Phase transition thresholds for Gödel incompleteness
Andreas Weiermann

Universiteit Utrecht
email: weiermann@math.uu.nl

Introduction
We shall survey surprising advances in classifying the phase transition thresholds from provability to unprovability (indepen-
dence) for natural mathematical assertions.

Phase transitions
Phase transition is a type of behaviour wherein small changes of a parameter of a system cause dramatic shifts in some globally observed behaviour of the system, such shifts being usually marked by a sharp ‘threshold point’. (An everyday life example of such thresholds are ice melt-
ing and water boiling temperatures.) This kind of phenom-
ena nowadays occurs throughout many mathematical and computational disciplines: statistical physics, evolutionary graph theory, percolation theory, Markov chains, computa-
tional complexity, artificial intelligence etc.

The last few years have seen an unexpected series of results that bring together independence results in logic, analytic combinatorics and Ramsey Theory. These results can be described intuitively as phase transitions from provability to unprovability by varying a threshold pa-
rameter.

In the poster we shall survey recent advances on phase transition phenomena which are related to Gödel’s incomple-
teness theorems. We treat two spectacular results on the mathem-
atical relevance of the Gödel incompleteness theore-
oms. We will indicate that our advances are based on cross-fertilization between Ramsey theory, analytic combi-
natorics (which finds its main applications in the average case analysis of computer algorithms) and mathematical logic.

Gödel’s results
The Peano axioms have been designed to provide a com-
plete axiomatization of the properties of the natural num-
bers. However, Gödel showed that there are statements about the natural numbers which do neither follow nor can be refuted from the Peano axioms. Moreover he showed that the consistency assertion for the Peano axioms is one such statement.

For a long time it has then been open whether Gödel’s re-
sult applies to assertions which are mathematically relevant or mathematically ‘interesting’. It would have indeed been possible that Gödel’s theorem only applies to somewhat arti-
tifical statements. A first breakthrough in showing that Gödel’s theorem matters to mathematics has been obtained around 1977 by Jeff Paris (and Leo Harrington) [1] and a second around 1980 by Harvey Friedman [2].

Ramsey’s Theorem
The Paris Harrington theorem is about Ramsey theory which is a branch of mathematics dedicated to the propo-
sition that complete disorder is impossible. Ramsey’s theo-

rem in its finite version says that given positive integers k, p
and n you can always find a number r := R(k, p, n) such that for any set S of cardinality not below r and for any mapping P from the k-element subsets of S into a set with p colours you will always find a subset S′ of S con-
taining at least n elements such that every k-element subset of S′ gets the same value under P. This theorem is clearly about finite objects and it is no big surprise that it follows easi-
ly from the Peano axioms. Indeed, Erdős and Rado pro-
vided 1952 an explicit bound in terms of iterated exponent-
ential functions of the size of S to guarantee the conclusion of Ramsey’s theorem.

The Paris Harrington Theorem
Let us now consider a slight modification of Ramsey’s theo-
orem. Given a function f : N → N let us call a set S of finite natural numbers f-large if the number of elements in S is not below f(\min(S)).

Following Paris and Harrington let \( f^{PH} \) be the assertion that given numbers \( k, p, r \) you can find a natural number \( r \) so large that for any mapping \( P \) of the k-element sub-
sets of \( \{1, \ldots, r\} \) with range contained in \( \{1, \ldots, p\} \) you

will always find an \( f^{PH} \)-large subset \( H \) of \( \{1, \ldots, r\} \) contain-
ing at least \( n \) elements, such that such that every k-element subset of \( H \) gets the same value under \( P \).

If \( f \) is a constant function then \( f^{PH} \) is clearly a consequence of Ramsey’s theorem. The finite version of Ramsey’s theorem (stating that for any \( p \)-coloring of the k-element subsets of an infinite set \( S \) there exists an infinite monochromatic subset \( S' \) of \( S \)) yields that \( f^{PH} \) is true for any \( f \). The famous Paris Harrington theorem is that \( f^{PH} \) is not provable from the Peano axioms where \( id \) denotes the identity function. Thus in-between constant functions and the identity function there must be a phase transition threshold for the Paris Harrington assertion \( f^{PH} \).

