Classifying the phase transition threshold for unordered regressive Ramsey numbers

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Abstract

Following ideas of Richer (2000) we introduce the notion of unordered regressive Ramsey numbers or unordered Kanamori-McAloon numbers. We show that these are of Ackermannian growth rate. For a given number-theoretic function f we consider unordered f-regressive Ramsey numbers and classify exactly the threshold for f which gives rise to the Ackermannian growth rate of the induced f-regressive Ramsey numbers. This threshold coincides with the corresponding threshold for the standard regressive Ramsey numbers. Our proof is based on an extension of an argumtent from a corresponding proof in a paper by Kojman,Lee,Omri and Weiermann 2007.

1 Introduction

There exist three basic and well known infinitary Ramsey principles which give rise to independence results for PA: Ramsey's theorem, the canonical Ramsey theorem (Erdös Rado) and the regressive Ramsey theorem (Kanamori McAloon). The latter two principles make essential use of a preexisting order on the natural numbers in order to speak about min-colorings, max-colorings and min-homogeneous sets, etc. The three basic infinitary Ramsey-principles give rise to finitary Ramsey principles which can be formulated in the language of arithmetic: the finite Ramsey principle with a suitable largeness condition (i.e. the Paris-Harrington principle), the Kanamori McAloon principle and the finite canonical Ramsey principle with a suitable largeness condition. It turns out that all these finite versions make essential use of the standard $<$-relation and one might wonder if it is possible to find strong principles which do not depend so intrinsically on the less than relation.

An interesting approach to this question can be gotten from a recent paper by Richer about unordered canonical Ramsey numbers and their asymptotic classification. It is quite natural to extend Richer's approach to the context of strong Ramsey principles and in this paper we extend Richer's approach to the Kanamori McAloon Ramsey theorem.

For simplicity we consider only colourings of pair (graphs) and expect that the result extends to higher dimensions (hypergraphs) using appropriate bounds from the fast growing hierarchy. We also expect that our results generalize to the unordered canonical Ramsey theorem with a suitable largeness condition. This and other related questions will be investigated jointly in a bigger research project with Bovykin, Carlucci, Lee et al.

In a first step we establish that unordered regressive Ramsey numbers are Ackermannian/ Here a colouring c of pairs of natural numbers is called regressive if $c(u,v) \leq u$ for all u, v with $u < v$. Given m there exists a least number $R := uKM(m)$ such that for all regressive colourings $c : [R]^2 \to \mathbb{N}$ there exists an $H \subseteq R$ and a linear ordering $\prec$ on H such that H is c-$\prec$-min-homogeneous, i.e. if $u, v, w \in H$ and $u \prec v$ and $u \prec w$ than $c(\{u,v\}) = c(\{u,w\})$.

Our first result will be that the function $m \mapsto uKM(m)$ is Ackermannian, so that it in particular will eventually dominate every primitive recursive function.

In a second step we consider the phase transition for unordered regressive Ramsey numbers. Investigations of this kind form part of a bigger research project of the second author This contributes to a general research program of the second author about phase transitions in logic and combinatorics. (See, for example, [3, 4, 5, 6, 7, 8, **?**] for more information.)

Given a number-theoretic function f let us call a colouring c of ordered pairs f-regressive if if $c(u,v) \leq u$ for all u, v with $u < v$. Then given m there exists a least number $R := uKM_f(m)$ such that for all f-regressive colourings $c : [R]^2 \to \mathbb{N}$ there exists an $H \subseteq R$ and a linear ordering $\prec$ on H such that H is c-$\prec$-min-homogeneous.

It is easy to see that for constant functions f the function uKM_f is primitive recursive and so inbetween the constant function and the identity function there will be phase transition from being primitive recursive to Ackermannian of uKM_f. Roughly speaking, for $f(i) = \sqrt[k]{i}$, the function uKM_f is Ackermannian whereas for $f(i) = \log(i)$ the function uKM_f is still elementary recursive. In a final step we let k in $\sqrt[k]{i}$ depend on i. We show that function uKM_f is still primitive recursive if $f(i) \leq \sqrt[A_d^{-1}(i)]{i}$ but becomes Ackermannian if $f(i) \geq \sqrt[A_{-1}(i)]{i}$.

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