d+1} \to 2^{d-1}k \cdot m$) be a partition such that every min-homogeneous set for R_d (or max-homogeneous set for R'_d) is of cardinality $< g_{d-2}(2m+4)$.

We define then $R_{d+1} : [2^{k^m}_d]^{d+2} \to 2^d k \cdot m$ as follows. $R_{d+1}(x_1, \dots, x_{d+2}) :=$

$$\begin{cases} R_d(d(x_1,x_2),\dots,d(x_{d+1},x_{d+2})) & \text{if } d(x_1,x_2) < \dots < d(x_{d+1},x_{d+2}), \\ 2^{d-1}k\cdot m + R'_d(d(x_{d+2},x_{d+1}),\dots,d(x_2,x_1)) & \text{if } d(x_1,x_2) > \dots > d(x_{d+1},x_{d+2}), \\ 0 & \text{if } d(x_1,x_2) < d(x_2,x_3) > d(x_3,x_4) \\ 2^{d-1}k\cdot m & \text{else.} \end{cases}$$

And $R'_{d+1} : [2^{k^m}_d]^{d+2} \to 2^d k \cdot m$ is defined similarly by interchanging R_d and R'_d. Now we can argue analogously to Lemma 4.12. □

We now state the key result of the present section, the Sparseness Lemma. Let $f(i) := \lfloor \sqrt[c]{\log_{d-1}(i)} \rfloor$. We show how an f-regressive function P of dimension $d+1$ can be defined such that all large min-homogeneous sets are $(2^{(\log_{d-1}(\cdot))^c}_{d-1}, 3)$-sparse.

Lemma 4.15 (Sparseness Lemma). Given $c \geq 2$ and $d \geq 1$ let $f(i) := \lfloor \sqrt[c]{\log_{d-1}(i)} \rfloor$. And define $m := 2c^2$, $n := 2^{d-1} \cdot m$, and $\ell := 2_{d-1}((n \cdot (n+1)+1)^c)$. There exists an f-regressive partition $P_{c,d} : [\mathbb{N}]^{d+1} \to \mathbb{N}$ such that, if Y is

- min-homogeneous for $P_{c,d}$ and
- $card(Y) \geq 3\ell - 1$,

then we have $2^{(\log_{d-1}(a))^c}_{d-1} \leq b$ for all $a, b \in \bar{Y}/4$, where

$$\bar{Y} := Y \setminus (\{\text{the first } \ell \text{ elements of } Y\} \cup \{\text{the last } \ell - 2 \text{ elements of } Y\}).$$

Proof. Let $u_0 := 0, u_1 := \ell$ and $u_{i+1} := \mathrm{MIN}^{d+1}_{f(u_i)-1}(\ell+1)-1$. Notice that $u_i < u_{i+1}$. This is because $u_i \geq \ell$ implies by Lemma 4.14

$$\begin{array}{rcl} u_{i+1} & = & \mathrm{MIN}^{d+1}_{f(u_i)-1}(\ell+1)-1 \\ & \geq & \mathrm{MIN}^{d+1}_{f(u_i)-1}(g_{d-2}(2m+4))-1 \\ & \geq & 2_{d-1}^{\lfloor \frac{f(u_i)-1}{2^{d-1}\cdot m} \rfloor^m} \\ & > & 2_{d-1}^{f(u_i)^{m/2}} \\ & = & 2^{\log(u_i)^c} \\ & \geq & u_i \end{array}$$

Note that $\ell > g_{d-2}(2m+4)$. Let $G_0 : [u_1]^{d+1} \to 1$ be the constant function with value 0 and for $i > 0$ choose $G_i : [u_{i+1}]^{d+1} \to f(u_i) - 1$ such that every G_i-min-homogeneous set $Y \subseteq u_{i+1}$ satisfies $card(Y) \leq \ell$. Let $P : [\mathbb{N}]^{d+1} \to \mathbb{N}$ be defined as follows:

$$P_{c,d}(x_0, \ldots, x_d) := \begin{cases} G_i(x_0, \ldots, x_d) + 1 & \text{if } u_i \leq x_0 < \cdots < x_d < u_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

Then $P_{c,d}$ is f-regressive by choice of the G_i's. Assume $Y \subseteq \mathbb{N}$ is min-homogeneous for $P_{c,d}$ and $card(Y) \geq 3\ell - 1$. Let $\bar{Y}$ be as described, i.e. $card(\bar{Y}) \geq \ell + 1$. If $\bar{Y} \subseteq [u_i, u_{i+1}[$ for some i then $\bar{Y}$ is min-homogeneous for G_i, hence $card(\bar{Y}) \leq \ell$, which is impossible. Hence each interval $[u_i, u_{i+1}[$ contains at most two elements from $\bar{Y}$, since we have omitted the last $\ell - 2$ elements of Y.

Given $a, b \in \bar{Y}/4$ let $e_1, e_2, e_3 \in \bar{Y}$ such that $a < e_1 < e_2 < e_3 < b$. Then there exists an $i \geq 1$ such that $a \leq u_i < u_{i+1} \leq b$. Hence $b \geq u_{i+1} \geq 2^{f(u_i)^{m/2}} \geq 2^{\log(a)^c}$ as above by Lemma 4.14. $\square$

4.4. **Combinatorics.** Given $c \geq 2$ and $d \geq 1$ let $f_{c,d}(x) := \lfloor \sqrt[c]{\log_d(x)} \rfloor$. We first want to show that the regressive Ramsey function $R(\mu)^{d+1}_{f_{c,d-1}}$ eventually dominates $B_{\epsilon,c,d,\omega^c_{d-1}}$ (for suitable choices of ϵ). Now let $f_{\omega_d,d-1}$ be $\lfloor \sqrt[B^{-1}_{\omega_d}(\cdot)]{\log_{d-1}} \rfloor$. We will conclude that the regressive Ramsey function $R(\mu)^{d+1}_{f_{\omega_d,d-1}}$ eventually dominates B_{ω_d}. The latter fact will be seen to imply our main result of the present section, i.e.:

$$\mathrm{I}\Sigma_d \nvdash (\mathrm{KM})^{d+1}_{\lfloor \sqrt[F^{-1}_{\omega_d}(\cdot)]{\log_{d-1}} \rfloor}$$

4.4.1. $B_{\omega^c_d}$*-sparse min-homogeneous sets.* We begin by recalling the definition of the "step-down" relation on ordinals from [10] and some of its properties with respect to the hierarchies defined in Section 4.2.

