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Families of Intersecting Subspaces

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Contents

Preface

> *The way to get started is to quit talking and begin doing.*
>
> —Walt Disney

My engagement towards mathematics started at secondary school. I always liked to solve exercises, and I loved to accept the challenges the teachers gave me. After my graduation in secondary school, I really wanted to further discover the beautiful parts of mathematics. So it was clear that I wanted to study this. During the bachelor and master years, I enjoyed seeing all the different parts of mathematics. It gave me a broad view and a chance to sample every branch in mathematics. During my bachelor project, I got the opportunity to work on different topics in finite geometry. I really liked the freedom to think about some new things, and due to the combination of good ideas and excellent aid of my supervisors we discovered new characteristics about Sudoku Latin Squares. This was the start of my first publication. Together with prof. Klaus Metsch from the University of Gießen, we generalized the first results and continued the research on this topic. This was an interesting chance to start exploring the research world. In the last master year, I focused on the master thesis. During this period, I also got the opportunity to go abroad. With the Erasmus program, I went to the Technical University of Eindhoven. Here I got the chance to work together with prof. Aart Blokhuis on the Sunflower bound. Thanks to enriching conversations and discussions with prof. Aart Blokhuis and other researchers in Eindhoven, I discovered the advantages of working together with international academics. I realized for the second time that I enjoyed doing research and discovering new things. Beside that, I was aware, by reading lots of articles, of the fact that my knowledge at that time, only corresponds to the tip of the iceberg. That was the reason why I wanted to continue the research, to get a more fundamental understanding and to reach the bottom of the iceberg.

So, more or less 4 years ago, I got the great opportunity to start with a PhD in finite geometry. I got the chance to work on topics in finite geometry that interest me, such as Cameron-Liebler sets and intersection problems. The result of this research is collected in this thesis.

This thesis contains three main parts. The first part handles several intersection problems.

During the first months of the PhD, I started with the first intersection problem. I investigated sets of solids pairwise intersecting in at least a line. Later on, we could generalise this to a classification of the largest sets of k-spaces in $\mathrm{PG}(n,q)$, pairwise intersecting in at least a $(k-2)$-space. With the aid of dr. Giovanni Longobardi, dr. Ago Riet and prof. Leo Storme, we were able to classify the ten largest examples, see Chapter 3. Thorough this thesis, it will become clear that I like to classify different structures in finite geometries.

A second intersection problem handles a Hilton-Milner problem in projective and affine spaces. Here, I investigated large sets of k-spaces pairwise intersecting in at least a t-space in both $\mathrm{PG}(n,q)$ and $\mathrm{AG}(n,q)$. A straightforward example of these sets is a t-pencil; the set of all k-spaces containing a fixed t-space. In this research, I classified the largest examples of pairwise t-intersecting sets in both $\mathrm{PG}(n,q)$ and $\mathrm{AG}(n,q)$, different from a t-pencil. This classification result can be found in Chapter 4.

Recall that, in my master thesis, I started investigating the Sunflower bound in projective spaces. For this, I studied large sets of k-spaces in a projective space, pairwise intersecting in *precisely* a point. A classical example of such a set is the *sunflower*, where all subspaces pass through the same point. The Sunflower bound states that a set S of k-spaces, pairwise intersecting in a point must be a sunflower if $|S|$ surpasses the Sunflower bound. Prof. Aart Blokhuis, dr. Maarten De Boeck and I could lower this Sunflower bound significantly. How we succeeded in this, can be read in Chapter 5.

In spring 2020, I got the opportunity to visit prof. Klaus Metsch in Gießen. Together with dr. Daniel Werner, we investigated the chromatic number of q-Kneser graphs of flags in projective spaces. This problem can be translated to the following research problem: finding a partition of flags such that every two flags in a partition class *intersect*. We found the chromatic number of the q-Kneser graph of line-solid flags and of line-plane flags in $\mathrm{PG}(4, q)$. Furthermore, if we assume that structural information on the large intersecting sets of $\{d-1, d\}$-flags in $\mathrm{PG}(2d, q)$ is known, then we were also able to generalize our results. Hence, given a Hilton-Milner type conjecture, we found the chromatic number of $\{d-1, d\}$-flags in $\mathrm{PG}(2d, q)$. These results are written in Chapter 6, which concludes the first main part.

In the second main part of this thesis, I describe several Cameron-Liebler results in different contexts.

In [28], Cameron and Liebler introduced specific line classes in $\mathrm{PG}(3, q)$ when investigating the orbits of the projective groups $\mathrm{PGL}(n+1, q)$. These line sets $\mathcal{L}$ have the property that every line spread $\mathcal{S}$ in $\mathrm{PG}(3, q)$ has the same number of lines in common with $\mathcal{L}$. One of the main reasons for studying Cameron-Liebler sets is that there are several equivalent definitions for them, some algebraic, some geometrical or combinatorial in nature. The main question, independent of the context where Cameron-Liebler sets are investigated, is always the same: for which values of the parameter x do there exist Cameron-Liebler sets and which examples correspond to a given parameter x?

In the first year of my PhD, I started defining and investigating Cameron-Liebler sets of k-spaces in $\mathrm{PG}(n, q)$. Prof. Aart Blokhuis, dr. Maarten De Boeck and I found many equivalent definitions, and we could prove a classification result. These results are described in Chapter 8.

During this first Cameron-Liebler project, my interest grew, and I was curious to discover Cameron-Liebler sets in different contexts.

In a second Cameron-Liebler project, Cameron-Liebler sets of generators in finite classical polar spaces were investigated. Dr. Maarten De Boeck and I introduced *degree one Cameron-Liebler sets* in finite classical polar spaces. These sets are Cameron-Liebler sets with an extra assumption, and they give a link between Boolean degree one functions (see [59]) and Cameron-Liebler sets of generators in finite classical polar spaces (see [36]). These results can be found in Chapter 10.

In summer 2019, prof. Morgan Rodgers found a new, non-trivial example of a Cameron-Liebler set of generators in $Q^+(5, 3)$ by using a computer search. Dr. Maarten De Boeck and I investigated this example, and generalized it. In this way, we found a non-trivial example of a degree one Cameron-Liebler set of generators in $Q^+(5, q)$. The construction for this example is described in Section 10.5.

In the second year of my PhD, I got the opportunity to mentor the master thesis of Jonathan Mannaert. Prof. Leo Storme suggested to investigate Cameron-Liebler sets in an affine context. During this research, we first defined Cameron-Liebler line sets in $\mathrm{AG}(3, q)$. We found many equivalent definitions, and some classification results. In a second step, we generalized these Cameron-Liebler line sets in $\mathrm{AG}(3,q)$ to Cameron-Liebler k-sets in $\mathrm{AG}(n,q)$. These results are described in Chapter 9.

The last main part of this thesis discusses Linear sets of pseudoregulus type. In spring 2019, I visited my co-supervisor dr. Geertrui Van de Voorde in Christchurch, New-Zealand, where she immersed me in the world of linear sets. In [7], a characterisation for translation hyperovals in $\mathrm{PG}(4,q)$, q even, was given. Originally our research goal was to generalize these results for $\mathrm{PG}(2k,q)$, q even. While investigating this topic, we could characterise the point sets defined by translation hyperovals in the André/Bruck-Bose representation. We showed that the affine point sets of translation hyperovals in $\mathrm{PG}(2,q^k)$ are precisely those that have a scattered $\mathbb{F}_2$-linear set of pseudoregulus type in $\mathrm{PG}(2k-1,q)$ as set of directions. These results are described in Chapter 11.

I hope that this introduction could engage you for reading this thesis. I already want to thank you for the interest and I hope you enjoy reading this exciting math story. ☺

Jozefien D'haeseleer
March 2021

1 Preliminaries

> “ *La mathématique est l’art de donner le même nom à des choses différentes.* ”
>
> —Henri Poincaré

In this first chapter, we introduce important concepts and known results that will be used throughout the thesis. We suppose that the reader is familiar with the basic notions in finite geometry, combinatorics, linear algebra and graph theory.

1.1 Incidence geometries

Several geometries, such as projective geometries, affine geometries and finite classical polar spaces, are investigated in this thesis. These geometries all are incidence geometries, and therefore we start with introducing the notion of a general incidence geometry.

Definition 1.1.1. An *incidence geometry* $\mathcal{S}$ is a quadruple $\mathcal{S} = (\mathcal{V}, \omega_n, t, \mathcal{I})$, with $\mathcal{V}$ a non-empty set, $\omega_n = \{0, 1, \ldots, n-1\}$, t a surjective map from $\mathcal{V}$ to ω_n and $\mathcal{I}$ a symmetric incidence relation on $\mathcal{V}$, such that $(v_1, v_2) \in \mathcal{I}$, implies that $t(v_1) \neq t(v_2)$, for all $v_1, v_2 \in \mathcal{V}$.

The elements of $\mathcal{V}$ are called the *varieties* of $\mathcal{S}$. Varieties of type 0 and 1 are called the *points* and *lines* respectively. The map t is called the *type map* and in this thesis, this map will always be the dimension map. The integer n is called the *rank* of the geometry $\mathcal{S}$. If $(v_1, v_2) \in \mathcal{I}$, then these elements v_1 and v_2 are called *incident*. Moreover, if $t(v_1) < t(v_2)$, then we say that v_1 is *contained in* v_2, that v_2 *contains* v_1 or that v_2 goes through v_1. A set of points, incident with a fixed line, is said to be *collinear*, and a set of lines incident with a fixed point, is said to be *concurrent*.

If the rank of the incidence geometry is 2, then the set $\mathcal{V}$ of varieties consists of points and lines. This geometry is called a *point-line geometry*. For this, we use the notation $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, with $\mathcal{I}$ the incidence relation such that $\mathcal{I} \subseteq (\mathcal{P} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{P})$. In this geometry, the elements of $\mathcal{P}$ are the points and the elements of $\mathcal{B}$ are the lines. The elements of $\mathcal{B}$ are sometimes also called the *blocks* of $\mathcal{S}$.

The *dual* of an incidence geometry $\mathcal{S} = (\mathcal{V}, \omega_n, t, \mathcal{I})$ is the incidence geometry $\mathcal{S}' = (\mathcal{V}, \omega_n, t', \mathcal{I})$ with $t' = \mathcal{V} \to \omega_n : v \mapsto n - t(v) - 1$. Note that the dual of a point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ can be obtained by interchanging the roles of points and lines. Hence, the dual of the point-line geometry $\mathcal{S}$ is the point-line geometry $\mathcal{S}' = (\mathcal{B}, \mathcal{P}, \mathcal{I})$.

Let $\mathcal{S}_1 = (\mathcal{V}_1, \omega_n, t_1, \mathcal{I}_1)$ and $\mathcal{S}_2 = (\mathcal{V}_2, \omega_n, t_2, \mathcal{I}_2)$ be two incidence geometries of the same rank n. A bijection $\alpha : \mathcal{V}_1 \to \mathcal{V}_2$ with the property that $(v, v') \in \mathcal{I}_1 \Leftrightarrow (\alpha(v), \alpha(v')) \in \mathcal{I}_2, \forall v, v' \in \mathcal{V}_1$, and $t_1(v) = t_2(\alpha(v)), \forall v \in \mathcal{V}_1$, is an *isomorphism* between $\mathcal{S}_1$ and $\mathcal{S}_2$. In the case that $\mathcal{S}_1 = \mathcal{S}_2$, then α is called an *automorphism* of $\mathcal{S}_1$. If $\mathcal{S}_1$ is the dual of $\mathcal{S}_2$, then α is called a *duality*.

Definition 1.1.2. The *incidence matrix* H of a point-line geometry $(\mathcal{P},\mathcal{B},\mathcal{I})$, with $\mathcal{P}$ the set of points $\{p_1,p_2,\dots,p_m\}$ and $\mathcal{B}$ the set of blocks $\{b_1,b_2,\dots,b_n\}$ is the $m\times n$ matrix over the field $\mathbb{R}$, in which the rows are labeled by the points and the columns are labeled by the blocks, so that $H_{ij}=1$ if $(p_i,b_j)\in\mathcal{I}$, and $H_{ij}=0$ otherwise.

In this thesis, we denote the $n\times n$ identity matrix by I_n, the $n\times n$ all one matrix by J_n and the all one column vector of dimension n by $\boldsymbol{j}_n$. If the size n is clear from the context, we also use the notations I, J, and $\boldsymbol{j}$ respectively. In general, all vectors in this thesis are regarded as column vectors.

For a subset S of a finite set Ω, which can consist of points or blocks, we will often use the corresponding *characteristic vector* χ_S.

Definition 1.1.3. Consider a set $\Omega=\{x_1,\dots,x_n\}$ of size n. Then we define for every subset S of Ω a characteristic vector $\chi_S\in\mathbb{R}^n$ as a $\{0,1\}$-valued column vector that has a one on position i if and only $x_i\in S$.

We end this section with a first example of an incidence geometry.

Definition 1.1.4. A $t-(v,k,\lambda)$ *design*, $v>k>1, k\geq t\geq 1, \lambda>0$, is a point-line geometry $\mathcal{D}=(\mathcal{P},\mathcal{B},\mathcal{I})$ with incidence matrix $\mathcal{I}$ with the following properties:

- $|\mathcal{P}|=v$,
- every element of $\mathcal{B}$ contains k points of $\mathcal{P}$,
- every set of t distinct points of $\mathcal{P}$ is contained in precisely λ different lines of $\mathcal{B}$,
- no two lines of $\mathcal{B}$ are incident with the same k points of $\mathcal{P}$.

In this thesis, we will often investigate $2-(v,k,\lambda)$ designs, or in short, 2-designs. For these designs, we give a classical result in design theory, which follows from the proof of Fisher's inequality by Bose [19].

Result 1.1.5. *The incidence matrix of a* 2*-design has full row rank over* $\mathbb{R}$.

1.2 Finite projective spaces

Consider the finite field $\mathbb{F}_q$ of order q, with $q=p^h$, p prime and $h>0$. Let $V(n+1,q)$ denote the vector space of dimension $n+1$ over $\mathbb{F}_q$: $V(n+1,q)=\mathbb{F}_q^{n+1}$.

Let $D(V)$ be the set of non-trivial subspaces of $V(n+1,q)$. Define the incidence relation $\mathcal{I}$ as follows: $(U,W)\in\mathcal{I}$ if $U\subseteq W$ or $W\subseteq U$. Let $\dim : D(V)\to\{0,1,\dots,n-1\}$ be the map such that $\dim(\pi)$ is the vector dimension of π minus one. Then the incidence geometry $(D(V),\{0,1,\dots,n-1\},\dim,\mathcal{I})$ is by definition the *projective space* corresponding with $V(n+1,q)$. This projective space has projective dimension n and is denoted by $\mathrm{PG}(n,q)$. Note that the projective dimension $\dim(\pi)$ of a subspace π is its vector dimension minus one. In this thesis we will always use the projective dimension for subspaces of a projective geometry. Recall that the subspaces of $\mathrm{PG}(n,q)$ of dimension 0 and 1 are the points and lines of the projective space. The subspaces of dimension $2,3$ and $n-1$ are called the planes, solids, and hyperplanes, respectively. We will consider the empty set as the subspace with dimension -1. Often, a k-dimensional subspace is called a k-space, and we will sometimes consider a k-space as its set of points.

In this thesis, we will count many objects. For the notation of these countings, we will use *Gaussian binomial coefficients* $\begin{bmatrix} a \\ b \end{bmatrix}_q$ for $a, b \in \mathbb{N} \setminus \{0\}$, $a \geq b$, and prime power $q \geq 2$:

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \frac{(q^a - 1) \cdots (q^{a-b+1} - 1)}{(q^b - 1) \cdots (q - 1)}.$$

Furthermore, we define $\begin{bmatrix} a \\ b \end{bmatrix}_q = 1$ if $b = 0$, and $\begin{bmatrix} a \\ b \end{bmatrix}_q = 0$ if $b < 0$ or $b > a$.

The Gaussian binomial coefficient $\begin{bmatrix} a \\ b \end{bmatrix}_q$ is equal to the number of b-spaces of the vector space $\mathbb{F}_q^a$, or in the projective context, the number of $(b-1)$-spaces in the projective space $\mathrm{PG}(a-1, q)$. Moreover, we will denote the number $\begin{bmatrix} n+1 \\ 1 \end{bmatrix}_q$ of points in $\mathrm{PG}(n, q)$ by the symbol $\theta_n(q)$. If the field size q is clear from the context, we will write $\begin{bmatrix} a \\ b \end{bmatrix}$ and θ_n instead of $\begin{bmatrix} a \\ b \end{bmatrix}_q$ and $\theta_n(q)$, respectively.

The *intersection* of two subspaces U and W of $\mathrm{PG}(n, q)$, is the subspace of $\mathrm{PG}(n, q)$ containing all points that are contained in both U and W, and is denoted by $U \cap W$. The *span* of two subspaces U and W of $\mathrm{PG}(n, q)$, is the smallest subspace of $\mathrm{PG}(n, q)$ containing the points of both U and W, and is denoted by $\langle U, W \rangle$.

A frequently used identity in this thesis is the *Grassmann identity* for subspaces of a projective space:

$$\dim(U) + \dim(V) = \dim(\langle U, V \rangle) + \dim(U \cap V),$$

for all subspaces U and V of $\mathrm{PG}(n, q)$.

We started introducing projective spaces by using vector spaces. On the other side, we want to mention that a projective space can also be defined by axioms. A projective space is a point-line geometry $(\mathcal{P}, \mathcal{B}, \mathcal{I})$ that satisfies the following three axioms.

1. Through every two points of $\mathcal{P}$, there is exactly one line of $\mathcal{B}$.
2. If P, Q, R, S are distinct points of $\mathcal{P}$ and the lines PQ and RS intersect, then so do the lines PR and QS.
3. There are at least 3 points on a line.

Veblen and Young proved in [111] that if the dimension of the projective space is at least 3, then every finite projective space (defined by the three axioms above) of dimension $n \geq 3$, is derived from a vector space, and so, it is isomorphic with $\mathrm{PG}(n, q)$, with q a prime power.

For finite projective planes, the classification is more complicated, as not all of them are isomorphic to $\mathrm{PG}(2, q)$. We continue with the definition of a Desarguesian plane.

Definition 1.2.1. A *Desarguesian plane* is an (axiomatic) projective plane Π such that for all two triangles of points P_1, P_2, P_3 and Q_1, Q_2, Q_3 in Π, with the property that the lines P_1Q_1, P_2Q_2 and P_3Q_3 are concurrent, it holds that points $P_1P_2 \cap Q_1Q_2$, $P_2P_3 \cap Q_2Q_3$ and $P_1P_3 \cap Q_1Q_3$ are collinear.

The Desarguesian planes are precisely the planes coming from a three-dimensional vector space over a division ring, see [70]. Since we know, by Wedderburn [88], that a finite division ring is a (finite) field, it follows that a finite Desarguesian projective plane is a projective plane $\mathrm{PG}(2, q)$.

Many non-Desarguesian projective planes are known, for example the *Hall planes*, *Moulton planes* and *Figueroa planes*, see [75].

In this thesis we will only consider the projective spaces coming from a vector space.

1.3 Collineations of $\mathrm{PG}(n,q)$

A *linear map* on a vector space $V = V(n+1,q)$ is a mapping $f_A : V \to V : x \mapsto Ax$, with A a non-singular $(n+1) \times (n+1)$-matrix over $\mathbb{F}_q$. We identify this matrix with the corresponding linear map. The set of all linear maps on $V(n+1,q)$ corresponds to the set of all non-singular $(n+1)\times(n+1)$-matrices over $\mathbb{F}_q$ and they form the *general linear group*, denoted by $\mathrm{GL}(n+1,q)$.

A *semi-linear map* on a vector space $V = V(n+1,q)$ is a mapping $f_{A,\sigma} : V \to V : x \mapsto Ax^{\sigma}$, with $x \in V$ again a column vector, A a non-singular $(n+1) \times (n+1)$-matrix over $\mathbb{F}_q$ and σ an automorphism of the field $\mathbb{F}_q$. The automorphisms of the field $\mathbb{F}_q$, $q = p^r$, p prime, are precisely the maps $\phi^k : \mathbb{F}_{p^r} \to \mathbb{F}_{p^r} : x \mapsto x^{p^k}, 0 \leq k < r$. The group of all semi-linear maps on $V(n+1,q)$ is denoted by $\Gamma\mathrm{L}(n+1,q)$.

An automorphism of the projective space $\mathrm{PG}(n,q), n \geq 2$, is called a *collineation*. The set of all collineations of $\mathrm{PG}(n,q)$ forms the group $\mathrm{Aut}(\mathrm{PG}(n,q))$. Let $V(n+1,q)$ be the corresponding vector space of the projective space $\mathrm{PG}(n,q)$. The *fundamental theorem of projective geometry* states that each collineation of $\mathrm{PG}(n,q), n \geq 2$, arises from an invertible semi-linear map $f_{A,\sigma}$ of the points of $\mathrm{PG}(n,q)$ (and so of the 1-dimensional subspaces of $V = V(n+1,q)$): $f_{A,\sigma} : V \to V : x \mapsto Ax^{\sigma}$. The set of semi-linear maps on $\mathrm{PG}(n,q)$ forms a group and is denoted by $\mathrm{P}\Gamma\mathrm{L}(n+1,q)$. Hence, it follows that $\mathrm{P}\Gamma\mathrm{L}(n+1,q) \simeq \mathrm{Aut}(\mathrm{PG}(n,q))$. If we consider a linear map on $V(n+1,q)$, then the corresponding collineation of $\mathrm{PG}(n,q)$ is called a *projectivity*. The group of all projectivities of $\mathrm{PG}(n,q)$ is called the *projective (general) linear group* $\mathrm{PGL}(n+1,q)$.

A *perspectivity* of $\mathrm{PG}(n,q)$ with *axis* the hyperplane H is an element of $\mathrm{P}\Gamma\mathrm{L}(n+1,q)$ that fixes all points of H. Let α be a perspectivity of $\mathrm{PG}(n,q)$ with axis H, then a point P is called a *center* if α fixes every hyperplane through P. It can be proven that every perspectivity, different from the identity map, contains precisely one axis and precisely one center.

An *elation* with *axis* a hyperplane H and *center* a point P of $\mathrm{PG}(n,q)$ is a perspectivity whose center is contained in its axis; $P \in H$.

1.4 Affine geometries

Definition 1.4.1. Let H_∞ be a hyperplane of an n-dimensional projective space $\mathrm{PG}(n,q)$, and let Δ_A be the set of subspaces of $\mathrm{PG}(n,q)$ that are not contained in H_∞. Let $\mathcal{I}_A$ and $\dim_A$ be the restriction of the natural incidence relation and the type map of $\mathrm{PG}(n,q)$ to Δ_A, respectively. Then the incidence geometry using the subspaces of Δ_A, the type map $\dim_A$ and the incidence relation $\mathcal{I}_A$ defines the n-dimensional affine space $\mathrm{AG}(n,q)$. We call H_∞ the *hyperplane at infinity* of $\mathrm{AG}(n,q)$.

We introduced affine geometries by using projective geometries. The affine spaces used in this thesis, will always arise from a vector space. We want to note that, similar to the projective spaces, affine spaces can also be defined by axioms, see Theorem 2.4 and Theorem 2.6 in [73] for dimension 2 and dimension $n \geq 3$, respectively. Similar to the projective space, every axiomatic affine space of dimension n arise from a vector space for $n \geq 3$. For $n = 2$ this is not the case.

1.5 Finite classical polar spaces

Finite classical polar spaces play an important role in finite geometries. We start introducing these structures in vector spaces, but we will translate them to projective spaces later. Let $\mathbb{F}$ be a field, and let σ be a field automorphism. Let V be a vector space over $\mathbb{F}$. A *sesquilinear form* is a map $f : V \times V \to \mathbb{F}$ that is linear in its first argument and semi-linear in its second argument, hence for all $u_1, u_2, v_1, v_2 \in V, a \in \mathbb{F} : f(au_1 + u_2, v_1) = af(u_1, v_1) + f(u_2, v_1)$ and $f(u_1, av_1 + v_2) = a^\sigma f(u_1, v_1) + f(u_1, v_2)$. A *bilinear form* is a map $f : V \times V \to \mathbb{F}$ that is linear in both arguments. A *quadratic form* Q on a vector space V is a map $Q : V \to F$ that is homogeneous of degree two, and with the property that $f : V \times V \to \mathbb{F} : (v, w) \mapsto Q(v + w) - Q(v) - Q(w)$ is a bilinear form.

A sesquilinear form f on V is *reflexive* if $f(u, v) = 0$ implies that $f(v, u) = 0,\ \forall u, v \in V$. It is called *symplectic* if $f(v, v) = 0,\ \forall v \in V$, and called *Hermitian* if the corresponding field automorphism σ is a non-trivial involution, so σ^2 is the identity, and if $f(v, w) = f(w, v)^\sigma, \forall v, w \in V$. We note that every non-trivial reflexive sesquilinear form is a bilinear form or a non-zero scalar multiple of a Hermitian form.

A reflexive sesquilinear form f is called *degenerate* if there exists a vector $v \in V \backslash \{0\}$ with $f(v, w) = 0, \forall w \in V$. A quadratic form is *degenerate* if there exists a vector $v \in V \setminus \{0\}$ with $Q(v) = 0$ and with $f(v, w) = 0, \forall w \in V$.

A subspace is called *totally isotropic* with respect to a sesquilinear or quadratic form, when the form is trivial on this subspace.

Now we are able to describe the *classical polar spaces.*

Definition 1.5.1. Let Δ be the set of subspaces in a vector space $V(n + 1, \mathbb{F})$, that are totally isotropic with respect to a quadratic, symplectic or Hermitian form on V, and let d be the maximum of the vector dimensions of the elements of Δ. Furthermore, let $\mathcal{I}_\mathcal{P}$ be the restriction of the natural incidence relation of $V(n + 1, \mathbb{F})$ to Δ, and let $\dim_\mathcal{P}$ be the map such that $\dim_\mathcal{P}(\pi)$ is the vector dimension of π minus one. Then, the incidence geometry $\mathcal{P} = (\Delta, \{0, 1, \ldots, d - 1\}, \dim_\mathcal{P}, \mathcal{I}_\mathcal{P})$ is a *classical polar space.*

These classical polar spaces can be seen as substructures in the projective geometry $\mathrm{PG}(n, \mathbb{F})$. If $\mathbb{F}$ is the finite field $\mathbb{F}_q$, then, these polar spaces are called *the finite classical polar spaces.* Note that we will always consider the finite classical polar spaces through their embedding in the projective space.

In this thesis, all polar spaces we will handle are finite classical polar spaces, so we will refer to them as the polar spaces. Although there is a broad theory linked to these geometrical structures, we will briefly discuss the most important properties and definitions, which will be of importance in the following chapters. For an extensive introduction to finite classical polar spaces, we refer to [74].

A polar space arising from a quadratic form is called a *quadric.* Consider a non-degenerate quadratic form Q on the vector space $V = V(n + 1, q)$. If n is even, we can find an appropriate basis for V, so that Q can be written as

$$Q(X_0, \ldots, X_n) = X_0^2 + X_1X_2 + \cdots + X_{n-1}X_n.$$

This non-degenerate quadratic form is called *parabolic*. If n is odd, then we can again, by using an appropriate basis for V, write Q as

$$Q(X_0, \dots, X_n) = X_0X_1 + X_2X_3 + \cdots + X_{n-1}X_n, \tag{1.1}$$

$$\text{or as } Q(X_0, \dots, X_n) = X_0X_1 + X_2X_3 + \cdots + X_{n-3}X_{n-2} + h(X_{n-1}, X_n), \tag{1.2}$$

with h an irreducible homogeneous polynomial over $\mathbb{F}_q$ of degree 2. The non-degenerate quadratic form in (1.1) is called *hyperbolic* ; and the non-degenerate quadratic form in (1.2) is called *elliptic*. The polar spaces arising from a non-degenerate parabolic, hyperbolic or elliptic quadratic form are called a non-degenerate *parabolic, hyperbolic or elliptic quadric*, respectively. Embedded in $\mathrm{PG}(n, q)$, they are denoted by $Q(n, q)$, $Q^+(n, q)$ and $Q^-(n, q)$ respectively.

A polar space arising from a symplectic form is called a *symplectic polar space*. A non-degenerate symplectic form f on $V(m, q)$ only exists if m is even. Let $m = 2n$, then we can find an appropriate basis $\{e_1, \dots, e_n, e'_1, \dots, e'_n\}$ for $V(2n, q)$, so that $f(e_i, e_j) = f(e'_i, e'_j) = 0$ and $f(e_i, e'_j) = \delta_{i,j}$, $\forall i, j \in \{1, 2, \dots, n\}$. Embedded in $\mathrm{PG}(2n-1, q)$, this symplectic polar space is denoted by $W(2n-1, q)$. Note that a symplectic polar space contains all points of $\mathrm{PG}(2n-1, q)$, but not all subspaces of dimension at least one.

A polar space arising from a Hermitian form is called a *Hermitian polar space*. The construction of a Hermitian form over $\mathbb{F}_{q'}$ requires an involutory field automorphism of $\mathbb{F}_{q'}$, which only exists for q' a square, $q' = q^2$. The only involutory field automorphism of $\mathbb{F}_{q^2}$ is the map $\sigma : \mathbb{F}_{q^2} \to \mathbb{F}_{q^2} : x \mapsto x^q$. Let f be a non-degenerate Hermitian form on the vector space $V(n+1, q^2)$. An appropriate basis $\{e_0, ..., e_n\}$ for $V(n+1, q^2)$ can be found, such that $f(e_i, e_j) = \delta_{i,j}, \forall i, j \in \{0, 1, 2, \dots, n\}$.

Note that quadrics and Hermitian varieties are completely determined by their point sets, and can be described as a set of points satisfying the corresponding quadratic or Hermitian form. This is not the case for the symplectic polar spaces.

We continue with the definition of the rank and the parameter of a polar space.

Definition 1.5.2. A *generator* of a polar space is a subspace of maximal dimension and the rank d of a polar space is the projective dimension of a generator plus 1. The parameter e of a polar space $\mathcal{P}$ of rank d over $\mathbb{F}_q$ is defined as the number so that the number of generators through a $(d-2)$-space of $\mathcal{P}$ equals $q^e + 1$.

In Table 1.1, we give the parameter e of the polar spaces of rank d.

Polar space	e
$Q^+(2d-1, q)$	0
$H(2d-1, q)$	1/2
$W(2d-1, q)$	1
$Q(2d, q)$	1
$H(2d, q)$	3/2
$Q^-(2d+1, q)$	2

Table 1.1: The parameter e of the polar spaces

Another important notion are the polarities associated to a polar space. Consider a non-degenerate Hermitian form, or the bilinear form f, based on a non-degenerate quadratic form Q on the vector space $V = V(n+1, q)$. Recall that $f(v, w) = Q(v+w) - Q(v) - Q(w)$. For a subspace W of V, we can define its orthogonal complement regarding f:

$$W^{\perp} = \{v \in V \mid \forall w \in W : f(v, w) = 0\}.$$

If we see the subspaces of V as subspaces of $\mathrm{PG}(n, q)$, then the map β that maps the subspace W onto the subspace $W^{\perp}$, is an involutory duality. This map β is called a polarity. For q odd, the subspaces of a quadric or Hermitian variety in $\mathrm{PG}(n, q)$ are precisely the subspaces that are contained in their image under the polarity. Geometrically, for q odd, the image of a subspace on the polar space under the corresponding polarity, is its tangent space.

Consider now a quadric or a Hermitian variety $\mathcal{F} \subseteq \mathrm{PG}(n, q)$. A *tangent line* in a point P to $\mathcal{F}$ is a line ℓ through this point such that $\ell \cap \mathcal{F}$ is $\{P\}$ or the whole line ℓ. A point $P \in \mathrm{PG}(n, q)$ is *singular* for $\mathcal{F}$, if every line through P is a tangent line, or equivalently, if for every line ℓ through P: $\ell \cap \mathcal{F} = \{P\}$ or $\ell \cap \mathcal{F} = \ell$. The polar space $\mathcal{F}$ is *singular* if it contains a singular point. For a non-singular point P of $\mathcal{F}$, we define the tangent space as the union of the tangent lines of $\mathcal{F}$ in P. This tangent space forms a hyperplane, which we call the tangent hyperplane $T_P(\mathcal{F})$ in P. For q odd, this tangent hyperplane is the image of the point P under the corresponding polarity, as mentioned above.

It is known that all singular points of a singular quadric or Hermitian variety $\mathcal{F}$ form a subspace. In this case, $\mathcal{F}$ is a *cone* $\pi_{n-r-1}\mathcal{F}'$. The *vertex* π_{n-r-1} of this cone is the $(n-r-1)$-space of singular points of $\mathcal{F}$, $n > r$, and the *basis* of the cone is a non-singular quadric or Hermitian variety (depending on the type of $\mathcal{F}$), in a subspace $\mathrm{PG}(r, q)$ of $\mathrm{PG}(n, q)$ that is disjoint from π_{n-r-1}.

A symplectic polar space can also be singular. Similar to the quadrics and Hermitian varieties, a singular symplectic polar space in $\mathrm{PG}(n, q)$ is a cone. The vertex of this cone is an s-dimensional subspace π_s, and the basis of the cone is a non-singular symplectic polar space in an $(n-s-1)$-dimensional subspace, disjoint from π_s. Note that $n-s-1$ must be odd, since non-singular symplectic polar spaces only exist in a projective space with odd dimension. The singular points of a singular symplectic polar space are the points contained in the vertex of the cone. For more information, we refer to [73, 74].

We continue with some important counting results and remarks on some specific finite classical polar spaces.

Lemma 1.5.3 ([23, Lemma 9.4.1]). *The number of k-spaces in a finite classical polar space $\mathcal{F}$ of rank d and with parameter e, embedded in a projective space over the field $\mathbb{F}_q$, is given by*

$$\begin{bmatrix} d \\ k+1 \end{bmatrix} \prod_{i=1}^{k+1} (q^{d+e-i} + 1).$$

Hence, the number of points in $\mathcal{F}$ is $\begin{bmatrix} d \\ 1 \end{bmatrix} (q^{d+e-1} + 1)$. The number of generators in $\mathcal{F}$ is $\prod_{i=1}^{d} (q^{d+e-i} + 1)$.

Example 1.5.4. *The non-singular parabolic quadric $Q(2, q)$ is a set of $q+1$ points in a plane $\mathrm{PG}(2, q)$, such that no three points are collinear. This parabolic quadric is also called a* conic.

Remark 1.5.5. For q even, there exists a special point N, not belonging to the parabolic quadric $Q(2k, q)$, $k \geq 1$, such that every line through N in $\mathrm{PG}(2k, q)$ meets the quadric in a unique point. Hence, every such line is a tangent line to the quadric. This point N is called the *nucleus* of the quadric.

Example 1.5.6. *Consider a hyperbolic quadric* $Q = Q^{+}(2n+1,q)$*. The set of generators* Ω *of* Q *can be partitioned into two equivalence classes* Ω_1 *and* Ω_2*. The corresponding equivalence relation* $\sim$ *is defined as follows:* $\pi_1 \sim \pi_2 \Leftrightarrow \dim(\pi_1 \cap \pi_2) \equiv n \ (mod\, 2)$*, for any two generators* π_1 *and* π_2 *in* $Q^{+}(2n+1,q)$*. The two equivalence classes* Ω_1 *and* Ω_2 *are called the Latin and Greek generators. In Section 1.6, we will see that for* $n = 1$*, the equivalence classes in* $Q^{+}(3,q)$ *are two opposite reguli.*

Remark 1.5.7 *([74]).* The polar spaces $Q(2d,q)$ and $W(2d-1,q)$ are isomorphic for q even. We find $W(2d-1,q)$, for q even, by a projection of $Q(2d,q)$ from the nucleus N of $Q(2d,q)$ to a hyperplane not through N in the ambient projective space $\mathrm{PG}(2d,q)$. In this way, there is a one-to-one connection between the generators of $W(2d-1,q)$ and the generators of $Q(2d,q)$.

We finish this section with the *Klein correspondence*, which is a map from the lines of $\mathrm{PG}(3,q)$ to the points of the hyperbolic quadric $Q^{+}(5,q)$.

Definition 1.5.8. Let l be a line in $\mathrm{PG}(3,q)$, and let $Y(y_0,y_1,y_2,y_3)$ and $Z(z_0,z_1,z_2,z_3)$ be two different points of l. The ordered set $(p_{01},p_{02},p_{03},p_{23},p_{31},p_{12})$, with

$$p_{ij} = y_i z_j - y_j z_i,$$

is called the set of *Plücker coordinates* of l. The *Klein correspondence* maps a line l to the point P_l in $\mathrm{PG}(5,q)$, such that the set of coordinates of P_l is $(p_{01},p_{02},p_{03},p_{23},p_{31},p_{12})$.

Note that all points P_l in $\mathrm{PG}(5,q)$, with l a line in $\mathrm{PG}(3,q)$, are contained in the hyperbolic quadric $Q^{+}(5,q)$, defined by the equation $x_0x_3 + x_1x_4 + x_2x_5 = 0$. We also denote this quadric by the Klein quadric. This correspondence has the advantage that constructions in $\mathrm{PG}(3,q)$ can lead to good constructions of subspaces in $\mathrm{PG}(5,q)$. In Section 10.5, we use this correspondence to give a new, non-trivial Cameron-Liebler example in $Q^{+}(5,q)$.