Phase transition threshold for \( f^{PH} \)

The immediate question is to motivate the largeness condi-
tion in the assertion \( f^{PH} \) and there has been some discussion in the FOM newsgroup on this topic. (The (as we would like to argue) intrinsic motivation provided by the Erdős Rado Residue as \( \log f^{HR}(n) \) is not as large as \( n \).)

In practice for large \( k \) the functions \( \log f^{HR}(k, n) \) and \( \log f^{HR}(n) \) grow very similar to constant functions (at least as calcula-
tions on a computer are concerned.) Then the Erdős Rado bound is no longer applicable to prove \( f^{PH} \). Moreover the jump is now a surprisingly big one since as proved in [4] the assertion \( f^{PH} \) does not follow from the Peano axioms. In fact the threshold from the provable versions of \( f^{PH} \) to the unprovable versions of \( f^{PH} \) has been classified in-between \( \log^3 \) and \( \log^n \) in greater detail (by letting \( k \) de-
pend on the input argument) but a complete explanation would take as too far into technicalities [4].

Phase transition principle 1:
How to obtain an unprovable version of Ramsey’s the-
om? Just escape by an extra condition the bounds dictated by finite combinatorics (here Otter’s tree enumeration result)!

Analog in physics this phase transition might be considered as discontinuous or being of first order.

Friedman’s Finite Form of Kruskal’s Theorem
Our second example for a phase transition concerns Fried-
man’s finite form of Kruskal’s tree theorem. To fix termi-
ne weiermann@math.uu.nl

nological bounds for \( r \) and \( \rho \).

References
[1] Paris and L. Harrington: A mathematical incomple-
teness in Peano arithmetic, Handbook of Mathemat-

torial properties of \( \omega_1 \) in an additive setting and a multiplicative setting, and for various other statements in Ramsey theory, WQO-theory, and the theory of well-orders. Similar to sta-
tistical physics resulting phase transitions share features of renormalization and universality.

Some project coauthors:
A. Bovykin and L. Carlucci (both have been finalists in the young scholars competition), G. Lee, E. Omri, M. Kojman, H. Kotlarski, B. Piekart, A. den Boer, G. Moser.

Further results
Using logic, combinatorics and analytic number theory we have obtained sharp phase transition thresholds for hydra
dattles, the Goodstein sequences, subrecursvie hierarchies, the Kanamori McAloon result, the combinatorial well-

foundedness of \( \epsilon_0 \) in an additive setting and a multiplicative setting, and for various other statements in Ramsey theory, WQO-theory, and the theory of well-orders. Similar to sta-
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References
teness in Peano arithmetic, Handbook of Mathemat-
torial properties of finite trees. Harvey Friedman’s research on the foundations of mathematics, North-
Holland, 1985, 87–117.
function for \( \text{FFF} \). Matusseck and Loebl showed subsequently the following re-
finement of Friedman’s result. Let \( \log \) be the binary loga-

rithm. Then \( \text{FFF}_{n+1} \) follows from the Peano axioms but \( \text{FFF}_{n+k} \) does not.

Again it is an obvious question whether it is possible to lo-
cate the phase transition threshold in the interval from \( 0 \) to \( 1 \). In particular it seems of interest to see whether there is a continuous phase transition from provability to unprovabil-
ity or whether there is a sharp threshold (somewhat similar to results in random graph theory). Using classical results from analytic combinatorics we have shown in [3] that there is indeed an extremely sharp threshold. For a certain real number \( r \approx 0.6377079 \ldots \) the following dichotomy holds: If \( 0 < r \) then \( \text{FFF}_{n+k} \) does follow elementarily from Otter’s ex-
ponential bounds on the number of trees, hence from the Peano axioms. If \( r > 0 \) then \( \text{FFF}_{n+k} \) is unprovable from the Peano axioms [3].

Phase transition principle 2: How to obtain an unprovable version of Friedman’s finite form of Kruskal’s theorem? Just escape by an extra con-
dition the bounds dictated by finitary combinatorics (here Otter’s tree enumeration result)!

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