Phase transition thresholds for Gödel incompleteness

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Introduction

We shall survey surprising advances in classifying the phase transition thresholds from provability to unprovability (independence) for natural mathematical assertions.

Phase transitions

Phase transition is a type of behaviour wherein small changes of a parameter of a system cause dramatic shifts in some globally observed behaviour of the system, such shifts being usually marked by a sharp 'threshold point'. (An everyday life example of such thresholds are ice melting and water boiling temperatures.) This kind of phenomena nowadays occurs throughout many mathematical and computational disciplines: statistical physics, evolutionary graph theory, percolation theory, Markov chains, computational complexity, artificial intelligence etc.

The last few years have seen an unexpected series of results that bring together independence results in logic, analytic combinatorics and Ramsey Theory. These results can be described intuitively as phase transitions from provability to unprovability of an assertion by varying a threshold parameter.

In the poster we shall survey recent advances on phase transition phenomena which are related to Gödel's incompleteness theorems. We treat two spectacular results on the mathematical relevance of the Gödel incompleteness theorems. We will indicate that our advances are based on cross-fertilization between Ramsey theory, analytic combinatorics (which finds its main applications in the average case analysis of computer algorithms) and mathematical logic.

Gödel's results

The Peano axioms have been designed to provide a complete axiomatization of the properties of the natural numbers. However, Gödel showed that there are statements about the natural numbers which do neither follow nor can be refuted from the Peano axioms. Moreover he showed that the consistency assertion for the Peano axioms is one such statement.

For a long time it has then been open whether Gödel's result applies to assertions which are mathematically relevant or mathematically 'interesting'. It would have indeed been possible that Gödel's theorem only applies to somewhat artificial statements. A first breakthrough in showing that Gödel's theorem matters to mathematics has been obtained around 1977 by Jeff Paris (and Leo Harrington) [1] and a second around 1980 by Harvey Friedman [2].

Ramsey's Theorem

The Paris Harrington theorem is about Ramsey theory which is a branch of mathematics dedicated to the proposition that complete disorder is impossible. Ramsey's theorem in its finite version says that given positive integers k, p and n you can always find a number $r =: R(k, p, n)$ such that for any finite set S of cardinality not below r and for any mapping P from the k-element subsets of S into a set with p colours you will always find a subset S' of S containing at least n elements such that every k-element subset of S' gets the same value under P. This theorem is clearly about finite objects and it is no big surprise that it follows easily from the Peano axioms. Indeed, Erdös and Rado provided 1952 an explicit bound in terms of iterated exponential functions of the size of S to guarantee the conclusion of Ramsey's theorem.

The Paris Harrington Theorem

Let us now consider a slight modification of Ramsey's theorem. Given a function $f : \mathbb{N} \to \mathbb{N}$ let us call a set S of natural numbers f-large if the number of elements in S is not below $f(\min(S))$.

Following Paris and Harrington let PH_f be the assertion that given numbers p, k and n you can find a find a natural number r so large that for any mapping P of the k-element subsets of $\{1, \dots, r\}$ with range contained in $\{1, \dots, p\}$ you will always find an f-large subset H of $\{1, \dots, r\}$ containing at least n elements, such that such that every k-element subset of H gets the same value under P.

If f is a constant function then PH_f is clearly a consequence of Ramsey's theorem.

The infinite version of Ramsey's theorem (stating that for any p-coloring of the k-element subsets of an infinite set S there exists an infinite monochromatic subset S' of S) yields that PH_f is true for any f. The famous Paris Harrington theorem is that PH_{id} is not provable from the Peano axioms where id denotes the identity function. Thus in-between constant functions and the identity function there must be a phase transition threshold for the Paris Harrington assertion PH.

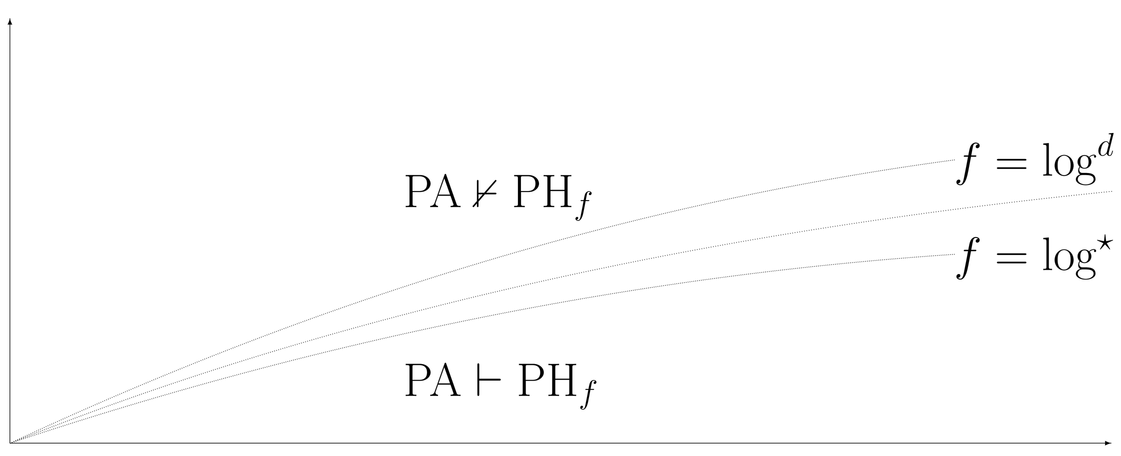
Phase transition threshold for PH

The immediate question is to motivate the largeness condition in the assertion PH and there has been some discussion in the FOM newsgroup on this topic. The (as we would like to argue) intrinsic motivation provided by the Erdös Rado result is as follows. Let $\log^*$ denote the inverse function of the super-exponential function. Then, essentially $\log^*(R(n, n, n))$ is as big as n. A small calculation therefore yields that $\mathrm{PH}_{\log^*}$ is provable from the Peano axioms. Let $\log^k$ denote the k-th iteration of the binary logarithm function. [In practice for large k the functions $\log^k$ and $\log^*$ are very similar to constant functions (at least as calculations on a computer are concerned).] Then the Erdös Rado bound is no longer applicable to prove $\mathrm{PH}_{\log^k}$. Moreover the jump is now a surprisingly big one since as proved in [4] the assertion $\mathrm{PH}_{\log^k}$ does not follow from the Peano axioms.