Definition 4.16. Let $\alpha < \beta \leq \varepsilon_0$ Then $\beta \longrightarrow_n \alpha$ if for some sequence $\gamma_0, \ldots, \gamma_k$ of ordinals we have $\gamma_0 = \beta, \gamma_{i+1} = \gamma_i[n]$ for $0 \leq i < k$ and $\gamma_k = \alpha$.

We first recall the following property of the $\longrightarrow_n$ relation. It is stated and proved as Corollary 2.4 in [10].

Lemma 4.17. Let $\beta < \alpha < \varepsilon_0$. Let $n > i$. If $\alpha \longrightarrow_i \beta$ then $\alpha \longrightarrow_n \beta$.

Proposition 4.18. Let $\alpha \leq \varepsilon_0$. For all $c \geq 2, d \geq 1$, let $f(x) = \lfloor \sqrt[c]{\log_d(x)} \rfloor$. Let $0 < \epsilon \leq 1$. Then we have the following.

(1) If $f(n) > f(m)$ then $B_{\epsilon,c,d,\alpha}(n) > B_{\epsilon,c,d,\alpha}(m)$.
(2) If $\alpha = \beta + 1$ then $B_{\epsilon,c,d,\alpha}(n) \geq B_{\epsilon,c,d,\beta}(n)$; if $\epsilon \cdot f(n) \geq 1$ then $B_{\epsilon,c,d,\alpha}(n) > B_{\epsilon,c,d,\beta}(n)$.
(3) If $\alpha \longrightarrow_{\lfloor \epsilon \cdot f(n) \rfloor} \beta$ then $B_{\epsilon,c,d,\alpha}(n) \geq B_{\epsilon,c,d,\beta}(n)$.

Proof. Straightforward from the proof of Proposition 2.5 in [10]. □

We denote by $T_{\omega_d^c,n}$ the set $\{\alpha : \omega_d^c \longrightarrow_n \alpha\}$. We recall the following bound from [10], Proposition 2.10.

Lemma 4.19. Let $n \geq 2$ and $c, d \geq 1$. Then

$$card(T_{\omega_d^c,n}) \leq 2_{d-1}(n^{6c}).$$

Observe that, by straightforward adaptation of the proof of Lemma 4.19 (Proposition 2.10 in [10]), we accordingly have $card(T_{\omega_d^c,f(n)}) \leq 2_{d-1}(f(n)^{6c})$ for f a non-decreasing function and all n such that $f(n) \geq 2$.

Definition 4.20. Let τ be a function of type k. We say that τ is *weakly monotonic on first arguments* on X (abbreviated *w.m.f.a.*) if for all $s, t \in [X]^k$ such that $\min(s) < \min(t)$ we have $\tau(s) \leq \tau(t)$.

In the rest of the present section, when ϵ, c, d are fixed and clear from the context, B_α stands for $B_{\epsilon,c,d,\alpha}$ for brevity.

Proposition 4.21 (Capturing). Given $c, d \geq 2$ let $\epsilon = \sqrt[6c]{1/3}$. Put

$$\begin{aligned} f(x) &:= \left\lfloor \sqrt[c]{\log_{d-1}(x)} \right\rfloor \\ g(x) &:= \left\lfloor \sqrt[6c^2]{\log_{d-1}(x)} \right\rfloor \\ h(x) &:= \left\lfloor \sqrt[6c]{1/3} \cdot \sqrt[6c^2]{\log_{d-1}(x)} \right\rfloor \end{aligned}$$

Then there are functions $\tau_1 : [\mathbb{N}]^2 \to \mathbb{N}$ $2_{d-2}\left(\frac{1}{3}f\right)$-regressive, $\tau_2 : [\mathbb{N}]^2 \to \mathbb{N}$ f-regressive, $\tau_3 : [\mathbb{N}]^2 \to 2$ so that the following holds: If $H \subseteq \mathbb{N}$ is of cardinality > 2 and s.t.

(a) H is min-homogeneous for τ_1,
(b) $\forall s, t \in [H]^2$ if $\min(s) < \min(t)$ then $\tau_1(s) \leq \tau_1(t)$ (i.e. τ_1 is w.m.f.a. on H),
(c) H is $2_{d-1}^{\lfloor \log_{d-1}(\cdot)^c \rfloor}$-sparse (i.e. $B_{\epsilon,c,d-1,0}$-sparse),
(d) $\min(H) \geq h^{-1}(2)$,
(e) H is min-homogeneous for τ_2, and
(f) H is homogeneous for τ_3,

then for any $x < y$ in H we have $B_{\epsilon,c,d-1,\omega_{d-1}^c}(x) \leq y$ (i.e. H is $B_{\epsilon,c,d-1,\omega_{d-1}^c}$-sparse).