In Table 1.2, we give an overview of the most important correspondences.

$\mathbf{PG(3,q)}$	$\mathbf{Q^{+}(5,q)}$
Line	Point
Two intersecting lines	Two points, contained in a common line
The set of lines through a fixed point P and in a fixed plane π with $P \in \pi$	Line
The set of lines in a fixed plane	Greek plane
The set of lines through a fixed point	Latin plane
Lines in a regulus	Points of a conic, not contained in a Latin or Greek plane
Lines of a hyperbolic quadric	Points of two conics, contained in two planes that are each others image under the polarity of $Q^{+}(5,q)$.

Table 1.2: The image of sets of subspaces under the Klein correspondence.

1.6 Arcs, reguli, spreads and pencils

A line meeting a point set $\mathcal{A}$ in $0, 1$ or 2 points, is called an *external line,* a *tangent line* or a *bisecant* to $\mathcal{A}$, respectively. In general, a line, meeting $\mathcal{A}$ in i points, is called an i*-secant.*

Definition 1.6.1. A set $\mathcal{S}$ of k-spaces in $\mathrm{PG}(n,q)$, $\mathrm{AG}(n,q)$ or in a polar space $\mathcal{P}$, that pairwise have no point in common, is called a *partial k-spread* in $\mathrm{PG}(n,q)$, $\mathrm{AG}(n,q)$ or $\mathcal{P}$ respectively. If $\mathcal{S}$ cannot be extended to a larger partial k-spread, then $\mathcal{S}$ is called *maximal.* A partial k-spread $\mathcal{S}$ such that every point of $\mathrm{PG}(n,q)$, $\mathrm{AG}(n,q)$ or $\mathcal{P}$ is contained in an element of $\mathcal{S}$, is called a *k-spread.* The elements of a $(d-1)$-spread in a polar space $\mathcal{P}$ of rank d are generators of $\mathcal{P}$. A $(d-1)$-spread is also called a *spread* in $\mathcal{P}$. For $k=1$, a (partial) k-spread is called a (partial) line spread.

It is known that not every projective space $\mathrm{PG}(n,q)$ contains a k-spread.

Theorem 1.6.2 ([109]). *There exists a k-spread in $\mathrm{PG}(n,q)$ if and only if $k+1$ is a divisor of $n+1$.*

Since $\mathrm{PG}(n,q)$ contains $\frac{q^{n+1}-1}{q-1}$ points, and a k-space contains $\frac{q^{k+1}-1}{q-1}$ points, it follows that a k-spread only can exist if $k+1$ is a divisor of $n+1$. It is also a sufficient condition, which follows from the construction of a Desarguesian spread, see for example [73, Theorem 4.1]. For this construction, *field reduction* is used to determine the spread elements. Let $r=\frac{n+1}{k+1}$. The points of $\mathrm{PG}(r-1,q^{k+1})$ correspond to 1-dimensional subspaces of $V(r,q^{k+1})$. By considering this vector space over $\mathbb{F}_q$, we obtain a vector space isomorphic to $V(r(k+1),q)=V(n+1,q)$, such that the 1-dimensional subspaces of $V(r,q^{k+1})$ correspond to $(k+1)$-dimensional subspaces of $V(n+1,q)$. This is the concept of *field reduction.* In this way, the point set of $\mathrm{PG}(r-1,q^{k+1})$ corresponds to a set $\mathcal{D}$ of k-dimensional subspaces of $\mathrm{PG}(n,q)$, which partitions the point set of $\mathrm{PG}(n,q)$. Hence, these subspaces form a k-spread in $\mathrm{PG}(n,q)$. More specifically, this set $\mathcal{D}$ is called a *Desarguesian spread*, and we have a one-to-one correspondence between the points of $\mathrm{PG}(r-1,q^{k+1})$ and the elements of $\mathcal{D}$.

We will also introduce *regular spreads.* For this, we first give the definition of a regulus.

Definition 1.6.3. A *regulus* in $\mathrm{PG}(2k+1,q)$ is a set $\mathcal{S}$ of $q+1$ pairwise disjoint k-spaces, such that every line that meets three elements of $\mathcal{S}$, meets all elements of $\mathcal{S}$.

It is known that every three pairwise disjoint k-spaces S_1,S_2,S_3 in $\mathrm{PG}(2k+1,q)$ are contained in a unique regulus, see [72, Lemma 15.1.1, Theorem 15.3.12]. For $k=1$, a regulus consists of $q+1$ lines in $\mathrm{PG}(3,q)$. For every three lines l_1,l_2,l_3 in a regulus R, the $q+1$ lines, meeting l_1,l_2 and l_3, also form a regulus, which we call the *opposite regulus.* A regulus and its opposite regulus in $\mathrm{PG}(3,q)$ form a hyperbolic quadric $Q^+(3,q)$, see Section 1.5.

Definition 1.6.4. A k-spread $\mathcal{S}$ in $\mathrm{PG}(2k+1,q)$ is *regular* if for every three elements S_1,S_2,S_3 in $\mathcal{S}$, it holds that all k-spaces of the regulus, determined by these subspaces, are also contained in $\mathcal{S}$.

For $q=2$, every k-spread in $\mathrm{PG}(2k+1,2)$ is regular. For $q>2$, a spread $\mathcal{S}$ is regular if and only if $\mathcal{S}$ is Desarguesian [25].

Definition 1.6.5. A k-spread $\mathcal{S}$ in $\mathrm{PG}(r(k+1)-1,q)$ is *normal* if the subspace spanned by any two spread elements is partitioned by elements of $\mathcal{S}$.

For $r\leq 2$, every k-spread in $\mathrm{PG}(r(k+1)-1,q)$ is normal. For $r>2$, it can be proven that $\mathcal{S}$ is normal, if and only if $\mathcal{S}$ is Desarguesian, see [4].

Definition 1.6.6. A *k-arc* in $\mathrm{PG}(n,q)$ is a set of k points such that every subset of $n+1$ points spans the whole space $\mathrm{PG}(n,q)$. A k-arc is called *complete* if it is not contained in a $(k+1)$-arc.

It is known that an arc in $\mathrm{PG}(2,q)$ has at most $q+1$ elements for q odd, and at most $q+2$ elements for q even, see [101]. A $(q+1)$-arc in $\mathrm{PG}(2,q)$ is called an *oval* and a $(q+2)$-arc a *hyperoval.* A hyperoval can only exist for q even. In this case, a hyperoval is a complete arc. For q odd, an oval is a complete arc. It can be proven that, for q even, every oval is contained in a hyperoval, and hence, is not complete [18]. For q even, the $q+1$ tangent lines to an oval are concurrent, see [18]. The intersection point of the tangent lines is called the *nucleus* of the oval. In this case, for q even, the union of an oval and its nucleus is a hyperoval.

It is clear that a non-singular parabolic quadric in $\mathrm{PG}(2,q)$, so a conic $Q(2,q)$, is an oval, see Example 1.5.4. Moreover, Segre [106] could prove the converse for q odd.

Theorem 1.6.7 ([106]). *Every oval in* $\mathrm{PG}(2,q)$*,* q *odd, is a conic.*

For $\mathrm{PG}(2,q)$, q even, this result is not true. A counterexample for this can be found by considering a hyperoval which is a conic together with its nucleus. If we delete a point, different from the nucleus, then we find an oval. This set is not a conic if $q \geq 8$.

In an unpublished manuscript from Penttila, a characterisation for ovals in $\mathrm{PG}(2,q^2)$, q even, is given.

Result 1.6.8 ([98]). *Let* O *be an oval of* $\mathrm{PG}(2,q^2)$*,* q *even. Then* O *is a conic if and only if every triple of distinct points of* O*, together with the nucleus of* O*, lies in a Baer subplane that meets* O *in* $q+1$ *points.*

A set $\mathcal{S}$ of points in $\mathrm{PG}(2,q)$ is called a *translation* set, with respect to a line ℓ, if the group of elations with axis ℓ, fixing $\mathcal{S}$, acts transitively on the points of $\mathcal{S} \setminus \ell$. The line ℓ is called the *translation line.* If a hyperoval H in $\mathrm{PG}(2,q)$ is a translation set, then it is called a *translation hyperoval.* To avoid the trivial and special cases, we suppose that $q = 2^h, h > 2$. It is known that the translation line must be a bisecant of H, and that every translation hyperoval in $\mathrm{PG}(2,q)$ is PGL-equivalent to the point set $H_i = \{(1,t,t^{2^i}) | t \in \mathbb{F}_q\} \cup \{(0,1,0),(0,0,1)\}$, for a certain $i < \frac{h}{2}$ and $\gcd(i,h) = 1$ (see e.g. [73, Theorem 8.5.4], [97]). For $i = 1$, the hyperoval H_i corresponds to a conic and its nucleus. All hyperovals, equivalent with H_1, are called *regular.* In this case, every bisecant of H_1, through the nucleus of the conic, is a translation line for the hyperoval, and so the translation line is not unique.

The hyperovals H_i, with $1 < i < \frac{h}{2}$, were the first examples of irregular hyperovals, and were determined by Segre in [107]. The translation line of these hyperovals is unique: $\ell : X = 0$. In this case, the group G of elations with axis the line ℓ, that fixes H_i, is the translation group containing all elements of the form

$$M_a = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ a^{2^i} & 0 & 1 \end{bmatrix},$$

with $a \in \mathbb{F}_q$. From this representation of the group, it is clear that $G \cong (\mathbb{F}_q, +)$.

Ovoids can be defined in several incidence geometries, but in this thesis, we only use them in the context of polar spaces.

Definition 1.6.9. A *partial ovoid* in a polar space $\mathcal{P}$ is a set of points in $\mathcal{P}$ such that each generator contains at most one point of this set. It is called an *ovoid* if each generator contains precisely one point of the set.

To end this section, we also give the definition of a pencil and a sunflower in $\mathrm{PG}(n,q)$, in $\mathrm{AG}(n,q)$ and in a polar space $\mathcal{P}$.

Definition 1.6.10. The set of all k-spaces through a fixed t-space τ, $k \geq t$, is called a t*-pencil* of k-spaces with vertex τ, and, in particular, a *point-pencil* if $t = 0$ and a *line-pencil* if $t = 1$.

Note that for all k-spaces U, V in a t-pencil with vertex τ, it holds that $\tau \subseteq U \cap V$. In this thesis, we will always use the notation *vertex*, except in Chapter 6. In this chapter, graphs are involved, and to avoid confusion, we will denote the vertex of a point-pencil by the *base point*.

We use the notation *Star*(P) for all lines through the point P, *Lines*(π) for all lines in the subspace π, and *Pencil*(P, π) for all lines through the point P contained in the subspace π.

Definition 1.6.11. A *sunflower* $\mathcal{S}$, with vertex τ, is a set of subspaces through τ, such that for every two distinct subspaces $U, V \in \mathcal{S}$ it holds that $U \cap V = \tau$.

1.7 Graph theory

1.7.1 General graph theory

In this thesis, we will use graphs to model some incidence geometries.

Definition 1.7.1. A *graph* $\Gamma = (V(\Gamma), E(\Gamma))$ consists of a set $V(\Gamma)$ of *vertices* and a set $E(\Gamma)$ of unordered pairs of $V(\Gamma)$, which are called *edges*. If we only use one graph Γ, then we use the notation V and E, instead of $V(\Gamma)$ and $E(\Gamma)$. A vertex v and an edge e are *incident* if the vertex v is contained in the edge e. Two vertices v and w are *adjacent* if there is an edge containing both vertices. We denote this by $v \sim w$. The vertices adjacent to a fixed vertex v are called the *neighbours* of v. Two edges are *adjacent* if they have a vertex in common.

Definition 1.7.2. A *path* of length l, from a vertex v_0 to a vertex v_l in a graph Γ is a sequence of (distinct) vertices $(v_0, v_1, v_2, \ldots, v_{l-1}, v_l)$, such that the vertices v_{i-1} and v_i are adjacent for all $i, 1 \leq i \leq l$. The *distance* $d(x, y)$ between two vertices x and y is the minimal length of a path $(v_0, \ldots, v_l)$ with $v_0 = x, v_l = y$. For a given vertex $v \in V$, the set of vertices in Γ at distance i from v is denoted by $\Gamma_i(v)$. A graph Γ is *connected* if there exists a path between every two vertices of Γ. The maximal distance that occurs between two vertices of a connected graph Γ is called the *diameter* of the graph.

In this thesis, we suppose that every pair of vertices can be contained in at most one edge and that every edge contains two different vertices. We also suppose that every two vertices can be connected by a path. In other words, we will only consider connected, simple graphs.

Definition 1.7.3. The *degree* of a vertex v in a graph $\Gamma = (V, E)$ is the number of vertices in V adjacent with v, or equivalently the number of edges in E that are incident with v. The graph Γ is k*-regular*, or *regular of degree* $k \in \mathbb{N}$ if every edge of E has degree k.

Let d be the diameter of Γ. If there exist integers $c_1, \ldots, c_d, a_0, \ldots, a_d, b_0, \ldots, b_{d-1}$, such that for all vertices v and w in V, we have that

- $a_i = |\Gamma_i(v) \cap \Gamma_1(w)|$ if $i = d(v, w)$,
- $b_i = |\Gamma_{i+1}(v) \cap \Gamma_1(w)|$ if $i = d(v, w) < d$,
- $c_i = |\Gamma_{i-1}(v) \cap \Gamma_1(w)|$ if $i = d(v, w) > 0$,

then Γ is a *distance-regular graph* with *intersection array* $\{c_1, \ldots, c_d; a_0, \ldots, a_d; b_0, \ldots, b_{d-1}\}$.

Note that all distance-regular graphs are regular.

Definition 1.7.4. A graph Γ is *strongly regular* if Γ is k-regular and if there exist integers λ and $\mu > 0$ such that

- every two adjacent vertices have λ common neighbours,
- every two non-adjacent vertices have μ common neighbours.

1.7.2 Algebraic graph theory

We continue with introducing some aspects in algebraic graph theory. These topics will be useful in the context of Cameron-Liebler sets.

Let $\Gamma = (V, E)$ be a graph, and let $V = \{v_1, v_2, \dots, v_n\}, n \geq 1$.

Definition 1.7.5. The *adjacency matrix* of Γ is the matrix $A = (a_{ij})_{1 \leq i,j \leq n}$, with $a_{ij} = 1$ if the vertices v_i and v_j are adjacent and $a_{ij} = 0$ if the vertices v_i and v_j are non-adjacent. The elements a_{ii} are zero for all i.

Note that the adjacency matrix of a graph depends on the order of the vertices.

Definition 1.7.6. The *characteristic polynomial* of a graph Γ is the characteristic polynomial of its adjacency matrix A, i.e. the polynomial $p(\lambda) = \det(\lambda I_n - A)$. Likewise, the *eigenvalues* of Γ are the eigenvalues of its adjacency matrix, i.e. the (complex) roots of the characteristic polynomial of the graph Γ. If Γ is k-regular, then $A\boldsymbol{j} = k\boldsymbol{j}$, and so, we have that k is an eigenvalue of Γ. This eigenvalue is often called the *trivial eigenvalue*. The *multiplicity* of an eigenvalue is the algebraic multiplicity as a root of the characteristic polynomial. As A is a real symmetric matrix, we know that all eigenvalues of A, and so of Γ, are real.

We end with the definition of intriguing and tight sets, which have a strong link with Cameron-Liebler sets.

Definition 1.7.7. Let $\Gamma = (V, E)$ be a connected k-regular graph. A set Y of vertices of Γ is an *intriguing set* if there are integers y and y' such that every vertex of Y is adjacent to y' vertices of Y and every vertex of $V \setminus Y$ is adjacent to y vertices of Y.

Note that $\emptyset$ and V are examples of intriguing sets in $\Gamma = (V, E)$. An intriguing set, different from $\emptyset$ and V, is called non-trivial.

Lemma 1.7.8. *Let $\Gamma = (V, E)$ be a connected k-regular graph. A set Y of vertices, with $Y \neq \emptyset, V$, is intriguing if and only if its characteristic vector lies in the span of the all-one vector and an eigenvector v_θ of Γ such that $y' - y = \theta$.*

If θ is the largest or smallest non-trivial eigenvalue of Γ, then Y is called a *tight set* of type 1 or 2 respectively.

1.7.3 Graph colorings

Many problems in finite geometry can be translated to finding specific families or partitions of vertices in a certain graph. To see this, we start with the definition of a clique and coclique.

Definition 1.7.9. Let $\Gamma = (V, E)$ be a graph.

- A set $\mathcal{S}$ of vertices in V is called a *clique* if every two vertices in $\mathcal{S}$ are adjacent.
- A set $\mathcal{S}$ of vertices in V is an *independent set* if no two vertices in $\mathcal{S}$ are adjacent. An independent set is also called a *coclique*.

A clique or coclique is maximal if it is not contained in a larger clique or coclique, respectively. The size of the largest clique or coclique in a graph Γ is called the *clique number* $\omega(\Gamma)$ and *independence number* $\alpha(\Gamma)$ respectively.

We end this section on graphs with the definition of a coloring.

Definition 1.7.10. A *coloring* of a graph Γ is an assignment of colors to the vertices of Γ, such that every vertex has one color and such that adjacent vertices get different colors. The sets of vertices with the same color are called the *color classes*.

The *chromatic number* $\chi(\Gamma)$ of a graph Γ is the smallest number c such that there exists a coloring of Γ with c colors.

1.8 Tactical decompositions

The first exploration of Cameron-Liebler sets, by Cameron and Liebler [28], uses the theory of tactical decompositions. Tactical decompositions were first introduced by Dembowski [42]. This section is based on the notes in [38].

Definition 1.8.1. Let $(\mathcal{P}, \mathcal{B}, I)$ be an incidence geometry with $\mathcal{P}$ a set of points and $\mathcal{B}$ a set of blocks. Let $\{P_1, P_2, \dots, P_s\}, P_i \neq \emptyset$, be a partition of $\mathcal{P}$, and let $\{B_1, B_2, \dots, B_r\}, B_i \neq \emptyset$, be a partition of $\mathcal{B}$.

- If there exists an $(s \times r)-$matrix X with $|\{p \in P_i| \; p \; I \; b\}| = X_{ij}, \forall b \in B_j$, then the decomposition is called *block-tactical.*
- If there exists an $(s \times r)-$matrix Y with $|\{b \in B_i| \; p \; I \; b\}| = Y_{ij}, \forall p \in P_j$, then the decomposition is called *point-tactical.*

The decomposition is called *tactical* if it is both block- and point-tactical.

Lemma 1.8.2. *Let $(\mathcal{P}, \mathcal{B}, I)$ be an incidence geometry with $\mathcal{P}$ a set of points, $\mathcal{B}$ a set of blocks and A the point-block incidence matrix. Let $\{P_1, P_2, \dots, P_s\}, P_i \neq \emptyset$, be a partition of $\mathcal{P}$, and let $\{B_1, B_2, \dots, B_r\}, B_i \neq \emptyset$, be a partition of $\mathcal{B}$.*

- *If the partition is block-tactical with corresponding matrix X, then*

$$A^T \chi_{\mathcal{P}_i} = \sum_{l=1}^{r} X_{il} \chi_{\mathcal{B}_l}, \forall i \in \{1, \dots, s\}.$$

- *If the partition is point-tactical with corresponding matrix Y, then*

$$A \chi_{\mathcal{B}_i} = \sum_{l=1}^{s} Y_{lj} \chi_{\mathcal{P}_l}, \forall i \in \{1, \dots, r\}.$$

The action of (a subgroup of) the automorphism group of an incidence geometry gives rise to a tactical decomposition of the point- and block-set.

Lemma 1.8.3. *Let $(\mathcal{P}, \mathcal{B}, I)$ be an incidence geometry, with $\mathcal{P}$ the set of points and $\mathcal{B}$ the set of blocks. Consider a subgroup G of the automorphism group of $(\mathcal{P}, \mathcal{B}, I)$, with orbits $\{P_1, P_2, \dots, P_s\}$ on the points and orbits $\{B_1, B_2, \dots, B_r\}$ on the blocks. Then these partitions form a tactical decomposition.*

1.9 Association schemes

In this section, we give a short introduction on association schemes. We rely on [23, Chapter 2]. For more details, we refer to [22, Section 2], [23, Chapter 2] and [20].

Definition 1.9.1 ([22, Section 2.1]). Let X be a finite set of size n, whose members are known as vertices. A *d-class association scheme* is a pair $(X, \mathcal{R})$, where $\mathcal{R} = \{\mathcal{R}_0, \mathcal{R}_1, \dots, \mathcal{R}_d\}$ is a set of binary symmetric relations with the following properties:

1. $\{\mathcal{R}_0, \mathcal{R}_1, \dots, \mathcal{R}_d\}$ is a partition of $X \times X$,
2. $\mathcal{R}_0$ is the identity relation,
3. there are constants p^l_{ij} such that for all $(x, y) \in \mathcal{R}_l$, there are exactly p^l_{ij} elements $z \in X$ such that $(x, z) \in \mathcal{R}_i$ and $(z, y) \in \mathcal{R}_j$. These constants are called the *intersection numbers* of the association scheme.

Note that the association schemes defined above are sometimes also called symmetrical association schemes. Since the relations $\mathcal{R}_i$ are symmetric, we have that $p^l_{ij} = p^l_{ji}$, $\forall\, 0 \leq i, j, l \leq d$.

We now investigate the (binary) adjacency matrices A_i corresponding to the relations $\mathcal{R}_i$.

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x, y) \in \mathcal{R}_i, \\ 0 & \text{else.} \end{cases}$$

Property 1.9.2. *For all values $0 \leq i, j \leq d$, it holds that:*

1. $\sum_{i=0}^{d} A_i = J$,
2. $A_0 = I$,
3. $A_i = A_i^T$,
4. $A_i A_j = \sum_{k=0}^{d} p^k_{ij} A_k = A_j A_i$.

From the first property, it follows that the matrices A_i are linearly independent, and from the third and fourth property we find that these matrices generate a $(d+1)$-dimensional commutative algebra $\mathcal{A}$ of symmetric matrices, which is called the *Bose-Mesner algebra.*

Since the matrices A_i commute, they can be diagonalized simultaneously. This gives the following result, which was originally proven in [41].

Result 1.9.3. *Consider a d-class association scheme $(X, \mathcal{R})$, with adjacency matrices A_i corresponding to the relations $\mathcal{R}_i$, $0 \leq i \leq d$, and with $|X| = n$. Then, there is an orthogonal decomposition of $\mathbb{R}^n$ as a direct sum of $d + 1$ orthogonal eigenspaces of the matrices A_i, corresponding to the common eigenvectors. Hence, we have that $\mathbb{R}^n = V_0 \perp V_1 \perp \cdots \perp V_d$, with $V_0, \dots, V_d$ the common spaces of eigenvectors with associated eigenvalues P_{ji}, with P_{ji} the eigenvalue of A_i on V_j. Note that one of the spaces of eigenvectors, w.l.o.g. V_0, will be 1-dimensional since $J \in \mathcal{A}$ has eigenvalue n with multiplicity 1.*

Let $(\Delta_k, \mathcal{R})$ be an association scheme linked to a geometrical structure, such as a projective space, an affine space or a finite classical polar space. The elements Δ_k of the association scheme correspond to the k-spaces in the geometrical structure. For these schemes, a classical ordering of the eigenspaces $V_0, \dots, V_d$ is imposed; V_0 is the 1-dimensional eigenspace $\langle \boldsymbol{j} \rangle$ and V_1 is the eigenspace such that $\mathrm{im}(A^T) = V_0 \perp V_1$, with A the point-k-space incidence matrix.

We end this subsection with two well-known association schemes. For more information on these, we refer to [41], [23, Section 9.1 and 9.3] and [67, Section 6 and 9].

Example 1.9.4 (The Johnson scheme). *Let* X *be a finite set of size* n*, and let* $\mathcal{F}_k$*,* $k < n$*, be the set of all subsets of size* k*. The* Johnson graph $J(n,k)$ *is the graph whose vertices are the elements of* $\mathcal{F}_k$*, and two vertices are adjacent if they have* $k-1$ *elements in common. The relations of the corresponding association scheme are* $\mathcal{R}_i = \left\{(\Pi_1,\Pi_2) \in \mathcal{F}_k \times \mathcal{F}_k \middle| |\Pi_1 \cap \Pi_2| = k-i\right\}$*, with* $i \in \{0,\dots,k\}$.

Example 1.9.5 (The Grassmann scheme). *Consider the* n*-dimensional projective space* $\mathrm{PG}(n,q)$ *over the field* $\mathbb{F}_q$*, and let* Δ_k*,* $k < n$*, be the set of all* k*-dimensional subspaces. The* Grassmann graph $J_q(n+1,k+1)$ *is the graph whose vertices are the elements of* Δ_k*, and two vertices are adjacent if the corresponding subspaces intersect in a* $(k-1)$*-space. The relations of the corresponding association scheme are* $\mathcal{R}_i = \{(\pi_1,\pi_2) \in \Delta_k \times \Delta_k | \dim(\pi_1 \cap \pi_2) = k-i\}$*, with* $i \in \{0,\dots,k+1\}$*. This scheme is also called the* q-analogue *of the Johnson scheme.*

There are many other mathematical structures that can be linked to an association scheme, for example polar spaces, affine spaces and groups, see [114, Introduction]. In Chapter 10, we will often use the association schemes on the generators of finite classical polar spaces.

1.10 Useful countings and bounds

In this thesis, we will frequently use counting arguments to find classification results. For this, we will often use the following lemma.

Lemma 1.10.1 ([108, Section 170]). *The number of* j*-spaces disjoint from a fixed* m*-space in* $\mathrm{PG}(n,q)$ *equals* $q^{(m+1)(j+1)}\begin{bmatrix}n-m\\j+1\end{bmatrix}$.

Furthermore, we will use bounds on the Gaussian binomial coefficients found in [77, Lemma 2.1] and [78, Lemma 34, Lemma 37].

Lemma 1.10.2. *Let* $n \geq k \geq 0$.

1. *Let* $q \geq 3$*. Then* $\begin{bmatrix}n\\k\end{bmatrix} \leq 2q^{k(n-k)}$.
2. *Let* $q \geq 4$*. Then* $\begin{bmatrix}n\\k\end{bmatrix} \leq \left(1+\frac{2}{q}\right)q^{k(n-k)}$.
3. *Let* $q \geq 2$ *and* $n \geq 1$*. Then* $\theta_n \leq \frac{q^{n+1}}{q-1}$.
4. *Let* $n > k > 0$*. Then* $\begin{bmatrix}n\\k\end{bmatrix} \geq \left(1+\frac{1}{q}\right)q^{k(n-k)}$.

We end with another result on the Gaussian binomial coefficients. First, we formulate the (double) q-analogue of Pascal's rule:

Result 1.10.3 (Pascal's Rule).

$$q^b\begin{bmatrix}a-1\\b\end{bmatrix} + \begin{bmatrix}a-1\\b-1\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}a-1\\b\end{bmatrix} + q^{a-b}\begin{bmatrix}a-1\\b-1\end{bmatrix}. \tag{1.3}$$

Lemma 1.10.4. *For integers* a, b, c*, with* $0 \leq b, c \leq a$*, we have that*

$$\begin{bmatrix}a\\b\end{bmatrix} = \sum_{i=0}^{c}\begin{bmatrix}a-c\\b-i\end{bmatrix}\begin{bmatrix}c\\i\end{bmatrix}q^{(b-i)(c-i)}. \tag{1.4}$$

Proof. We use induction on c. For $c = 0$, the statement is trivial, so suppose that (1.4) is true for a value $c - 1$. Then we will prove that it is also true for the value c. We first use the left equality of (1.3). In the second last step, we use the right equality of (1.3).

$$\begin{aligned}
\begin{bmatrix} a \\ b \end{bmatrix} &= \sum_{i=0}^{c-1} \begin{bmatrix} a-c+1 \\ b-i \end{bmatrix} \begin{bmatrix} c-1 \\ i \end{bmatrix} q^{(b-i)(c-1-i)} \\
&= \sum_{i=0}^{c-1} \left(\begin{bmatrix} a-c \\ b-i \end{bmatrix} q^{b-i} + \begin{bmatrix} a-c \\ b-i-1 \end{bmatrix} \right) \begin{bmatrix} c-1 \\ i \end{bmatrix} q^{(b-i)(c-1-i)} \\
&= \sum_{i=0}^{c-1} \begin{bmatrix} a-c \\ b-i \end{bmatrix} \begin{bmatrix} c-1 \\ i \end{bmatrix} q^{(b-i)(c-i)} + \sum_{i=0}^{c-1} \begin{bmatrix} a-c \\ b-i-1 \end{bmatrix} \begin{bmatrix} c-1 \\ i \end{bmatrix} q^{(b-i)(c-1-i)} \\
&= \sum_{i=0}^{c-1} \begin{bmatrix} a-c \\ b-i \end{bmatrix} \begin{bmatrix} c-1 \\ i \end{bmatrix} q^{(b-i)(c-i)} + \sum_{j=1}^{c} \begin{bmatrix} a-c \\ b-j \end{bmatrix} \begin{bmatrix} c-1 \\ j-1 \end{bmatrix} q^{(b-j+1)(c-j)} \\
&= \begin{bmatrix} a-c \\ b \end{bmatrix} q^{bc} + \sum_{k=1}^{c-1} \begin{bmatrix} a-c \\ b-k \end{bmatrix} q^{(b-k)(c-k)} \left(\begin{bmatrix} c-1 \\ k \end{bmatrix} + \begin{bmatrix} c-1 \\ k-1 \end{bmatrix} q^{(c-k)} \right) + \begin{bmatrix} a-c \\ b-c \end{bmatrix} \\
&= \begin{bmatrix} a-c \\ b \end{bmatrix} q^{bc} + \sum_{k=1}^{c-1} \begin{bmatrix} a-c \\ b-k \end{bmatrix} \begin{bmatrix} c \\ k \end{bmatrix} q^{(b-k)(c-k)} + \begin{bmatrix} a-c \\ b-c \end{bmatrix} \\
&= \sum_{k=0}^{c} \begin{bmatrix} a-c \\ b-k \end{bmatrix} \begin{bmatrix} c \\ k \end{bmatrix} q^{(b-k)(c-k)}. \qquad \blacksquare
\end{aligned}$$

Part I

Intersection problems for subspaces in projective and affine spaces

2 Introduction

> *Life without geometry is pointless.*
>
> —Unknown

One of the classical problems in extremal set theory is to determine the size of the largest sets of pairwise non-trivially intersecting subsets. This problem was solved in 1961 by Erdős, Ko and Rado [55], and their result was improved by Wilson in 1984.

Theorem 2.0.1 ([113]). *Let n, k and t be positive integers and suppose that $k \geq t \geq 1$ and $n \geq (t+1)(k-t+1)$. If $\mathcal{S}$ is a family of subsets of size k in a set Ω with $|\Omega| = n$, such that the elements of $\mathcal{S}$ pairwise intersect in at least t elements, then $|\mathcal{S}| \leq \binom{n-t}{k-t}$.*
Moreover, if $n \geq (t+1)(k-t+1)+1$, then $|\mathcal{S}| = \binom{n-t}{k-t}$ holds if and only if $\mathcal{S}$ is the set of all the subsets of size k through a fixed subset of Ω of size t.

Note that if $t = 1$, then $\mathcal{S}$ is a collection of subsets of size k of an arbitrary set, which are pairwise not disjoint. In the literature, a family of subsets that are pairwise not disjoint, is called an *Erdős-Ko-Rado set*, in short *EKR set* and the classification of the largest Erdős-Ko-Rado sets is called the *Erdős-Ko-Rado problem.* Furthermore, as new families of any size can be found by deleting elements, the research is focused on *maximal* families: these are families of pairwise intersecting subsets, not extendable to a larger family with the same property.

Hilton and Milner [71] described the largest Erdős-Ko-Rado sets $\mathcal{S}$ with the property that there is no element contained in all elements of $\mathcal{S}$.

Theorem 2.0.2 ([71]). *Let Ω be a set of size n and let $\mathcal{S}$ be an Erdős-Ko-Rado set of k-subsets in Ω, $k \geq 3$ and $n \geq 2k+1$. If there is no element in Ω which is contained in all subsets in $\mathcal{S}$, then*

$$|\mathcal{S}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.$$

Moreover, equality holds if and only if

- *$\mathcal{S}$ is the union of $\{F\}$, for some fixed k-subset F, and the set of all k-subsets G of Ω containing a fixed element $x \notin F$, such that $G \cap F \neq \emptyset$, or*
- *$k = 3$ and $\mathcal{S}$ is the set of all subsets of size 3 having an intersection of size at least 2 with a fixed subset F of size 3.*

The classification of the second largest maximal EKR set is often called a *Hilton-Milner* result.

This set-theoretical problem can be generalized in a natural way to many other structures such as designs [102], permutation groups [66], affine spaces and projective geometries [37]. In this thesis, we work in the projective and affine setting, where this problem is known as the *q-analogue* of the Erdős-Ko-Rado problem. Frankl and Wilson classified the largest set of k-spaces, pairwise intersecting in at least a t-space in $\mathrm{PG}(n, q)$.

Theorem 2.0.3 ([60]). *Let t and k be integers, with $0 \leq t \leq k$. Let $\mathcal{S}$ be a set of k-spaces in $\mathrm{PG}(n, q)$, pairwise intersecting in at least a t-space.*

(i) If $n \geq 2k+1$, then $|\mathcal{S}| \leq \begin{bmatrix} n-t \\ k-t \end{bmatrix}$. Equality holds if and only if $\mathcal{S}$ is the set of all the k-spaces, containing a fixed t-space of $\mathrm{PG}(n,q)$, or $n = 2k+1$ and $\mathcal{S}$ is the set of all the k-spaces in a fixed $(2k-t)$-space.

(ii) If $2k-t \leq n \leq 2k$, then $|\mathcal{S}| \leq \begin{bmatrix} 2k-t+1 \\ k-t \end{bmatrix}$. Equality holds if and only if $\mathcal{S}$ is the set of all the k-spaces in a fixed $(2k-t)$-space.

Corollary 2.0.4. *Let $\mathcal{S}$ be an Erdős-Ko-Rado set of k-spaces in $\mathrm{PG}(n,q)$, so $t = 0$. If $n \geq 2k+1$, then $|\mathcal{S}| \leq \begin{bmatrix} n \\ k \end{bmatrix}$. Equality holds if and only if $\mathcal{S}$ is the set of all the k-spaces, containing a fixed point of $\mathrm{PG}(n,q)$, or $n = 2k+1$ and $\mathcal{S}$ is the set of all the k-spaces in a fixed hyperplane.*

Note that in Theorem 2.0.3, the condition $n \geq 2k-t$ is not a restriction, since any two k-dimensional subspaces in $\mathrm{PG}(n,q)$, with $n \leq 2k-t$, meet in at least a t-dimensional subspace.

Related to this question, we report the q-analogue of the Hilton-Milner result on the second largest maximal Erdős-Ko-Rado sets of subspaces in a finite projective space, due to Blokhuis *et al.*

Theorem 2.0.5 ([12]). *Let $\mathcal{S}$ be a maximal set of pairwise intersecting k-spaces in $\mathrm{PG}(n,q)$, with $n \geq 2k+2$, $k \geq 2$ and $q \geq 3$ (or $n \geq 2k+4$, $k \geq 2$ and $q = 2$). If $\mathcal{S}$ is not a point-pencil, then*

$$|\mathcal{S}| \leq \begin{bmatrix} n \\ k \end{bmatrix} - q^{k(k+1)} \begin{bmatrix} n-k-1 \\ k \end{bmatrix} + q^{k+1}.$$

Moreover, if equality holds, then

(i) either $\mathcal{S}$ consists of all the k-spaces through a fixed point P, meeting a fixed $(k+1)$-space τ, with $P \in \tau$, in a j-space, $j \geq 1$, and all the k-spaces in τ; or

(ii) $k = 2$ and $\mathcal{S}$ is the set of all the planes meeting a fixed plane π in at least a line.

The Erdős-Ko-Rado problem for $k = 1$ has been solved completely. Indeed, in $\mathrm{PG}(n,q)$ with $n \geq 3$, a maximal Erdős-Ko-Rado set of lines is either the set of all the lines through a fixed point or the set of all the lines contained in a fixed plane. It is possible to generalize this result for a maximal family $\mathcal{S}$ of k-spaces, pairwise intersecting in a $(k-1)$-space, in a projective space $\mathrm{PG}(n,q)$, $n \geq k+2$.

Theorem 2.0.6 ([23, Section 9.3]). *Let $\mathcal{S}$ be a set of projective k-spaces, pairwise intersecting in a $(k-1)$-space in $\mathrm{PG}(n,q)$, $n \geq k+2$. Then, all the k-spaces of $\mathcal{S}$ contain a fixed $(k-1)$-space or they are contained in a fixed $(k+1)$-space.*

All intersection problems we discuss in this part, can be linked to the Erdős-Ko-Rado problem.

In Chapter 3, we classify the largest examples of k-spaces, pairwise intersecting in at least a $(k-2)$-space in $\mathrm{PG}(n,q)$. In Chapter 4, we investigate the second largest Erdős-Ko-Rado sets of k-spaces in both a projective and affine context. This Hilton-Milner result classifies large sets $\mathcal{S}$ of k-spaces pairwise intersecting in a t-space, such that $\mathcal{S}$ is not a t-pencil.