In fact the threshold from the provable versions of PH_f to the unprovable versions of PH_f has been classified inbetween $\log^*$ and all $\log^k$ in greater detail (by letting k depending on the input argument) but a complete explanation would take as too far into technicalities [4].

Phase transition principle 1:

How to obtain an unprovable version of Ramsey's theorem? Just escape by an extra condition the bounds dictated by finite combinatorics (here the Erdös Rado result)! In analogy with physics this phase transition might be considered as continuous or being of second order.



Friedman's Finite Form of Kruskal's Theorem

Our second example for a phase transition concerns Friedman's finite form of Kruskal's tree theorem. To fix terminology we agree that a finite tree is a finite partial order such that for every element in the tree the set of predecessors is linearly ordered. Moreover we require trees to have exactly one minimal element (the root). We say that one tree is embeddable into another tree if there exists an inf preserving one to one mapping of the first tree into the second.

The story of Friedman's finite form of Kruskal's theorem runs now as follows. We say that the vertex-growth of a sequence $T_0, \dots, T_N$ of finite trees is controlled by $k \in \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$ if for all $i \le N$ the number of nodes in T_i does not exceed $k + f(i)$. Let FFF_f (Friedman's finite form of Kruskal's theorem) be the assertion that for every k there is a finite number N which is so large that for every sequence $T_0, \dots, T_N$ of finite trees whose vertex-growth is controlled by k and f there will always be two numbers i and j below N so that $i < j$ and T_i is embeddable into T_j. Friedman proved that neither FFF_{id} nor its negation follow from the Peano axioms (and not even from predicative analysis).

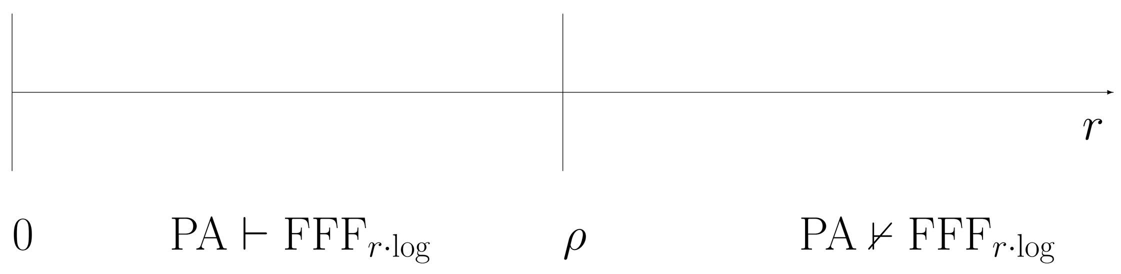
Phase transition threshold for FFF

Matousek and Loebl showed subsequently the following refinement of Friedman's result. Let $\log$ be the binary logarithm. Then $\mathrm{FFF}_{\frac{1}{2}\cdot\log}$ follows from the Peano axioms but $\mathrm{FFF}_{4\cdot\log}$ does not.

Again it is an obvious question whether it is possible to locate the phase transition threshold in the interval from 0.5 to 4. In particular it seems of interest to see whether there is a continuous phase transition from provability to unprovability or whether there is a sharp threshold (somewhat similar to results in random graph theory). Using classical results from analytic combinatorics we have shown in [3] that there is indeed an extremely sharp threshold. For a certain real number $\rho \approx 0.63957769\dots$ the following dichotomy holds: If $r \le \rho$ then $\mathrm{FFF}_{r\cdot\log}$ does follow elementarily from Otter's exponential bounds on the number of trees, hence from the Peano axioms. If $r > \rho$ then $\mathrm{FFF}_{r\cdot\log}$ is unprovable from the Peano axioms [3].

Phase transition principle 2:

How to obtain an unprovable version of Friedman's finite form of Kruskal's theorem? Just escape by an extra condition the bounds dictated by finitary combinatorics (here Otter's tree enumeration result)! In analogy with physics this phase transition might be considered as discontinuous or being of first order.



Manifesto

1. For all existing independence results, find the sharp versions (establish exact unprovability thresholds).
2. Seek new unprovable statements following a new methodology: given a known mathematical theorem with a bound, introduce a parameter (maybe functional) and try to supersede the existing bounds.
3. Cross-fertilize Gödel incompleteness with proof theory, recursion theory, non standard models, logical limit laws, Ramsey theory, analytic combinatorics, number theory, ergodic theory, dynamical systems and statistical physics.

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Further results

Using logic, combinatorics and analytic number theory we have obtained sharp phase transition thresholds for hydra battles, the Goodstein sequences, subrecursive hierarchies, the Kanamori McAloon result, the combinatorial well-foundedness of ε_0 in an additive setting and a multiplicative setting, and for various other statements in Ramsey theory, WQO-theory, and the theory of well-orders. Similar to statistical physics resulting phase transitions share features of renormalization and universality.

Some project coauthors:

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