Proof. Define a function τ_1 as follows.

$$\tau_1(x,y) := \begin{cases} 0 & \text{if } B_{\omega_{d-1}^c}(x) \leq y \text{ or } h(x) < 2, \\ \xi \dot{-} 1 & \text{otherwise, where } \xi = \min\{\alpha \in T_{\omega_{d-1}^c,h(x)} : y < B_\alpha(x)\}. \end{cases}$$

$\xi \dot{-} 1$ means 0 if $\xi = 0$ and β if $\xi = \beta + 1$. We have to show that τ_1 is well-defined. First observe that the values of τ_1 can be taken to be in $\mathbb{N}$ since, by Lemma 4.19, we can assume an order preserving bijection between $T_{\omega_{d-1}^c,h(x)}$ and $2_{d-2}^{h(x)^{6c}}$:

$$\tau_1(x,y) < 2_{d-2}(h(x)^{6c}) = 2_{d-2}^{\left(\sqrt[6c]{\frac{1}{3}}\sqrt[6c^2]{\log_{d-1}(x)}\right)^{6c}} = 2_{d-2}^{\left(\frac{1}{3}\sqrt[c]{\log_{d-1}(x)}\right)}.$$

In the following we will only use properties of values of τ_1 that can be inferred by this assumption.

Let $\xi = \min\{\alpha \in T_{\omega^c_{d-1},h(x)} : y < B_\alpha(x)\}$. Note that $\xi \neq 0$ by condition (c). Suppose that the minimum ξ is a limit ordinal, call it λ. Then, by definition of the hierarchy, we have

$$B_\lambda(x) = B_{\lambda[h(x)]}(x) > y.$$

But $\lambda[h(x)] < \lambda$ and $\lambda[h(x)] \in T_{\omega^c_{d-1},h(x)}$, against the minimality of λ.

Define a function τ_2 as follows.

$$\tau_2(x,y) := \begin{cases} 0 & \text{if } B_{\omega^c_{d-1}}(x) \leq y \text{ or } h(x) < 2, \\ k-1 & \text{otherwise, where } B^{k-1}_{\tau_1(x,y)}(x) \leq y < B^k_{\tau_1(x,y)}(x). \end{cases}$$

If $\xi = \min\{\alpha \in T_{\omega^c_{d-1},h(x)} : y < B_\alpha(x)\} = 0$, i.e., $B_0(x) > y$, then $\tau_2(x,y) = 0$. On the other hand, if $\xi > 0$ then one observes that $k - 1 < \epsilon \cdot \sqrt[c]{\log_{d-1}(x)}$ by definition of τ_1 and of B, so that τ_2 is f-regressive.

Define a function τ_3 as follows.

$$\tau_3(x,y) := \begin{cases} 0 & \text{if } B_{\omega^c_{d-1}}(x) \leq y \text{ or } h(x) < 2, \\ 1 & \text{otherwise.} \end{cases}$$

Suppose H is as hypothesized. We show that τ_3 takes constant value 0. This implies the $B_{\omega^c_{d-1}}$-sparseness since $h(\min(H)) \geq 2$. Assume otherwise and let $x < y < z$ be in H. Note first that by the condition (c)

$$\min\{\alpha \in T_{\omega^c_{d-1},h(x)} : y < B_\alpha(x)\} > 0 \quad \text{and hence} \quad \tau_2(x,y) > 0.$$

By hypotheses on H, $\tau_1(x,y) = \tau_1(x,z)$, $\tau_2(x,y) = \tau_2(x,z)$, $\tau_1(x,z) \leq \tau_1(y,z)$. We have the following, by definition of τ_1, τ_2.

$$B^{\tau_2(x,z)}_{\tau_1(x,z)}(x) \leq y < z < B^{\tau_2(x,z)+1}_{\tau_1(x,z)}(x).$$

This implies that $B^{\tau_2(x,z)+1}_{\tau_1(x,z)}(x) \leq B_{\tau_1(x,z)}(y)$, by one application of $B_{\tau_1(x,z)}$.

We now show that $\tau_1(y,z) \longrightarrow_{h(y)} \tau_1(x,z)$. We know $\tau_1 \in T_{\omega^c_{d-1},h(x)}$, i.e. $\omega^c_{d-1} \longrightarrow_{h(x)} \tau_1(x,z)$. Since $x < y$ implies $h(x) \leq h(y)$ we have $\omega^c_{d-1} \longrightarrow_{h(y)} \tau_1(x,z)$. But since $\tau_1(y,z) \in T_{\omega^c_{d-1},h(y)}$ and $\tau_1(y,z) \geq \tau_1(x,z)$ by hypotheses on H, we can conclude that $\tau_1(y,z) \longrightarrow_{h(y)} \tau_1(x,z)$.

Hence, by Lemma 4.17 and Proposition 4.18.(3), we have $B_{\tau_1(x,z)}(y) \leq B_{\tau_1(y,z)}(y)$, and we know that $B_{\tau_1(y,z)}(y) \leq z$ by definition of τ_1. So we reached the contradiction $z < z$.

□

A comment about the utility of Proposition 4.21. If, assuming $(\mathrm{KM})^{d+1}_{\lfloor \sqrt[c]{\log_{d-1}} \rfloor}$, we are able to infer the existence of a set H satisfying the conditions of Proposition 4.21, then we can conclude that $R(\mu)^{d+1}_{\lfloor \sqrt[c]{\log_{d-1}} \rfloor}$ has eventually the same growth rate of $B_{\omega^c_{d-1}}$. In fact, suppose that there exists a M such that for almost all x there exists a set H satisfying the conditions of Proposition 4.21 and such that $H \subseteq R(\mu)^{d+1}_{\lfloor \sqrt[c]{\log_{d-1}} \rfloor}(x + M)$, which means that such an H can be found as a consequence of $(\mathrm{KM})^{d+1}_{\lfloor \sqrt[c]{\log_{d-1}} \rfloor}$. Also suppose that, for almost all x we can find such an H of cardinality $\geq x + 2$. Then for such an $H = \{h_0, \ldots, h_k\}$ we have $k \geq x + 1$,

$h_{k-1} \geq x$ and, by Proposition 4.21 $h_k \geq B_{\omega^c_{d-1}}(h_{k-1})$. Hence we can show that $R(\mu)^{d+1}_{\lfloor \sqrt[c]{\log_{d-1}} \rfloor}$ has eventually the same growth rate as $B_{\omega^c_{d-1}}$:

$$R(\mu)^{d+1}_{\lfloor \sqrt[c]{\log_{d-1}} \rfloor}(x+M) \geq h_k \geq B_{\omega^c_{d-1}}(h_{k-1}) \geq B_{\omega^c_{d-1}}(x)$$

In the following we show how to obtain a set H as in Proposition 4.21 from $(\mathrm{KM})^{d+1}_{\lfloor \sqrt[c]{\log_{d-1}} \rfloor}$.