Note that in Chapters 3 and 4, we investigate subspaces pairwise intersecting in *at least* a subspace of a certain dimension. However, in Chapter 5, we investigate sets $\mathcal{S}$ of k-spaces in $\mathrm{PG}(n,q)$ pairwise intersecting in *precisely* a point. The *Sunflower bound* states that if the number of elements in such a set $\mathcal{S}$ surpasses the Sunflower bound, then $\mathcal{S}$ must be a sunflower. We were able to lower this bound for $k \geq 3$ and $q \geq 9$.

In Chapter 6, we do not investigate subspaces in $\mathrm{PG}(n,q)$, but flags of subspaces. By definition, two flags are intersecting if they are not in *general position.* Hence, an Erdős-Ko-Rado set of flags, is a set of flags that are pairwise not in general position. In this thesis, we investigate how we can cover all flags of a specific type in $\mathrm{PG}(n,q)$, by using as few Erdős-Ko-Rado sets as possible. We discuss this question for line-solid flags in $\mathrm{PG}(4,q)$ and for flags containing a $(d-1)$- and a d-space in $\mathrm{PG}(2d,q)$, $d \geq 2$.

3

Subspaces of dimension k, pairwise intersecting in at least a $(k-2)$-space

> *Not everything that counts can be counted, and not everything that can be counted counts.*
>
> —Albert Einstein

The results in this chapter are joint work with dr. Giovanni Longobardi, dr. Ago-Erik Riet and prof. Leo Storme, and will appear in [45].

3.1 Introduction and preliminaries

In this chapter, we investigate large sets of k-spaces, pairwise intersecting in at least a $(k-2)$-space in $\mathrm{PG}(n,q)$. For $k=2$, this corresponds to large sets of planes, pairwise intersecting in at least a point. This Erdős-Ko-Rado problem for sets of projective planes is trivial if $n \leq 4$. For $n=5$, Blokhuis, Brouwer and Szőnyi classified the six largest examples [13, Section 6].
De Boeck investigated the maximal Erdős-Ko-Rado sets of planes in $\mathrm{PG}(n,q)$ with $n \geq 5$, see [33]. He characterized those sets with sufficiently large size and showed that they belong to one of the 11 known examples, explicitly described in his work.

In [53], a classification of the largest examples of sets of k-spaces in $\mathrm{PG}(n,q)$ pairwise intersecting in *precisely* a $(k-2)$-space is given. In [21], Brouwer and Hemmeter investigated sets of generators, pairwise intersecting in at least a space with codimension 2, in quadrics and symplectic polar spaces. In this chapter, we will study the projective analogue of this question. Let $f(k,q) = \max\{3q^4 + 6q^3 + 5q^2 + q + 1, \theta_{k+1} + q^4 + 2q^3 + 3q^2\}$ and so

$$f(k,q) = \begin{cases} 3q^4 + 6q^3 + 5q^2 + q + 1 & \text{if } k = 3, q \geq 2 \text{ or } k = 4, q = 2 \\ \theta_{k+1} + q^4 + 2q^3 + 3q^2 & \text{if } k = 4, q > 2 \text{ or } k > 4. \end{cases}$$

We analyze the sets of k-spaces in $\mathrm{PG}(n,q)$ pairwise intersecting in *at least* a $(k-2)$-space and with more than $f(k,q)$ elements. We will suppose that these sets $\mathcal{S}$ of subspaces are maximal, and during this discussion, we will give bounds on the size of the largest examples.

In [54], and in Chapter 4, families of subspaces pairwise intersecting in at least a t-space were investigated. More specifically, the largest non-trivial examples of a set of k-spaces, pairwise intersecting in at least a t-space in $\mathrm{PG}(n,q)$ were given.

Theorem 3.1.1 ([54] and Theorem 4.4.7). *Let $\mathcal{F}$ be a set of k-spaces pairwise intersecting in at least a t-space in $\mathrm{PG}(n,q)$, $k > t+1, t > 0, n > 2k+3+t, q \geq 3$, of maximum size, with $\mathcal{F}$ not a t-pencil, then $\mathcal{F}$ is one of the following examples:*

- $i)$ *the set of k-spaces, meeting a fixed $(t+2)$-space in at least a $(t+1)$-space,*
- $ii)$ *the set of k-spaces in a fixed $(k+1)$-space ξ together with the set of k-spaces through a t-space $\delta \subset \xi$, that have at least a $(t+1)$-space in common with ξ.*

Note that the two examples in the previous theorem correspond to Example 3.1.2(ii) and (iii) for $t = k-2$ respectively (see below). While, in [54] and in Chapter 4, the largest non-trivial example for all values of t is classified, here, for $t = k-2$ we improve on this result by classifying the ten largest examples, see Main Theorem 3.5.1.

We end this section with some examples of maximal sets $\mathcal{S}$ of k-spaces in $\mathrm{PG}(n,q)$ pairwise intersecting in at least a $(k-2)$-space, $n \geq k+2$ and $k \geq 3$. We add a proof of maximality for the examples for which it is not straightforward.

Example 3.1.2. *Examples of maximal sets $\mathcal{S}$ of k-spaces in $\mathrm{PG}(n,q)$ pairwise intersecting in at least a $(k-2)$-space.*

- (i) $(k-2)$-pencil: *the set $\mathcal{S}$ is the set of all k-spaces that contain a fixed $(k-2)$-space. Then $|\mathcal{S}| = \begin{bmatrix} n-k+2 \\ 2 \end{bmatrix}$.*
- (ii) Star: *there is a k-space ζ such that $\mathcal{S}$ contains all k-spaces that have at least a $(k-1)$-space in common with ζ. Then $|\mathcal{S}| = q\theta_k\theta_{n-k-1} + 1$.*
- (iii) Generalized Hilton-Milner example: *there is a $(k+1)$-space ν and a $(k-2)$-space $\pi \subset \nu$ such that $\mathcal{S}$ consists of all k-spaces in ν (type 1), together with all k-spaces of $\mathrm{PG}(n,q)$, not in ν, through π that intersect ν in a $(k-1)$-space (type 2). Then $|\mathcal{S}| = \theta_{k+1} + q^2(q^2+q+1)\theta_{n-k-2}$.*
- (iv) *There is a $(k+2)$-space ρ, a k-space $\alpha \subset \rho$ and a $(k-2)$-space $\pi \subset \alpha$ so that $\mathcal{S}$ contains all k-spaces in ρ that meet α in a $(k-1)$-space not through π (type 1), all k-spaces in ρ through π (type 2), and all k-spaces in $\mathrm{PG}(n,q)$, not in ρ, that contain a $(k-1)$-space of α through π (type 3). Then $|\mathcal{S}| = (q+1)\theta_{n-k} + q^3(q+1)\theta_{k-2} + q^4 - q$.*

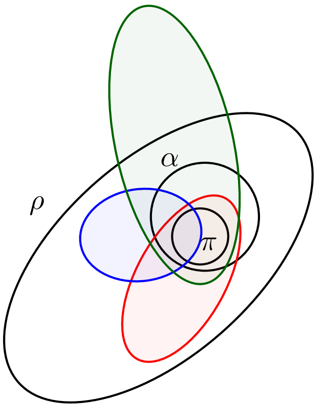


Figure 3.1: Example (iv): the blue, red and green k-spaces correspond to the k-spaces of type 1, 2 and 3, respectively.

Lemma 3.1.3. *The set $\mathcal{S}$ from Example 3.1.2 (iv) is maximal.*

Proof. Suppose there is a k-space $E \notin \mathcal{S}$, meeting all elements of $\mathcal{S}$ in at least a $(k-2)$-space. We start with the case $\pi \not\subset E$. If $\dim(E \cap \alpha) \leq k-2$, then there is a $(k-1)$-space μ through π in α with $\dim((E\cap\alpha)\cap\mu) \leq k-3$. There are elements of type 3 through μ that meet E in a subspace of dimension at most $k-3$, which gives a contradiction. Hence, E contains a $(k-1)$-space $\sigma_E \subset \alpha$. Let G be an element of $\mathcal{S}$ of type 2 such that $\langle G, \alpha\rangle = \rho$, and so $G \cap \alpha = \pi$. We have

$$\begin{aligned}\dim(E\cap\rho) \geq \dim(\langle E\cap G, E\cap\alpha\rangle) &\geq \dim(E\cap\alpha) + \dim(E\cap G) - \dim(E\cap G\cap\alpha)\\ &\geq (k-1)+(k-2)-(k-3) \geq k.\end{aligned}$$

So, $E \subset \rho$, which implies that $E \in \mathcal{S}$ (type 1), a contradiction. Now, we suppose that $\pi \subset E$. Let F_1 and F_2 be two elements of $\mathcal{S}$ of type 1, with $\langle F_1, F_2 \rangle = \rho$ and $\dim(\pi \cap F_1 \cap F_2) = k-4$. First we show that their existence is assured. Indeed, let π_1 and π_2 be two different $(k-3)$-spaces in π and let α_i be a $(k-1)$-space in α through π_i, $i = 1, 2$. Let P_1 be a point in $\rho \setminus \alpha$ and let $F_1 = \langle P_1, \alpha_1 \rangle$. Finally, consider P_2 to be a point in $\rho \setminus \langle \alpha, F_1 \rangle$ and let $F_2 = \langle P_2, \alpha_2 \rangle$. Since $E \notin \mathcal{S}$ and $\pi \subset E$, we know that E cannot contain a $(k-1)$-space of α, and so, $E \cap \alpha = \pi$. Hence, from $F_1 \cap F_2 \subset \alpha$, it follows that $\dim(E \cap F_1 \cap F_2) = \dim(\pi \cap F_1 \cap F_2)$. Then

$$\begin{aligned}\dim(E \cap \rho) &= \dim(E \cap \langle F_1, F_2 \rangle) \\ &\geq \dim(E \cap F_1) + \dim(E \cap F_2) - \dim(E \cap F_1 \cap F_2) \\ &\geq (k-2) + (k-2) - (k-4) \geq k.\end{aligned}$$

Hence, $E \subset \rho$ which implies that $E \in \mathcal{S}$, type 2, again a contradiction. ■

(v) *There is a $(k+2)$-space ρ, and a $(k-1)$-space $\alpha \subset \rho$ such that $\mathcal{S}$ contains all k-spaces in ρ that meet α in at least a $(k-2)$-space (type 1), and all k-spaces in $\mathrm{PG}(n, q)$, not in ρ, through α (type 2). Note that all k-spaces in $\mathrm{PG}(n, q)$ through α are contained in $\mathcal{S}$.*

Then $|\mathcal{S}| = \theta_{n-k} + q^2(q^2+q+1)\theta_{k-1}$.

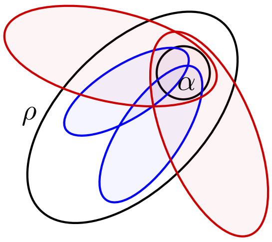


Figure 3.2: Example(v): the blue and red k-spaces correspond to the k-spaces of type 1, 2, respectively.

Lemma 3.1.4. *The set $\mathcal{S}$ from Example 3.1.2 (v) is maximal.*

Proof. Suppose there is a k-space $E \notin \mathcal{S}$, meeting all elements of $\mathcal{S}$ in at least a $(k-2)$-space. Then E contains a $(k-2)$-space σ_E in α, since E meets all elements of $\mathcal{S}$ of type 2. Note that E cannot contain α, since then, E would be a k-space in $\mathcal{S}$. Let σ_1 and σ_2 be two distinct $(k-2)$-spaces in α with $\dim(\sigma_1 \cap \sigma_2 \cap \sigma_E) = k-4$. Consider F_1 and F_2, two elements of $\mathcal{S}$ of type 1 through σ_1 and σ_2, respectively, with $\dim(F_1 \cap F_2) = k-2$. Note that $\dim(E \cap F_1 \cap F_2) = k-4$. Indeed,

$$k-4 \leq \dim(E \cap F_1 \cap F_2) \leq k-2.$$

(a) If $\dim(E \cap F_1 \cap F_2) = k-2$, then $E \cap F_1 \cap F_2 \cap \alpha = F_1 \cap F_2 \cap \alpha$, a contradiction.

(b) If $\dim(E \cap F_1 \cap F_2) = k-3$, there exists a point $P \in F_1 \cap F_2 \cap E$ not in α and $\dim(E \cap \rho) \geq k-1$. Since $E \notin \mathcal{S}$, then $E \not\subset \rho$. The only possibility is $\dim(E \cap \rho) = k-1$, but then we can find a k-space F of type 1 such that $E \cap F$ is a $(k-3)$-space, again a contradiction.

Hence, $\dim(E \cap F_1 \cap F_2) = k - 4$ and

$$\begin{aligned}\dim(E \cap \rho) &= \dim(E \cap \langle F_1, F_2 \rangle) \\ &\geq \dim(E \cap F_1) + \dim(E \cap F_2) - \dim(E \cap F_1 \cap F_2) \\ &\geq (k-2) + (k-2) - (k-4) \geq k.\end{aligned}$$

So, $E \subset \rho$, which implies that $E \in \mathcal{S}$, a contradiction. ■

(vi) *There are two $(k+2)$-spaces ρ_1, ρ_2 intersecting in a $(k+1)$-space $\alpha = \rho_1 \cap \rho_2$. There are two $(k-1)$-spaces $\pi_A, \pi_B \subset \alpha$ with $\pi_A \cap \pi_B$ the $(k-2)$-space λ, there is a point $P_{AB} \in \alpha \backslash \langle \pi_A, \pi_B \rangle$, and let $\lambda_A, \lambda_B \subset \lambda$ be two different $(k-3)$-spaces. Then $\mathcal{S}$ contains*

type 1. all k-spaces in α,

type 2. all k-spaces of $\mathrm{PG}(n,q)$ through $\langle P_{AB}, \lambda \rangle$, not in ρ_1 and not in ρ_2.

type 3. all k-spaces in ρ_1, not in α, through P_{AB} and a $(k-2)$-space in π_A through λ_A,

type 4. all k-spaces in ρ_1, not in α, through P_{AB} and a $(k-2)$-space in π_B through λ_B,

type 5. all k-spaces in ρ_2, not in α, through P_{AB} and a $(k-2)$-space in π_A through λ_B,

type 6. all k-spaces in ρ_2, not in α, through P_{AB} and a $(k-2)$-space in π_B through λ_A.

Then $|\mathcal{S}| = \theta_{n-k} + q^2\theta_{k-1} + 4q^3$.

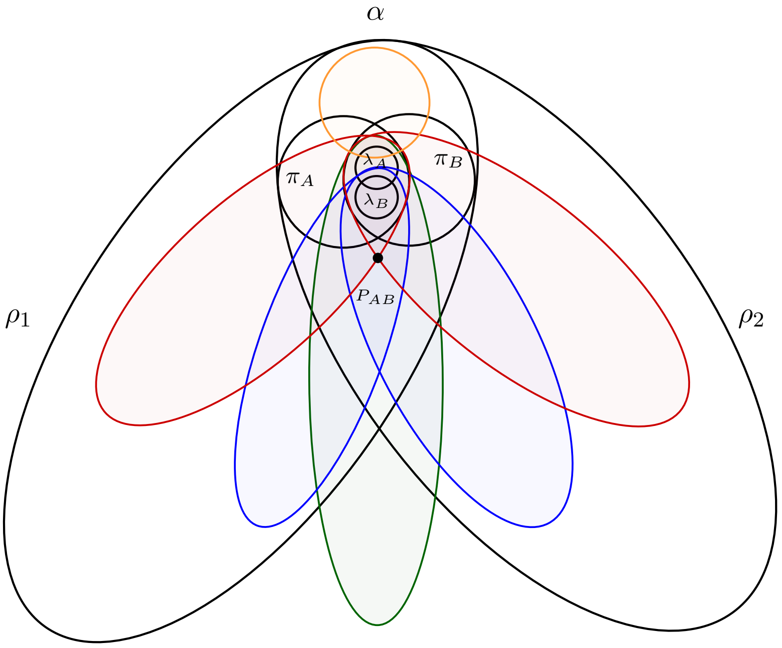


Figure 3.3: Example(vi): the orange k-space is of type 1, the green one of type 2, the red ones of type 3 and 6, and the blue ones of type 4 and 5.

Lemma 3.1.5. *The set $\mathcal{S}$ from Example 3.1.2 (vi) is maximal.*

Proof. Suppose there is a k-space $E \notin \mathcal{S}$, meeting all elements of $\mathcal{S}$ in at least a $(k-2)$-space. Suppose first that $P_{AB} \notin E$. As E contains at least a $(k-2)$-space of all elements of $\mathcal{S}$, type 1 and 2, E contains a $(k-1)$-space β in α such that β contains a $(k-2)$-space of $\langle P_{AB}, \lambda\rangle$, not through P_{AB}. Consider now the elements F and G of $\mathcal{S}$, type 3 and 4 respectively, with $F \cap G \cap \alpha = \langle P_{AB}, \lambda_A \cap \lambda_B\rangle$. If $E \not\subset \rho_1$, then $\dim(E \cap F \cap G) \leq k-4$ and

$$\begin{aligned} k-1 = \dim(E \cap \alpha) &= \dim(E \cap \rho_1) = \dim(E \cap \langle F, G\rangle) \\ &\geq \dim(E \cap F) + \dim(E \cap G) - \dim(E \cap F \cap G) \\ &\geq (k-2) + (k-2) - (k-4) \geq k, \end{aligned}$$

a contradiction. Hence, $E \subset \rho_1$. Analogously, we find that $E \subset \rho_2$, using two elements of $\mathcal{S}$ of type 5 and 6. And so, $E \subset \rho_1 \cap \rho_2 = \alpha$, which implies that $E \in \mathcal{S}$, type 1, a contradiction. So now we may suppose that $P_{AB} \in E$. Then E contains a $(k-1)$-space of α that meets λ in a $(k-3)$-space. This follows since E meets the elements of $\mathcal{S}$ of type 1 and 2 in at least a $(k-2)$-space. Note that the dimension of $E \cap \pi_A$ and $E \cap \pi_B$ is $k-2$ or $k-3$ as $E \cap \lambda$ is a $(k-3)$-space. Moreover, the latter spaces do not both have the same dimension. Indeed, if $\dim(E \cap \pi_A) = \dim(E \cap \pi_B) = k-2$, then $E \subset \alpha$, type 1, a contradiction. Moreover, since E contains P_{AB}, and since $\dim(E \cap \alpha) = k-1$, we know that $\dim(E \cap \langle \pi_A, \pi_B\rangle) = k-2$. If $\dim(E \cap \pi_A) = \dim(E \cap \pi_B) = k-3$, then w.l.o.g. we may suppose that $E \cap \lambda \neq \lambda_A$. Consider now an element X of type 3 such that $\lambda \not\subseteq X$. Then $\dim(X \cap E \cap \alpha) = k-3$, and so, $E \cap X \not\subseteq \alpha$. Hence, E and X also share points in $\rho_1 \setminus \alpha$ and so, $E \subset \rho_1$. Similarly, $E \subset \rho_2$ and so $E \subset \rho_1 \cap \rho_2 = \alpha$ which cannot occur.
By a similar argument, we find that the dimension of $E \cap \lambda_A$ and $E \cap \lambda_B$ is $k-3$ or $k-4$, both not the same dimension. Then E contains a $(k-2)$-space of π_A or π_B, and E contains λ_A or λ_B. W.l.o.g. we may suppose that E contains λ_A and a $(k-2)$-space of π_A, and meets π_B in λ_A.
Let H be an element of type 1 of $\mathcal{S}$, and let G be an element of type 4 of $\mathcal{S}$ through a $(k-2)$-space $\sigma \neq \lambda$ in π_B with $H \cap G = \sigma$. Then, since $\dim(E \cap G \cap H) = k-4$,

$$\begin{aligned} \dim(E \cap \rho_1) &= \dim(E \cap \langle G, H\rangle) \\ &\geq \dim(E \cap G) + \dim(E \cap H) - \dim(E \cap G \cap H) \\ &\geq (k-2) + (k-2) - (k-4) \geq k, \end{aligned}$$

and so $E \subset \rho_1$. Hence, $E \in \mathcal{S}$, type 3, a contradiction. ■

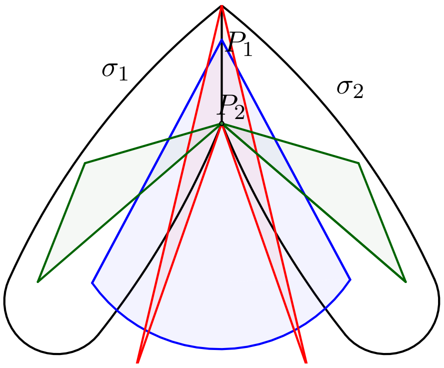


Figure 3.4: Example(vii): the red, blue and green planes correspond to the k-spaces of type 1, 2 and 3 in $\mathrm{PG}(n,q)/\gamma$, respectively.

(vii) *There is a* $(k-3)$*-space* γ *contained in all* k*-spaces of* $\mathcal{S}$*. In the quotient space* $\mathrm{PG}(n,q)/\gamma$*, the set of planes corresponding to the elements of* $\mathcal{S}$ *is the set of planes of example* $VIII$ *in*

[33]: let Ψ be an $(n-k+2)$-space, disjoint from γ, in $\mathrm{PG}(n,q)$. Consider two solids σ_1 and σ_2 in Ψ, intersecting in a line l. Take the points P_1 and P_2 on l. Then $\mathcal{S}$ is the set containing all k-spaces through $\langle\gamma,l\rangle$ (type 1), all k-spaces through $\langle\gamma,P_1\rangle$ that contain a line in σ_1 and a line in σ_2 (type 2), and all k-spaces through $\langle\gamma,P_2\rangle$ in $\langle\gamma,\sigma_1\rangle$ or in $\langle\gamma,\sigma_2\rangle$ (type 3). Then $|\mathcal{S}|=\theta_{n-k}+q^4+2q^3+3q^2$.

In Lemma 3.4.2, we prove that the set $\mathcal{S}$ is maximal.

$(viii)$ *There is a $(k-3)$-space γ contained in all k-spaces of $\mathcal{S}$. In the quotient space $\mathrm{PG}(n,q)/\gamma$, the set of planes corresponding to the elements of $\mathcal{S}$ is the set of planes of example IX in [33]: let Ψ be an $(n-k+2)$-space, disjoint from γ, in $\mathrm{PG}(n,q)$, and let l be a line and σ a solid skew to l, both in Ψ. Denote $\langle l,\sigma\rangle$ by ρ. Let P_1 and P_2 be two points on l and let $\mathcal{R}_1$ and $\mathcal{R}_2$ be a regulus and its opposite regulus in σ. Then $\mathcal{S}$ is the set containing all k-spaces through $\langle\gamma,l\rangle$ (type 1), all k-spaces through $\langle\gamma,P_1\rangle$ in the $(k+1)$-space generated by γ,l and a fixed line of $\mathcal{R}_1$ (type 2), and all k-spaces through $\langle\gamma,P_2\rangle$ in the $(k+1)$-space generated by γ,l and a fixed line of $\mathcal{R}_2$ (type 3). Then $|\mathcal{S}|=\theta_{n-k}+2q^3+2q^2$.*

In Lemma 3.4.3, we prove that the set $\mathcal{S}$ is maximal.

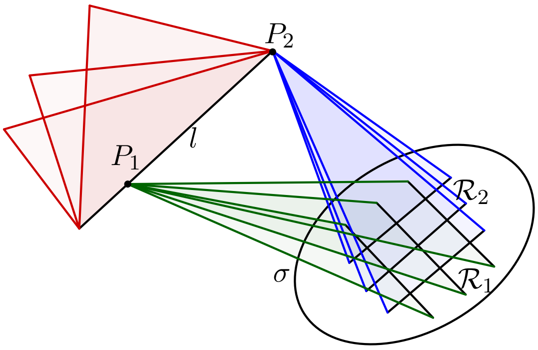


Figure 3.5: Example$(viii)$: the red, green and blue planes correspond to the k-spaces of type 1, 2, 3 in $\mathrm{PG}(n,q)/\gamma$, respectively.

(ix) *There is a $(k-3)$-space γ contained in all k-spaces of $\mathcal{S}$. In the quotient space $\mathrm{PG}(n,q)/\gamma$, the set of planes corresponding to the elements of $\mathcal{S}$ is the set of planes of example VII in [33]: let Ψ be an $(n-k+2)$-space, disjoint from γ in $\mathrm{PG}(n,q)$ and let ρ be a 5-space in Ψ. Consider a line l and a 3-space σ disjoint from l, both in ρ. Choose three points P_1, P_2, P_3 on l and choose four non-coplanar points Q_1, Q_2, Q_3, Q_4 in σ. Denote $l_1=Q_1Q_2$, $\bar{l}_1=Q_3Q_4$, $l_2=Q_1Q_3$, $\bar{l}_2=Q_2Q_4$, $l_3=Q_1Q_4$, and $\bar{l}_3=Q_2Q_3$. Then $\mathcal{S}$ is the set containing all k-spaces through $\langle\gamma,l\rangle$ (type 0) and all k-spaces through $\langle\gamma,P_i\rangle$ in $\langle\gamma,l,l_i\rangle$ or in $\langle\gamma,l,\bar{l}_i\rangle$, $i=1,2,3$ (type i). Then $|\mathcal{S}|=\theta_{n-k}+6q^2$.*

In Lemma 3.4.1, we prove that the set $\mathcal{S}$ is maximal.

(x) *$\mathcal{S}$ is the set of all k-spaces contained in a fixed $(k+2)$-space ρ. Then $|\mathcal{S}|=\begin{bmatrix}k+3\\2\end{bmatrix}$.*

From now on, let $\mathcal{S}$ be a maximal set of k-spaces pairwise intersecting in at least a $(k-2)$-space in the projective space $\mathrm{PG}(n,q)$ with $n\geq k+2$.

We will focus on the sets $\mathcal{S}$ such that $|\mathcal{S}|>f(k,q)$. In Section 3.2, we investigate the sets $\mathcal{S}$ of k-spaces in $\mathrm{PG}(n,q)$ such that there is no point contained in all elements of $\mathcal{S}$ and such that $\mathcal{S}$ contains a set of three k-spaces that meet in a $(k-4)$-space. In Section 3.3, we assume again

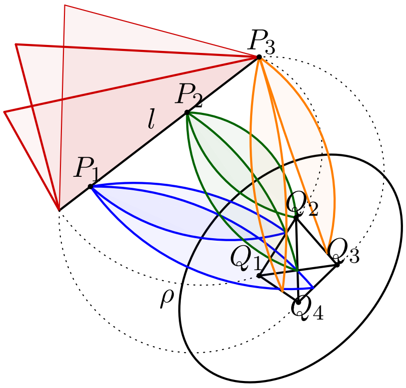


Figure 3.6: Example(ix): the red, blue, green and orange planes correspond to the k-spaces of type $0, 1, 2$ and 3 respectively.

that there is no point contained in all elements of $\mathcal{S}$ and that for any three k-spaces X, Y, Z in $\mathcal{S}$, $\dim(X \cap Y \cap Z) \geq k-3$. In Section 3.4, we investigate the maximal sets $\mathcal{S}$ of k-spaces such that there is at least a point contained in all elements of $\mathcal{S}$. We end this chapter with the Main Theorem 3.5.1 that classifies all sets of k-spaces pairwise intersecting in at least a $(k-2)$-space with size larger than $f(k,q)$.

3.2 There are three elements of $\mathcal{S}$ that meet in a $(k-4)$-space

Note that for three k-spaces A, B, C in $\mathcal{S}$, it holds that $\dim(A \cap B \cap C) \geq k-4$. Suppose there exist three k-spaces A, B, C in $\mathcal{S}$ with $\dim(A \cap B \cap C) = k-4$, and suppose that there is no point contained in all elements of $\mathcal{S}$. If all k-spaces are contained in a $(k+2)$-space, then we find Example 3.1.2(x), so we may assume that the elements of $\mathcal{S}$ span at least a $(k+3)$-space. In this subsection, we will use the following notation.

Notation 3.2.1. *Let $\mathcal{S}$ be a maximal set of k-spaces in $\mathrm{PG}(n,q)$ pairwise intersecting in at least a $(k-2)$-space. Let A, B and C in $\mathcal{S}$ be three k-spaces with $\pi_{ABC} = A \cap B \cap C$ a $(k-4)$-space. Let $\pi_{AB} = A \cap B$, $\pi_{AC} = A \cap C$ and $\pi_{BC} = B \cap C$. Let $\mathcal{S}'$ be the set of k-spaces of $\mathcal{S}$ not contained in $\langle A, B \rangle$, and let α be the span of all subspaces $D' := D \cap \langle A, B \rangle$, $D \in \mathcal{S}'$.*

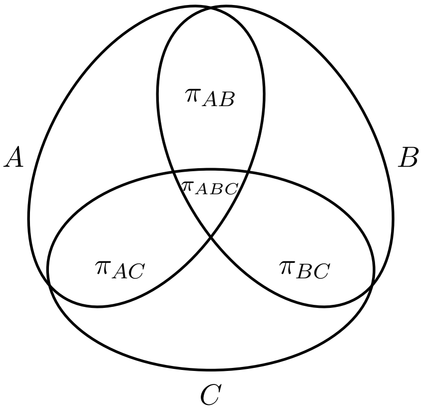


Figure 3.7: Notation 3.2.1

Note, by the Grassmann dimension property, that π_{AB}, π_{BC} and π_{AC} are $(k-2)$-spaces and $\langle A, B\rangle = \langle B, C\rangle = \langle A, C\rangle$.

We first present a lemma that will be useful for the later classification results in this section.

Lemma 3.2.2. [Using Notation 3.2.1] *If there exist three k-spaces A, B and C in $\mathcal{S}$, with $\dim(A \cap B \cap C) = k-4$, then a k-space of $\mathcal{S}'$ meets $\langle A, B\rangle$ in a $(k-1)$-space. More specifically, it contains π_{ABC} and meets π_{AB}, π_{AC} and π_{BC}, each in a $(k-3)$-space through π_{ABC}.*

Proof. Consider a k-space E of $\mathcal{S}'$. Clearly,

$$k-2 \leq \dim(E \cap \langle A, B\rangle) \leq k-1.$$

If $\dim(E \cap \langle A, B\rangle) = k-2$, then this $(k-2)$-space has to lie in A, B and C, and so in the $(k-4)$-space π_{ABC}, a contradiction. Hence, we know that $\dim(E \cap \langle A, B\rangle) = k-1$. By the symmetry of the k-spaces A, B and C, it suffices to prove that E contains π_{ABC} and meets π_{AB} in a $(k-3)$-space through π_{ABC}. Using the Grassmann dimension property we find that

$$\begin{aligned}\dim(E \cap \pi_{AB}) &\geq \dim(E \cap A) + \dim(E \cap B) - \dim(E \cap \langle A, B\rangle)\\ &= (k-2) + (k-2) - (k-1) = k-3,\end{aligned}$$

and so, $\dim(E \cap \pi_{AB})$ is $k-2$ or $k-3$. If $\dim(E \cap \pi_{AB}) = k-2$, then

$$\begin{aligned}\dim(E \cap C) &\leq \dim(E \cap \pi_{ABC}) + \dim(E \cap \langle C, \pi_{AB}\rangle) - \dim(E \cap \pi_{AB})\\ &\leq (k-4) + (k-1) - (k-2) = k-3,\end{aligned}$$

a contradiction since any two elements of $\mathcal{S}$ meet in at least a $(k-2)$-space. Hence, $\dim(E \cap \pi_{AB})$ is $k-3$, and so

$$\begin{aligned}\dim(E \cap \pi_{ABC}) &\geq \dim(E \cap C) + \dim(E \cap \pi_{AB}) - \dim(E \cap \langle C, \pi_{AB}\rangle)\\ &\geq (k-2) + (k-3) - (k-1) = k-4.\end{aligned}$$

This implies that the $(k-4)$-space π_{ABC} is contained in E. ■

Let D be a k-space of $\mathcal{S}'$. By Lemma 3.2.2, we know that $D \cap \langle A, B\rangle$ is a $(k-1)$-space. For the remaining part of this chapter, we will denote this $(k-1)$-space by D'.

Corollary 3.2.3. [Using Notation 3.2.1] *Suppose $\mathcal{S}$ contains three elements A, B and C, meeting in a $(k-4)$-space, and α is a $(k+i)$-space. Up to a suitable labelling of A, B and C, we have the following results.*

a) If $i = -1$, then $\alpha = D \cap \langle A, B\rangle$ for every $D \in \mathcal{S}'$.

b) If $i = 0$, then $\alpha = \langle \rho_1, \rho_2, \rho_3\rangle$, with ρ_1 a $(k-3)$-space in π_{AB}, ρ_2 a $(k-3)$-space in π_{BC}, $\rho_3 = \pi_{AC}$ and $\pi_{ABC} \subset \rho_j, j = 1, 2, 3$. In this case, all elements of $\mathcal{S}'$ contain the $(k-2)$-space $\langle \rho_1, \rho_2\rangle$.

c) If $i = 1$, then $\alpha = \langle \rho_1, \rho_2, \rho_3\rangle$, with ρ_1 a $(k-3)$-space in π_{AB}, $\rho_2 = \pi_{BC}$, $\rho_3 = \pi_{AC}$ and $\pi_{ABC} \subset \rho_j, j = 1, 2, 3$. In this case, all elements of $\mathcal{S}'$ contain the $(k-3)$-space ρ_1.

d) If $i = 2$, then $\alpha = \langle A, B\rangle$.

Proof. For $i=-1$ and $i=2$, the corollary follows immediately from Lemma 3.2.2. Hence, we start with the case that α is a k-space. Consider two elements of $\mathcal{S}'$, say D_1, D_2, meeting $\langle A,B\rangle$ in two different $(k-1)$-spaces D_1', D_2'. These two elements of $\mathcal{S}'$ exist, as otherwise $\dim(\alpha)=k-1$. Since D_1' and D_2' span the k-space α, they meet in a $(k-2)$-space. By Lemma 3.2.2, this $(k-2)$-space $D_1'\cap D_2'$ contains π_{ABC}, together with a $(k-3)$-space ρ_1 through π_{ABC} in π_{XY} and a $(k-3)$-space ρ_2 through π_{ABC} in π_{YZ}, with $\{X,Y,Z\}=\{A,B,C\}$. By Lemma 3.2.2, every other element of $\mathcal{S}'$ will meet $\langle A,B\rangle$ in a $(k-1)$-space through this $(k-2)$-space $D_1'\cap D_2'=\langle\rho_1,\rho_2\rangle$, which proves the statement.

Suppose now that α is a $(k+1)$-space. Then, we consider two elements D_3, D_4 of $\mathcal{S}'$ meeting $\langle A,B\rangle$ in two $(k-1)$-spaces D_3', D_4' such that $\dim(D_3'\cap D_4')=k-3$. These elements of $\mathcal{S}'$ exist as otherwise all elements of $\mathcal{S}'$ correspond to $(k-1)$-spaces pairwise intersecting in a $(k-2)$-space. But then, since these $(k-1)$-spaces span a $(k+1)$-space, they form a $(k-2)$-pencil (see Theorem 2.0.6). Using Lemma 3.2.2, and the proof above of the case $\dim(\alpha)=k$ or $i=0$, it follows that α would be a k-space. Now, again by Lemma 3.2.2, we see that $D_3'\cap D_4'$ contains π_{ABC} and a $(k-3)$-space ρ_1 through π_{ABC} in π_{XY}, with $\{X,Y,Z\}=\{A,B,C\}$. Using dimension properties and the fact that $D_3'\cap D_4'=\rho_1$, we see that every other element of $\mathcal{S}'$ will contain ρ_1, which proves the statement. ■

We will now use Corollary 3.2.3 to explicitly describe the possibilities, depending on the dimension of $\alpha=\langle\, D\cap\langle A,B\rangle\,|D\in\mathcal{S}'\rangle$.