4.4.2. *Glueing and logarithmic compression of f-regressive functions.* We here collect some tools that are needed to combine or glue distinct f-regressive functions in such a way that a min-homogeneous set (or a subset thereof) for the resulting function is min-homogeneous for each of the component functions. Most of these tools are straightforward adaptations of analogous results for regressive partitions from [9].

The first simple lemma (Lemma 4.22 below) will help us glue the partition ensuring sparseness obtained by the Sparseness Lemma 4.15 with some other relevant function introduced below. Observe that one does not have to go to an higher dimension if one is willing to give up one square root in the regressiveness condition.

Lemma 4.22. Let $P : [\mathbb{N}]^n \to \mathbb{N}$ be $Q : [\mathbb{N}]^n \to \mathbb{N}$ be $\lfloor \sqrt[2c]{\log_k} \rfloor$-regressive functions. And define $(P \otimes Q) : [\mathbb{N}]^n \to \mathbb{N}$ as follows:

$$(P \otimes Q)(x_1, \ldots, x_n) := P(x_1, \ldots, x_n) \cdot \lfloor \sqrt[2c]{\log_k(x_1)} \rfloor + Q(x_1, \ldots, x_n)$$

Then $(P \otimes Q)$ is $\lfloor \sqrt[c]{\log_k} \rfloor$-regressive and if H is min-homogeneous for $(P \otimes Q)$ then H is min-homogenous for P and for Q.

Proof. We show that $(P \otimes Q)$ is $\sqrt[c]{\log_k}$-regressive:

$$\begin{aligned}
(P \otimes Q)(\vec{x}) &= P(\vec{x}) \cdot \lfloor \sqrt[2c]{\log_k(x_1)} \rfloor + Q(\vec{x}) \\
&\leq (\sqrt[2c]{\log_k(x_1)} - 1) \cdot \sqrt[2c]{\log_k(x_1)} + (\sqrt[2c]{\log_k(x_1)} - 1) \\
&= \sqrt[c]{\log_k(x_1)} - 1 \\
&< \lfloor \sqrt[c]{\log_k(x_1)} \rfloor
\end{aligned}$$

We show that if H is min-homogeneous for $(P \otimes Q)$ then H is min-homogeneous for both P and Q. Let $x < y_2 \cdots < y_n$ and $x < z_2 < \cdots < z_n$ be in H. Then $(P \otimes Q)(x, \vec{y}) = (P \otimes Q)(x, \vec{z})$. Then we show $a := P(x, \vec{y}) = P(x, \vec{z}) =: c$ and $c := Q(x, \vec{y}) = Q(x, \vec{z}) =: d$.

If $w := \lfloor \sqrt[2c]{\log_k(x_1)} \rfloor = 0$ then it is obvious since $a = b = 0$. Assume now $w > 0$. Then $a \cdot w + b = c \cdot w + d$. This, however, implies that $a = c$ and $b = d$, since $a, b, c, d < w$. □

The next two results are adaptations of Lemma 3.3 and Proposition 3.6 of Kanamori-McAloon [9] for f-regressiveness (for any choice of f). Lemma 4.23 is used in [9] for a different purpose, and it is quite surprising how well it fits in the present investigation. Essentially, it will be used to obtain, from an 2^f_{d-2}-regressive of dimension 2, an f-regressive function of dimension $d-2$ such that both have almost same min-homogeneous sets. Each iteration of the following Lemma costs one dimension.

Lemma 4.23. If $P : [\mathbb{N}]^n \to \mathbb{N}$ is f-regressive, then there is a $\bar{P} : [\mathbb{N}]^{n+1} \to \mathbb{N}$ f-regressive s.t.

- (i) $\bar{P}(s) < 2\log(f(\min(s))) + 1$ for all $s \in [\mathbb{N}]^{n+1}$, and
- (ii) if $\bar{H}$ is min-homogeneous for $\bar{P}$ then $H = \bar{H} - (f^{-1}(7) \cup \{\max(\bar{H})\})$ is min-homogeneous for P.

Proof. Write $P(s) = (y_0(s), \ldots, y_{d-1}(s))$ where $d = \log(f(\min(s)))$. Define $\bar{P}$ on $[N]^{n+1}$ as follows.

$$\bar{P}(x_0, \ldots, x_n) := \begin{cases} 0 & \text{if either } f(x_0) < 7 \text{ or } \{x_0, \ldots, x_n\} \\ & \text{is min-homogeneous for } P, \\ 2i + y_i(x_0, \ldots, x_{n-1}) + 1 & \text{otherwise, where } i < \log(f(x_0)) \\ & \text{is the least s.t. } \{x_0, \ldots, x_n\} \\ & \text{is not min-homogeneous for } y_i. \end{cases}$$

Then $\bar{P}$ is f-regressive and satisfies (i). We now verify (ii). Suppose that $\bar{H}$ is min-homogeneous for $\bar{P}$ and H is as described. If $\bar{P}|[H]^{n+1} = \{0\}$ then we are done, since then all $\{x_0, \ldots, x_n\} \in [H]^{n+1}$ are min-homogeneous for P. Suppose then that there are $x_0 < \cdots < x_n$ in H s.t. $\bar{P}(x_0, \ldots, x_n) = 2i + y_i(x_0, \ldots, x_{n-1}) + 1$. Given $s, t \in [\{x_0, \ldots, x_n\}]^n$ with $\min(s) = \min(t) = x_0$ we observe that

$$\bar{P}(s \cup \max(\bar{H})) = \bar{P}(x_0, \ldots, x_n) = \bar{P}(t \cup \max(\bar{H}))$$

by min-homogeneity. But then $y_i(s) = y_i(t)$, a contradiction. □

The next proposition allows one to glue together a finite number of f-regressive functions into a single f-regressive. This operation costs one dimension.