3.2.1 α is a $(k-1)$-space

Proposition 3.2.4. [Using Notation 3.2.1] *If $\mathcal{S}$ contains three k-spaces that meet in a $(k-4)$-space and $\dim(\alpha)=k-1$, then $\mathcal{S}$ is Example 3.1.2(v).*

Proof. From Corollary 3.2.3, we have that for all $D\in\mathcal{S}'$, $D\cap\langle A,B\rangle=\alpha$, so all the k-spaces in $\mathcal{S}'$ meet $\langle A,B\rangle$ in α. As a k-space of $\mathcal{S}$ in $\langle A,B\rangle$ needs to have at least a $(k-2)$-space in common with every $D\in\mathcal{S}'$, we find that every k-space of $\mathcal{S}$ in $\langle A,B\rangle$ meets α in at least a $(k-2)$-space. Note that the condition that every two k-spaces of $\mathcal{S}$ in $\langle A,B\rangle$ meet in at least a $(k-2)$-space is fulfilled. Hence, $\mathcal{S}$ is Example 3.1.2(v) with $\rho=\langle A,B\rangle$. ■

3.2.2 α is a k-space

Proposition 3.2.5. [Using Notation 3.2.1] *If $\mathcal{S}$ contains three k-spaces that meet in a $(k-4)$-space and $\dim(\alpha)=k$, then $\mathcal{S}$ is Example 3.1.2(iv).*

Proof. If α is a k-space, we may suppose w.l.o.g., by Corollary 3.2.3, that $\alpha=\langle\pi_{AB},P_{AC},P_{BC}\rangle$ with P_{AC} and P_{BC} points in $\pi_{AC}\setminus\pi_{ABC}$ and $\pi_{BC}\setminus\pi_{ABC}$, respectively. We also know that all the k-spaces $D\in\mathcal{S}'$ have a $(k-1)$-space D' in common with α and they contain the $(k-2)$-space $\pi=\langle\pi_{ABC},P_{AC}P_{BC}\rangle$. So, every pair of k-spaces in $\mathcal{S}'$ meets in a $(k-2)$-space inside $\langle A,B\rangle$. Consider a k-space E of $\mathcal{S}$ in $\langle A,B\rangle$, not having a $(k-1)$-space in common with α, and let D_1 and D_2 be k-spaces of $\mathcal{S}'$ with $D_1'\cap D_2'=\pi$, and so $\langle D_1',D_2'\rangle=\alpha$. If E does not contain π, then

$$\dim(E\cap\alpha)\geq\dim\langle E\cap D_1',E\cap D_2'\rangle\geq k-2+k-2-\dim(E\cap\pi)\geq k-1.$$

This is a contradiction. Hence, every k-space of $\mathcal{S}\setminus\mathcal{S}'$ contains π or has a $(k-1)$-space in common with α. From the maximality of $\mathcal{S}$, it follows that $\mathcal{S}$ is Example 3.1.2(iv) with $\rho=\langle A,B\rangle$ and $\pi=\langle\pi_{ABC},P_{AC}P_{BC}\rangle$. ■

3.2.3 α is a $(k+1)$-space

To understand the structure of these sets of k-spaces, we will first investigate the case $k = 3$ and then we will generalize our results to $k \geq 3$.

$k = 3$ and α is a 4-space

Note that for $k = 3$, the spaces π_{AB}, π_{BC} and π_{AC} are pairwise disjoint lines and π_{ABC} is the empty space. By Corollary 3.2.3, we may suppose w.l.o.g. that $\alpha = \langle P_{AB}, \pi_{AC}, \pi_{BC} \rangle$, with P_{AB} a point in $\pi_{AB} \setminus \pi_{ABC}$. Hence, each of the planes $D' = D \cap \langle A, B \rangle$, $D \in \mathcal{S}'$, contains P_{AB} and the set of all these planes D' span the 4-space α.

From now on, let $\mathcal{L}$ be the set of lines $D \cap C$, $D \in \mathcal{S}'$.

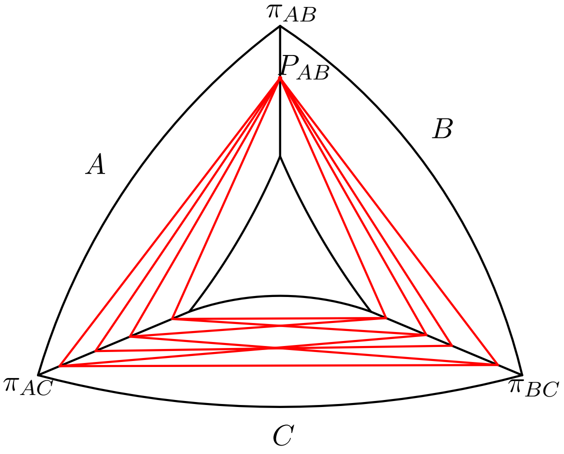


Figure 3.8: There are three solids A, B, C in $\mathcal{S}$, with $A \cap B \cap C = \emptyset$ and $\dim(\alpha) = 4$

Proposition 3.2.6. [Using Notation 3.2.1] *If $\mathcal{S}$ contains three solids such that there is no point contained in the three of them, and if* $\dim(\alpha) = 4$, *then a solid of $\mathcal{S}$ in $\langle A, B \rangle$ either*

i) is contained in α, or

ii) contains P_{AB} and a line r of C, intersecting all lines of $\mathcal{L}$.

Proof. Recall that each of the intersection planes $D \cap \langle A, B \rangle$ contains P_{AB} and that the set of all these planes span the $(k+1)$-space α. Hence, we can see that there exist solids $D_1, D_2 \in \mathcal{S}'$, such that their intersection planes D'_1 and D'_2 with α, meet exactly in the point P_{AB}. Indeed, by Theorem 2.0.6, if all the planes $D \cap \langle A, B \rangle$, $D \in \mathcal{S}'$, would pairwise intersect in a line, then these planes lie in a fixed solid or contain a fixed line. Neither possibility can occur since α is a 4-space, and P_{AB} is the only point contained in all intersection planes.

Suppose first that E is a solid of $\mathcal{S}$ in $\langle A, B \rangle$, not containing P_{AB}. As E needs to contain at least a line of every plane $D' = D \cap \langle A, B \rangle$, $D \in \mathcal{S}'$, we have that E contains at least a line $l_1 \subset D'_1 \subset \alpha$ and a line $l_2 \subset D'_2 \subset \alpha$. Note that l_1 and l_2 are disjoint as they do not contain the point P_{AB}. Hence, $E = \langle l_1, l_2 \rangle \subset \alpha$.

So now we may suppose that E contains the point P_{AB} and meets α in precisely the plane γ. The plane γ is the span of P_{AB} and the line $r = \gamma \cap C$. As $E \cap D$ is at least a line of the plane

$D' = D \cap \langle A, B\rangle$ for every $D \in \mathcal{S}'$, and since every two lines in the plane γ meet each other, we have that r has to intersect all the lines of $\mathcal{L}$. Hence, we find the second possibility. ■

In the previous proposition, we proved that there are two types of solids of $\mathcal{S}$ contained in $\langle A, B\rangle$. One of them are the solids containing P_{AB} and a line $r \subset C$, intersecting all lines of $\mathcal{L}$. The number of these solids depends on the number of lines r meeting all lines of $\mathcal{L}$.

We first investigate the case that there is a line $l \in \mathcal{L}$ that intersects all the lines of $\mathcal{L}$. Note that there cannot be two lines in $\mathcal{L}$ intersecting all the lines of $\mathcal{L}$, since then all lines of $\mathcal{L}$ would lie in a plane or go through a fixed point in C. This gives a contradiction as the lines of $\mathcal{L}$ span C and at least two points of both π_{AB} and π_{BC} are covered by the lines of $\mathcal{L}$.

Proposition 3.2.7. *If there is a line $l \in \mathcal{L}$ that intersects all the lines of $\mathcal{L}$, then $\mathcal{S}$ is Example 3.1.2(vi) for $k = 3$.*

Proof. Let $P_A = l \cap \pi_{AC}$, $P_B = l \cap \pi_{BC}$, $\pi_A = \langle \pi_{AC}, l\rangle$ and $\pi_B = \langle \pi_{BC}, l\rangle$. Since every line $m \neq l$ of $\mathcal{L}$ intersects the lines π_{AC}, π_{BC} and l, it follows that m contains the point P_A and is contained in π_B, or m contains the point P_B and is contained in π_A. Note that since $\dim(\alpha) = 4$, there is at least one line $m_1 \neq l$ in $\mathcal{L}$ through P_A and there is at least one line $m_2 \neq l$ in $\mathcal{L}$ through P_B. As a consequence of Proposition 3.2.6, we have that a solid of $\mathcal{S}$ in $\langle A, B\rangle$, not contained in α, contains P_{AB} and it meets C in a line r that meets all lines of $\mathcal{L}$. Hence, r is a line of the plane π_A through P_A or in a line of π_B through P_B. Consider now the set $\mathcal{F}$ of solids of $\mathcal{S}'$, not through $\langle P_{AB}, l\rangle$. We will prove that these solids lie in a 5-space that meets $\langle A, B\rangle$ in α. Let $E_A, E_B \in \mathcal{F}$ be two solids through $m_1 \ni P_A$ and $m_2 \ni P_B$ respectively. Since the planes $E_A \cap \alpha$ and $E_B \cap \alpha$ meet in precisely the point P_{AB}, the solids E_A and E_B have precisely a line in common, and so, they span a 5-space ρ_2 through α. Then every other solid $F \in \mathcal{F}$ is contained in ρ_2 as it meets $E_A \cap \alpha$, or $E_B \cap \alpha$, precisely in one point, namely P_{AB}, and so it must contain at least a point of E_A, or E_B respectively, in $\rho_2 \setminus \alpha$. This point, together with the plane $F \cap \alpha$, spans F and so $F \subset \rho_2$. Hence, $\mathcal{S}$ is Example 3.1.2(vi), with $\rho_1 = \langle A, B\rangle$, $\pi_A = \langle \pi_{AC}, l\rangle$, $\pi_B = \langle \pi_{BC}, l\rangle$, $\lambda_A = P_A$, $\lambda_B = P_B$ and $\lambda = l$. ■

Hence, in this case, we find that $\mathcal{S}$ has the following size

$$|\mathcal{S}| = \theta_{n-3} + q^2\theta_2 + 4q^3 = \theta_{n-3} + q^4 + 5q^3 + q^2. \tag{3.1}$$

Suppose now that there is no line in $\mathcal{L}$ that intersects all the lines of $\mathcal{L}$. Hence, for every line in $\mathcal{L}$, there exists another line in $\mathcal{L}$ disjoint from the given line. We will prove that

$$|\mathcal{S}| \leq 2q^4 + 3q^3 + 4q^2 + q + 1. \tag{3.2}$$

Since this number is smaller than $f(3, q) = 3q^4 + 6q^3 + 5q^2 + q + 1$, we will not consider these maximal sets of solids in our classification result for $k = 3$ (Main Theorem 3.5.1).

For every intersection plane D' in α, there are at most $\begin{bmatrix}3\\1\end{bmatrix} - \begin{bmatrix}2\\1\end{bmatrix} = q^2$ ways to extend the plane to a solid $D \in \mathcal{S}'$, as this solid also has to meet several solids of $\mathcal{S}'$ in a point $Q \notin \langle A, B\rangle$. And since the number of planes D' equals the number of lines in $\mathcal{L}$, there are at most $q^2 \cdot |\mathcal{L}|$ solids outside of $\langle A, B\rangle$. Let R be the set of lines meeting all lines of $\mathcal{L}$. For the solids inside $\langle A, B\rangle$, there are $\begin{bmatrix}5\\1\end{bmatrix} = \theta_4$ solids in α and $|R| \cdot q^2$ solids of the second type of Proposition 3.2.6, respectively. We find this number by multiplying the number $|R|$ of possibilities for the line r and the number q^2 of 3-spaces through a plane in $\langle A, B\rangle$, not contained in α. So, in total, we have that $|\mathcal{S}| \leq q^2|\mathcal{L}| + \theta_4 + Rq^2 = \theta_4 + q^2(|\mathcal{L}| + R)$. For every possible set of lines $\mathcal{L}$, we prove that $|\mathcal{S}| \leq 2q^4 + 3q^3 + 4q^2 + q + 1$, or equivalently, that $|\mathcal{L}| + |R| \leq q^2 + 2q + 3$.

Since every element of $\mathcal{L}$ meets both π_{AC} and π_{BC}, we know that $|\mathcal{L}| \leq (q+1)^2$. If $R = \{\pi_{AC}, \pi_{BC}\}$, then we have that $|\mathcal{L}| + |R| = |\mathcal{L}| + 2 \leq (q+1)^2 + 2$. Hence, we may assume that $\{\pi_{AC}, \pi_{BC}\} \subsetneq R$, and so $|R| \geq 3$.

Suppose first that $\mathcal{L}$ contains three pairwise disjoint lines l_1, l_2 and l_3. These three lines are contained in a unique regulus $\mathcal{R}$, and the lines, meeting l_1, l_2 and l_3, are contained in the opposite regulus $\mathcal{R}'$. Hence, $R \subseteq \mathcal{R}'$, and since R contains at least three pairwise disjoint lines, we know that $\mathcal{L}$ must be contained in the regulus $\mathcal{R}$, opposite to $\mathcal{R}'$. In this way, we find that $|\mathcal{L}| \leq q+1$ and $|R| \leq q+1$, and so $|\mathcal{L}| + |R| \leq 2q+2 < q^2+2q+3$.

For the other case, so if $\mathcal{L}$ contains no three pairwise disjoint lines, we may suppose that $\mathcal{L}$ contains at least two disjoint lines l_1, l_2, since the lines of $\mathcal{L}$ span the solid C. In this case, we prove the following lemma.

Lemma 3.2.8. *The set $\mathcal{L}$ is contained in the union of two point-pencils such that their vertices are contained either in π_{AC} or in π_{BC}.*

Proof. Let $P_i = \pi_{AC} \cap l_i$ and $Q_i = \pi_{BC} \cap l_i$, for $i = 1, 2$. As there are no three pairwise disjoint lines in $\mathcal{L}$ we see that every line $l \in \mathcal{L}$ contains at least one of the points P_i and Q_i, with $i = 1, 2$, and so $\mathcal{L}$ is contained in the union of 4 point-pencils with vertices P_1, P_2, Q_1, Q_2. If $|\mathcal{L}| \leq 4$, then it is easy to see that $\mathcal{L}$ is contained in the union of two point-pencils. Suppose now that $|\mathcal{L}| \geq 5$ and that $\mathcal{L}$ is not contained in the union of two of these point-pencils. Due to the symmetry, we may suppose that $\mathcal{L} \setminus \{l_1, l_2, P_1Q_2\}$ contains three lines l_3, l_4, l_5, such that $P_1 \in l_3, Q_2 \in l_4$ and $P_2 \in l_5$. Let $Q_3 = l_3 \cap \pi_{BC}$ and $P_4 = l_4 \cap \pi_{AC}$. Then l_5 contains the point Q_3 as otherwise l_3, l_4 and l_5 would be pairwise disjoint. So $l_5 = P_2Q_3$, but then we see that l_1, l_4 and l_5 are three pairwise disjoint lines, a contradiction. Hence, $\mathcal{L}$ is contained in the union of two point-pencils. ■

Hence, $|\mathcal{L}| \leq 2q+2$. If $|\mathcal{L}| = 2$, then there are at most $(q+1)^2$ lines meeting both l_1 and l_2, and so $|\mathcal{L}| + |R| \leq 2 + (q+1)^2$.

If $3 \leq |\mathcal{L}| \leq 2q+2$ then we may assume that $\mathcal{L}$ contains a line $l_0 \neq l_1, l_2$ with $P_1 \in l_0$. Every line r of R must meet both lines l_0, l_1, and so, it contains $P_1 = l_0 \cap l_1$ or it is contained in $\langle l_0, l_1 \rangle$. Taking into account that r must meet l_2 as well, we find that there are $q+1$ possibilities for the line r, containing the point P_1 and a point of l_2. Furthermore, if r does not contain P_1, then r is contained in the plane $\langle l_0, l_1 \rangle$, and meets $l_2 \cap \langle l_0, l_1 \rangle$. Since $l_2 \nsubseteq \langle l_0, l_1 \rangle$, we find that $l_2 \cap \langle l_0, l_1 \rangle = Q_2$, and so there are q possibilities for the line r in $\langle l_0, l_1 \rangle$ through the point Q_2, not through P_1. This implies that $|\mathcal{L}| + |R| \leq (2q+2) + (q+1+q) = 4q+3 \leq q^2+2q+3$.

General case $k > 3$ and α is a $(k+1)$-space

By Corollary 3.2.3, we may suppose w.l.o.g., that α is spanned by π_{AC}, π_{BC} and a point P_{AB} of π_{AB} outside of π_{ABC}, and that all $(k-1)$-spaces $D' = D \cap \langle A, B \rangle, D \in \mathcal{S}'$, contain $\langle P_{AB}, \pi_{ABC} \rangle$.

Proposition 3.2.9. [Using Notation 3.2.1] *If $\mathcal{S}$ contains three k-spaces that meet in a $(k-4)$-space and $\dim(\alpha) = k+1$, then a k-space of $\mathcal{S}$ in $\langle A, B \rangle$ is contained in α or contains π_{ABC}. More specifically, if $|\mathcal{S}| > f(k,q)$, then $\mathcal{S}$ is Example 3.1.2(vi).*

Proof. We suppose that E is a k-space of $\mathcal{S}$ in $\langle A, B \rangle$, not through π_{ABC}. As E contains at least a $(k-2)$-space of all the $(k-1)$-spaces D', with $D \in \mathcal{S}'$, we find that E contains a hyperplane τ_0 of π_{ABC}, a $(k-4)$-space τ_1 of $\alpha \cap \pi_{AB}$, a $(k-3)$-space τ_2 of π_{AC} and a $(k-3)$-space τ_3 of π_{BC}. As $\tau_1 \cap \tau_2 = \tau_1 \cap \tau_3 = \tau_2 \cap \tau_3 = \tau_0$, and by the Grassmann dimension property, we see that $E \subset \alpha$.

For the k-spaces through π_{ABC}, we can investigate the solids E/π_{ABC}, $E \in \mathcal{S}$, in the quotient space $\mathrm{PG}(n,q)/\pi_{ABC}$, and use the results for $k=3$ in the first part of Section 3.2.3. These results imply that a k-space in $\langle A, B\rangle$ through π_{ABC} is contained in α or contains $\langle P_{AB}, \pi_{ABC}\rangle$ and a line in $C \setminus \pi_{ABC}$ that meets all the $(k-2)$-spaces $D \cap C$, $D \in \mathcal{S}'$. Then there are two cases:

- Case 1. If there is a line $l \in C \setminus \pi_{ABC}$ meeting the subspaces $D \cap C$ for all $D \in \mathcal{S}'$, then we can use (3.1) in the quotient space $\mathrm{PG}(n,q)/\pi_{ABC} \cong \mathrm{PG}(n-k+3,q)$. Hence, there are $\theta_{n-k}+q^4+5q^3+q^2$ k-spaces of $\mathcal{S}$ that contain π_{ABC}.
- Case 2. If there is no line $l \in C \setminus \pi_{ABC}$ meeting the subspaces $D \cap C$ for all $D \in \mathcal{S}'$, then we use (3.2). Hence, there are at most $2q^4+3q^3+4q^2+q+1$ k-spaces of $\mathcal{S}$ that contain π_{ABC}.

It is clear that two elements of $\mathcal{S}$ in α meet in at least a $(k-1)$-space. From the investigation of the quotient space $\mathrm{PG}(n,q)/\pi_{ABC}$, it follows that two elements of $\mathcal{S}$ through π_{ABC}, not in α, meet in at least a $(k-2)$-space. A k-space E_1 of $\mathcal{S}$ in α and a k-space E_2 of $\mathcal{S}$ not in α, but through π_{ABC}, will also meet in a $(k-2)$-space. This follows since E_2 contains the $(k-3)$-space $\langle P_{AB}, \pi_{ABC}\rangle \subset \alpha$ and a line in $C \setminus \pi_{ABC} \subset \alpha$. Hence, E_2 meets α in a $(k-1)$-space. Since E_1 is contained in α, it follows that E_1 and E_2 meet in at least a $(k-2)$-space.

Now, as every element of $\mathcal{S}$, not through π_{ABC}, is contained in α, there are $\theta_{k+1}-\theta_4$ elements of $\mathcal{S}$ not through π_{ABC}. Hence, in Case 1, $\mathcal{S}$ is Example 3.1.2(vi) and $|\mathcal{S}| = \theta_{n-k}+\theta_{k+1}+4q^3-q-1$. In Case 2, $|\mathcal{S}| \leq \theta_{k+1}+q^4+2q^3+3q^2$, which proves the proposition. ■

3.2.4 α is a $(k+2)$-space

Here again, we first consider the case $k=3$.

$k=3$ and α is a 5-space

We start with a lemma that will often be used in this subsection.

Lemma 3.2.10. [Using Notation 3.2.1] *If $\mathcal{S}$ contains three solids A, B, C, with $A \cap B \cap C = \emptyset$, then every two intersection planes D'_1 and D'_2, with $D'_i = D_i \cap \langle A, B\rangle, D_i \in \mathcal{S}', i=1,2$, share a point on π_{AB}, π_{AC} or π_{BC}.*

Proof. Consider two solids D_1 and D_2 in $\mathcal{S}'$, with corresponding intersection planes D'_1 and D'_2 in $\langle A, B\rangle$. Since D_1 and D_2 meet in at least a line, D'_1 and D'_2 have to meet in at least a point. If D'_1 and D'_2 do not meet in a point of π_{AB}, π_{AC} or π_{BC}, then these planes define 6 different intersection points $P_1, \ldots, P_6$ on the lines π_{AB}, π_{AC} and π_{BC}. As $\langle D'_1, D'_2\rangle = \langle P_1, \ldots, P_6\rangle = \langle \pi_{AB}, \pi_{AC}, \pi_{BC}\rangle$, we find that D'_1 and D'_2 span a 5-space, so these planes are disjoint, a contradiction. ■

If α is a 5-space, we distinguish two cases, depending on the planes $D' = D \cap \langle A, B\rangle$, $D \in \mathcal{S}'$.

Lemma 3.2.11. [Using Notation 3.2.1] *If $\mathcal{S}$ contains three solids A, B, C, with $A \cap B \cap C = \emptyset$, and if* $\dim(\alpha)=5$, *then we have one of the following possibilities for the planes $D' = D \cap \langle A, B\rangle, D \in \mathcal{S}'$:*

i) There are four possibilities for the planes D': $\langle P_1, P_3, P_6\rangle$, $\langle P_1, P_4, P_5\rangle$, $\langle P_2, P_4, P_6\rangle$ and $\langle P_2, P_3, P_5\rangle$, where $P_1P_2 = \pi_{AB}$, $P_3P_4 = \pi_{BC}$ and $P_5P_6 = \pi_{AC}$. Each of them appears as an intersection plane D' for a solid D.

ii) *There are three points $P \in \pi_{AB}, Q \in \pi_{BC}$ and $R \in \pi_{AC}$ so that every plane D' contains at least two of the three points of $\{P, Q, R\}$. For every two different points in $\{P, Q, R\}$, there exists a plane D' containing these two points, but not the remaining point.*

Proof. We prove the Lemma by construction and we start with a plane, we say D'_1, intersecting π_{AB}, π_{BC} and π_{AC} in the points P, Q and R' respectively.
Case (a): there exists a plane D'_2 such that $D'_1 \cap D'_2$ is a point (w.l.o.g. P, see Lemma 3.2.10) *and let $D'_2 \cap \pi_{BC}$ be Q' and $D'_2 \cap \pi_{AC}$ be R.* In this case we know that there exists a third plane D'_3 intersecting π_{AB} in a point P' different from P (as $\dim(\alpha) = 5$). Then D'_3 needs to have at least a point of D'_2 and D'_1. This implies that D'_3 contains Q and R or Q' and R' (w.l.o.g. Q and R) by Lemma 3.2.10. Now there are two possibilities:

$i)$ There exists a plane $D'_4 = \langle P', Q', R' \rangle$, and then, by construction, we cannot add another plane D'_i. (In the formulation of the lemma $P = P_1, P' = P_2, Q = P_3, Q' = P_4, R = P_5, R' = P_6$.)

$ii)$ There does not exist a plane $D'_4 = \langle P', Q', R' \rangle$, then, by construction, we see that all the planes need to contain at least two of the three points P, Q, R by Lemma 3.2.10.

Case (b): all the planes D'_i intersect pairwise in a line. Then all these planes have to lie in a solid (contradiction since they span a 5-space) or they go through a fixed line l. In the latter, l cannot be one of the lines $\pi_{AB}, \pi_{AC}, \pi_{BC}$ and also, l cannot intersect one of these lines, as otherwise all the planes D'_i would contain the intersection point of this line and l (which gives a contradiction since $\dim(\alpha) = 5$). Consider now the disjoint lines l and π_{AB}. Then all the planes D'_i would contain l and a point of π_{AB}, but this implies that $\dim(\alpha) = 3$ which also gives a contradiction. We conclude that this case does not occur. ■

We start with the case that there are four intersection planes D'.

In this situation, using the notation from Lemma 3.2.11, there are four possibilities for the planes $D' = D \cap \langle A, B \rangle, D \in \mathcal{S}'$: $\langle P_1, P_3, P_6 \rangle, \langle P_1, P_4, P_5 \rangle, \langle P_2, P_4, P_6 \rangle$ and $\langle P_2, P_3, P_5 \rangle$, where $P_1, P_2 \in \pi_{AB}, P_3, P_4 \in \pi_{BC}$ and $P_5, P_6 \in \pi_{AC}$. We show that the only solids of $\mathcal{S}$ in $\langle A, B \rangle$ are A, B and C.

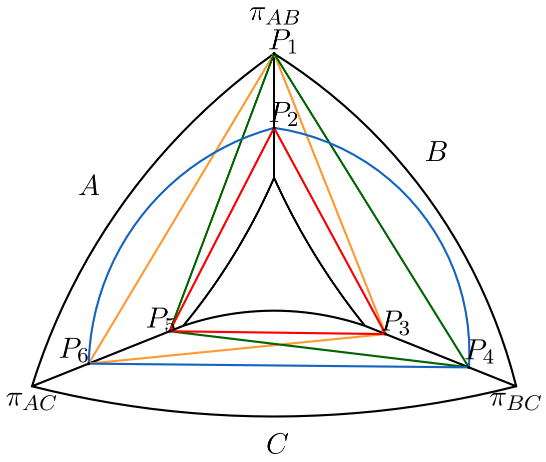


Figure 3.9: There are three elements A, B, C in $\mathcal{S}$ with $A \cap B \cap C = \emptyset$ and $\dim(\alpha) = 5$

Proposition 3.2.12. [Using Notation 3.2.1] *If $\mathcal{S}$ contains three solids A, B, C, with $A \cap B \cap C = \emptyset$, $\dim(\alpha) = 5$, and so that there are exactly four intersection planes D', see Lemma 3.2.11(i), then the only solids of $\mathcal{S}$ in $\langle A, B\rangle$ are A, B and C.*

Proof. Let $P_1, \dots, P_6$ be the intersection points of $D \cap \langle A, B\rangle$, $D \in \mathcal{S}'$, with the lines $\pi_{AB}, \pi_{AC}, \pi_{BC}$, and let E be a solid in $\langle A, B\rangle$ different from A, B, C. The solid E cannot contain all the points $P_1, \dots, P_6$, by its dimension so we may suppose that $P_1 \notin E$. We will first show that E contains the point P_2. As E has a line in common with every plane intersection $D' = D \cap \langle A, B\rangle$, with $D \in \mathcal{S}'$, E has at least a point in common with every line of these planes D'. This implies that E has at least a point in common with P_1P_3, P_1P_4, P_1P_5, and P_1P_6 or equivalently, a line l_A in common with $\langle P_1, \pi_{AC}\rangle$ and a line l_B in common with $\langle P_1, \pi_{BC}\rangle$. Hence, $E = \langle l_A, l_B\rangle$ and so $E \subset \langle P_1, C\rangle$. If $P_2 \notin E$ then we find by symmetry that $E \subset \langle P_2, C\rangle$, and so that $E \subseteq \langle P_1, C\rangle \cap \langle P_2, C\rangle$ and $E = C$, a contradiction. Then $P_2 \in E$; furthermore E cannot contain $P_2, \dots, P_6$, by the dimension, and so we may suppose that $P_6 \notin E$. Then, by the previous arguments and symmetry, we know that P_5 lies in E. In A, the solid E needs an extra point P of P_1P_6 since E shares a line with $\langle P_1, P_3, P_6\rangle$. This gives that E contains the plane $\gamma = \langle P, P_2, P_5\rangle$ of A. As E also needs to have at least a point of each line P_1P_3, P_1P_4, E needs at least one extra line, disjoint from γ. This gives the contradiction, again by the dimension, and so E cannot be different from A, B, C. ■

There are at most $4 \cdot \left(\begin{bmatrix}3\\1\end{bmatrix} - \begin{bmatrix}2\\1\end{bmatrix}\right)$ solids in $\mathcal{S}'$. The first factor of this number follows since every solid in $\mathcal{S}'$ meets $\langle A, B\rangle$ in one of the four intersection planes. The second factor follows as each of these intersection planes is contained in at most $\begin{bmatrix}3\\1\end{bmatrix} - \begin{bmatrix}2\\1\end{bmatrix}$ solids of $\mathcal{S}'$: any two solids, intersecting $\langle A, B\rangle$ in different intersection planes, have to intersect in at least a point Q outside of $\langle A, B\rangle$. There are only 3 solids, A, B, C, in $\langle A, B\rangle$. Hence $|\mathcal{S}| \leq 4q^2 + 3$.

The second possibility is that every intersection plane D' contains at least two of the points P, Q, R, and for every two different points in $\{P, Q, R\}$, there exists a plane D' containing these two points, but not the remaining point. Note that in this situation we have at least the red, green and blue plane (see Figure 3.10) as intersection planes $D' = D \cap \langle A, B\rangle$, $D \in \mathcal{S}'$. In the following proposition, we prove how the solids in $\langle A, B\rangle$ lie with respect to the points P, Q, R.

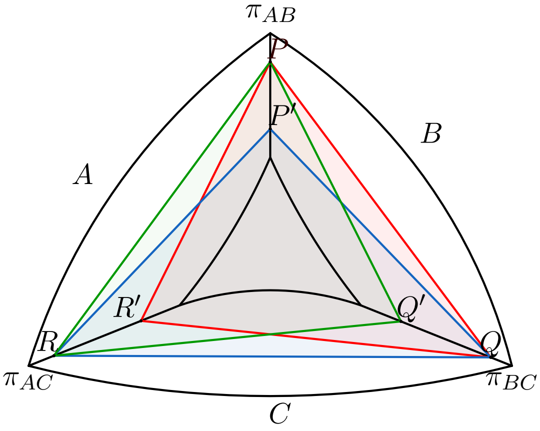


Figure 3.10: There are three elements A, B, C in $\mathcal{S}$ with $A \cap B \cap C = \emptyset$ and $\dim(\alpha) = 5$

Proposition 3.2.13. [Using Notation 3.2.1] *Suppose that $\mathcal{S}$ contains three solids A, B, C, with $A \cap B \cap C = \emptyset$, $\dim(\alpha) = 5$, and so that every intersection plane D' contains at least two of the points*

P, Q, R, such that for every two different points in $\{P, Q, R\}$, there exists a plane D' containing these two points, but not the remaining point (see Lemma 3.2.11(ii)). Then all the solids of $\mathcal{S}$ in $\langle A, B\rangle$, also contain at least two of the points P, Q, R.

Proof. Let E be a solid of $\mathcal{S}$ in $\langle A, B\rangle$, different from A, B and C. Suppose $P \notin E$, then we have to prove that E contains the points R and Q. We find that $E \cap A$ and $E \cap B$ are subspaces that meet the lines PR, PR', $P'R$ and $PQ, PQ', P'Q$, respectively, as E meets every intersection plane D' in at least a line. Hence, E meets A in a line l_{AE} through R and a point of PR', or E has a plane γ_{AE} in common with A. By symmetry, E meets B in a line l_{BE} through Q and a point of PQ', or E has a plane γ_{BE} in common with B.

$a)$ If $\dim(A \cap E) = \dim(B \cap E) = 2$, then the planes γ_{AE} and γ_{BE} meet in a point of π_{AB} as they cannot contain the line π_{AB} since $P \notin E$. Hence, E contains two planes meeting in a point, which gives a contradiction since $\dim(E) = 3$.

$b)$ If $\dim(A \cap E) = 2$ and $\dim(B \cap E) = 1$, then $\gamma_{AE} \cap \pi_{AB} = l_{BE} \cap \pi_{AB}$. First note that $l_{BE} \cap \pi_{AB}$ is not empty by the dimension of E. Now, if $\gamma_{AE} \cap \pi_{AB} \neq l_{BE} \cap \pi_{AB}$, then $\pi_{AB} \subset E$, which gives a contradiction as $P \notin E$. Since l_{BE} can only meet π_{AB} in the point P, we find a contradiction, again as $P \notin E$. Clearly, by symmetry, an analogous argument holds also if $\dim(A \cap E) = 1$ and $\dim(B \cap E) = 2$.

Hence, we know that E contains a line $l_{AE} \subset A$ through R and a line $l_{BE} \subset B$ through Q, which proves the proposition. ■

There are at most $\left(3 \cdot \begin{bmatrix}2\\1\end{bmatrix} - 2\right)\left(\begin{bmatrix}3\\1\end{bmatrix} - \begin{bmatrix}2\\1\end{bmatrix}\right)$ solids not in $\langle A, B\rangle$. This follows as two solids D_1, D_2, intersecting $\langle A, B\rangle$ in the intersection planes D_1' and D_2' meeting in a point, then D_1 and D_2 have to intersect in at least a point not in $\langle A, B\rangle$. And there are at most $3 \cdot \begin{bmatrix}2\\1\end{bmatrix} - 2$ intersection planes D'. There are at most $\begin{bmatrix}3\\1\end{bmatrix} + 3q\begin{bmatrix}3\\1\end{bmatrix}$ solids in $\langle A, B\rangle$, namely all the solids through the plane $\langle P, Q, R\rangle$ and all solids through precisely two of the three points P, Q, R in $\langle A, B\rangle$. Hence, $|\mathcal{S}| \leq 6q^3 + 5q^2 + 4q + 1$.

Remark 3.2.14. Note that if $\mathcal{S}$ contains three elements A, B, C, with $A \cap B \cap C = \emptyset$, and if $\dim(\alpha) = 5$, then the number of elements of $\mathcal{S}$ is at most $f(3, q) = 3q^4 + 6q^3 + 5q^2 + q + 1$, and so we will not consider these maximal sets of solids in our classification.

General case $k > 3$ and α is a $(k + 2)$-space

In this case, we prove that all the k-spaces of $\mathcal{S}$ contain π_{ABC}. This implies that we will be able to investigate this case by considering the quotient space of π_{ABC} and use the previous results for $k = 3$.

Proposition 3.2.15. [Using Notation 3.2.1] *If $\mathcal{S}$ contains three k-spaces A, B, C, with* $\dim(A \cap B \cap C) = k - 4$, *and* $\dim(\alpha) = k + 2$, *then every k-space in $\mathcal{S}$ contains π_{ABC}.*

Proof. By Lemma 3.2.2, we know that all the k-spaces of $\mathcal{S}$ outside of $\langle A, B\rangle$ contain π_{ABC}. It is also clear that A, B and C contain π_{ABC}.
Suppose that there is a k-space E in $\langle A, B\rangle$, not through π_{ABC}. As E has to meet all the $(k-1)$-spaces D_i' in at least a $(k-2)$-space, E has to meet π_{ABC} in a $(k-5)$-space γ and π_{AB}, π_{BC}, π_{AC} in three distinct $(k-3)$-spaces such that they meet pairwise in γ. This would imply that $\dim(E) = k + 1$, which gives a contradiction. ■

Clearly, the previous proposition implies that in order to have an estimate of the number of k-spaces in and outside of $\langle A, B\rangle$, we can use the results for $k = 3$ in the first part of Section 3.2.4: $|\mathcal{S}| \leq 4 \cdot \left(\begin{bmatrix}3\\1\end{bmatrix} - \begin{bmatrix}2\\1\end{bmatrix}\right) + 3$ or $|\mathcal{S}| \leq \left(3 \cdot \begin{bmatrix}2\\1\end{bmatrix} - 2\right)\left(\begin{bmatrix}3\\1\end{bmatrix} - \begin{bmatrix}2\\1\end{bmatrix}\right) + \begin{bmatrix}3\\1\end{bmatrix}\left(3q^2+1\right)$. In both cases, $|\mathcal{S}| < \theta_{k+1} + q^4 + 2q^3 + 3q^2 = f(k,q)$.

To conclude this section, we give a theorem which summarizes Proposition 3.2.4, Proposition 3.2.5, Proposition 3.2.9 and Proposition 3.2.15, and so, it gives an overview of the different cases studied in this section.

Proposition 3.2.16. [Using Notation 3.2.1] *In the projective space* $\mathrm{PG}(n,q)$*, with* $n \geq k+2$ *and* $k \geq 3$*, let* $\mathcal{S}$ *be a maximal set of* k*-spaces pairwise intersecting in at least a* $(k-2)$*-space such that* $\mathcal{S}$ *contains three* k*-spaces* A, B, C*, with* $\dim(A \cap B \cap C) = k-4$*, and such that* $|\mathcal{S}| \geq f(k,q)$*. Then we have one of the following possibilities:*

- *i)* *there are no* k*-spaces of* $\mathcal{S}$ *outside of* $\langle A, B\rangle$ *and* $\mathcal{S}$ *is Example 3.1.2*(x)*,*
- *ii)* $\dim(\alpha) = k-1$ *and* $\mathcal{S}$ *is Example 3.1.2*(v)*,*
- *iii)* $\dim(\alpha) = k$ *and* $\mathcal{S}$ *is Example 3.1.2*(iv)*,*
- *iv)* $\dim(\alpha) = k+1$ *and* $\mathcal{S}$ *is Example 3.1.2*(vi)*.*

3.3 Every three elements of $\mathcal{S}$ meet in at least a $(k-3)$-space

Throughout this section, we suppose that every three elements of $\mathcal{S}$ meet in at least a $(k-3)$-space. Moreover, to avoid trivial cases, we may suppose that there exist two k-spaces in $\mathcal{S}$ intersecting in precisely a $(k-2)$-space. We can find those two k-spaces as otherwise all subspaces would pairwise intersect in a $(k-1)$-space and the classification in this case is known: all the k-spaces go through a fixed $(k-1)$-space or all the k-spaces lie in a $(k+1)$-dimensional space, see Theorem 2.0.6. We also suppose that $\mathcal{S}$ is not a $(k-2)$- or a $(k-3)$-pencil as in this case either $\mathcal{S}$ is Example 3.1.2(i) or we can investigate the quotient space and use the known Erdős-Ko-Rado results on planes intersecting in at least a point [33]. We begin this section with a useful lemma.

Lemma 3.3.1. *Let* $\mathcal{S}$ *be a maximal set of* k*-spaces in* $\mathrm{PG}(n,q)$ *pairwise intersecting in at least a* $(k-2)$*-space such that for every* $X, Y, Z \in \mathcal{S}$*,* $\dim(X \cap Y \cap Z) \geq k-3$*, and such that there is no point contained in all elements of* $\mathcal{S}$*. Then there exist three elements* A, B, C *of* $\mathcal{S}$ *such that*

- *a)* $\pi = A \cap B \cap C$ *is a* $(k-3)$*-space,*
- *b)* *at least two of the three subspaces* $\pi_{AB} = A \cap B, \pi_{BC} = B \cap C, \pi_{AC} = A \cap C$ *have dimension* $k-2$*, and at most one of them has dimension* $k-1$*.*
- *c)* $\zeta = \langle \pi_{AB}, \pi_{BC}, \pi_{AC}\rangle$ *has dimension* k *or* $k+1$*.*

Every k*-space in* $\mathcal{S}$ *not through* π *meets the space* $\zeta = \langle \pi_{AB}, \pi_{BC}, \pi_{AC}\rangle$ *in at least a* $(k-1)$*-space.*

Proof. If every three k-spaces in $\mathcal{S}$ meet (at least) in a $(k-2)$-space, then $\mathcal{S}$ is a $(k-2)$-pencil, and so there is a point contained in all the k-spaces of $\mathcal{S}$. Therefore, there exist three elements $A, B, C \in \mathcal{S}$ such that $\pi = A \cap B \cap C$ is a $(k-3)$-space. Let $\pi_{AB} = A \cap B$, $\pi_{BC} = B \cap C$ and $\pi_{AC} = A \cap C$, and let $\zeta = \langle \pi_{AB}, \pi_{BC}, \pi_{AC}\rangle$. Note that at least two of the three subspaces $\pi_{AB}, \pi_{BC}, \pi_{AC}$ have dimension $k-2$. Otherwise, if, for example, $\dim(\pi_{AB}) = \dim(\pi_{AC}) = k-1$, then the k-space A contains two $(k-1)$-spaces, π_{AB} and π_{AC}, meeting in at most a $(k-3)$-space, a contradiction.

W.l.o.g. we may suppose that $\dim(\pi_{AB}) = \dim(\pi_{AC}) = k-2$ and $\dim(\pi_{BC}) \in \{k-1, k-2\}$. This also implies that the dimension of ζ is at most $k+1$. On the other hand, note that ζ has at least dimension k. Otherwise, if $\zeta = \langle \pi_{AB}, \pi_{BC}, \pi_{AC} \rangle$ is a $(k-1)$-space, then $\zeta = \langle \pi_{AB}, \pi_{AC} \rangle$ and so $\zeta \subset A$. By the same argument, $\zeta \subset B$, and $\zeta \subset C$. Hence, $\zeta \subset A \cap B \cap C = \pi$, a contradiction.

Case 1. Suppose that π_{AB}, π_{AC} and π_{BC} are $(k-2)$-spaces. Then, ζ is a k-space. Since there is no point contained in all elements of $\mathcal{S}$, we know that not all elements of $\mathcal{S}$ contain π. Let G be such a k-space G in $\mathcal{S}$ not through π. Since any three elements of $\mathcal{S}$ meet in at least a $(k-3)$-space and $\pi \nsubseteq G$, we have that G meets π in a $(k-4)$-space π_G and it contains at least a $(k-3)$-space of π_{AB}, π_{BC} and π_{AC}. Since the three subspaces $G \cap \pi_{AB}, G \cap \pi_{BC}$ and $G \cap \pi_{AC}$ have dimension at least $k-3$, since they pairwise meet in the $(k-4)$-space π_G, and since π_{AB}, π_{AC} and π_{BC} span at least a k-space, G contains the subspace $\langle G \cap \pi_{AB}, G \cap \pi_{BC}, G \cap \pi_{AC} \rangle$, with at least dimension $k-1$, in ζ.

Case 2. Suppose that $\dim(\pi_{AB}) = \dim(\pi_{AC}) = k-2$ and $\dim(\pi_{BC}) = k-1$. They meet in the $(k-3)$-space π. Now, ζ is a $(k+1)$-space and consider a k-space G not through π. As before G meets π in a $(k-4)$-space; the spaces $G \cap \pi_{AB}$ and $G \cap \pi_{AC}$ are $(k-3)$-spaces otherwise G goes through π and finally $\dim(G \cap \pi_{BC}) \in \{k-3, k-2\}$.

Case 2a. $\dim(G \cap \pi_{BC}) = k-3$. Then $G \cap \pi_{AC}$ and $G \cap \pi_{BC}$ cannot be contained in π_{AB} otherwise $\dim(G \cap \pi) = k-3$. Hence, $G \cap \pi_{AC}, G \cap \pi_{BC}$ and $G \cap \pi_{AB}$ are linearly independent $(k-3)$-spaces (i.e. the span of two of them does not meet the other space) pairwise intersecting in $G \cap \pi$. Therefore,

$$\dim\langle \pi_{AB} \cap G, \pi_{AC} \cap G, \pi_{BC} \cap G \rangle = k-1.$$

Case 2b. $\dim(G \cap \pi_{BC}) = k-2$. Note that $G \cap \pi_{BC}$ cannot meet π_{AB} in a $(k-3)$ space, otherwise G goes through π. Then, again, $G \cap \pi_{XY}$, with $\{X, Y\} \subset \{A, B, C\}$, are linearly independent $(k-3)$-spaces pairwise intersecting in $G \cap \pi$ and

$$\dim\langle \pi_{AB} \cap G, \pi_{AC} \cap G, \pi_{BC} \cap G \rangle = k.$$

Hence, the k-space G is inside of ζ.
So, in any case, we get that a k-space not through π meets ζ in at least a $(k-1)$-space. ■

Theorem 3.3.2. *Let $\mathcal{S}$ be a maximal set of k-spaces pairwise intersecting in at least a $(k-2)$-space in $\mathrm{PG}(n,q)$. If for every three elements X, Y, Z of $\mathcal{S}$: $\dim(X \cap Y \cap Z) \geq k-3$, and if there is no point contained in all elements of $\mathcal{S}$, then $\mathcal{S}$ is one of the following examples:*

(i) Example 3.1.2(ii): Star.

(ii) Example 3.1.2(iii): Generalized Hilton-Milner example.

Proof. From Lemma 3.3.1, it follows that we may suppose that there are three k-spaces A, B, C with $\dim(A \cap B \cap C) = k-3$, $\dim(\pi_{AB}) = \dim(\pi_{AC}) = k-2$ and $\dim(\pi_{BC}) \in \{k-1, k-2\}$.

Case 1. $\dim(\pi_{BC}) = k-2$. In this case we know, again from Lemma 3.3.1, that $\zeta = \langle \pi_{AB}, \pi_{AC}, \pi_{BC} \rangle$ has dimension k and that any element of $\mathcal{S}$, not through $\pi = A \cap B \cap C$, meets ζ in at least a $(k-1)$-space.

Case 1.1. Suppose that there exists a k-space D, not containing π, with $\dim(D \cap A) = \dim(D \cap B) = \dim(D \cap C) = k-2$.
Let π_{AD}, π_{BD} and π_{CD} be these $(k-2)$-spaces. Note that each of them contains the $(k-4)$-space $\pi_D = D \cap \pi$ and that they are contained in ζ. We prove that all elements of $\mathcal{S}$ meet ζ in at least a $(k-1)$-space. From Lemma 3.3.1, it follows that we only have to check that all elements of $\mathcal{S}$ through π have this property. Let E be a k-space in $\mathcal{S}$ through π. Then E contains a $(k-3)$-space

of π_{AD}, π_{BD} and π_{CD}. At least two of these $(k-3)$-spaces are different, since π is not contained in D, and span together with π at least a $(k-1)$-space contained in the k-space ζ. Hence, every k-space of $\mathcal{S}$ meets ζ in at least a $(k-1)$-space. Then $\mathcal{S}$ is Example 3.1.2(ii).

Case 1.2. For every k-space $D \in \mathcal{S}$, it holds that $\pi \subset D$ or at least one of the dimensions $\dim(D \cap A)$, $\dim(D \cap B), \dim(D \cap C)$ *is larger than $k-2$.*
In this case, we will prove that if not every k-space of $\mathcal{S}$ meets ζ in a $(k-1)$-space, then $\mathcal{S}$ is the second example described in the theorem. Let D be a k-space of $\mathcal{S}$ not containing π and meeting A, B or C in a $(k-1)$-space. W.l.o.g. we may suppose that $C \cap D$ is the $(k-1)$-space π_{CD} and that $A \cap D$ and $B \cap D$ are $(k-2)$-spaces (π_{AD} and π_{BD} respectively). Note that these subspaces $\pi_{AD}, \pi_{BD}, \pi_{CD}$ contain the $(k-4)$-space $\pi_D = D \cap \pi$ and that $\pi_{AD}, \pi_{BD} \subset \zeta$. This follows since D meets $\pi_{AB}, \pi_{AC}, \pi_{BC}$ in a $(k-3)$-space, and $D \cap \pi_{AB}$ and $D \cap \pi_{AC}$ span π_{AD}. The same argument holds for the space B. Suppose that $\mathcal{S}$ is not a Star, then there does not exist a k-space γ such that each element of $\mathcal{S}$ meets γ in at least a $(k-1)$-space. In particular, there exists a k-space $F \in \mathcal{S}$ that meets ζ in (at most) a $(k-2)$-space. As every k-space in $\mathcal{S}$, not containing π, meets ζ in a $(k-1)$-space (Lemma 3.3.1), we see that F contains π. Now, since every three elements of $\mathcal{S}$ meet in a $(k-3)$-space, F also contains a $(k-3)$-space of the two $(k-2)$-spaces π_{AD} and π_{BD} in ζ (π_{ADF}, π_{BDF} respectively). As F has no $(k-1)$-space in common with ζ, and since $\pi_{AD}, \pi_{BD} \subset \zeta$, $\pi_{CD} \not\subseteq \zeta$, we find that $\pi_{ADF} = \pi_{BDF} = \pi_{AB} \cap D$ and that $\pi_{CDF} \not\subseteq \zeta$. Hence, $F \cap \zeta = \pi_{AB}$ and $C \cap F = \langle \pi_{CDF}, \pi \rangle$. Let $\nu = \langle \zeta, C \rangle$. Then we prove that every k-space in $\mathcal{S}$ is contained in ν or contains π_{AB} and meets ν in a $(k-1)$-space. Every k-space in $\mathcal{S}$ containing π_{AB} must contain at least a $(k-2)$-space of C. Hence, this k-space meets ν in at least a $(k-1)$-space. Consider now a k-space $E \in \mathcal{S}$ not through π_{AB}. From the arguments above, it follows that, if $\pi \subset E$, then $E \subset \nu$. Moreover, if $\pi \not\subseteq E$, then, by Lemma 3.3.1, E contains a $(k-1)$-space in ζ and a point in $C \setminus \zeta$ as otherwise we have *Case* 1.1, and so $\mathcal{S}$ would be a Star, a contradiction. Hence, $E \subset \nu$.

CASE 2. *For every three k-spaces $X, Y, Z \in \mathcal{S}$, we have that* $\dim(X \cap Y \cap Z) \geq k-2$ *or two of these spaces meet in a $(k-1)$-space.* Since we suppose that there is no point contained in all elements of $\mathcal{S}$, we see that not every three elements meet in a fixed $(k-2)$-space. Recall that $A \cap B = \pi_{AB}$ is a $(k-2)$-space. Hence, every other element of $\mathcal{S}$ contains π_{AB} or meets A or B in a $(k-1)$-space. Note that the elements of $\mathcal{S}$, not through π_{AB}, are contained in $\langle A, B \rangle$. By Example 3.1.2(x), we may suppose that not all elements of $\mathcal{S}$ are contained in $\langle A, B \rangle$. Hence, let $D \in \mathcal{S}$ be a k-space not contained in $\langle A, B \rangle$.
If $D \cap A = D \cap B = \pi_{AB}$, then, by symmetry, it follows that every element of $\mathcal{S}$, not through π_{AB}, meets two of the three elements A, B, D in a $(k-1)$-space. This is a contradiction since a k-space cannot contain two $(k-1)$-spaces, meeting in a $(k-3)$-space.
Hence, every k-space in $\mathcal{S}$, not in $\langle A, B \rangle$, meets A or B in a $(k-1)$-space through π_{AB}. W.l.o.g. we suppose that $B \cap D = \pi_{BD}$ is a $(k-1)$-space, and so $A \cap D = \pi_{AD} = \pi_{AB}$. Consider now an element $E \in \mathcal{S}$ not through π_{AB}. Then, $E \subset \langle A, B \rangle$, and since both A, B and A, D meet in a $(k-2)$-space, E contains a $(k-1)$-space in A or E contains a $(k-1)$-space in both D and B. Note that E cannot contain a $(k-1)$-space of D, since $E \subset \langle A, B \rangle$, but $D \cap \langle A, B \rangle$ is a $(k-1)$-space through $\pi_{AB} \not\supseteq E$. Hence, E must contain a $(k-1)$-space of A and a $(k-2)$-space of $B \cap D$ and so every element of $\mathcal{S}$, not through π_{AB}, is contained in $\nu = \langle A, \pi_{BD} \rangle$.
To conclude this proof, we show that every element of $\mathcal{S}$, through π_{AB}, meets $\nu = \langle A, \pi_{BD} \rangle$ in at least a $(k-1)$-space, which proves that $\mathcal{S}$ is the Generalized Hilton-Milner example. So, consider a k-space $F \in \mathcal{S}$, $\pi_{AB} \subset F$. Then F must contain a $(k-2)$-space π_{EF} of E. Hence, F contains the $(k-1)$-space $\langle \pi_{EF}, \pi_{AB} \rangle \subset \langle A, \pi_{BD} \rangle$. ■

3.4 There is at least a point contained in all k-spaces of $\mathcal{S}$

To classify all maximal sets of k-spaces pairwise intersecting in at least a $(k-2)$-space, we also have to investigate the families of k-spaces such that there is a subspace contained in all its elements. More precisely, in this section, we will consider a set $\mathcal{S}$ of k-spaces of $\mathrm{PG}(n,q)$ such that there is at least a point contained in all elements of $\mathcal{S}$. So, let g, with $0 \leq g \leq k-3$, be the dimension of the maximal subspace γ contained in all elements of $\mathcal{S}$, and let $k' = k-g-1$. In the quotient space of $\mathrm{PG}(n,q)$ with respect to γ, the set $\mathcal{S}$ of k-spaces corresponds to a set $\mathcal{T}$ of k'-spaces in $\mathrm{PG}(n-g-1,q)$ that pairwise intersect in at least a $(k'-2)$-space, and so that there is no point contained in all elements of $\mathcal{T}$. Since we are interested in sets $\mathcal{S}$ of k-spaces with $|\mathcal{S}| > f(k,q)$, this corresponds with sets $\mathcal{T}$ of k'-spaces with $|\mathcal{T}| > f(k,q)$.

Since $f(k,q) \geq f(k',q) = f(k-g-1,q)$, we can use Theorem 3.2.16 and Theorem 3.3.2 for the sets $\mathcal{T}$ in $\mathrm{PG}(n-g-1,q)$, in the case that $k-g-1 > 2$. For each example, we show that it can be extended to one of the examples discussed in the previous sections.

1. $\mathcal{T}$ is the set of k'-spaces of Theorem 3.2.16(i), so that $\mathcal{T}$ is Example 3.1.2(x) : there exists a $(k'+2)$-space ρ' such that $\mathcal{T}$ is the set of all k'-spaces in ρ. Then $\mathcal{S}$ can be extended to Example 3.1.2(x) in $\mathrm{PG}(n,q)$, with $\rho = \langle \rho', \gamma \rangle$.

2. $\mathcal{T}$ is the set of k'-spaces of Theorem 3.2.16(ii), so that $\mathcal{T}$ is Example 3.1.2(v) : there are a $(k'+2)$-space ρ', and a $(k'-1)$-space $\alpha' \subset \rho'$ so that $\mathcal{T}$ contains all k'-spaces in ρ' that meets α' in at least a $(k'-2)$-space, and all k'-spaces in $\mathrm{PG}(n-g-1,q)$ through α'. Then $\mathcal{S}$ can be extended to Example 3.1.2(v) in $\mathrm{PG}(n,q)$, with $\rho = \langle \rho', \gamma \rangle$ and $\alpha = \langle \alpha', \gamma \rangle$.

3. $\mathcal{T}$ is the set of k'-spaces of Theorem 3.2.16(iii), so that $\mathcal{T}$ is Example 3.1.2(iv) : there are a $(k'+2)$-space ρ', a k'-space $\alpha' \subset \rho'$ and a $(k'-2)$-space $\pi' \subset \alpha'$ so that $\mathcal{T}$ contains all k'-spaces in ρ' that meet α' in at least a $(k'-1)$-space, all k'-spaces in ρ' through π', and all k'-spaces in $\mathrm{PG}(n-g-1,q)$ that contain a $(k'-1)$-space of α' through π'. Then $\mathcal{S}$ can be extended to Example 3.1.2(iv) in $\mathrm{PG}(n,q)$, with $\pi = \langle \pi', \gamma \rangle$, $\rho = \langle \rho', \gamma \rangle$ and $\alpha = \langle \alpha', \gamma \rangle$.

4. $\mathcal{T}$ is the set of k'-spaces of Theorem 3.2.16(iv). Since we suppose that $|\mathcal{S}| = |\mathcal{T}| > f(k,q)$, we know that $\mathcal{T}$ is Example 3.1.2(vi): there are two $(k'+2)$-spaces ρ'_1, ρ'_2 intersecting in a $(k'+1)$-space $\alpha' = \rho'_1 \cap \rho'_2$. There are two $(k'-1)$-spaces $\pi'_A, \pi'_B \subset \alpha'$, with $\pi'_A \cap \pi'_B$ the $(k'-2)$-space l', there is a point $P' \in \alpha' \setminus \langle \pi'_A, \pi'_B \rangle$, and let $P'_A, P'_B \subset l'$ be two different $(k'-3)$-spaces. Then $\mathcal{T}$ contains

    - all k'-spaces in α',
    - all k'-spaces through $\langle P', l' \rangle$,
    - all k'-spaces in ρ'_1 through P' and a $(k'-2)$-space in π'_A through P'_A,
    - all k'-spaces in ρ'_1 through P' and a $(k'-2)$-space in π'_B through P'_B,
    - all k'-spaces in ρ'_2 through P' and a $(k'-2)$-space in π'_A through P'_B,
    - all k'-spaces in ρ'_2 through P' and a $(k'-2)$-space in π'_B through P'_A.

    Then $\mathcal{S}$ can be extended to Example 3.1.2(vi) in $\mathrm{PG}(n,q)$, with $P_A = \langle P'_A, \gamma \rangle$, $P_B = \langle P'_B, \gamma \rangle$, $\pi_A = \langle \pi'_A, \gamma \rangle$, $\pi_B = \langle \pi'_B, \gamma \rangle$, $l = \langle l', \gamma \rangle$, $\alpha = \langle \alpha', \gamma \rangle$, $\rho_1 = \langle \rho'_1, \gamma \rangle$, $\rho_2 = \langle \rho'_2, \gamma \rangle$ and $P_{AB} = P'$.

5. $\mathcal{T}$ is the set of k'-spaces of Theorem 3.3.2(i): there exists a k'-space ζ' such that $\mathcal{T}$ is the set of all k'-spaces that have a $(k'-1)$-space in common with ζ'. Then $\mathcal{S}$ can be extended to example (i) in Theorem 3.3.2, and so to Example 3.1.2(ii), with $\zeta = \langle \zeta', \gamma \rangle$.

6. $\mathcal{T}$ is the set of k'-spaces of Theorem 3.3.2(ii): there exists a $(k'+1)$-space ν' and a $(k'-2)$-space $\pi' \subset \nu$ such that $\mathcal{T}$ consists of all k'-spaces in ν', together with all k'-spaces through π' that intersect ν' in at least a $(k'-1)$-space. Then $\mathcal{S}$ can be extended to example (ii) in Theorem 3.3.2, and so to Example 3.1.2(iii), with $\nu = \langle \nu', \gamma \rangle$, $\pi = \langle \pi', \gamma \rangle$.

We note that if $\mathcal{T}$ is one of the set of k'-spaces described in Section 3.2.4, then $\mathcal{S}$ can be extended to a set $\mathcal{S}'$ of k-spaces pairwise intersecting in a $(k-2)$-space such that $\mathcal{S}'$ contains three k-spaces that meet in a $(k-4)$-space with $\dim(\alpha) = k+2$. Hence, $|\mathcal{S}'| < f(k,q)$ and so these sets $\mathcal{T}$ do not lead to large examples of $\mathcal{S}$.

If $k-g-1=2$, the set $\mathcal{T}$ is a set of planes in $\mathrm{PG}(n-k+2,q)$ pairwise intersecting in at least a point, i.e. an Erdős-Ko-Rado set of planes. In [13, Section 6], Blokhuis *et al.* classified the maximal Erdős-Ko-Rado sets $\mathcal{T}$ of planes in $\mathrm{PG}(5,q)$ with $|\mathcal{T}| \geq 3q^4+3q^3+2q^2+q+1$. In [33], De Boeck generalized these results and classified the largest examples of sets of planes pairwise intersecting in at least a point in $\mathrm{PG}(n,q), n \geq 5$. Below we retrace the examples in [13] and [33] with size at least $f(k,q)$ and such that there is no point contained in all their elements. For each example, we show that it can be extended to one of the examples discussed in the previous sections, or that it gives rise to a new maximal example.

1. $\mathcal{T}$ is the set of planes of Example II in [33]: consider a 3-space σ and a point $P_0 \in \sigma$. Let $\mathcal{T}$ be the set of all planes that either are contained in σ or else intersect σ in a line through P_0. Then $\mathcal{S}$ can be extended to Example 3.1.2(iii), with ζ the $(k+1)$-space spanned by σ and γ, and $\pi_{AB} = \langle \gamma, P_0 \rangle$.

2. $\mathcal{T}$ is the set of planes of Example III in [33]: consider a plane π, then $\mathcal{T}$ is the set of planes meeting π in at least a line. Then $\mathcal{S}$ can be extended to Example 3.1.2(ii), with ζ the k-space spanned by π and γ.

3. $\mathcal{T}$ is the set of planes of Example IV in [33]: consider a 4-space τ, a plane $\delta \subset \tau$ and a point $P_0 \in \delta$. Then $\mathcal{T}$ is the set containing the planes in τ intersecting δ in a line, the planes intersecting δ in a line through P_0 and the planes in τ through P_0. Then we can refer to Subsection 3.2.2 and so $\mathcal{S}$ can be extended to Example 3.1.2(iv), with $\rho = \langle \gamma, \tau \rangle$, $\alpha = \langle \gamma, \delta \rangle$ and $\pi = \langle \gamma, P_0 \rangle$.

4. $\mathcal{T}$ is the set of planes of Example V in [33]: consider a 4-space τ, and a line $l \subset \tau$. Then $\mathcal{T}$ is the set containing the planes through l and all planes in τ containing a point of l. Then we can refer to Subsection 3.2.1 and $\mathcal{S}$ can be extended to Example 3.1.2(v), with $\rho = \langle \gamma, \tau \rangle$ and $\alpha = \langle \gamma, l \rangle$.

5. $\mathcal{T}$ is the set of planes of Example VI in [33]: let τ_1 and τ_2 be two 4-spaces such that $\sigma = \tau_1 \cap \tau_2$ is a 3-space. Let π_1 and π_2 be two planes in σ with intersection line l_0 and let P_1 and P_2 be two different points on l_0. Then $\mathcal{T}$ is the set of planes through l_0, the planes in σ, the planes in τ_1 containing a line through P_1 in π_1 or a line through P_2 in π_2, and the planes in τ_2 containing a line through P_1 in π_2 or a line through P_2 in π_1. Then by using Section 3.2.3, Case 1, $\mathcal{S}$ can be extended to Example 3.1.2(vi) with $\rho_i = \langle \gamma, \tau_i \rangle$, $\alpha = \langle \gamma, \sigma \rangle$, $\pi_A = \langle \gamma, \pi_1 \rangle$, $\pi_B = \langle \gamma, \pi_2 \rangle$, $\lambda = \langle \gamma, l_0 \rangle$, $\lambda_A = \langle \gamma, P_1 \rangle$, $\lambda_B = \langle \gamma, P_2 \rangle$ and P_{AB} a point in γ.

6. $\mathcal{T}$ is the set of planes of Example VII in [33]: let ρ be a 5-space. Consider a line $l \subset \rho$ and a 3-space $\sigma \subset \rho$ disjoint from l. Choose three points P_1, P_2, P_3 on l and choose four non-coplanar points Q_1, Q_2, Q_3, Q_4 in σ. Denote $l_1 = Q_1Q_2$, $\bar{l}_1 = Q_3Q_4$, $l_2 = Q_1Q_3$, $\bar{l}_2 = Q_2Q_4$, $l_3 = Q_1Q_4$, and $\bar{l}_3 = Q_2Q_3$. Then $\mathcal{T}$ is the set containing all planes through l and all planes through P_i in $\langle l, l_i\rangle$ or in $\langle l, \bar{l}_i\rangle$, $i = 1, 2, 3$. Note that this set $\mathcal{S}$ is the set described in Example 3.1.2(ix). We can prove the following lemma.

   Lemma 3.4.1. *The set $\mathcal{S}$ of k-spaces described in Example 3.1.2(ix) is a maximal set of k-spaces pairwise intersecting in at least a $(k-2)$-space.*

   Proof. We have to prove that there does not exist a k-space E in $\mathrm{PG}(n,q)$, with $\gamma \nsubseteq E$ and so that E meets all elements of $\mathcal{S}$ in at least a $(k-2)$-space. Suppose there exists such a k-space E. As $\mathcal{S}$ contains all k-spaces through the $(k-1)$-space $\langle\gamma, l\rangle$, E contains a $(k-2)$-space π_0 of $\langle\gamma, l\rangle$, not through γ. Hence, $\dim(E\cap\gamma) = g-1 = k-4$. As $\mathcal{S}$ contains all k-spaces through $\langle\gamma, P_i\rangle$ in the $(k+1)$-space $\langle\gamma, l, l_i\rangle$ (or $\langle\gamma, l, \bar{l}_i\rangle$), E contains a $(k-1)$-space of each of those $(k+1)$-spaces. Consider now the quotient space $\mathrm{PG}(n,q)/\gamma$, and let $E' = \langle\gamma, E\rangle/\gamma$, $Q'_i = \langle Q_i, \gamma\rangle/\gamma$, $P'_i = \langle P_i, \gamma\rangle/\gamma$, and $l' = \langle l, \gamma\rangle/\gamma$. Then E' is a solid in $\mathrm{PG}(n,q)/\gamma$ through l' that contains a point of each of the lines $Q'_iQ'_j$, $1 \le i < j \le 4$, but this gives a contradiction as $\dim(E') = 3$. ■

7. $\mathcal{T}$ is the set of planes of Example $VIII$ in $\mathrm{PG}(n-k+2,q)$ in [33]: consider two solids σ_1 and σ_2, intersecting in a line l. Take the points P_1 and P_2 on l. Then $\mathcal{T}$ is the set containing all planes through l, all planes through P_1 that contain a line in σ_1 and a line in σ_2, and all planes through P_2 in σ_1 of σ_2. Note that this set $\mathcal{S}$ is the set described in Example3.1.2(vii). We can prove that the set $\mathcal{S}$ of k-spaces is not extendable.

   Lemma 3.4.2. *The set $\mathcal{S}$ of k-spaces described in Example 3.1.2(vii) is a maximal set of k-spaces pairwise intersecting in at least a $(k-2)$-space.*

   Proof. We have to prove that there does not exist a k-space E in $\mathrm{PG}(n,q)$, with $\gamma \nsubseteq E$ and so that E meets all elements of $\mathcal{S}$ in at least a $(k-2)$-space. Suppose there exists such a k-space E. As $\mathcal{S}$ contains all k-spaces through the $(k-1)$-space $\langle\gamma, l\rangle$, E contains a $(k-2)$-space π_0 of $\langle\gamma, l\rangle$, not through γ. Hence, $\dim(\gamma\cap E) = k-4$. As $\mathcal{S}$ contains all k-spaces through $\langle\gamma, P_2\rangle$ in the $(k+1)$-space $\langle\gamma, \sigma_1\rangle$ (or $\langle\gamma, \sigma_2\rangle$), E contains a $(k-1)$-space of each of those $(k+1)$-spaces. These two $(k-1)$-spaces, α_1 and α_2 respectively, span E and meet in a $(k-2)$-space π_0. Then we show that there exists a k-space $A \in \mathcal{S}$, containing γ, that meets E in precisely a $(k-3)$-space. Consider the quotient space $\mathrm{PG}(n,q)/\gamma$, and let $E' = \langle\gamma, E\rangle/\gamma$, $\sigma'_i = \langle\sigma_i, \gamma\rangle/\gamma$, $P'_i = \langle P_i, \gamma\rangle/\gamma$, $A' = \langle A, \gamma\rangle/\gamma$ and $l' = \langle l, \gamma\rangle/\gamma = \langle\pi_0, \gamma\rangle/\gamma$. Then E' is a solid in $\mathrm{PG}(n,q)/\gamma$ through l' that contains planes α'_1, α'_2 in σ'_1 and σ'_2 respectively. Note that $\alpha'_1\cap\alpha'_2 = l'$. Let $l_1 \in \sigma'_1$ and $l_2 \in \sigma'_2$ be two lines containing P'_1 so that $l_1\cap\alpha'_1 = l_2\cap\alpha'_2 = P'_1$, and let A' be the plane spanned by l_1 and l_2. Then $E'\cap A'$ is a point in $\mathrm{PG}(n,q)/\gamma$. Since $\gamma \subseteq A$ and $\gamma \nsubseteq E$, we find that $E\cap A$ is a $(k-3)$-space of $\langle\gamma, P_1\rangle$ in $\mathrm{PG}(n,q)$, and so these elements of $\mathcal{S}$ meet in a $(k-3)$-space, a contradiction. ■

8. $\mathcal{T}$ is the set of planes of Example IX in $\mathrm{PG}(n-k+2,q)$ in [33]: let l be a line and σ a solid skew to l. Denote $\langle l, \sigma\rangle$ by ρ. Let P_1 and P_2 be two points on l and let $\mathcal{R}_1$ and $\mathcal{R}_2$ be a regulus and its opposite regulus in σ. Then $\mathcal{T}$ is the set containing all planes through l, all planes through P_1 in the solid generated by l and a line of $\mathcal{R}_1$, and all planes through P_2 in the solid generated by l and a line of $\mathcal{R}_2$. Note that this set $\mathcal{S}$ is the set described in Example 3.1.2$(viii)$. We can prove the following lemma.

   Lemma 3.4.3. *The set $\mathcal{S}$ of k-spaces described in Example 3.1.2$(viii)$ is a maximal set of k-spaces pairwise intersecting in at least a $(k-2)$-space.*

Proof. We have to prove that there does not exist a k-space E in $\mathrm{PG}(n,q)$, with $\gamma \nsubseteq E$, and so that E meets all elements of $\mathcal{S}$ in at least a $(k-2)$-space. Suppose there exists such a k-space E. Let $\mathcal{R}_1 = \{l_1, l_2, \dots, l_{q+1}\}$ and $\mathcal{R}_2 = \{\bar{l}_1, \bar{l}_2, \dots, \bar{l}_{q+1}\}$. As $\mathcal{S}$ contains all k-spaces through the $(k-1)$-space $\langle\gamma, l\rangle$, E contains a $(k-2)$-space π_0 of $\langle\gamma, l\rangle$, not through γ. Hence, $\dim(\gamma \cap E) = k-4$. As $\mathcal{S}$ contains all k-spaces through $\langle\gamma, P_i\rangle$ in the $(k+1)$-spaces $\langle\gamma, l, l'\rangle$ (or $\langle\gamma, l, \bar{l}'\rangle$), with $l' \in \mathcal{R}_i$, E contains a $(k-1)$-space of each of those $(k+1)$-spaces. Consider now the quotient space $\mathrm{PG}(n,q)/\gamma$, and let $E' = \langle\gamma, E\rangle/\gamma$, $l'_i = \langle l_i, \gamma\rangle/\gamma$, $\bar{l}'_i = \langle\bar{l}_i, \gamma\rangle/\gamma$, $P'_i = \langle P_i, \gamma\rangle/\gamma$, and $l' = \langle l, \gamma\rangle/\gamma = \langle\pi_0, \gamma\rangle/\gamma$. Then E' is a solid in $\mathrm{PG}(n,q)/\gamma$ through l' that contains a point of each of the lines l'_i and $\bar{l}'_i$, $1 \leq i \leq q+1$, but this gives a contradiction as $\dim(E') = 3$. ■

We see that example (f), (g) and (h) give rise to maximal examples of sets $\mathcal{S}$ of k-spaces pairwise intersecting in at least a $(k-2)$-space, described in Example 3.1.2(ix), (vii), $(viii)$ respectively. From [33], it follows that the number of elements in $\mathcal{S}$ equals $\theta_{n-k} + 6q^2$, $\theta_{n-k} + q^4 + 2q^3 + 3q^2$ and $\theta_{n-k} + 2q^3 + 2q^2$ respectively.

Finally, if $k-g-1 = 1$, then $g = k-2$ and so, there is a $(k-2)$-space contained in all solids of $\mathcal{S}$. This case gives rise to Example 3.1.2(i).

3.5 Main Theorem

By collecting the results from Propositions 3.2.16, Theorem 3.3.2 and Section 3.4, we find the following result.

Main Theorem 3.5.1. *Let $\mathcal{S}$ be a maximal set of k-spaces pairwise intersecting in at least a $(k-2)$-space in $\mathrm{PG}(n,q)$, $n \geq 2k$, $k \geq 3$. Let*

$$f(k,q) = \begin{cases} 3q^4 + 6q^3 + 5q^2 + q + 1 & \text{if } k = 3, q \geq 2 \text{ or } k = 4, q = 2 \\ \theta_{k+1} + q^4 + 2q^3 + 3q^2 & \text{else.} \end{cases}$$

If $|\mathcal{S}| > f(k,q)$, then $\mathcal{S}$ is one of the families described in Example 3.1.2. Note that for $n > 2k+1$, the examples $(i) - (ix)$ are stated in decreasing order of the sizes.

Proof.

- If there is no point contained in all elements of $\mathcal{S}$ and $\mathcal{S}$ contains three k-spaces A, B, C with $\dim(A \cap B \cap C) = k-4$, then we distinguished the possibilities for $\mathcal{S}$ depending on the dimension of $\alpha = \langle D \cap \langle A, B\rangle \,|\, D \in \mathcal{S}'\rangle$, where $\mathcal{S}' = \{D \in \mathcal{S} \,|\, D \not\subset \langle A, B\rangle\}$, see Section 3.2. By Proposition 3.2.16, it follows that $\mathcal{S}$ is one of the examples (iv), (v), (vi), (x) in Example 3.1.2.
- If there is no point contained in all elements of $\mathcal{S}$ and if for every three elements A, B, C in $\mathcal{S}$, we have that $\dim(A \cap B \cap C) \geq k-3$, then the only possibilities for $\mathcal{S}$ are described in Example 3.1.2 (ii) and (iii), see Theorem 3.3.2.
- If there is at least a point contained in all k-spaces of $\mathcal{S}$, then we refer to Section 3.4. Let γ be the maximal subspace contained in all k-spaces of $\mathcal{S}$, with $\dim(\gamma) = g$. Then $\mathcal{T} = \{D/\gamma \,|\, D \in \mathcal{S}\}$ is a set of $(k-g-1)$-spaces of $\mathrm{PG}(n-g-1,q) \simeq \mathrm{PG}(n,q)/\gamma$ pairwise intersecting in at least a $(k-g-3)$-space. The only examples of sets $\mathcal{T}$ that give rise to maximal examples of sets of k-spaces are described in Section 3.4 in the examples (f), (g), (h). In these examples, $g = k-3$. They correspond to Example 3.1.2(i), (ix), (vii), $(viii)$.

■

4 Hilton–Milner problems in $\mathrm{PG}(n,q)$ and $\mathrm{AG}(n,q)$

> *Equations are just the boring part of mathematics. I attempt to see things in terms of geometry.*
>
> —Stephen Hawking

The results in this chapter will appear in [43].