Proposition 4.24. There is a primitive recursive function $p : \mathbb{N} \to \mathbb{N}$ such that for any $n, e \in \mathbb{N}$, if $P_i : [\mathbb{N}]^n \to \mathbb{N}$ is f-regressive for every $i \leq e$ and $P : [\mathbb{N}]^{n+1} \to \mathbb{N}$ is f-regressive, there are $\rho_1 : [\mathbb{N}]^{n+1} \to \mathbb{N}$ f-regressive and $\rho_2 : [\mathbb{N}]^{n+1} \to 2$ such that if $\bar{H}$ is min-homogeneous for ρ_1 and homogeneous for ρ_2, then

$$H = \bar{H} \setminus (\max\{f^{-1}(7), p(e)\} \cup \{\max(\bar{H})\})$$

is min-homogeneous for each P_i and for P.

Proof. Note that given any $k \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that for all $x \geq m$

$$(2\log(f(x)) + 1)^{k+1} \leq f(x).$$

Let $p(k)$ be the least such m.

For each P_i, let $\bar{P}_i$ be obtained by an application of the Lemma 4.23. Define $\rho_2 : [\mathbb{N}]^{n+1} \to 2$ as follows.

$$\rho_2(s) := \begin{cases} 0 & \text{if } \bar{P}_i(s) \neq 0 \text{ for some } i \leq e, \\ 1 & \text{otherwise.} \end{cases}$$

Define $\rho_1 : [\mathbb{N}]^{n+1} \to \mathbb{N}$ f-regressive as follows.

$$\rho_1(s) := \begin{cases} \langle \bar{P}_0(s), \ldots, \bar{P}_e(s) \rangle & \text{if } \rho_2(s) = 0 \text{ and } \min(s) \geq p(e), \\ P(s) & \text{otherwise.} \end{cases}$$

Observe that ρ_1 can be coded as a f-regressive function by choice of $p(\cdot)$.

Suppose $\bar{H}$ is as hypothesized and H is as described. If ρ_2 on $[H]^{n+1}$ were constantly 0, we can derive a contradiction as in the proof of the previous Lemma.

Thus ρ_2 is constantly 1 on $[H]^{n+1}$ and therefore $\rho_1(s) = P(s)$ for $s \in [H]^{n+1}$ and the proof is complete. $\square$

The following proposition is an f-regressive version of Proposition 3.4 in Kanamori-McAloon [9]. It is easily seen to hold for any choice of f, but we include the proof for completeness. This proposition will allow us to find a min-homogeneous set on which τ_1 from Proposition 4.21 is weakly monotonic increasing on first arguments. The cost for this is one dimension.

Proposition 4.25. If $P : [\mathbb{N}]^n \to \mathbb{N}$ is f-regressive, then there are $\sigma_1 : [\mathbb{N}]^{n+1} \to \mathbb{N}$ f-regressive and $\sigma_2 : [\mathbb{N}]^{n+1} \to 2$ such that if H is of cardinality $> n+1$, min-homogeneous for σ_1 and homogeneous for σ_2, then $H \setminus \{\max(H)\}$ is min-homogeneous for P and for all $s, t \in [H]^n$ with $\min(s) < \min(t)$ we have $P(s) \leq P(t)$.

Proof. Define $\sigma_1 : [\mathbb{N}]^{n+1} \to \mathbb{N}$ as follows:

$$\sigma_1(x_0, \ldots, x_n) := \min(P(x_0, \ldots, x_{n-1}), P(x_1, \ldots, x_n))$$

Obviously σ_1 is f-regressive since P is f-regressive. Define $\sigma_2 : [\mathbb{N}]^{n+1} \longrightarrow \mathbb{N}$ as follows:

$$\sigma_2(x_0, \ldots, x_n) := \begin{cases} 0 & \text{if } P(x_0, \ldots, x_{n-1}) \leq P(x_1, \ldots, x_n), \\ 1 & \text{otherwise} \end{cases}$$

Now let H be as hypothesized. Suppose first that σ_2 is constantly 0 on $[H]^{n+1}$. Then weak monotonicity is obviously satisfied. We show that $H \setminus \{\max(H)\}$ is min-homogeneous for P as follows. Let $x_0 < x_1 \cdots < x_{n-1}$ and $x_0 < y_1 < \cdots < y_{n-1}$ be in $H \setminus \{\max(H)\}$. Since σ_2 is constantly 0 on H, we have $F(x_0, x_1, \ldots, x_{n-1}) \leq F(x_1, \ldots, x_{n-1}, \max(H))$, and $F(x_0, y_1, \ldots, y_{n-1}) \leq F(y_1, \ldots, y_{n-1}, \max(H))$. Since H is also min-homogeneous for σ_1, we have

$$\sigma_1(x_0, x_1, \ldots, x_{n-1}, \max(H)) = \sigma_1(x_0, y_1, \ldots, y_{n-1}, \max(H)).$$

Thus, $F(x_0, x_1, \ldots, x_{n-1}) = F(x_0, y_1, \ldots, y_{n-1})$.