4.1 Introduction

Before we start with the introduction, we would like to indicate how this chapter came about. We started investigating the Hilton-Milner problem in the affine context: we studied the second largest examples of sets of affine k-spaces pairwise intersecting in at least a t-space in $\mathrm{AG}(n,q)$. Thanks to prof. Tamás Szőnyi, we received notes of David Ellis about the projective analogue of this problem [54]. In these notes, he studied the second largest families of projective k-spaces, pairwise intersecting in at least a t-space in $\mathrm{PG}(n,q)$. These notes helped me to shorten my, affine, arguments. Since these notes are not published, we integrate them in this chapter. The results that are mostly influenced by the ideas in the notes of David Ellis are Lemmas 4.4.3, 4.4.4, 4.4.5 and 4.4.6. In his notes, David Ellis used the kernel method *[67, Section* 15.1*].*
While finishing the last details of this project, the paper [29] appeared on Arxiv. In that paper, the authors deduce similar results as ours in the vector space setting. It is worth noting that our results were obtained independently, and our paper deals with both the affine and projective case at once. A comparison between the results of this chapter and the results in [29] is given in Remark 4.4.8.

In [69], Guo and Xu investigated the Erdős-Ko-Rado problem in affine spaces. They proved that the largest t-intersecting family of k-spaces in $\mathrm{AG}(n,q)$, $n \geq 2k+t+2$, is the set of all k-spaces through a fixed t-space. In Section 4.4.2, we give a shorter proof for their result and improve their bound on n to $n \geq 2k+1$. For $t=0$, the second largest t-intersecting set of k-spaces in $\mathrm{PG}(n,q)$ and $\mathrm{AG}(n,q)$ were already described in [12] (see Theorem 2.0.5) and [68] respectively. We describe the result from [68] in Theorem 4.4.10. The main goal in this chapter is to describe the second largest Erdős-Ko-Rado sets for $t \geq 1$, for both $\mathrm{PG}(n,q)$ and $\mathrm{AG}(n,q)$.

In Section 4.2 and in Section 4.3, we give two examples of maximal sets of k-spaces in $\mathrm{PG}(n,q)$ and $\mathrm{AG}(n,q)$, respectively, pairwise intersecting in at least a t-space, which are not t-pencils. In Section 4.4, we prove the Hilton-Milner results.

4.2 Two examples in $\mathrm{PG}(n,q)$

We start by giving two examples of maximal sets of k-spaces in $\mathrm{PG}(n,q)$, pairwise meeting in at least a t-space. Note that for $n \leq 2k-t$, all projective k-spaces in $\mathrm{PG}(n,q)$ are pairwise intersecting in at least a t-space. Hence, we may suppose that $n \geq 2k-t+1$.

Example 4.2.1. *Let δ be a t-space, $t \leq k-1$, in $\mathrm{PG}(n,q)$, $n \geq 2k-t+1$, and let ξ be a $(k+1)$-space in $\mathrm{PG}(n,q)$ with $\delta \subset \xi$. Let S_1 be the set of all k-spaces in ξ. Let S_2 be the set of all k-spaces through δ and meeting ξ in at least a $(t+1)$-space. Let $\mathcal{S}$ be the union of the sets S_1 and S_2.*

Lemma 4.2.2. *The set $\mathcal{S}$, described in Example 4.2.1, is a maximal set of k-spaces in $\mathrm{PG}(n,q)$, $n \geq 2k-t+1$, pairwise intersecting in at least a t-space, of size*

$$|\mathcal{S}| = \theta_{k+1} - \theta_{k-t} + \begin{bmatrix} n-t \\ k-t \end{bmatrix} - q^{(k-t+1)(k-t)} \begin{bmatrix} n-k-1 \\ k-t \end{bmatrix}.$$

Proof. We start with determining the size of $\mathcal{S}$. First note that the number of elements of $S_1 \setminus S_2$ is equal to the number of k-spaces in the $(k+1)$-space ξ, not containing δ. Hence, $|S_1 \setminus S_2| = \theta_{k+1} - \theta_{k-t}$.

All elements of S_2 contain δ. To determine $|S_2|$, we consider the quotient space $\mathrm{PG}(n,q)/\delta$, which is isomorphic to $\mathrm{PG}(n-t-1,q)$. Let σ_0 be the projective $(k-t)$-space in $\mathrm{PG}(n,q)/\delta$, corresponding to ξ. A $(k-t-1)$-space, corresponding to an element of S_2 in $\mathrm{PG}(n,q)/\delta$ has at least a point in common with σ_0. Hence, $|S_2|$ is the number of $(k-t-1)$-spaces in $\mathrm{PG}(n-t-1)$, minus the number of $(k-t-1)$-spaces, disjoint from σ_0. From Lemma 1.10.1, we have that $|S_2| = \begin{bmatrix} n-t \\ k-t \end{bmatrix} - q^{(k-t+1)(k-t)} \begin{bmatrix} n-k-1 \\ k-t \end{bmatrix}$. Hence,

$$|\mathcal{S}| = \theta_{k+1} - \theta_{k-t} + \begin{bmatrix} n-t \\ k-t \end{bmatrix} - q^{(k-t+1)(k-t)} \begin{bmatrix} n-k-1 \\ k-t \end{bmatrix}. \tag{4.1}$$

It is clear that all elements of S_2 pairwise meet in at least the t-space δ. Every two elements of S_1 meet in a $(k-1)$-space, since they are contained in a $(k+1)$-space. Note that $k-1 \geq t$. Consider now a k-space π_1 in S_1 and a k-space π_2 in S_2. Note that $\pi_1 \subset \xi$, and π_2 meets ξ in at least a $(t+1)$-space. Again, from the Grassmann dimension property, it follows that they meet in at least a t-space.

Now we prove that $\mathcal{S}$ cannot be extended to a larger set of k-spaces pairwise intersecting in at least a t-space. Suppose that $\alpha \notin \mathcal{S}$ is a k-space that meets every element of $\mathcal{S}$ in at least a t-space. If $\delta \subset \alpha$, then, since $\alpha \notin \mathcal{S}$, α meets ξ only in δ. Hence, there is an element π of S_1 such that $\dim(\pi \cap \delta) = t-1$, and so, $\dim(\pi \cap \alpha) < t$. This gives a contradiction with the fact that α meets all elements of $\mathcal{S}$ in at least a t-space. Hence, we may suppose that $\delta \nsubseteq \alpha$. So, α meets δ in a d-space with $d \leq t-1$. Note that $\dim(\alpha \cap \xi) \geq t+1$ since α meets all elements of S_1 in at least a t-space. Let $\pi_0 \subset \xi$ be a $(k-t-1)-$space disjoint from δ. For every point $P \in \pi_0$, consider the set $\mathcal{S}_P$ of elements of $\mathcal{S}$ that meet ξ in $\langle \delta, P \rangle$. If $\dim(\alpha \cap \langle \delta, P \rangle) < t$, then α must meet all elements of $\mathcal{S}_P$ in a subspace outside of ξ. We now prove that this gives a contradiction since $n \geq 2k-t+1$. Let $\alpha \cap \langle P, \delta \rangle = \nu$ and suppose that $\dim(\nu) = r < t$. We investigate the quotient space $\mathrm{PG}(n,q)/\nu$, and let α' be the subspace in this quotient space corresponding to α. Let β be a k-space through $\langle P, \delta \rangle$ with β' be the corresponding subspace in $\mathrm{PG}(n,q)/\nu$, such that $\dim(\langle \alpha', \beta' \rangle)$ is maximal. Hence, $\dim(\langle \alpha', \beta' \rangle) = \min\{n-r-1, 2k-2r-1\}$. From the Grassmann dimension property, and since α and β have at least a $(t-r-1)$-space in common in the quotient space, we then have that

$$\begin{aligned} \dim(\alpha' \cap \beta') &= \dim(\alpha') + \dim(\beta') - \dim(\langle \alpha', \beta' \rangle) \\ \Rightarrow t-r-1 &\leq 2k-2r-2-\min\{n-r-1, 2k-2r-1\}. \end{aligned}$$

This gives a contradiction since $r < t$ and $n \geq 2k-t+1$. Hence, $\dim(\alpha \cap \langle \delta, P \rangle) = t$ for all points $P \in \pi_0$. This implies that $\dim(\alpha \cap \delta) = t-1$, and α must have a t-space in common with all $(t+1)$-spaces $\langle \delta, P \rangle$ with $P \in \pi_0$. Hence, $\alpha \subseteq \xi$, and so $\alpha \in S_1$, a contradiction. ■

Example 4.2.3. *Suppose* $k \geq t+1$ *and let* ω *be a* $(t+2)$*-space in* $\mathrm{PG}(n,q)$*,* $n \geq 2k-t+1$*. Let* $\mathcal{S}$ *be the set of all* k*-spaces in* $\mathrm{PG}(n,q)$*, meeting* ω *in at least a* $(t+1)$*-space.*

Lemma 4.2.4. *The set* $\mathcal{S}$*, described in Example 4.2.3, is a maximal set of* k*-spaces in* $\mathrm{PG}(n,q)$*, pairwise intersecting in at least a* t*-space, of size*

$$|\mathcal{S}| = \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} + \theta_{t+2} \cdot \left(\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} - \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \right).$$

Proof. The number of elements in $\mathcal{S}$ is the number of k-spaces through ω, together with the number of k-spaces, meeting ω in a $(t+1)$-space:

$$|\mathcal{S}| = \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} + \theta_{t+2} \cdot \left(\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} - \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \right).$$

Consider two elements $\pi_1, \pi_2 \in \mathcal{S}$. Then $\pi_1 \cap \omega$ and $\pi_2 \cap \omega$ are two subspaces with dimension at least $t+1$ in a $(t+2)$-space, and so, they meet in at least a t-space.

Now we prove that $\mathcal{S}$ cannot be extended to a larger set of k-spaces pairwise intersecting in at least a t-space. Suppose that $\alpha \notin \mathcal{S}$ is a k-space that meets every element of $\mathcal{S}$ in at least a t-space. Since $\alpha \notin \mathcal{S}$, we know that $\dim(\alpha \cap \omega) \leq t$. Let γ be a $(t+1)$-space in ω such that $\dim(\alpha \cap \omega \cap \gamma) \leq t-1$. Then α must meet all elements of $\mathcal{S}$ through γ in a subspace outside of ω. Since $n \geq 2k-t+1$, this is not possible. Hence, $\mathcal{S}$ cannot be extended. ■

Remark 4.2.5. Note that for $k = t+1$, Example 4.2.1 and Example 4.2.3 coincide. In that case, $\mathcal{S}$ is the set of all $(t+1)$-spaces in a fixed $(t+2)$-space in $\mathrm{PG}(n,q)$, see Theorem 2.0.6.

Remark 4.2.6. In the previous chapter, k-spaces pairwise intersecting in at least a $(k-2)$-space in $\mathrm{PG}(n,q)$ were investigated. For $t = k-2$, Example 4.2.1 coincides with Example 3.1.2(iii), and Example 4.2.3 coincides with Example 3.1.2(ii).

4.3 Two examples in $\mathrm{AG}(n,q)$

We also give two examples of maximal sets of k-spaces in $\mathrm{AG}(n,q)$, pairwise meeting in at least a t-space. For the remainder of this chapter, we suppose that $n \geq 2k-t+1$ and $t \geq 1$. In Section 4.4, we prove that the largest non-trivial sets of k-spaces, pairwise meeting in at least a t-space, in $\mathrm{AG}(n,q)$ are given by Examples 4.3.1 and 4.3.3. If $k \geq 2t+2$, Example 4.3.1 is the largest set, whereas if $k \leq 2t+1$, Example 4.3.3 is the largest one.

For an affine subspace α, we denote the projective extension of α by $\tilde{\alpha}$, and let $H_\infty = \mathrm{PG}(n,q) \setminus \mathrm{AG}(n,q)$ be the hyperplane at infinity. Similarly, if $\mathcal{S} = \{\pi_1, \pi_2, \dots, \pi_m\}$ is a set of affine spaces, then we denote the corresponding set of projective spaces by $\tilde{\mathcal{S}} = \{\tilde{\pi}_1, \tilde{\pi}_2, \dots, \tilde{\pi}_m\}$.

Example 4.3.1. *Let* δ *be a* t*-space,* $t \leq k-1$*, in* $\mathrm{AG}(n,q)$*, and let* ξ *be a* $(k+1)$*-space in* $\mathrm{AG}(n,q)$ *with* $\delta \subset \xi$*. Let* S_1 *be a maximal set of affine* k*-spaces in* ξ*, such that for any two elements* π_1, π_2 *of* S_1*,* $\tilde{\pi}_1 \cap H_\infty \neq \tilde{\pi}_2 \cap H_\infty$*, and such that for every* $\pi_1 \in S_1$*:* $\tilde{\delta} \cap H_\infty \not\subseteq \tilde{\pi}_1$*. Let* S_2 *be the set of all* k*-spaces through* δ *and meeting* ξ *in at least a* $(t+1)$*-space. Let* $\mathcal{S}$ *be the union of the sets* S_1 *and* S_2*.*

Note that this example corresponds to the affine case of Example 4.2.1.

Lemma 4.3.2. *The set $\mathcal{S}$, described in Example 4.3.1, is a maximal set of k-spaces in $\mathrm{AG}(n,q)$, $n \geq 2k-t+1$, pairwise intersecting in at least a t-space, of size*

$$|\mathcal{S}| = \theta_k - \theta_{k-t} + \begin{bmatrix} n-t \\ k-t \end{bmatrix} - q^{(k-t+1)(k-t)} \begin{bmatrix} n-k-1 \\ k-t \end{bmatrix}. \tag{4.2}$$

Proof. We start with determining the size of $\mathcal{S}$. Note first that the number of elements of S_1 is equal to the number of $(k-1)$-spaces in $H_\infty \cap \xi$, not containing $\delta \cap H_\infty$. Hence, $|S_1| = \theta_k - \theta_{k-t}$.

Let $\tilde{\sigma_0}$ be the projective $(k-t)$-space, corresponding to $\tilde{\xi}$ in the quotient space $\mathrm{PG}(n,q)/\tilde{\delta}$. An extended element of S_2 to $\mathrm{PG}(n,q)$, corresponds to a $(k-t-1)$-space in $\mathrm{PG}(n,q)/\tilde{\delta}$, that has at least a point in common with $\tilde{\sigma_0}$. Hence, $|S_2|$ is the number of projective $(k-t-1)$-spaces in $\mathrm{PG}(n,q)/\tilde{\delta} \cong \mathrm{PG}(n-t-1,q)$, minus the number of $(k-t-1)$-spaces, disjoint from $\tilde{\sigma_0}$. By Lemma 1.10.1, we have that $|S_2| = \begin{bmatrix} n-t \\ k-t \end{bmatrix} - q^{(k-t+1)(k-t)} \begin{bmatrix} n-k-1 \\ k-t \end{bmatrix}$. Hence,

$$|\mathcal{S}| = \theta_k - \theta_{k-t} + \begin{bmatrix} n-t \\ k-t \end{bmatrix} - q^{(k-t+1)(k-t)} \begin{bmatrix} n-k-1 \\ k-t \end{bmatrix}. \tag{4.3}$$

It is clear that all elements of S_2 pairwise meet in at least a t-space (δ). Consider now two elements $\pi_1, \pi_2 \in S_1$. It follows, from the Grassmann dimension property, that $\tilde{\pi}_1 \cap \tilde{\pi}_2$ is a $(k-1)$-space in the $(k+1)$-space $\tilde{\xi}$. This $(k-1)$-space is not contained in H_∞ by the definition of S_1. Let π_1 be a k-space in S_1 and let π_3 be a k-space in S_2. Since $\pi_1 \subset \xi$, and $\dim(\pi_3 \cap \xi) \geq t+1$, we know, again by the Grassmann dimension property, that $\tilde{\pi}_1 \cap \tilde{\pi}_3$ meet in at least a projective t-space. Now, $\tilde{\pi}_1 \cap \tilde{\pi}_3$ is not contained in H_∞, since there is an affine $(t-1)$-space contained in both π_1 and π_3.

Now we prove that $\mathcal{S}$ cannot be extended to a larger set of k-spaces pairwise intersecting in at least an affine t-space. Suppose that $\alpha \notin \mathcal{S}$ is an affine k-space that meets every element of $\mathcal{S}$ in at least an affine t-space. If α contains δ, then, since $\alpha \notin \mathcal{S}$, we know that $\alpha \cap \xi = \delta$. Let $\pi \in S_1$ with $\delta \not\subseteq \pi$. Then α meets π only in a $(t-1)$-space, and so, there is an element of $\mathcal{S}$ that meets α not in a t-space, which contradicts the statement. Hence, we may suppose that $\delta \not\subseteq \alpha$, and this implies that $\dim(\alpha \cap \delta) \leq t-1$. Note that there is no affine t-space contained in all elements of S_1, as $t \geq 1$. Hence, we have that $\dim(\alpha \cap \xi) \geq t+1$ as α meets all elements of S_1 in at least a t-space. Let π_0 be a projective $(k-t)$-space in $\tilde{\xi} \setminus \tilde{\delta}$. For every point $P \in \pi_0$, consider the set $\mathcal{S}_P$ of elements of $\tilde{\mathcal{S}}$ that meet $\tilde{\xi}$ in $\langle \tilde{\delta}, P \rangle$. If $\dim(\tilde{\alpha} \cap \langle \tilde{\delta}, P \rangle) < t$, then $\tilde{\alpha}$ must meet all elements of $\mathcal{S}_P$ in a subspace outside of $\tilde{\xi}$. This gives a contradiction since $n \geq 2k-t+1$. Hence, $\dim(\tilde{\alpha} \cap \langle \tilde{\delta}, P \rangle) = t$ for all points $P \in \pi_0$. This implies that $\dim(\alpha \cap \delta) = t-1$, and $\tilde{\alpha}$ must have a t-space in common with all $(t+1)$-spaces $\langle \tilde{\delta}, P \rangle$, with $P \in \pi_0$. Hence, $\alpha \subseteq \xi$, and so $\alpha \in S_1$, a contradiction, since we supposed that $\alpha \notin \mathcal{S}$. ■

Example 4.3.3. *Suppose $k \geq t+1$. Let ω be an affine $(t+2)$-space in $\mathrm{AG}(n,q)$, and let $\mathcal{R}$ be a set of θ_{t+1} affine $(t+1)$-spaces in ω such that $\mathcal{R}$ contains precisely one element through every t-space in $H_\infty \cap \tilde{\omega}$. Note that every two different elements of $\mathcal{R}$ meet in an affine t-space. Let $\mathcal{S}$ be the set of all k-spaces in $\mathrm{AG}(n,q)$, containing ω or meeting ω in an element of $\mathcal{R}$.*

Note that this example corresponds to the affine case of Example 4.2.3.

Lemma 4.3.4. *The set $\mathcal{S}$, described in Example 4.3.3, is a maximal set of k-spaces in $\mathrm{AG}(n,q)$, $n \geq 2k-t+1$, pairwise intersecting in at least a t-space, of size*

$$|\mathcal{S}| = \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} + \theta_{t+1} \cdot \left(\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} - \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \right).$$

Proof. Since $\mathcal{R}$ is a maximal set, we have that $|\mathcal{R}|$ is the number of all t-spaces in $\tilde{\omega} \cap H_\infty$. Hence, $|\mathcal{R}| = \theta_{t+1}$. The number of elements in $\mathcal{S}$ is the number of k-spaces through ω, together with the number of k-spaces, meeting ω in an element of $\mathcal{R}$:

$$|\mathcal{S}| = \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} + \theta_{t+1} \cdot \left(\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} - \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \right).$$

Consider two elements $\pi_1, \pi_2 \in \mathcal{S}$. If π_1 or π_2 contains ω, then π_1 and π_2 intersect in at least a $(t+1)$-dimensional space. Hence, we suppose that $\pi_1 \cap \omega$ and $\pi_2 \cap \omega$ are two $(t+1)$-spaces of $\mathcal{R}$ in a $(t+2)$-space. Since every two elements of $\mathcal{R}$ meet in an affine space with dimension at least t, we have that π_1 and π_2 meet in at least an affine t-space.

Now we prove that $\mathcal{S}$ cannot be extended to a larger set of k-spaces pairwise intersecting in at least a t-space. Suppose that $\alpha \notin \mathcal{S}$ is an affine k-space that meets every element of $\mathcal{S}$ in at least a t-space. Consider an element $\sigma \in \mathcal{R}$. Since α must meet all affine k-spaces through σ in at least a t-space, we find that α contains a t-space of σ, as $n \geq 2k - t + 1$. As σ is an arbitrary element of $\mathcal{R}$, we see that α must meet every element of $\mathcal{R}$ in at least an affine t-space. As $t \geq 1$, there cannot be an affine t-space contained in all elements of $\mathcal{R}$. This implies that α meets ω in a $(t+1)$-space α_ω. Now, α_ω must meet every element of $\mathcal{R}$ in an affine t-space. From the maximality of $\mathcal{R}$, we have that $\alpha_\omega \in \mathcal{R}$, and so that $\alpha \in \mathcal{S}$, a contradiction. ■

4.4 Classification results

We start with a classification result on maximal sets of k-spaces pairwise intersecting in a $(k-1)$-space. In the projective case, we know that a set of k-spaces, pairwise intersecting in a $(k-1)$-space in $\mathrm{PG}(n,q)$, $n \geq k+2$, is a set of k-spaces through a fixed $(k-1)$-space or a set of k-spaces such that each element is contained in a fixed $(k+1)$-space, see Theorem 2.0.6.

We use this classification to deduce the classification of maximal sets of k-spaces pairwise intersecting in a $(k-1)$-space in $\mathrm{AG}(n,q)$.

Theorem 4.4.1. *Let $\mathcal{S}$ be a set of k-spaces in $\mathrm{AG}(n,q)$, $n \geq k+1$, pairwise intersecting in a $(k-1)$-space such that $\mathcal{S}$ is not a $(k-1)$-pencil, then $|\mathcal{S}| \leq \theta_k$, and equality occurs if and only if $\mathcal{S}$ is Example 4.3.3 for $t = k-1$. Hence, all elements of $\mathcal{S}$ are contained in a $(k+1)$-space.*

Proof. As before, let $\tilde{\mathcal{S}}$ be the set of projective extensions of the elements in $\mathcal{S}$. So, $\tilde{\mathcal{S}}$ is a set of projective k-spaces pairwise intersecting in a $(k-1)$-space, and such that there is no $(k-1)$-space contained in all these elements. Hence, $\tilde{\mathcal{S}}$ is contained in a $(k+1)$-space Π, by Theorem 2.0.6. Now, every two elements of $\mathcal{S}$ must meet in $\mathrm{AG}(n,q)$. So, for every two elements $\pi_1, \pi_2 \in \mathcal{S}$, $\tilde{\pi}_1 \cap \tilde{\pi}_2 \nsubseteq H_\infty$. This implies that every k-space in $\Pi \cap H_\infty$ is contained in precisely one element of $\tilde{\mathcal{S}}$. This is Example 4.3.3, for $k = t+1$, which proves the theorem. ■

Remark 4.4.2. Note that for $t = k-1$, the set of all examples described in Example 4.3.1 is a subset of the set of examples in Example 4.3.3. This follows since for $t = k-1$, the k-spaces of a set $\mathcal{S}$ from Example 4.3.1, are contained in a fixed $(t+2)$- or, $(k+1)$-space (ξ). Moreover, the set of examples in Example 4.3.1 and 4.3.3 are not equal, since in Example 4.3.1, an extra condition is imposed. For these sets, all k-spaces $\pi \in \tilde{\mathcal{S}}$ through $\tilde{\delta} \cap H_\infty$ contain δ.

For $t = k-1$, the number of elements of Example 4.3.3 (and so of Example 4.3.1), is θ_k, while, the number of affine subspaces in $\mathrm{AG}(n,q)$ through a fixed affine $(k-1)$-space is θ_{n-k}. Hence, for

$n < 2k$, Example 4.3.3, is the largest example of a set of affine k-spaces, pairwise intersecting in at least a $(k-1)$-space.

From now on, we suppose that $k \geq t+2$. In Section 4.4.1 and Section 4.4.3, we classify the largest non-trivial t-intersecting sets of k-spaces in $\mathrm{PG}(n,q)$ and $\mathrm{AG}(n,q)$, respectively. In Section 4.4.2, we give a shorter proof of the classification result for the largest t-intersecting sets of k-spaces in $\mathrm{AG}(n,q)$, which was first proven in [69]. We will also improve the bound on n in their result to $n \geq 2k+1$. As mentioned in the introduction, several ideas in the following subsection are based on the notes of David Ellis [54].

4.4.1 Classification result in $\mathrm{PG}(n,q)$

Let $\mathcal{S}_p$ be a maximal set of k-spaces in $\mathrm{PG}(n,q)$, $n \geq 2k-t+1$, $k \geq t+2$, and $t \geq 1$, pairwise meeting in at least a t-space. Let

$$\psi(\mathcal{S}_p) = \min\{\, \dim(T) \mid T \subset \mathrm{PG}(n,q),\ \dim(T \cap \alpha) \geq t,\ \forall \alpha \in \mathcal{S}_p\}.$$

Note that $\psi(S_p)$ is well-defined. Every element $\beta \in \mathcal{S}_p$ is an example of a subspace such that $\dim(\beta \cap \alpha) \geq t, \forall \alpha \in \mathcal{S}_p$. Let $\mathcal{T}$ be the collection of all $\psi(\mathcal{S}_p)$-dimensional spaces in $\mathrm{PG}(n,q)$, that meet every element of $\mathcal{S}_p$ in at least a t-space.

Lemma 4.4.3. *We have the following properties for $\psi(\mathcal{S}_p)$ and $\mathcal{T}$.*

1. *We have that $t \leq \psi(\mathcal{S}_p) \leq k$, and if $\psi(\mathcal{S}_p) = t$, then $\mathcal{S}_p$ is a t-pencil.*
2. *If $T \in \mathcal{T}$, then all k-spaces through T are contained in $\mathcal{S}_p$.*
3. *The elements of $\mathcal{T}$ are t-intersecting in $\mathrm{PG}(n,q)$.*

Proof.

1. Let $\pi_1 \in \mathcal{S}_p$. Since every element of $\mathcal{S}_p$ meets π_1 in at least a t-space, we have that $\psi(\mathcal{S}_p) \leq k$. Let $T \in \mathcal{T}$. Since all elements of $\mathcal{S}_p$ meet T in at least a t-space, we have that $\psi(\mathcal{S}_p) \geq t$. If $\psi(\mathcal{S}_p) = t$, then all elements of $\mathcal{S}_p$ contain the t-space T, and, hence, $\mathcal{S}_p$ is a t-pencil.
2. This property follows from the maximality of $\mathcal{S}_p$.
3. Suppose that there are two elements $T_1, T_2 \in \mathcal{T}$, with $\dim(T_1 \cap T_2) = l < t$. Since $n \geq 2k-t+1$, there are two k-spaces π_1 and π_2 through T_1 and T_2, respectively, such that $\dim(\pi_1 \cap \pi_2) < t$. From the second item, we have that $\pi_1, \pi_2 \in \mathcal{S}_p$, a contradiction since they have no t-space in common. ■

Lemma 4.4.4. *Let $\psi(\mathcal{S}_p) = t+x$, $x \geq 1$, $k \geq t+2$, $t \geq 1$ and $n \geq 2k-t+1$. Then the number of elements of $\mathcal{S}_p$ through a projective $(t+x-j)$-space, with $j \in \{0,1,2,\dots,x\}$, is at most $(\theta_{k-t})^j \begin{bmatrix} n-t-x \\ k-t-x \end{bmatrix}$.*

Proof. Let $\psi(\mathcal{S}_p) = t+x,\ x \geq 2$. We prove, by induction on $j \in \{0,1,2,\dots,x\}$, that the number of k-spaces of $\mathcal{S}_p$ through a $(t+x-j)$-space is at most $\begin{bmatrix} n-t-x \\ k-t-x \end{bmatrix} (\theta_{k-t})^j$. Note that the statement is true for $j=0$. Let $j \in \{1,2,3,\dots,x\}$ and suppose now that the number of k-spaces of $\mathcal{S}_p$ through a projective $(t+x-j_0)$-space, is at most $(\theta_{k-t})^{j_0} \begin{bmatrix} n-t-x \\ k-t-x \end{bmatrix}$, for all $j_0 < j$. Then we prove that this also holds for j. Consider a projective $(t+x-j)$-space γ_j. Since $\psi(\mathcal{S}_p) = t+x$, we know that there exists a k-space π_j of $\mathcal{S}_p$, meeting γ_j in at most a $(t-1)$-space. Let $\max\{\dim(\gamma_j \cap \pi) | \pi \in \mathcal{S}_p, \dim(\gamma_j \cap \pi) < t\} = t-l$, then $l \geq 1$, and suppose that $\pi_j \in \mathcal{S}_p$ is an element such that $\dim(\pi_j \cap \gamma_j) = t-l$. Let $\pi_{j\gamma}$ be a projective $(k-t+l-1)$-space in $\pi_j \setminus \gamma_j$. Then every element of $\mathcal{S}_p$ through γ_j contains at least an $(l-1)$-space of $\pi_{j\gamma}$. Since there are $\begin{bmatrix} k-t+l \\ l \end{bmatrix}$ subspaces of dimension $l-1$ in $\pi_{j\gamma}$, and since the number of projective k-spaces through a $(t+x-j+l)$-space is at most $(\theta_{k-t})^{j-l}\begin{bmatrix}n-t-x\\k-t-x\end{bmatrix}$, we find that the number of elements of $\mathcal{S}_p$ through γ_j is at most $\begin{bmatrix}k-t+l\\l\end{bmatrix}(\theta_{k-t})^{j-l}\begin{bmatrix}n-t-x\\k-t-x\end{bmatrix}$. Note that

$$\begin{aligned}\begin{bmatrix}k-t+l\\l\end{bmatrix}(\theta_{k-t})^{j-l} &= \frac{(q^{k-t+l}-1)\dots(q^{k-t+1}-1)}{(q^l-1)\dots(q-1)}(\theta_{k-t})^{j-l}\\ &\leq \left(\frac{(q^{k-t+1}-1)}{(q-1)}\right)^l(\theta_{k-t})^{j-l}=(\theta_{k-t})^j.\end{aligned}$$

Hence, we find that the number of elements of $\mathcal{S}_p$ through γ_j is at most $(\theta_{k-t})^j\begin{bmatrix}n-t-x\\k-t-x\end{bmatrix}$. ■

Lemma 4.4.5. *Let $\mathcal{S}_p$ be a maximal set of k-spaces, pairwise intersecting in at least a t-space in* $\mathrm{PG}(n,q)$. *If $\psi(\mathcal{S}_p)=t+x$, $x\geq 2$, $k\geq t+2$, $t\geq 1$, and $n\geq 2k-t+1$, then $|\mathcal{S}_p|\leq(\theta_{k-t})^x\begin{bmatrix}n-t-x\\k-t-x\end{bmatrix}\begin{bmatrix}t+x+1\\t+1\end{bmatrix}$.*

Proof. Suppose that $\psi(\mathcal{S}_p)=t+x,\ x\geq 2$. By Lemma 4.4.4, we know, for $j\in\{0,1,2,\dots,x\}$, that the number of k-spaces of $\mathcal{S}_p$ through a $(t+x-j)$-space is at most $\begin{bmatrix}n-t-x\\k-t-x\end{bmatrix}(\theta_{k-t})^j$.

Consider now an element $T\in\mathcal{T}$. Then every element of $\mathcal{S}_p$ meets T in at least a t-space. Since there are $\begin{bmatrix}t+x+1\\t+1\end{bmatrix}$ projective t-spaces in T and since every t-space is contained in at most $(\theta_{k-t})^x\begin{bmatrix}n-t-x\\k-t-x\end{bmatrix}$ elements of $\mathcal{S}_p$, we find that $\mathcal{S}_p$ has at most $(\theta_{k-t})^x\begin{bmatrix}n-t-x\\k-t-x\end{bmatrix}\begin{bmatrix}t+x+1\\t+1\end{bmatrix}$ elements. ■

Lemma 4.4.6. *Let $\mathcal{S}_p$ be a maximal set of k-spaces, pairwise intersecting in at least a t-space in* $\mathrm{PG}(n,q)$, *$n\geq 2k-t+1, k\geq t+1$ and $t\geq 1$. If $\psi(\mathcal{S}_p)=t+1$ and $|\mathcal{T}|\leq 2$, then*

$$|\mathcal{S}_p|\leq 2\begin{bmatrix}n-t-1\\k-t-1\end{bmatrix}+(\theta_{t+1}\theta_{k-t}-\theta_{t+1}-1)\theta_{k-t}\begin{bmatrix}n-t-2\\k-t-2\end{bmatrix}.$$

Proof. Let T be a $(t+1)$-space of $\mathcal{T}$. Since $\mathcal{S}_p$ is a maximal set, we know that all $\begin{bmatrix}n-t-1\\k-t-1\end{bmatrix}$ subspaces of dimension k, through T, are contained in $\mathcal{S}_p$. Now we determine the size of the set $\mathcal{S}_{p0}$ of k-spaces of $\mathcal{S}_p$ not through T. For every $\pi\in\mathcal{S}_{p0}$, $\dim(\pi\cap T)=t$. Let E be a t-space in T, then there exists an element $\alpha\in\mathcal{S}_{p0}$ not through E, and so $\dim(\alpha\cap E)=t-1$. Hence, every element π of $\mathcal{S}_{p0}$ through E must contain a $(t+1)$-space τ, different from T, such that $E\subset\tau$ and $\tau\cap(\alpha\setminus E)\neq\emptyset$. Note that there are $\theta_{k-t}-1$ possibilities for τ. Fix such a $(t+1)$-space τ.

- If $\mathcal{T}=\{T\}$, we know that $\tau\notin\mathcal{T}$, and hence there exists an element σ of $\mathcal{S}_p$, meeting τ in at most a $(t-1)$-space. Hence, every element of $\mathcal{S}_{p0}$ through τ meets $\sigma\setminus\tau$, and so the number of elements of $\mathcal{S}_{p0}$ through τ is at most $\theta_{k-t}\begin{bmatrix}n-t-2\\k-t-2\end{bmatrix}$. Since there are θ_{t+1} possibilities for E, and at most $\theta_{k-t}-1$ for τ, we have that

$$|\mathcal{S}_p|\leq\begin{bmatrix}n-t-1\\k-t-1\end{bmatrix}+\theta_{t+1}(\theta_{k-t}-1)\theta_{k-t}\begin{bmatrix}n-t-2\\k-t-2\end{bmatrix}.$$

- Suppose $|\mathcal{T}|=2$, and let $\mathcal{T}=\{T,\Psi\}$. If $\tau=\Psi$, then $\mathcal{S}_p$ contains all $\begin{bmatrix}n-t-1\\k-t-1\end{bmatrix}$ k-spaces through τ. If $\tau\neq\Psi$, then we can follow the argument in the previous item, and we find that the number of elements of $\mathcal{S}_{p0}$ through τ is at most $\theta_{k-t}\begin{bmatrix}n-t-2\\k-t-2\end{bmatrix}$. Note that there are $\theta_{t+1}-1$ possibilities for $E\neq T\cap\Psi$. If $E\neq T\cap\Psi$, there are at most $\theta_{k-t}-1$ possibilities for $\tau\notin\{\Psi,T\}$, through E. Furthermore, if $E=T\cap\Psi$, there are at most $\theta_{k-t}-2$ possibilities for $\tau \notin \{\Psi, T\}$ through $E = T \cap \Psi$. Hence, we have that

$$\begin{aligned}
|\mathcal{S}_p| \leq & \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + \sum_{E \subset T} \sum_{\tau \supseteq E} |\{\pi \in \mathcal{S}_{p0} | \tau \subset \pi\}| \\
\leq & \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + \sum_{E \neq T \cap \Psi} \sum_{\tau \supseteq E} \theta_{k-t} \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} + \sum_{\tau \supset T \cap \Psi} |\{\pi \in \mathcal{S}_{p0} | \tau \subset \pi\}| \\
\leq & \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + (\theta_{t+1} - 1)(\theta_{k-t} - 1)\theta_{k-t} \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \\
& \qquad + \sum_{\tau \neq \Psi, T} \theta_{k-t} \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} + \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} \\
\leq & 2 \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + (\theta_{t+1} - 1)(\theta_{k-t} - 1)\theta_{k-t} \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} + (\theta_{k-t} - 2)\theta_{k-t} \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \\
= & 2 \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + (\theta_{t+1}\theta_{k-t} - \theta_{t+1} - 1)\theta_{k-t} \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}.
\end{aligned}$$

The lemma follows since

$$\begin{aligned}
2 \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} & + (\theta_{t+1}\theta_{k-t} - \theta_{t+1} - 1)\theta_{k-t} \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \\
& \geq \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + \theta_{t+1}(\theta_{k-t} - 1)\theta_{k-t} \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}
\end{aligned}$$

for $n \geq 2k - t + 1, k \geq t + 1, q \geq 2$ (see Lemma 4.5.3). ■

From now on, we define $f_p(q, n, k, t)$ as the maximum of the number of elements in the sets described in Example 4.2.1 and Example 4.2.3.

$$\begin{aligned}
f_p(q, n, k, t) = \max \Bigg\{ & \theta_{k+1} - \theta_{k-t} + \begin{bmatrix} n-t \\ k-t \end{bmatrix} - q^{(k-t+1)(k-t)} \begin{bmatrix} n-k-1 \\ k-t \end{bmatrix}, \\
& \theta_{t+2} \cdot \left(\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} - \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \right) + \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \Bigg\}.
\end{aligned}$$

From Lemma 4.5.5, 4.5.6 and 4.5.7, we find, for $n \geq 2k - t + 1, k \geq t + 2, q \geq 3$, that

$$f_p(q, n, k, t) = \begin{cases} \theta_{k+1} - \theta_{k-t} + \left[\begin{smallmatrix} n-t \\ k-t \end{smallmatrix}\right] - q^{(k-t+1)(k-t)} \left[\begin{smallmatrix} n-k-1 \\ k-t \end{smallmatrix}\right] & \text{if } k \geq 2t + 3 \\ \theta_{t+2} \cdot \left(\left[\begin{smallmatrix} n-t-1 \\ k-t-1 \end{smallmatrix}\right] - \left[\begin{smallmatrix} n-t-2 \\ k-t-2 \end{smallmatrix}\right] \right) + \left[\begin{smallmatrix} n-t-2 \\ k-t-2 \end{smallmatrix}\right] & \text{if } k \leq 2t + 2. \end{cases}$$

Theorem 4.4.7. *Let $\mathcal{S}_p$ be a maximal set of k-spaces, pairwise intersecting in at least a t-space in $\mathrm{PG}(n, q)$, $k \geq t + 2$, $t \geq 1$, with $q \geq 3$, and $n \geq 2k + t + 3$,. If $\mathcal{S}_p$ is not a t-pencil, then*

$$|\mathcal{S}_p| \leq f_p(q, n, k, t).$$

Equality occurs if and only if $\mathcal{S}_p$ is Example 4.2.1 for $k \geq 2t + 3$ or Example 4.2.3 for $k \leq 2t + 2$.

Proof. Let $\mathcal{S}_p$ be a maximal set of k-spaces, pairwise intersecting in at least a t-space, in $\mathrm{PG}(n, q)$, with $\mathcal{S}_p$ not a t-pencil, and suppose that $|\mathcal{S}_p| \geq f_p(q, n, k, t)$. From Lemma 4.4.5 and Lemma 4.5.13, it follows that $\psi(\mathcal{S}_p) < t + 2$. Since $\mathcal{S}_p$ is not a t-pencil, $\psi(\mathcal{S}_p) > t$, and so $\psi(\mathcal{S}_p) = t + 1$. From Lemma 4.4.6, it follows that if $|\mathcal{T}| \leq 2$, then $|\mathcal{S}_p| \leq 2\left[\begin{smallmatrix} n-t-1 \\ k-t-1 \end{smallmatrix}\right] + (\theta_{t+1}\theta_{k-t} - \theta_{t+1} - 1)\theta_{k-t}\left[\begin{smallmatrix} n-t-2 \\ k-t-2 \end{smallmatrix}\right]$, a contradiction, by Lemma 4.5.16. Hence, $|\mathcal{T}| > 2$. From Lemma 4.4.3(3), it follows that $\mathcal{T}$ is a t-intersecting set of $(t+1)$-spaces. Hence, $\mathcal{T}$ is contained in a t-pencil or all elements of $\mathcal{T}$ are contained in a $(t+2)$-space (see Theorem 2.0.6).

We first suppose that there is no t-space contained in all elements of $\mathcal{T}$. Hence, we know that all elements of $\mathcal{T}$ are contained in a $(t+2)$-space ω. This implies that every element of $\mathcal{S}_p$ must meet ω in at least a $(t+1)$-space. Since $\mathcal{S}_p$ is maximal, we know that $\mathcal{S}_p$ contains all k-spaces meeting ω in at least a $(t+1)$-space, which is Example 4.2.3. Hence, $|\mathcal{S}_p| = \theta_{t+2} \cdot \left(\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} - \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \right) + \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}$, if there is no t-space contained in all elements of $\mathcal{T}$. This number is larger than $\theta_{k+1} - \theta_{k-t} + \begin{bmatrix} n-t \\ k-t \end{bmatrix} - q^{(k-t-1)(k-t)} \begin{bmatrix} n-k-1 \\ k-t \end{bmatrix}$, if and only if $k \leq 2t+2$. So, for $k \geq 2t+3$, we find a contradiction.

It follows that we may suppose that the elements of $\mathcal{T}$ are contained in a t-pencil with vertex the t-space δ. Let Z be the span of all elements of $\mathcal{T}$ and let $\dim(Z) = t+x$, $x \geq 2$. Since $\mathcal{S}_p$ is not a t-pencil, we know that there are k-spaces in $\mathcal{S}_p$ that do not contain δ. These elements of $\mathcal{S}_p$, not through δ, meet δ in a $(t-1)$-space, since they have a t-space in common with every $(t+1)$-space of $\mathcal{T}$. We can also check that each such element meets Z in a $(t+x-1)$-space: suppose to the contrary that there is an element α of $\mathcal{S}_p$, not through δ, that meets Z in the subspace $Z_0 = \alpha \cap Z$, with dimension at most $t+x-2$. Since α meets all $(t+1)$-spaces of $\mathcal{T}$ in a t-space different from δ, it follows that the span of all elements of $\mathcal{T}$ is equal to $\langle Z_0, \delta \rangle$, which has dimension at most $t+x-1$. This contradicts the assumption that the span of all elements of $\mathcal{T}$ has dimension $t+x$.

The dimension of the span Z of all the $(t+1)$-spaces in $\mathcal{T}$ is at most $k+1$: if $\dim(Z) > k+1$, then every k-space of $\mathcal{S}_p$, not through δ, would meet Z in a subspace with dimension $\dim(Z) - 1 > k$, a contradiction.

Let $\pi \in \mathcal{S}_p$ be an element that does not contain δ, and let $\xi = \langle \delta, \pi \rangle$. Note that every element of $\mathcal{S}_p$ through δ has at least a $(t+1)$-space in common with ξ. Now we claim that all elements of $\mathcal{S}_p$, not through δ, are contained in ξ. Suppose that this is not the case, then there exists an element $\pi_1 \in \mathcal{S}_p$ with $\delta \nsubseteq \pi_1$ and $\pi_1 \nsubseteq \xi$. Then every element of $\mathcal{S}_p$ through δ meets both $\pi \setminus \delta$ and $\pi_1 \setminus \delta$. Hence, the number of elements of $\mathcal{S}_p$, through δ, is at most $\theta_{k-t}^2 \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} + \theta_{k-t-1} \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix}$. Here, the first term is an upper bound on the number of elements meeting both $\pi \setminus \pi_1$ and $\pi_1 \setminus \pi$. The second term is an upper bound on the number of elements meeting $(\pi \cap \pi_1) \setminus \delta$. Since every element of $\mathcal{S}_p$ not through δ meets Z in a $(t+x-1)$-space, we find that $|\mathcal{S}_p| \leq \theta_{t+x} \begin{bmatrix} n-t-x+1 \\ k-t-x+1 \end{bmatrix} + \theta_{k-t}^2 \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} + \theta_{k-t-1} \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix}$. For $2 \leq x \leq k-t+1, k \geq 2t+3, n \geq 2k+t+3, t \geq 1$ and $q \geq 3$; this gives a contradiction by Lemma 4.5.20, since $|S| \geq f_p(q,n,k,t)$. Hence, we find that $\mathcal{S}_p$ is contained in Example 4.2.1. The theorem follows from the maximality of $\mathcal{S}_p$. ■

Remark 4.4.8. As already mentioned in the introduction of this chapter, a similar result was found independently by Cao, Lv, Wang and Zhou in [29]. They could prove the same result as in Theorem 4.4.7 for all values of q and $n \geq 2k+t+6$. Hence, the difference between the results is that they also covered the case for $q = 2$, but we found a better bound on the possible values of the dimension n: $n \geq 2k+t+3$.

4.4.2 Classification of the largest t-intersecting sets in $\mathrm{AG}(n,q)$

In [69], the authors prove that the largest t-intersecting set of k-spaces in $\mathrm{AG}(n,q)$, with $n \geq 2k+t+2$, is the set of all k-spaces through a fixed affine t-space. They use geometrical and combinatorial techniques, but they do not use the connection between $\mathrm{AG}(n,q)$ and $\mathrm{PG}(n,q)$. Below we give a shorter proof for this result, for $n \geq 2k+1$, by using Theorem 2.0.3.

Theorem 4.4.9. *Let* $\mathcal{S}$ *be a set of* k*-spaces in* $\mathrm{AG}(n,q)$, $n \geq 2k+1$, $k \geq t \geq 0$, *pairwise intersecting in at least a* t*-space. Then* $|\mathcal{S}| \leq \begin{bmatrix} n-t \\ k-t \end{bmatrix}$, *and equality occurs if and only if* $\mathcal{S}$ *is a* t*-pencil.*

Proof. Let $\mathcal{S}$ be a set of k-spaces in $\mathrm{AG}(n,q)$, pairwise intersecting in at least a t-space. Every affine element α in $\mathcal{S}$ can be extended to the corresponding projective k-space $\tilde{\alpha}$ in $\mathrm{PG}(n,q)$. Let $\tilde{\mathcal{S}}$ be the set of these extended k-spaces. Note that then, $\tilde{\mathcal{S}}$ is a t-intersecting set of k-spaces in $\mathrm{PG}(n,q)$. If there would exist such a set $\mathcal{S}$ with $|\mathcal{S}| > \begin{bmatrix} n-t \\ k-t \end{bmatrix}$, then $|\tilde{\mathcal{S}}| > \begin{bmatrix} n-t \\ k-t \end{bmatrix}$, which contradicts Theorem 2.0.3. Hence, $|\mathcal{S}| \leq \begin{bmatrix} n-t \\ k-t \end{bmatrix}$ for all t-intersecting sets $\mathcal{S}$ in $\mathrm{AG}(n,q)$.

Note that the set of all affine k-spaces through a fixed affine t-space is a t-intersecting set of k-spaces in $\mathrm{AG}(n,q)$ with size $\begin{bmatrix} n-t \\ k-t \end{bmatrix}$. Suppose now that there exists a t-intersecting set $\mathcal{S}$ of k-spaces in $\mathrm{AG}(n,q)$ with $\begin{bmatrix} n-t \\ k-t \end{bmatrix}$ elements, which is not a t-pencil. Then $\tilde{\mathcal{S}}$ is a t-intersecting set of k-spaces in $\mathrm{PG}(n,q)$ with $|\tilde{\mathcal{S}}| = \begin{bmatrix} n-t \\ k-t \end{bmatrix}$. It follows from Theorem 2.0.3 that $n = 2k+1$ and that all elements of $\tilde{\mathcal{S}}$ are contained in a projective $(2k-t)$-space. Since the number of affine k-spaces in a projective $(2k-t)$-space is $\begin{bmatrix} 2k-t+1 \\ k+1 \end{bmatrix} - \begin{bmatrix} 2k-t \\ k+1 \end{bmatrix}$, we see that in this case $|\mathcal{S}| < \begin{bmatrix} n-t \\ k-t \end{bmatrix}$. Hence, an affine t-pencil is the only example of a set of pairwise t-intersecting k-spaces in $\mathrm{AG}(n,q)$ with size $\begin{bmatrix} n-t \\ k-t \end{bmatrix}$. ■

4.4.3 Classification of the largest non-trivial t-intersecting sets in $\mathrm{AG}(n,q)$

In this subsection, we investigate the largest non-trivial sets of k-spaces in $\mathrm{AG}(n,q)$ pairwise intersecting in at least a t-space. For $t=0$, the largest non-trivial example was found in [68].

Theorem 4.4.10 ([68]). *Suppose* $k \geq 3$, $n \geq 2k+4$ *and* $(n,q) \neq (2k+4,2)$. *Let* $\mathcal{S}$ *be a non-trivial intersecting family in* $\mathrm{AG}(n,q)$, *then* $|\mathcal{S}| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} - q^{k(k-1)} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} + q^k$. *Equality holds if and only if*

1. $\mathcal{S}$ *is Example 4.3.1 for* $t=0$, *or*
2. $\mathcal{S}$ *is Example 4.3.3 for* $t=0$.

Many results and proofs in this affine setting are similar to the results and proofs in the projective setting, but because of some structural differences, we decided to discuss the Hilton-Milner problem, in the projective and affine context, in different subsections.

We again suppose that $k \geq t+2$ and $t \geq 1$. Let $\mathcal{S}_a$ be a maximal set of k-spaces in $\mathrm{AG}(n,q)$, $n \geq 2k-t+1$, pairwise meeting in at least a t-space. Let

$$\psi(\mathcal{S}_a) = \min\{\, \dim(T) \mid T \subset \mathrm{AG}(n,q),\ \dim(T \cap \alpha) \geq t,\ \forall \alpha \in \mathcal{S}_a \}.$$

Let $\mathcal{T}$ be the set of all $\psi(\mathcal{S}_a)$-dimensional spaces in $\mathrm{AG}(n,q)$ that meet every element of $\mathcal{S}_a$ in at least a t-space.

Lemma 4.4.11. *We have the following properties for* $\psi(\mathcal{S}_a)$ *and* $\mathcal{T}$.

1. $t \leq \psi(\mathcal{S}_a) \leq k$, *and if* $\psi(\mathcal{S}_a) = t$, *then* $\mathcal{S}_a$ *is a* t*-pencil.*
2. *Let* $T \in \mathcal{T}$, *then all* k*-spaces through* T *are contained in* $\mathcal{S}_a$.
3. *The elements of* $\mathcal{T}$ *are* t*-intersecting in* $\mathrm{AG}(n,q)$.

Proof. Analogous to the proof of Lemma 4.4.3. ■

Lemma 4.4.12. *Let* $\psi(\mathcal{S}_a) = t+x$, $x \geq 1$, $k \geq t+2$, $t \geq 1$, *and* $n \geq 2k-t+1$. *Then the number of elements of* $\mathcal{S}_a$ *through an affine* $(t+x-j)$*-space, with* $j \in \{0,1,2,\ldots,x\}$, *is at most* $(\theta_{k-t})^j \begin{bmatrix} n-t-x \\ k-t-x \end{bmatrix}$.

Proof. Suppose that $\psi(\mathcal{S}_a) = t + x,\ x \geq 2$. We prove, by induction on $j \in \{0, 1, 2, \dots, x\}$, that the number of k-spaces of $\mathcal{S}_a$ through an affine $(t + x - j)$-space is at most $\begin{bmatrix} n-t-x \\ k-t-x \end{bmatrix}(\theta_{k-t})^j$. Note that the statement is true for $j = 0$, by counting the total number of k-spaces through an affine $(t + x)$-space.

Let $j \in \{1, 2, 3, \dots, x\}$ and suppose now that the number of k-spaces of $\mathcal{S}_a$ through an affine $(t + x - j_0)$-space, is at most $(\theta_{k-t})^{j_0} \begin{bmatrix} n-t-x \\ k-t-x \end{bmatrix}$, for all $j_0 < j$. Then we prove that this also holds for j. Consider an affine $(t + x - j)$-space γ_j. Since $\psi(\mathcal{S}_a) = t + x$, we know that there exists a k-space π_j of $\mathcal{S}_a$, meeting γ_j in at most an affine $(t - 1)$-space.

Let $\max\{\dim(\gamma_j \cap \pi) | \pi \in \mathcal{S}_a, \dim(\gamma_j \cap \pi) < t\} = t - l$, then $l \geq 1$, and suppose that $\pi_j \in \mathcal{S}_a$ is an element such that $\dim(\pi_j \cap \gamma_j) = t - l$. Let $\pi_{j\gamma}$ be a projective $(k - t + l - 1)$-space in $\tilde{\pi}_j \setminus \tilde{\gamma}_j$. Then every element of $\tilde{\mathcal{S}}_a$ through $\tilde{\gamma}_j$ contains at least an $(l - 1)$-space of $\pi_{j\gamma}$. Since there are $\begin{bmatrix} k-t+l \\ l \end{bmatrix}$ $(l - 1)$-spaces in $\pi_{j\gamma}$, and since the number of affine k-spaces in $\mathcal{S}_a$ through a $(t + x - j + l)$-space is at most $(\theta_{k-t})^{j-l} \begin{bmatrix} n-t-x \\ k-t-x \end{bmatrix}$, we find that the number of elements of $\tilde{\mathcal{S}}_a$ through $\tilde{\gamma}_j$ is at most $\begin{bmatrix} k-t+l \\ l \end{bmatrix}(\theta_{k-t})^{j-l} \begin{bmatrix} n-t-x \\ k-t-x \end{bmatrix}$. Note that

$$\begin{aligned} \begin{bmatrix} k-t+l \\ l \end{bmatrix}(\theta_{k-t})^{j-l} &= \frac{(q^{k-t+l} - 1) \dots (q^{k-t+1} - 1)}{(q^l - 1) \dots (q - 1)}(\theta_{k-t})^{j-l} \\ &\leq \left(\frac{q^{k-t+1} - 1}{q - 1}\right)^l (\theta_{k-t})^{j-l} = (\theta_{k-t})^j \end{aligned}$$

Hence, also in this case, we find that the number of elements of $\tilde{\mathcal{S}}_a$ through $\tilde{\gamma}_j$, and so, the number of elements of $\mathcal{S}_a$ through γ_j is at most $(\theta_{k-t})^j \begin{bmatrix} n-t-x \\ k-t-x \end{bmatrix}$. ■

Lemma 4.4.13. *Let* $\mathcal{S}_a$ *be a set of* k*-spaces, pairwise intersecting in at least a* t*-space in* $\mathrm{AG}(n,q)$*. If* $\psi(\mathcal{S}_a) = t + x,\ x \geq 2, k \geq t + 2, t \geq 1,$ *and* $n \geq 2k - t + 1$*, then* $|\mathcal{S}_a| \leq q^x \begin{bmatrix} t+x \\ x \end{bmatrix}(\theta_{k-t})^x \begin{bmatrix} n-t-x \\ k-t-x \end{bmatrix}$*.*

Proof. Suppose that $\psi(\mathcal{S}_a) = t + x,\ x \geq 2$. By Lemma 4.4.12, we know, for $j \in \{0, 1, 2, \dots, x\}$, that the number of k-spaces of $\mathcal{S}_a$ through an affine $(t + x - j)$-space is at most $\begin{bmatrix} n-t-x \\ k-t-x \end{bmatrix}(\theta_{k-t})^j$.

Consider now an element $T \in \mathcal{T}$. Then every element of $\mathcal{S}_a$ meets T in at least a t-space. Since there are $q^x \begin{bmatrix} t+x \\ x \end{bmatrix}$ affine t-spaces in T and since every t-space is contained in at most $(\theta_{k-t})^x \begin{bmatrix} n-t-x \\ k-t-x \end{bmatrix}$ elements of $\mathcal{S}_a$, we find that $\mathcal{S}_a$ has at most $q^x \begin{bmatrix} t+x \\ x \end{bmatrix}(\theta_{k-t})^x \begin{bmatrix} n-t-x \\ k-t-x \end{bmatrix}$ elements. ■

Lemma 4.4.14. *Let* $\mathcal{S}_a$ *be a maximal set of* k*-spaces, pairwise intersecting in at least a* t*-space in* $\mathrm{AG}(n,q)$*,* $n \geq 2k - t + 1, k \geq t + 1$ *and* $t \geq 1$*. If* $\psi(\mathcal{S}_a) = t + 1$ *and* $|\mathcal{T}| \leq 2$*, then*

$$|\mathcal{S}_a| \leq 2\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + (\theta_{t+1}\theta_{k-t} - \theta_{t+1} - \theta_{k-t})\theta_{k-t}\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}.$$

Proof. Let T be an element of $\mathcal{T}$. Since $\mathcal{S}_a$ is a maximal set, we know that all $\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix}$ k-spaces through T are contained in $\mathcal{S}_a$. Now we count the size of the set $\mathcal{S}_{a0}$ of k-spaces of $\mathcal{S}_a$ not through T. For every $\pi \in \mathcal{S}_{a0}$, $\dim(\pi \cap T) = t$, and let E be an affine t-space in T. Then there exists an element $\alpha \in \mathcal{S}_{a0}$ not through E, and so $\dim(\alpha \cap E) = t - 1$. Hence, every element π of $\mathcal{S}_{a0}$, through E must contain a $(t + 1)$-space τ, different from T, such that $E \subset \tau$ and $\tau \cap (\alpha \setminus E) \neq \emptyset$. Note that there are $\theta_{k-t} - 1$ possibilities for τ. Fix such a $(t + 1)$-space τ.

- If $\mathcal{T} = \{T\}$, we know that $\tau \notin \mathcal{T}$, and hence there exists an element σ of $\mathcal{S}_a$, meeting τ in at most a $(t - 1)$-space. Hence, every element of $\mathcal{S}_{a0}$ through τ meets $\sigma \setminus \tau$, and so the number of elements of $\mathcal{S}_{a0}$ through τ is at most $\theta_{k-t} \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}$. Since there are $q\theta_t$ possibilities for E, and at most $\theta_{k-t} - 1$ for τ, we have that

$$|\mathcal{S}_a| \leq \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + q\theta_t(\theta_{k-t} - 1)\theta_{k-t}\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}.$$

- Suppose $|\mathcal{T}| = 2$, and let $\mathcal{T} = \{T, \Psi\}$. If $\tau = \Psi$, then $\mathcal{S}_a$ contains all $\left[\begin{smallmatrix} n-t-1 \\ k-t-1 \end{smallmatrix}\right]$ k-spaces through τ. If $\tau \neq \Psi$, then we can follow the argument in the previous item, and we find that the number of elements of $\mathcal{S}_{a0}$ through τ is at most $\theta_{k-t}\left[\begin{smallmatrix} n-t-2 \\ k-t-2 \end{smallmatrix}\right]$. Note that there are $q\theta_t - 1$ possibilities for $E \neq T \cap \Psi$, and at most $\theta_{k-t} - 1$ for $\tau \neq \Psi, T$, through $E \neq T \cap \Psi$. Moreover, there are at most $\theta_{k-t} - 2$ possibilities for $\tau \neq \Psi, T$ through $E = T \cap \Psi$. Hence, we have that

$$\begin{aligned}
|\mathcal{S}_a| \leq & \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + \sum_{E \subset T} \sum_{\tau \supseteq E} |\{\pi \in \mathcal{S}_{a0} | \tau \subset \pi\}| \\
\leq & \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + \sum_{E \neq T \cap \Psi} \sum_{\tau \supseteq E} \theta_{k-t} \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} + \sum_{\tau \supset T \cap \Psi} |\{\pi \in \mathcal{S}_{a0} | \tau \subset \pi\}| \\
\leq & \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + (q\theta_t - 1)(\theta_{k-t} - 1)\theta_{k-t} \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \\
& \qquad + \sum_{\tau \neq \Psi, T} \theta_{k-t} \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} + \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} \\
\leq & 2\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + (q\theta_t - 1)(\theta_{k-t} - 1)\theta_{k-t} \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} + (\theta_{k-t} - 2)\theta_{k-t} \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \\
= & 2\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + (\theta_{t+1}\theta_{k-t} - \theta_{t+1} - \theta_{k-t})\theta_{k-t} \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}.
\end{aligned}$$

The lemma follows since

$$\begin{aligned}
2\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} & + (\theta_{t+1}\theta_{k-t} - \theta_{t+1} - \theta_{k-t})\theta_{k-t} \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \\
& \geq \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + q\theta_t(\theta_{k-t} - 1)\theta_{k-t} \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}
\end{aligned}$$

for $k \geq t+1, n \geq 2k-t, q \geq 2$ (see Lemma 4.5.4). ■

From now on, we define $f_a(q,n,k,t)$ as the maximum of the number of elements in the sets described in Example 4.3.1 and Example 4.3.3 for $n \geq 2k-t+1$.

$$\begin{aligned}
f_a(q,n,k,t) = \max \Bigg\{ & \theta_k - \theta_{k-t} + \begin{bmatrix} n-t \\ k-t \end{bmatrix} - q^{(k-t+1)(k-t)} \begin{bmatrix} n-k-1 \\ k-t \end{bmatrix}, \\
& \theta_{t+1} \cdot \left(\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} - \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \right) + \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \Bigg\}.
\end{aligned}$$

From Lemma 4.5.8, 4.5.9 and 4.5.10, we find for $n \geq 2k-t+1, k \geq t+2, q \geq 3$ that

$$f_a(q,n,k,t) = \begin{cases} \theta_k - \theta_{k-t} + \left[\begin{smallmatrix} n-t \\ k-t \end{smallmatrix}\right] - q^{(k-t+1)(k-t)} \left[\begin{smallmatrix} n-k-1 \\ k-t \end{smallmatrix}\right] & \text{if } k \geq 2t+2 \\ \theta_{t+1} \cdot \left(\left[\begin{smallmatrix} n-t-1 \\ k-t-1 \end{smallmatrix}\right] - \left[\begin{smallmatrix} n-t-2 \\ k-t-2 \end{smallmatrix}\right] \right) + \left[\begin{smallmatrix} n-t-2 \\ k-t-2 \end{smallmatrix}\right] & \text{if } k \leq 2t+1. \end{cases}$$

Theorem 4.4.15. *Let $\mathcal{S}_a$ be a maximal set of k-spaces, pairwise intersecting in at least a t-space in $\mathrm{AG}(n,q)$, $k \geq t+2, t \geq 1$, with $q \geq 3$, and $n \geq 2k+t+3$. If $\mathcal{S}_a$ is not a t-pencil, then*

$$|\mathcal{S}_a| \leq f_a(q,n,k,t).$$

Equality occurs if and only if $\mathcal{S}_a$ is Example 4.3.1 for $k \geq 2t+2$ or Example 4.3.3 for $k \leq 2t+1$.

Proof. Let $\mathcal{S}_a$ be a maximal set of k-spaces, pairwise intersecting in at least a t-space, in $\mathrm{AG}(n, q)$, with $\mathcal{S}_a$ not a t-pencil, and suppose that $|\mathcal{S}_a| \geq f_a(q, n, k, t)$. From Lemma 4.4.13 and Lemma 4.5.15, it follows that $\psi(\mathcal{S}_a) < t + 2$. Since $\mathcal{S}_a$ is not a t-pencil, we find that $\psi(\mathcal{S}_a) > t$, and so $\psi(\mathcal{S}_a) = t + 1$.

From Lemma 4.4.14, it follows that if $|\mathcal{T}| \leq 2$, then $|\mathcal{S}_a| \leq 2\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + (\theta_{t+1}\theta_{k-t} - \theta_{t+1} - \theta_{k-t})\theta_{k-t}\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}$, a contradiction by Lemma 4.5.17. Hence, $|\mathcal{T}| > 2$. From Lemma 4.4.11(3), it follows that $\mathcal{T}$ is a t-intersecting set of $(t + 1)$-spaces. Hence, $\mathcal{T}$, is contained in a t-pencil or all elements of $\mathcal{T}$ are contained in a $(t + 2)$-space (see Theorem 4.4.1).

We first suppose that there is no t-space contained in all elements of $\mathcal{T}$. Hence, we know that all elements of $\mathcal{T}$ are contained in a $(t + 2)$-space ω. We also know that the elements of $\mathcal{T}$ are t-intersecting in the affine space, and so, every t-space in $\tilde{\omega} \cap H_\infty$ is contained in at most one element of $\mathcal{T}$. Moreover, we also find that every element π_1 of $\mathcal{S}_a$ must meet ω in at least a $(t + 1)$-space. This follows since π_1 must meet all elements of $\mathcal{T}$, that are contained in a $(t + 2)$-space, in at least a t-space, and that there is no t-space contained in all elements of $\mathcal{T}$.

In this case, we claim the following.
Claim $(*)$ The number of elements of $\mathcal{S}_a$ is at most $\theta_{t+1} \cdot \left(\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} - \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}\right) + \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}$, and equality holds if and only if $\mathcal{S}_a$ is Example 4.3.3 .

Proof of claim: We first of all note that all k-spaces through ω are contained in $\mathcal{S}_a$. Consider a projective t-space $\alpha_t \subset \tilde{\omega} \cap H_\infty$. Then we count the number of elements of $\tilde{\mathcal{S}_a}$ through α_t, not through ω. There are two possibilities.

- All these elements meet $\tilde{\omega}$ in the same affine $(t+1)$-space α_t^+ through α_t. Then the number of elements of $\tilde{\mathcal{S}_a}$ through α_t and not through ω is at most $\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} - \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}$. If this is the case for all t-spaces $\alpha_t \subset \tilde{\omega} \cap H_\infty$, then $|\mathcal{S}_a| \leq \theta_{t+1} \cdot \left(\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} - \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}\right) + \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}$. Note that two elements through the same t-space α_t meet in at least an affine $(t + 1)$-space; α_t^+. Two k-spaces through different t-spaces α_{t1} and α_{t2} will also have a t-space in common, since they both contain the affine t-space $\alpha_{t1}^+ \cap \alpha_{t2}^+$. Since $\mathcal{S}_a$ is a maximal set of t-intersecting k-spaces, we find that $|\mathcal{S}_a| = \theta_{t+1} \cdot \left(\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} - \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}\right) + \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}$, and that $\mathcal{S}_a$ has the form described in Example 4.3.3.

- There is a t-space $\alpha_t \subset \tilde{\omega} \cap H_\infty$, such that there are two elements $\pi_1, \pi_2 \in \mathcal{S}_a$, not contained in ω, with $\alpha_t \subset \tilde{\pi}_1 \cap \tilde{\pi}_2$, but $\pi_1 \cap \omega \neq \pi_2 \cap \omega$. Then every element π of $\tilde{\mathcal{S}_a}$ through α_t, not through $\pi_1 \cap \omega$, meets π_1 in an affine point outside of ω. For the elements of $\tilde{\mathcal{S}_a}$ through $\pi_1 \cap \omega$, but not through $\tilde{\omega}$, we can use the same argument by using π_2.

  Note that $\tilde{\pi}$ meets $\tilde{\omega}$ in one of the q affine $(t + 1)$-spaces in $\tilde{\omega}$ through α_t.

  - If $\tilde{\pi} \cap \tilde{\omega} \neq \tilde{\pi}_1 \cap \tilde{\omega}$, then there are $q^{k-t-1} - 1$ ways to extend this $(t + 1)$-space $\pi \cap \omega$ to a $(t + 2)$-space, meeting π_1 in an affine $(t + 1)$-space, not in ω. By investigating $\tilde{\pi}_1$ in the quotient space $\mathrm{PG}(n, q)/\alpha_t$, we find that there are q^{k-t-1} ways to extend $\tilde{\pi} \cap \tilde{\omega}$ to a $(t + 2)$-space meeting π_1 in an affine $(t + 1)$-space, and one of these extended $(t + 2)$-spaces is equal to ω.

  - If $\tilde{\pi} \cap \tilde{\omega} = \tilde{\pi}_1 \cap \tilde{\omega}$, then $\tilde{\pi} \cap \tilde{\omega} \neq \tilde{\pi}_2 \cap \tilde{\omega}$. Hence, we can use the same argument from the previous point to see that there are $q^{k-t-1} - 1$ ways to extend this $(t + 1)$-space to a $(t + 2)$-space, meeting π_2 in an affine $(t + 1)$-space, not in ω.

Hence, there are at most $q(q^{k-t-1}-1)\cdot\begin{bmatrix}n-t-2\\k-t-2\end{bmatrix}$ elements of $\tilde{\mathcal{S}_a}$ through α_t and not through ω, and as there are θ_{t+1} possibilities for α_t, and $\begin{bmatrix}n-t-2\\k-t-2\end{bmatrix}$ elements through ω, we find that $|\mathcal{S}_a| = |\tilde{\mathcal{S}_a}| \le \theta_{t+1}q(q^{k-t-1}-1)\cdot\begin{bmatrix}n-t-2\\k-t-2\end{bmatrix}+\begin{bmatrix}n-t-2\\k-t-2\end{bmatrix}$. We can check that this upper bound is smaller than $\theta_{t+1}\cdot\left(\begin{bmatrix}n-t-1\\k-t-1\end{bmatrix}-\begin{bmatrix}n-t-2\\k-t-2\end{bmatrix}\right)+\begin{bmatrix}n-t-2\\k-t-2\end{bmatrix}$, since

$$\begin{aligned}
&\theta_{t+1}(q^{k-t}-q)\begin{bmatrix}n-t-2\\k-t-2\end{bmatrix} < \theta_{t+1}\left(\begin{bmatrix}n-t-1\\k-t-1\end{bmatrix}-\begin{bmatrix}n-t-2\\k-t-2\end{bmatrix}\right)\\
\Leftrightarrow\quad &(q^{k-t}-q)\begin{bmatrix}n-t-2\\k-t-2\end{bmatrix} < \begin{bmatrix}n-t-1\\k-t-1\end{bmatrix}-\begin{bmatrix}n-t-2\\k-t-2\end{bmatrix}\\
\Leftrightarrow\quad &q^{k-t}-q < \frac{q^{n-t-1}-q^{k-t-1}}{q^{k-t-1}-1}\\
\Leftrightarrow\quad &q^{2k-2t-1}-2q^{k-t}+q < q^{n-t-1}-q^{k-t-1}\\
\Leftrightarrow\quad &0 < q^{2k-2t-1}\left(q^{n-2k+t}-1\right)+q^{k-t-1}(q-1)+q\left(q^{k-t-1}-1\right),
\end{aligned}$$

is valid for $n \ge 2k-t+1, k \ge t+2, q \ge 3$. This proves that if there exists a t-space $\alpha_t \in H_\infty \cap \tilde{\omega}$, such that not all elements of $\tilde{\mathcal{S}_a}$ through α_t meet ω in the same $(t+1)$-space, then the number of elements in $\mathcal{S}_a$ is smaller than the number of elements in Example 4.3.3.

This proves Claim $(*)$.

So $|\mathcal{S}_a| = \theta_{t+1}\cdot\left(\begin{bmatrix}n-t-1\\k-t-1\end{bmatrix}-\begin{bmatrix}n-t-2\\k-t-2\end{bmatrix}\right)+\begin{bmatrix}n-t-2\\k-t-2\end{bmatrix}$ if there is no t-space contained in all elements of $\mathcal{T}$. This number is larger than $\theta_k-\theta_{k-t}+\begin{bmatrix}n-t\\k-t\end{bmatrix}-q^{(k-t-1)(k-t)}\begin{bmatrix}n-k-1\\k-t\end{bmatrix}$, if and only if $k \le 2t+1$. So, for $k \ge 2t+2$, we find a contradiction.

Now we continue with the case that all elements of $\mathcal{T}$ are contained in a t-pencil with vertex the affine t-space δ. Let Z be the span of all elements of $\mathcal{T}$ and let $\dim(Z) = t+x$, $x \ge 2$. Since $\mathcal{S}_a$ is not a t-pencil, we know that there are k-spaces in $\mathcal{S}_a$ that do not contain δ. These elements of $\mathcal{S}_a$, not through δ, meet δ in a $(t-1)$-space, since they have an affine t-space in common with every $(t+1)$-space of $\mathcal{T}$. We can also check that each such element meets Z in a $(t+x-1)$-space: suppose to the contrary that there is an element α of $\mathcal{S}_a$, not through δ, that meets Z in the subspace $Z_0 = \alpha \cap Z$, with dimension at most $t+x-2$. Since α meets all $(t+1)$-spaces of $\mathcal{T}$ in a t-space different from δ, it follows that the span of all elements of $\mathcal{T}$ is equal to $\langle Z_0, \delta\rangle$, which has dimension at most $t+x-1$. This contradicts the assumption that the span of all elements of $\mathcal{T}$ has dimension $t+x$.

The dimension of the span Z of all the $(t+1)$-spaces in $\mathcal{T}$ is at most $k+1$: if $\dim(Z) > k+1$, then every k-space of $\mathcal{S}_a$, not through δ, would meet Z in a subspace with dimension $\dim(Z)-1 > k$, a contradiction.