Assume by way of contradiction that σ_2 is constantly 1 on $[H]^{n+1}$. Let $x_0 < \cdots < x_{n+1}$ be in H. Then, by two applications of σ_2 we have

$$F(x_0, \ldots, x_{n-1}) > F(x_1, \ldots, x_n) > F(x_2, \ldots, x_{n+1}),$$

so that $\sigma_1(x_0, \ldots, x_n) = F(x_1, \ldots, x_n)$ while $\sigma_1(x_0, x_2, \ldots, x_{n+1}) = F(x_2, \ldots, x_{n+1})$, against the min-homogeneity of H for σ_1. $\square$

4.4.3. *Putting things together.* Now we have all ingredients needed for the sharp unprovability result. Figure 1 below is a scheme of how we will put them together to get the desired result. It illustrates, besides the general structure of the argument, how the need for Kanamori-McAloon principle for hypergraphs of dimension $d+1$ arises when dealing with the ω_d-level of the fast-growing hierarchy (in other words, with $\mathrm{I}\Sigma_d$).

Given f let $\bar{f}_k$ be defined as follows: $\bar{f}_0(x) := f(x)$, $\bar{f}_{k+1}(x) := 2\log(\bar{f}_k(x)) + 1$. Thus,

$$\bar{f}_k(x) := 2\log(2\log(\ldots(2\log(f(x)) + 1)\ldots) + 1) + 1,$$

with k iterations of $2\log(\cdot) + 1$ applied to f.

Let $f(x) = \lfloor \sqrt[c]{\log_{d-1}} \rfloor$ and $f'(x) = 2_\ell(1/3 \cdot f(x))$, $\ell = d-2$. Observe then that $\bar{f}'_\ell$ is eventually dominated by f, so that an $\bar{f}'_\ell$-regressive function is also f-regressive

$$\begin{tikzcd}
 & \begin{array}{c}\text{Proposition 4.21}\\ \tau_1 : [\mathbb{N}]^2 \to \mathbb{N} \\ 2_{d-2}(1/3 \sqrt[2c]{\log_{d-1}})\text{-reg}\end{array} \arrow[d, "\text{Proposition 4.25}"] \\
 & \begin{array}{c}\sigma_1 : [\mathbb{N}]^3 \to \mathbb{N} \\ 2_{d-2}(1/3 \sqrt[2c]{\log_{d-1}})\text{-reg}\end{array} \arrow[d, "\begin{array}{l}\text{Lemma 4.23}\\ (d-2 \text{ times})\end{array}"] \\
 & \begin{array}{c}\sigma_1^* : [\mathbb{N}]^{d+1} \to \mathbb{N} \\ \text{eventually } \sqrt[2c]{\log_{d-1}}\text{-reg}\end{array} \arrow[d, "\text{Proposition 4.24}"] \\
\begin{array}{c}\text{Sparseness Lemma 4.15}\\ P_{2c,d} : [\mathbb{N}]^{d+1} \to \mathbb{N} \\ \sqrt[2c]{\log_{d-1}}\text{-reg}\end{array} \arrow[dr] & \begin{array}{c}\rho_1 : [\mathbb{N}]^{d+1} \to \mathbb{N} \\ \sqrt[2c]{\log_{d-1}}\text{-reg}\end{array} \arrow[d, "\text{Lemma 4.22}"] \\
 & \begin{array}{c}(P_{2c,d} \otimes \rho_1) : [\mathbb{N}]^{d+1} \to \mathbb{N} \\ \sqrt[c]{\log_{d-1}}\text{-reg}\end{array}
\end{tikzcd}$$

FIGURE 1. Scheme of the unprovability proof

if the arguments are large enough. Let m be such that $\lfloor \sqrt[c]{\log_{d-1}(x)} \rfloor \geq \bar{f}'_\ell(x)$ for all $x \geq m$. We have

$$R(\mu)^{d+1}_f(x+m) \geq R(\mu)^{d+1}_{\bar{f}'_\ell}(x).$$

Thus $(\mathrm{KM})^{d+1}_f$ implies $(\mathrm{KM})^{d+1}_{\bar{f}'_\ell}$. This will be used in the following. We summarize the above argument in the following Lemma.

Lemma 4.26. If h eventually dominates g then

$$R(\mu)^d_h(x+m) \geq R(\mu)^d_g(x)$$

where m is such that $h(x) \geq g(x)$ for all $x \geq m$.

Proof Sketch. If G is g-regressive then define G' on the same interval by letting $G'(i) = 0$ if $i \leq m$ and $G'(i) = G(i)$ otherwise. Then G' is h-regressive. If H' is min-homogeneous for G' and $card(H') \geq x+m$ then $H = H' - \{\text{first } m \text{ elements of } H'\}$ is min-homogeneous for G and of cardinality $\geq x$. □

The next Theorem shows that $R(\mu)_f^{d+1}$, with $f(x) = \lfloor \sqrt[c]{\log_{d-1}(x)} \rfloor$, has eventually the same growth rate of $B_{\epsilon,c,d-1,\omega_{d-1}^c}(x)$. As a consequence we will obtain the desired unprovability result.

Theorem 4.27 (in $\mathrm{I}\Sigma_1$). Given $c, d \geq 2$ let $f(x) = \lfloor \sqrt[c]{\log_{d-1}(x)} \rfloor$. Then for all x

$$R(\mu)_f^{d+1}(12x + K(c,d)) > B_{\epsilon,2c,d-1,\omega_{d-1}^{2c}}(x),$$

where $\epsilon = \sqrt[12c]{1/3}$ and $K : \mathbb{N}^2 \to \mathbb{N}$ is a primitive recursive function.

Proof. Let $\hat{f}(x) := \lfloor \sqrt[2c]{\log_{d-1}(x)} \rfloor$ and $q(x) := 2_{d-2}(\frac{1}{3}\hat{f}(x))$. Then $\bar{q}_{d-2}$ is eventually dominated by $\hat{f}$, so there is a number r such that for all $x \geq r$ we have $\bar{q}_{d-2}(x) \leq \hat{f}(x)$. Let $D(c,d)$ be the least such r. Notice that $D : \mathbb{N}^2 \to \mathbb{N}$ is primitive recursive.

Let $h(x) := \lfloor \sqrt[12c]{1/3} \cdot \sqrt[24c^2]{\log_{d-1}(x)} \rfloor$. Now we are going to show that for all x

$$R(\mu)_f^{d+1}(3\ell' - 1) > B_{\epsilon,2c,d-1,\omega_{d-1}^{2c}}(x),$$

where $\ell' = \ell + 4x + 4d + 4D(c,d) + 7$, $\ell = 2_{d-1}((n \cdot (n+1) + 1)^{2c})$, $n = 2^{d-1} \cdot m$, where m is the least number such that $m \geq 2(2c)^2$, and

$$\ell \geq \max(\{\hat{f}^{-1}(7), h^{-1}(2), p(0)\} \cup \{\bar{q}_k^{-1}(7) : k \leq d-3\}),$$

where $p(\cdot)$ is as in Proposition 4.24. The existence of such an m depends primitive recursively on c, d. Notice that the Sparseness Lemma 4.15 functions for any such m with respect to $\hat{f}$. We just remark that one should not wonder about how one comes to the exact numbers above. They just follows from the following construction of the proof.

Let τ_1, τ_2, τ_3 be the functions defined in Proposition 4.21 with respect to $\hat{f}$. Observe that τ_1 is $2_{d-2}(\frac{1}{3}\hat{f})$-regressive and τ_2 is $\hat{f}$-regressive.

Let σ_1, σ_2 be the functions obtained by Proposition 4.25 applied to τ_1. Observe that σ_1 is $2_{d-2}(\frac{1}{3}\hat{f})$-regressive, i.e. q-regressive.

Let $\sigma_1^* : [\mathbb{N}]^{d+1} \to \mathbb{N}$ be the function obtained by applying Proposition 4.23 to σ_1 $d-2$ times. Observe that σ_1^* is *eventually* $\hat{f}$-regressive by the same argument as above.

Define $\hat{\sigma}_1^* : [\mathbb{N}]^{d+1} \to \mathbb{N}$ as follows:

$$\hat{\sigma}_1^* := \begin{cases} 0 & \text{if } x < D(c,d), \\ \sigma_1^*(x) & \text{otherwise.} \end{cases}$$

Then $\hat{\sigma}_1^*$ is $\hat{f}$-regressive such that if H is min-homogeneous for $\hat{\sigma}_1^*$ then

$$H \setminus \{\text{first } D(c,d) \text{ elements of } H\}$$

is min-homogeneous for σ_1^*.

Let ρ_1 and ρ_2 be the functions obtained by applying Proposition 4.24 to the $\hat{f}$-regressive functions $\hat{\sigma}_1^*$ and τ_2 (the latter trivially lifted to dimension d). Observe that ρ_1 is $\hat{f}$-regressive.

Now let $(P_{2c,d} \otimes \rho_1)$ be obtained, as in Lemma 4.22, from ρ_1 and the partition $P_{2c,d} : [\mathbb{N}]^{d+1} \longrightarrow \mathbb{N}$ from the Sparseness Lemma 4.15 with respect to $\hat{f}$. Observe that, by Lemma 4.22, we have that $(P_{2c,d} \otimes \rho_1)$ is $\sqrt[c]{\log_{d-1}}$-regressive, i.e. f-regressive.

Now x be given. Let $H \subseteq R(\mu)^{d+1}_f(3\ell' - 1)$ be such that

$$card(H) > 3\ell' - 1$$

and H is min-homogeneous for $(P_{2c,d} \otimes \rho_1)$ and homogeneous for ρ_2, for σ_2 and for τ_3. This is possible since the Finite Ramsey Theorem is provable in $\mathrm{I}\Sigma_1$. Notice that H is then min-homogeneous for $P_{2c,d}$ and for ρ_1.

Now we follow the process just above in the reverse order to get a set which satisfies the conditions of the Capturing Proposition 4.21.

Define first H_0 and H_1 by:

$$\begin{aligned} H_0 &:= H \setminus (\{\text{first } \ell \text{ elements of } H\} \cup \{\text{last } \ell - 2 \text{ elements of } H\}), \\ H_1 &:= H_0/4 \end{aligned}$$

Then for all $a, b \in H_1$ such that $a < b$ we have $2_{d-1}^{(\log_{d-1}(a))^{2c}} \leq b$ by Lemma 4.15. Notice that

$$\begin{aligned} card(H_0) &\geq \ell' + 1, \\ card(H_1) &\geq \lfloor (\ell' + 1)/4 \rfloor + 1. \end{aligned}$$

Since H_1 is also min-homogeneous for ρ_1 (and ρ_2) we have by Proposition 4.24 that H_2 defined by

$$H_2 := H_1 \setminus (\max\{\hat{f}^{-1}(7), p(0)\} \cup \{\max(H_1)\}) = H_1 \setminus \{\max(H_1)\}$$

is min-homogeneous for $\hat{\sigma}_1^*$ and for τ_2, and

$$card(H_2) \geq \lfloor (\ell' + 1)/4 \rfloor .$$

Let

$$H_3 := H_2 \setminus \{ \text{ first } D(c, d) \text{ elements of } H_2\}.$$

Then H_3 is also min-homogeneous for σ_1^* (and obviously still min-homogeneous for τ_2, homogeneous for ρ_2, for σ_2 and for τ_3). Also, we have

$$card(H_3) \geq \lfloor (\ell' + 1)/4 \rfloor - D(c, d).$$

By Lemma 4.23 we have that H_4 defined by

$$\begin{aligned} H_4 &:= H_3 \setminus (\max\{\bar{q}_k^{-1}(7) : k \leq d - 3\} \cup \{\text{last } d - 2 \text{ elements of } H_3\}) \\ &= H_3 \setminus \{\text{last } d - 2 \text{ elements of } H_3\} \end{aligned}$$

is min-homogeneous for σ_1 (and σ_2), and

$$card(H_4) \geq \lfloor (\ell' + 1)/4 \rfloor - D(c, d) - d + 2.$$

Now define H^* as follows:

$$H^* := H_4 \setminus \{\max H_4\}$$

Notice that $card(H_4) > 3$. Then by Proposition 4.25 H^* is min-homogeneous for τ_1 which is weakly monotonic on first arguments on $[H^*]^2$, and

$$card(H^*) \geq \lfloor (\ell' + 1)/4 \rfloor - D(c, d) - d + 1 > x + 1.$$

The second inequality follows from the definition of ℓ'. Notice now that H^* satisfies all the conditions of the Capturing Proposition 4.21 with respect to $\hat{f}$.

Let $H^* = \{h_0, \dots, h_k\}$ ($k \geq x+1$, so that $h_{k-1} \geq x$). Then, by Proposition 4.21, for all $a, b \in H^*$ such that $a < b$ we have $B_{\omega^c_{d-1}}(a) \leq b$.

$$R(\mu)^{d+1}_f(3\ell' - 1) > h_k \geq B_{\epsilon,2c,d-1,\omega^{2c}_{d-1}}(h_{k-1}) \geq B_{\epsilon,2c,d-1,\omega^{2c}_{d-1}}(x),$$

where $\epsilon = \sqrt[12c]{1/3}$. The first inequality holds since we chose $H^* \subseteq R(\mu)^{d+1}_f(\ell' - 1)$. The second holds by Proposition 4.21. The third holds because $h_{k-1} \geq x$. $\square$

Let us restate Theorem 4.27 in a somewhat simplified form. Given $c, d \geq 2$ set, from now on,

$$\hat{g}_c(x) := \sqrt[c]{\log_{d-1}(x)}$$

Theorem 4.28. There are primitive recursive functions $h : \mathbb{N} \to \mathbb{N}$ and $K : \mathbb{N}^2 \to \mathbb{N}$ such that for all x and all $c, d \geq 2$

$$R(\mu)^{d+1}_{\hat{g}_c}(h(x) + K(c,d)) \geq B_{\epsilon,c,d-1,\omega^c_{d-1}}(x),$$

where $\epsilon = \sqrt[6c]{1/3}$.

Proof. By inspection of the proof of Theorem 4.27, and by the fact that, as proved in Theorem 4.7, $B_{c,d,\alpha}$ and $B_{2c,d,\alpha}$ have the same growth rate. $\square$

Remember the fast-growing hierarchy F_α defined in Section 2.

Theorem 4.29. Given $d \geq 2$ let $f(x) = \sqrt[F^{-1}_{\omega_d}(x)]{\log_{d-1}(x)}$. Then there is a primitive recursive function k such that the function $x \mapsto R(\mu)^{d+1}_f(k(x))$ eventually dominates every $\mathrm{I}\Sigma_d$-provably total function.

Proof. First remember that, by Lemma 4.6, there is a primitive recursive function $r : \mathbb{N}^2 \to \mathbb{N}$ such that

$$B_{\omega^c_{d-1}}(r(c,x)) \geq F_{\omega^c_{d-1}}(x).$$

On the other hand by Theorem 4.28, we have that for all x

$$R(\mu)^{d+1}_{\hat{g}_c}(h(x) + K(c,d)) > B_{\omega^c_{d-1}}(x)$$

for some primitive recursive functions h and K. Hence

$$R(\mu)^{d+1}_{\hat{g}_c}(h(r(c,x)) + K(c,d)) > B_{\omega^c_{d-1}}(r(c,x)) > F_{\omega^c_{d-1}}(x).$$

We claim that

$$R(\mu)^{d+1}_f(h(r(x,x)) + K(x,d)) > F_{\omega_d}(x)$$

for all x.

Assume it is false for some x and let

$$N(x) := R(\mu)^{d+1}_f(h(r(x,x)) + K(x,d)).$$

Then for all $i \leq N(x)$ we have $F^{-1}_{\omega_d}(i) \leq x$ and so

$$f(i) = \sqrt[F^{-1}_{\omega_d}(i)]{\log_{d-1}(i)} \geq \sqrt[x]{\log_{d-1}(i)} = \hat{g}_x(i).$$

This implies that

$$\begin{aligned} R(\mu)^{d+1}_f(h(r(x,x)) + K(x,d)) &\geq R(\mu)^{d+1}_{\hat{g}_x}(h(r(x,x)) + K(x,d)) \\ &> F_{\omega^x_{d-1}}(x) \\ &= F_{\omega_d}(x). \end{aligned}$$

Contradiction! $\square$

5. Conclusion

Putting together the results from Sections 3 and 4 we can state the main Theorem.

Theorem 5.1. Given $d \geq 1$ and $\alpha \leq \varepsilon_0$ let $f^d_\alpha(i) = \lfloor \sqrt[F_\alpha^{-1}(i)]{\log_d(i)} \rfloor$.

$$\mathrm{I}\Sigma_d \vdash (\mathrm{KM})^{d+1}_{f^{d-1}_\alpha} \Leftrightarrow \alpha < \omega_d.$$

Proof. The provability follows from the Theorem 3.7, and the unprovability parts are established by Theorems 4.4, 4.28 and 4.29. □

Observe that Theorem 5.1 also closes the gap between $d-2$ and d in Theorem 2.3, showing

$$\mathrm{I}\Sigma_d \vdash (\mathrm{KM})^{d+1}_{\lfloor \log_n \rfloor} \Leftrightarrow n \geq d.$$

The truth of Conjecture 2.4 is implied as well: for all $n \geq 1$ for all $d \geq 1$, we have that $\mathrm{I}\Sigma_d \nvdash (\mathrm{KM})^{d+1}_{\lfloor \sqrt[n]{\log_d} \rfloor}$.

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