Let $\pi \in \mathcal{S}_a$ be an element that does not contain δ, and let $\xi = \langle\delta, \pi\rangle$. Note that every element of $\mathcal{S}_a$ through δ has at least a $(t+1)$-space in common with ξ. Now we claim that all elements of $\mathcal{S}_a$, not through δ, are contained in ξ. Suppose that this is not the case, then there exists an element $\pi_2 \in \mathcal{S}_a$ with $\delta \not\subseteq \pi_2$ and $\pi_2 \not\subseteq \xi$. Then every element of $\mathcal{S}_a$ through δ meets both $\pi \setminus \delta$ and $\pi_2 \setminus \delta$. Hence, the number of elements of $\mathcal{S}_a$, through δ, is at most $\theta_{k-t}^2\begin{bmatrix}n-t-2\\k-t-2\end{bmatrix}+\theta_{k-t-1}\begin{bmatrix}n-t-1\\k-t-1\end{bmatrix}$. Here, the first term is an upper bound on the number of elements meeting both $\pi\setminus\pi_2$ and $\pi_2\setminus\pi$. The second term is an upper bound on the number of elements meeting $(\pi\cap\pi_2)\setminus\delta$, since $\dim((\pi\cap\pi_2)\setminus\delta) \le k-t-1$. Every element of $\mathcal{S}_a$ not through δ meets Z in a $(t+x-1)$-space. This implies that $|\mathcal{S}_a| \le \theta_{t+x}\begin{bmatrix}n-t-x+1\\k-t-x+1\end{bmatrix}+\theta_{k-t}^2\begin{bmatrix}n-t-2\\k-t-2\end{bmatrix}+\theta_{k-t-1}\begin{bmatrix}n-t-1\\k-t-1\end{bmatrix}$. For $n \ge 2k+t+3, k \ge 2t+2, t \ge 1, x \ge 3, q \ge 3$; this gives a contradiction by Lemma 4.5.21, since $|\mathcal{S}_a| \ge f_a(q,n,k,t)$. Now, in a last step, we also have to find a contradiction for $x = 2$, and so Z a $(t+2)$-space. In this situation, all k-spaces not through δ must meet Z in a $(t+1)$-space, not through δ. Now, every two elements of $\mathcal{S}$, not through δ, must meet in at least a t-space. The same argument, used to deduce Claim $(*)$, can be used to show the following. For every t-space $\alpha_t \subset \tilde{Z} \cap H_\infty$, $\tilde{\delta} \cap H_\infty \nsubseteq \alpha_t$, we have that all elements of $\tilde{\mathcal{S}_a}$ through α_t must meet Z in the same $(t+1)$-space. Hence, there are at most $\theta_{t+1} - \theta_1$ possibilities for the intersection $\pi \cap Z$, with $\pi \in \mathcal{S}_a$, $\delta \nsubseteq \pi$, and there are at most $\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix}$ k-spaces through a fixed $(t+1)$-space. Hence, we find that the number of elements of $\mathcal{S}_a$, not through δ, is at most $q^2\theta_{t-1}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix}$, and so $|\mathcal{S}_a| \leq q^2\theta_{t-1}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + \theta_{k-t}^2\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} + \theta_{k-t-1}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix}$. This gives a contradiction for $n \geq 2k+t+3$, $k \geq 2t+2$ and $q \geq 3$ by Lemma 4.5.22 since $|\mathcal{S}_a| \geq f_a(q,n,k,t)$. Hence, we find that every element of $\mathcal{S}_a$, not through δ, is contained in ξ, and so $\mathcal{S}_a$ is contained in Example 4.3.1. The theorem follows from the maximality of $\mathcal{S}_a$. ■

4.5 Appendix

In this appendix, we will often use the bounds on the binomial Gaussian coefficient, see Lemma 1.10.2.

We start with two lemmas that give two formulas for the number of elements in each of the Examples 4.2.1, 4.2.3, 4.3.1 and 4.3.3. We will use these different expressions of the number of elements of a set, depending on which formula simplifies the counting argument.

Lemma 4.5.1. *Let* $S_{2.1}$ *be the set of elements described in Example 4.2.1 and let* $S_{2.3}$ *be the set of elements described in Example 4.2.3, then we have that*

$$|S_{2.1}| = \theta_{k+1} - \theta_{k-t} + \begin{bmatrix} n-t \\ k-t \end{bmatrix} - q^{(k-t+1)(k-t)}\begin{bmatrix} n-k-1 \\ k-t \end{bmatrix} \tag{4.4}$$

$$= \theta_{k+1} + \sum_{j=0}^{k-t-2}\begin{bmatrix} k-t+1 \\ j+1 \end{bmatrix} q^{(k-t-j)(k-t-j-1)}\begin{bmatrix} n-k-1 \\ k-t-j-1 \end{bmatrix}, \tag{4.5}$$

$$< q^{k-t+1}\theta_t + \theta_{k-t}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} \tag{4.6}$$

$$|S_{2.3}| = \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} + \theta_{t+2}\cdot\left(\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} - \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}\right) \tag{4.7}$$

$$= \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}\left(1 + \theta_{t+2}q^{k-t-1}\frac{q^{n-k}-1}{q^{k-t-1}-1}\right). \tag{4.8}$$

Proof. We will use the notation from Examples 4.2.1 and 4.2.3. The first equality for $|S_{2.1}|$ follows from Lemma 4.2.2. For the second equality, we count the number of elements of $S_{2.1}$ in a different way. We have that $|S_{2.1}| = \theta_{k+1} + \sum_{j=0}^{k-t-2}|Q_j(n,k,t)|$, with $Q_j(n,k,t) = \{\beta \in S_{2.1} | \beta \nsubseteq \xi, \dim(\beta\cap\xi) = j+t+1\}$, $j \in \{0,1,\dots,k-t-2\}$. Let σ_0 be the $(k-t)$-space corresponding to ξ in the quotient space $\mathrm{PG}(n,q)/\delta$. Note that the first term in the sum is the number of k-spaces in ξ. Since an element in Q_j corresponds to a $(k-t-1)$-space in $\mathrm{PG}(n,q)/\delta$, meeting σ_0 in a j-space, and since there are $\begin{bmatrix} k-t+1 \\ j+1 \end{bmatrix}$ j-spaces in σ_0, we find, by using Lemma 1.10.1, that

$$|S_{2.1}| = \theta_{k+1} + \sum_{j=0}^{k-t-2}\begin{bmatrix} k-t+1 \\ j+1 \end{bmatrix} q^{(k-t-j)(k-t-j-1)}\begin{bmatrix} n-k-1 \\ k-t-j-1 \end{bmatrix}.$$

Inequality (4.6) follows since $q^{k-t+1}\theta_t$ is the number of elements of $S_{2.1}$ contained in ξ but not

containing δ. The second term $\theta_{k-t}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix}$ is the total number of k-spaces through a $(t+1)$-space in ξ through δ. Note that in this term, the k-spaces meeting ξ in a subspace with dimension more than $t+1$ are counted multiple times.
The first equality for $|S_{2.3}|$ follows from Lemma 4.2.4, and the second from the definition of the Gaussian coefficients. ■

Lemma 4.5.2. *Let $R_{3.1}$ be the set of elements described in Example 4.3.1 and let $R_{3.3}$ be the set of elements described in Example 4.3.3, then we have that*

$$
\begin{aligned}
|R_{3.1}| &= \theta_k - \theta_{k-t} + \begin{bmatrix} n-t \\ k-t \end{bmatrix} - q^{(k-t+1)(k-t)}\begin{bmatrix} n-k-1 \\ k-t \end{bmatrix} && (4.9)\\
&= \theta_k + \sum_{j=0}^{k-t-2}\begin{bmatrix} k-t+1 \\ j+1 \end{bmatrix} q^{(k-t-j)(k-t-j-1)}\begin{bmatrix} n-k-1 \\ k-t-j-1 \end{bmatrix}, && (4.10)\\
&< q^{k-t+1}\theta_t + \theta_{k-t}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} && (4.11)\\
|R_{3.3}| &= \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} + \theta_{t+1}\cdot\left(\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} - \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}\right) && (4.12)\\
&= \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}\left(1 + \theta_{t+1}q^{k-t-1}\frac{q^{n-k}-1}{q^{k-t-1}-1}\right). && (4.13)
\end{aligned}
$$

Proof. The first equality for $|R_{3.1}|$ follows from Lemma 4.2.2. For the second equality, we use the equality between the two formulas for $|S_{2.1}|$ in Lemma 4.5.1, since the formulas for $|S_{2.1}|$ and $|R_{3.1}|$ only differ in the first term. Inequality (4.11) follows from inequality (4.6) and the fact that $|R_{3.1}| < |S_{2.1}|$. The first equality for $|S_{2.3}|$ follows from Lemma 4.2.4, and the second from the definition of the Gaussian coefficients. ■

Lemma 4.5.3. *For $n \geq 2k-t+1$, $k \geq t+1$ and $q \geq 2$, we have that*

$$
\begin{aligned}
2\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} &+ (\theta_{t+1}\theta_{k-t} - \theta_{t+1} - 1)\theta_{k-t}\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}\\
&\geq \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + \theta_{t+1}(\theta_{k-t} - 1)\theta_{k-t}\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}.
\end{aligned}
$$

Proof. The inequality is equivalent to

$$
\begin{aligned}
& \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} \geq \theta_{k-t}\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}\\
\Leftrightarrow\quad & \frac{q^{n-t-1}-1}{q^{k-t-1}-1} \geq \frac{q^{k-t+1}-1}{q-1}\\
\Leftrightarrow\quad & q^{n-t} - q^{n-t-1} - q + 1 \geq q^{2k-2t} - q^{k-t+1} - q^{k-t-1} + 1\\
\Leftrightarrow\quad & q^{2k-2t}\left(q^{n-2k+t} - q^{n-2k+t-1} - 1\right) + q\left(q^{k-t} - 1\right) + q^{k-t-1} \geq 0
\end{aligned}
$$

The last inequality is valid since all terms in the left hand side of the last inequality are non-negative for $n \geq 2k-t+1$, $k \geq t+1$ and $q \geq 2$. ■

Lemma 4.5.4. *For $n \geq 2k-t$, $k \geq t+1$ and $q \geq 2$, we have that*

$$
\begin{aligned}
2\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} &+ (\theta_{t+1}\theta_{k-t} - \theta_{t+1} - \theta_{k-t})\theta_{k-t}\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}\\
&\geq \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + q\theta_t(\theta_{k-t} - 1)\theta_{k-t}\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}.
\end{aligned}
$$

Proof. The inequality follows by subtracting $(\theta_{k-t}-1)\theta_{k-t}\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}$ on both sides of the inequality from Lemma 4.5.3. ■

Lemma 4.5.5. *Let $n \geq 2k-t+1, q \geq 3$ and consider Example 4.2.1 and Example 4.2.3 in $\mathrm{PG}(n,q)$. The number of elements in Example 4.2.1 is larger than the number of elements in Example 4.2.3 if $k \geq 2t+3$.*

Proof. Suppose to the contrary that the number of elements in Example 4.2.3 is larger than the number of elements in Example 4.2.1 for $k \geq 2t+3$. By using (4.5) and (4.8), we have

$$\begin{aligned}
&\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}\left(1+\theta_{t+2}q^{k-t-1}\frac{q^{n-k}-1}{q^{k-t-1}-1}\right)\\
&\qquad \geq \theta_{k+1} + \sum_{j=0}^{k-t-2}\begin{bmatrix} k-t+1 \\ j+1 \end{bmatrix} q^{(k-t-j)(k-t-j-1)}\begin{bmatrix} n-k-1 \\ k-t-j-1 \end{bmatrix}\\
\xRightarrow{j=0}\quad &\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}\left(1+\theta_{t+2}q^{k-t-1}\frac{q^{n-k}-1}{q^{k-t-1}-1}\right) > \theta_{k-t}q^{(k-t)(k-t-1)}\begin{bmatrix} n-k-1 \\ k-t-1 \end{bmatrix}\\
\xRightarrow{L.1.10.2}\quad & 2q^{(k-t-2)(n-k)}\left(1+\theta_{t+2}q^{k-t-1}\frac{q^{n-k}-1}{q^{k-t-1}-1}\right)\\
&\qquad > \left(1+\frac{1}{q}\right)^2 q^{k-t+(k-t)(k-t-1)+(k-t-1)(n-2k+t)}\\
\Rightarrow\quad & 2+2\theta_{t+2}q^{k-t-1}\frac{q^{n-k}-1}{q^{k-t-1}-1} > \left(1+\frac{1}{q}\right)^2 q^{n-t}\\
\Rightarrow\quad & 2(q-1)(q^{k-t-1}-1)+2(q^{t+3}-1)q^{k-t-1}(q^{n-k}-1)\\
&\qquad > \left(1+\frac{1}{q}\right)^2 (q-1)(q^{k-t-1}-1)q^{n-t}\\
\Rightarrow\quad & 2q^{k-t}-2q^{k-t-1}-2q+2+2q^{n+2}-2q^{n-t-1}-2q^{k+2}+2q^{k-t-1}\\
&\qquad > \left(1+\frac{1}{q}\right)^2 (q^{n+k-2t}-q^{n+k-2t-1}-q^{n-t+1}+q^{n-t})\\
&\qquad = q^{n+k-2t}+q^{n+k-2t-1}-q^{n+k-2t-2}-q^{n+k-2t-3}-q^{n-t+1}-q^{n-t}\\
&\qquad\qquad + q^{n-t-1}+q^{n-t-2}\\
\Rightarrow\quad & q^{n-t+1}\left(-q^{k-t-1}+2q^{t+1}+1\right)+q^{n-t}\left(-q^{k-t-1}+q^{k-t-2}+q^{k-t-3}+1\right)\\
&\qquad + \left(-2q^{k+2}+2q^{k-t}+2\right)+(-2q-3q^{n-t-1}-q^{n-t-2})>0.
\end{aligned}$$

In the left hand side of the last inequality, all terms are at most zero for $k \geq 2t+3$ and $q \geq 3$. Hence, we find a contradiction which proves the statement. ■

Lemma 4.5.6. *Let $n \geq 2k-t+1, k \geq t+2, q \geq 3$, and consider Example 4.2.1 and Example 4.2.3 in $\mathrm{PG}(n,q)$. The number of elements in Example 4.2.3 is larger than the number of elements in Example 4.2.1 if $k \leq 2t+1$.*

Proof. Let $k \leq 2t+1$ and suppose to the contrary that the number of elements in Example 4.2.3 is at most the number of elements in Example 4.2.3. Then we have, by using (4.6) and (4.8) that

$$\begin{aligned}
&\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \left(1 + \theta_{t+2} q^{k-t-1} \frac{q^{n-k}-1}{q^{k-t-1}-1}\right) < q^{k-t+1}\theta_t + \theta_{k-t} \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} \\
\xRightarrow{L.1.10.2} \quad & \left(1+\frac{1}{q}\right) q^{(n-k)(k-t-2)} \left(\theta_{t+2} q^{k-t-1} \frac{q^{n-k}-1}{q^{k-t-1}-1}\right) < q^{k-t+1}\theta_t + 2\theta_{k-t} q^{(n-k)(k-t-1)} \\
\Rightarrow \quad & \left(1+\frac{1}{q}\right) (q^{t+3}-1)(q^{n-k}-1) q^{k-t-1} \\
& \quad < \frac{(q^{t+1}-1)(q^{k-t-1}-1)}{q^{(n-k)(k-t-2)-k+t-1}} + 2(q^{k-t+1}-1)(q^{k-t-1}-1)q^{n-k} \\
\Rightarrow \quad & q^{n+2} + q^{k-t-1} + q^{n+1} + q^{k-t-2} - q^{n-t-1} - q^{k+2} - q^{n-t-2} - q^{k+1} \\
& \quad < 2q^{n+k-2t} + 2q^{n-k} - 2q^{n-t-1} - 2q^{n-t+1} \\
& \qquad + q^{k-t+1-(n-k)(k-t-2)}(q^{t+1}-1)(q^{k-t-1}-1) \\
\Rightarrow \quad & \left(q^{n+2} - 2q^{n+k-2t} - q^{k-t+1-(n-k)(k-t-2)}(q^{t+1}-1)(q^{k-t-1}-1)\right) + q^{n-t-2}(q-1) \\
& \qquad + q^{k+1}(q^{n-k}-q-1) + q^{k-t-1}(2q^{n-k+2}+1-2q^{n-2k+t+1}) + q^{k-t-2} < 0
\end{aligned}$$

Now, the contradiction follows since all terms in the left hand side of the last inequality are positive. For the last four terms, this follows immediately since $n \geq 2k-t+1, k < 2t+2, k \geq t+2, q \geq 3$. We end this proof by proving that the first term is also positive. Since $k \geq t+2$ and $n \geq 2k-t+1 = k+(k-t)+1 \geq k+2$, we have that

$$\begin{aligned}
& 1 \leq (n-k-1)(k-t-1) \\
\Leftrightarrow \quad & n+1 \geq 2k-t+1-(n-k)(k-t-2) \\
\Rightarrow \quad & q^{n+2} \geq 2q^{n+1} + q^{n+1} \geq 2q^{n+k-2t} + q^{2k-t+1-(n-k)(k-t-2)} \\
& \qquad > 2q^{n+k-2t} + q^{k-t+1-(n-k)(k-t-2)}(q^{t+1}-1)(q^{k-t-1}-1).
\end{aligned}$$

■

Lemma 4.5.7. *Let* $n \geq 2k-t+1, q \geq 3$, *and consider Example 4.2.1 and Example 4.2.3 in* $\mathrm{PG}(n, q)$. *The number of elements in Example 4.2.3 is larger than the number of elements in Example 4.2.1 if* $k = 2t+2$.

Proof. Let $S_{2.1}$ and $S_{2.3}$ be the set of elements in Example 4.2.1 and in Example 4.2.3 respectively. Suppose that $k = 2t+2$, then we have to prove that $|S_{2.3}| > |S_{2.1}|$. From (4.7) and Lemma 1.10.4 for $\begin{bmatrix} a \\ b \end{bmatrix}$ equal to $\begin{bmatrix} n-t-1 \\ t+1 \end{bmatrix}$ and $\begin{bmatrix} n-t-2 \\ t \end{bmatrix}$, and with for both $c = t+1$, we find that

$$\begin{aligned}
|S_{2.3}| &= \begin{bmatrix} n-t-2 \\ t \end{bmatrix} + \sum_{j=0}^{t+1} \theta_{t+2} \begin{bmatrix} t+1 \\ j \end{bmatrix} \left(q^{(t-j+1)^2} \begin{bmatrix} n-2t-2 \\ t-j+1 \end{bmatrix} - q^{(t-j)(t-j+1)} \begin{bmatrix} n-2t-3 \\ t-j \end{bmatrix}\right) \\
&= \begin{bmatrix} n-t-2 \\ t \end{bmatrix} + \sum_{j=0}^{t} \theta_{t+2} \begin{bmatrix} t+1 \\ j \end{bmatrix} q^{(t-j+1)(t-j)} \begin{bmatrix} n-2t-3 \\ t-j \end{bmatrix} \frac{q^{n-t-j-1}-2q^{t-j+1}+1}{q^{t-j+1}-1} + \theta_{t+2}.
\end{aligned} \tag{4.14}$$

On the other hand, by (4.5), we have that

$$|S_{2.1}| = \theta_{2t+3} + \sum_{j=0}^{t} \begin{bmatrix} t+3 \\ j+1 \end{bmatrix} q^{(t+2-j)(t+1-j)} \begin{bmatrix} n-2t-3 \\ t+1-j \end{bmatrix}. \tag{4.15}$$

From (4.14) and (4.15), it follows that $|S_{2.3}| - |S_{2.1}|$ is equal to

$$\underbrace{\begin{bmatrix} n-t-2 \\ t \end{bmatrix} + \theta_{t+2} - \theta_{2t+3}}_{=w_1} + \sum_{j=0}^{t} q^{(t+1-j)(t-j)} \begin{bmatrix} n-2t-3 \\ t-j \end{bmatrix} \begin{bmatrix} t+1 \\ j \end{bmatrix} \frac{q^{t+3}-1}{q^{t-j+1}-1} w_2,$$

with

$$w_2 = \frac{q^{n-t-j-1} - 2q^{t-j+1} + 1}{q-1} - q^{2(t+1-j)} \frac{(q^{n-3t-3+j}-1)(q^{t+2}-1)}{(q^{j+1}-1)(q^{t+2-j}-1)}$$

We will prove that $w_1 \geq 0$ and $w_2 \geq 0$, which proves that $|S_{2.3}| \geq |S_{2.1}|$ for $k = 2t+2$.

$$\begin{aligned} w_1 = \begin{bmatrix} n-t-2 \\ t \end{bmatrix} + \theta_{t+2} - \theta_{2t+3} &\overset{L.1.10.2}{\geq} \left(1 + \frac{1}{q}\right) q^{(n-2t-2)t} + \theta_{t+2} - \frac{q^{2t+4}}{q-1} \\ &\geq \frac{1}{q(q-1)} \left(q^{(n-2t-2)t+2} - q^{(n-2t-2)t} - q^{2t+5}\right) + \theta_{t+2} \end{aligned}$$

As

$$q^{(n-2t-2)t+2} - q^{(n-2t-2)t} - q^{2t+5} \geq 3q^{(n-2t-2)t+1} - q^{(n-2t-2)t} - q^{2t+5} > q^{(n-2t-2)t+1} - q^{2t+5},$$

it is sufficient to prove that $q^{(n-2t-2)t+1} \geq q^{2t+5}$. This inequality is valid for $n \geq 2t+4+\frac{4}{t}$. For $t > 1$, this assumption holds since $n \geq 2k-t+1 = 3t+5$. If $t=1$ and $n \geq 10$, we also find that $q^{(n-2t-2)t+1} \geq q^{2t+5}$. For $n=9$ and $t=1$, we find that $w_1 = \theta_3 > 0$. In the last remaining case; $n=8, t=1$, we have that $w_1 < 0$. For this case, we used a computer algebra package to calculate both numbers $|S_{2.3}|, |S_{2.1}|$ to see that $|S_{2.3}| \geq |S_{2.1}|$.

$$\begin{aligned} w_2 &= \frac{q^{n-t-j-1} - 2q^{t-j+1} + 1}{q-1} - q^{2(t+1-j)} \frac{(q^{n-3t-3+j}-1)(q^{t+2}-1)}{(q^{j+1}-1)(q^{t+2-j}-1)} \\ &= \frac{q^{n-t-j}\left(q^{t+1} + 1 - q^{j} - q^{t-j+1}\right) + q^{2t-j+4}\left(q^{t-j+1} - q^{t-j} - 2\right)}{(q-1)(q^{j+1}-1)(q^{t+2-j}-1)} \\ &\qquad + \frac{2q^{t-j+1}\left(q^{j+1}-1\right) + \left(q^{t+3} - q^{j+1} - q^{t-j+2}\right) + q^{2t-2j+2} + q^{2t-2j+3} + 1}{(q-1)(q^{j+1}-1)(q^{t+2-j}-1)} \end{aligned}$$

For $0 \leq j \leq t$, we find that both the nominator and denominator are positive, since we have that $q \geq 3$. So $w_2 \geq 0$. Hence, we have that $|S_{2.3}| > |S_{2.1}|$. ■

Lemma 4.5.8. *Let* $n \geq 2k-t+1, q \geq 3$, *and consider Example 4.3.1 and Example 4.3.3 in* $\mathrm{AG}(n,q)$. *The number of elements in Example 4.3.1 is larger than the number of elements in Example 4.3.3 if* $k \geq 2t+2$.

Proof. Let $k \geq 2t+2$ and suppose to the contrary that the number of elements in Example 4.3.1 is at most the number of elements in Example 4.3.3. Then by using (4.10) and (4.13), we have that

$$\begin{aligned}
& \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \left(1+\theta_{t+1}q^{k-t-1}\frac{q^{n-k}-1}{q^{k-t-1}-1}\right) \\
& \qquad \geq \theta_k + \sum_{j=0}^{k-t-2} \begin{bmatrix} k-t+1 \\ j+1 \end{bmatrix} q^{(k-t-j)(k-t-j-1)} \begin{bmatrix} n-k-1 \\ k-t-j-1 \end{bmatrix} \\
\overset{j=0}{\Longrightarrow} \quad & \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \left(1+\theta_{t+1}q^{k-t-1}\frac{q^{n-k}-1}{q^{k-t-1}-1}\right) > \theta_{k-t}q^{(k-t)(k-t-1)} \begin{bmatrix} n-k-1 \\ k-t-1 \end{bmatrix} \\
\overset{L.1.10.2}{\Longrightarrow} \quad & 2q^{(k-t-2)(n-k)} \left(1+\theta_{t+1}q^{k-t-1}\frac{q^{n-k}-1}{q^{k-t-1}-1}\right) \\
& \qquad > \left(1+\frac{1}{q}\right)^2 q^{k-t+(k-t)(k-t-1)+(k-t-1)(n-2k+t)} \\
\Rightarrow \quad & 2+2\theta_{t+1}q^{k-t-1}\frac{q^{n-k}-1}{q^{k-t-1}-1} > \left(1+\frac{1}{q}\right)^2 q^{n-t} \\
\Rightarrow \quad & 2(q-1)(q^{k-t-1}-1)+2(q^{t+2}-1)q^{k-t-1}(q^{n-k}-1) \\
& \qquad > \left(1+\frac{1}{q}\right)^2 (q-1)(q^{k-t-1}-1)q^{n-t} \\
\Rightarrow \quad & 2q^{k-t}-2q^{k-t-1}-2q+2+2q^{n+1}-2q^{n-t-1}-2q^{k+1}+2q^{k-t-1} \\
& \qquad > \left(1+\frac{1}{q}\right)^2 (q^{n+k-2t}-q^{n+k-2t-1}-q^{n-t+1}+q^{n-t}) \\
& \qquad = q^{n+k-2t}+q^{n+k-2t-1}-q^{n+k-2t-2}-q^{n+k-2t-3}-q^{n-t+1}-q^{n-t} \\
& \qquad\quad + q^{n-t-1}+q^{n-t-2} \\
\Rightarrow \quad & q^{n-t+1}\left(-q^{k-t-1}+2q^{t}+1\right)+q^{n-t}\left(-q^{k-t-1}+q^{k-t-2}+q^{k-t-3}+1\right) \\
& \qquad + 2\left(-q^{k+1}+q^{k-t}+1\right)-(2q+3q^{n-t-1}+q^{n-t-2})>0.
\end{aligned}$$

In the left hand side of the last inequality, all terms are at most zero for $k \geq 2t+2$ and $q \geq 3$. Hence, we find a contradiction which proves the statement. ■

Lemma 4.5.9. *Let* $n \geq 2k-t+1, k \geq t+2, q \geq 3$, *and consider Example 4.3.1 and Example 4.3.3 in* $\mathrm{AG}(n,q)$. *The number of elements in Example 4.3.3 is larger than the number of elements in Example 4.3.1 if* $k \leq 2t$.

Proof. Let $k \leq 2t$ and suppose to the contrary that the number of elements in Example 4.3.1 is at least the number of elements in Example 4.3.3. Then we have, by using (4.11) and (4.13), that

$$\begin{aligned}
& \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \left(1+\theta_{t+1}q^{k-t-1}\frac{q^{n-k}-1}{q^{k-t-1}-1}\right) < q^{k-t+1}\theta_t+\theta_{k-t}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} \\
\overset{L.1.10.2}{\Longrightarrow} \quad & \left(1+\frac{1}{q}\right) q^{(n-k)(k-t-2)} \left(\theta_{t+1}q^{k-t-1}\frac{q^{n-k}-1}{q^{k-t-1}-1}\right) < q^{k-t+1}\theta_t+2\theta_{k-t}q^{(n-k)(k-t-1)} \\
\Rightarrow \quad & \left(1+\frac{1}{q}\right)(q^{t+2}-1)(q^{n-k}-1)q^{k-t-1} \\
& \qquad < \frac{(q^{t+1}-1)(q^{k-t-1}-1)}{q^{(n-k)(k-t-2)-k+t-1}}+2(q^{k-t+1}-1)(q^{k-t-1}-1)q^{n-k}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad & q^{n+1}+q^{k-t-1}+q^{n}+q^{k-t-2}-q^{n-t-1}-q^{k+1}-q^{n-t-2}-q^{k} \\
& \quad < 2q^{n+k-2t}+2q^{n-k}-2q^{n-t-1}-2q^{n-t+1}+\frac{q^{k-t+1}(q^{t+1}-1)(q^{k-t-1}-1)}{q^{(n-k)(k-t-2)}} \\
\Rightarrow \quad & \left(q^{n+1}-2q^{n+k-2t}-q^{k-t+1-(n-k)(k-t-2)}(q^{t+1}-1)(q^{k-t-1}-1)\right)+q^{n-t-2}\left(q-1\right) \\
& \quad +q^{k}\left(q^{n-k}-q^{n-2k}-q-1\right)+q^{k-t-2}\left(2q^{n-k+3}+q+1-q^{n-2k+t+2}\right)<0
\end{aligned}$$

Now, the contradiction follows since all terms in the left hand side of the last inequality are positive. For the last three terms, this follows immediately since $n \geq 2k-t+1, k < 2t+1, k \geq t+2, q \geq 3$. We end this proof by proving that the first term is also positive. Since $k \geq t+2$ and $n \geq 2k-t+1 = k+(k-t)+1 \geq k+3$, we have that

$$\begin{aligned}
& 2 \leq (n-k-1)(k-t-1) \\
\Leftrightarrow\ & n \geq 2k-t+1-(n-k)(k-t-2) \\
\Rightarrow\ & q^{n+1} \geq 2q^{n}+q^{n} \geq 2q^{n+k-2t}+q^{2k-t+1-(n-k)(k-t-2)} \\
& \qquad\qquad > 2q^{n+k-2t}+q^{k-t+1-(n-k)(k-t-2)}(q^{t+1}-1)(q^{k-t-1}-1).
\end{aligned}$$

■

Lemma 4.5.10. *Let* $n \geq 2k-t+1, q \geq 3$, *and consider Example 4.3.1 and Example 4.3.3 in* $\mathrm{AG}(n,q)$. *The number of elements in Example 4.3.3 is at least the number of elements in Example 4.3.1 if* $k = 2t+1$.

Proof. Let $R_{3.1}$ and $R_{3.3}$ be the set of elements in Example 4.3.1 and in Example 4.3.3 respectively. Suppose that $k = 2t+1$, then we have to prove that $|R_{3.3}| \geq |R_{3.1}|$. By (4.12) and Lemma 1.10.4 for $\begin{bmatrix} a \\ b \end{bmatrix}$ equal to $\begin{bmatrix} n-t-1 \\ t \end{bmatrix}$ and $\begin{bmatrix} n-t-2 \\ t-1 \end{bmatrix}$, and with for both $c = t$, we find that

$$\begin{aligned}
|R_{3.3}| &= \begin{bmatrix} n-t-2 \\ t-1 \end{bmatrix} + \sum_{j=0}^{t} \theta_{t+1} \begin{bmatrix} t \\ j \end{bmatrix} \left(q^{(t-j)^2} \begin{bmatrix} n-2t-1 \\ t-j \end{bmatrix} - q^{(t-j-1)(t-j)} \begin{bmatrix} n-2t-2 \\ t-j-1 \end{bmatrix} \right) \\
&= \begin{bmatrix} n-t-2 \\ t-1 \end{bmatrix} + \sum_{j=0}^{t-1} \theta_{t+1} \begin{bmatrix} t \\ j \end{bmatrix} q^{(t-j)(t-j-1)} \begin{bmatrix} n-2t-2 \\ t-j-1 \end{bmatrix} \frac{q^{n-t-j-1}-2q^{t-j}+1}{q^{t-j}-1} + \theta_{t+1}.
\end{aligned} \tag{4.16}$$

On the other hand, by (4.10), we have that

$$|R_{3.1}| = \theta_{2t+1} + \sum_{j=0}^{t-1} \begin{bmatrix} t+2 \\ j+1 \end{bmatrix} q^{(t+1-j)(t-j)} \begin{bmatrix} n-2t-2 \\ t-j \end{bmatrix}. \tag{4.17}$$

Hence, it follows that

$$|R_{3.3}| - |R_{3.1}| = \underbrace{\begin{bmatrix} n-t-2 \\ t-1 \end{bmatrix} + \theta_{t+1} - \theta_{2t+1}}_{=w_1} + \sum_{j=0}^{t-1} q^{(t-j)(t-j-1)} \begin{bmatrix} n-2t-2 \\ t-j \end{bmatrix} \begin{bmatrix} t \\ j \end{bmatrix} (q^{t+2}-1) w_2,$$

with

$$w_2 = \frac{q^{n-t-j-1}-2q^{t-j}+1}{(q-1)(q^{n-3t+j-1}-1)} - \frac{q^{t+1}-1}{(q^{j+1}-1)(q^{t-j+1}-1)} q^{2(t-j)}.$$

We will prove that $w_1 \geq 0$ and $w_2 \geq 0$, which proves that $|R_{3.3}| \geq |R_{3.1}|$ for $k = 2t + 1$.

$$\begin{aligned} w_1 &= \begin{bmatrix} n-t-2 \\ t-1 \end{bmatrix} + \theta_{t+1} - \theta_{2t+1} \\ &\geq \left(1 + \frac{1}{q}\right) q^{(n-2t-1)(t-1)} + \theta_{t+1} - \frac{q^{2t+2}}{q-1} \\ &= \frac{1}{q(q-1)} \left(q^{(n-2t-1)(t-1)+2} - q^{(n-2t-1)(t-1)} - q^{2t+3}\right) + \theta_{t+1}. \end{aligned}$$

Note that we used Lemma 1.10.2 for the inequality on the second line. Since

$$\begin{aligned} q^{(n-2t-1)(t-1)+2} - q^{(n-2t-1)(t-1)} - q^{2t+3} &\geq 3q^{(n-2t-1)(t-1)+1} - q^{(n-2t-1)(t-1)} - q^{2t+3} \\ &> q^{(n-2t-1)(t-1)+1} - q^{2t+3}, \end{aligned}$$

it is sufficient to prove that $q^{(n-2t-1)(t-1)+1} \geq q^{2t+3}$. This inequality is valid for $n \geq 2t + 3 + \frac{4}{t-1}$. For $t \geq 3$, this assumption holds since $n \geq 2k - t + 1 = 3t + 3$. For $t = 2$, the assumption holds for $n \geq 11$. For $t = 2, n = 10$, we have that $w_1 = \theta_3 > 0$. Since $n \geq 2k - t + 1 = 3t + 3$, the only remaining cases are $t = 2$ and $n = 9$, and $t = 1$ and $n \geq 6$. In these cases, we immediately calculate $|R_{3.3}| - |R_{3.1}|$. For $t = 2, n = 9$, we have that $|R_{3.3}| - |R_{3.1}| = q^9 + 2q^8 + 3q^7 + 2q^6 + q^5 > 0$. For $t = 1, n > 5$, we have that $|R_{3.3}| = |R_{3.1}| = 1 + q\theta_2\theta_{n-4}$.

Now we investigate w_2:

$$\begin{aligned} w_2 &= \frac{q^{n-t-j-1} - 2q^{t-j} + 1}{(q-1)(q^{n-3t+j-1} - 1)} - \frac{q^{t+1} - 1}{(q^{j+1} - 1)(q^{t-j+1} - 1)} q^{2(t-j)} \\ &= \frac{(q^{j+1} - 1)(q^{t-j+1} - 1)(q^{n-t-j-1} - 2q^{t-j} + 1) - (q-1)(q^{n-3t+j-1} - 1)(q^{t+1} - 1)q^{2(t-j)}}{(q-1)(q^{n-3t+j-1} - 1)(q^{j+1} - 1)(q^{t-j+1} - 1)} \\ &= \frac{q^{n-2j-t}\left(q^{j+t} - q^{2j} - q^t\right) + q^{2t-j+2}\left(q^{t-j} - q^{t-j-1} - 2\right)}{(q-1)(q^{n-3t+j-1} - 1)(q^{j+1} - 1)(q^{t-j+1} - 1)} \\ &\quad + \frac{\left(q^{t+2} + 2q^{t+1} - q^{t-j+1} - 2q^{t-j} - q^{j+1}\right) + q^{2t-2j+1} + q^{2t-2j} + 1 + q^{n-j-t}}{(q-1)(q^{n-3t+j-1} - 1)(q^{j+1} - 1)(q^{t-j+1} - 1)}. \end{aligned}$$

As $0 \leq j \leq t - 1$ and $q \geq 3$, we find that all terms in the nominator are at least 0, which proves that $w_2 \geq 0$. Hence, we find that $|R_{3.3}| \geq |R_{3.1}|$. ■

Lemma 4.5.11. *Suppose* $n \geq 2k + t + 3, q \geq 2, k \geq t + 2, t \geq 1$, *then*

$$(\theta_{k-t})^x \begin{bmatrix} n-t-x \\ k-t-x \end{bmatrix} \begin{bmatrix} t+x+1 \\ t+1 \end{bmatrix} < (\theta_{k-t})^2 \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \begin{bmatrix} t+3 \\ t+1 \end{bmatrix}$$

for all $2 < x \leq k - t$.

Proof. It is sufficient to prove that

$$(\theta_{k-t})^{x+1} \begin{bmatrix} n-t-x-1 \\ k-t-x-1 \end{bmatrix} \begin{bmatrix} t+x+2 \\ t+1 \end{bmatrix} < (\theta_{k-t})^x \begin{bmatrix} n-t-x \\ k-t-x \end{bmatrix} \begin{bmatrix} t+x+1 \\ t+1 \end{bmatrix}, \tag{4.18}$$

for all $x \geq 2$.

$$\begin{aligned}
&(\theta_{k-t})^{x+1}\begin{bmatrix} n-t-x-1 \\ k-t-x-1 \end{bmatrix}\begin{bmatrix} t+x+2 \\ t+1 \end{bmatrix} < (\theta_{k-t})^{x}\begin{bmatrix} n-t-x \\ k-t-x \end{bmatrix}\begin{bmatrix} t+x+1 \\ t+1 \end{bmatrix} \\
\Leftrightarrow \quad & \frac{q^{k-t+1}-1}{q-1}(q^{t+x+2}-1) < \frac{q^{n-t-x}-1}{q^{k-t-x}-1}(q^{x+1}-1) \\
\Leftrightarrow \quad & (q^{k-t+1}-1)(q^{k-t-x}-1)\left(q^{t+x+2}-1\right) < (q-1)\left(q^{n-t-x}-1\right)\left(q^{x+1}-1\right) \\
\Leftrightarrow \quad & (q^{n-t+2}-q^{n-t+1}-q^{n-t-x+1}-q^{2k-t+3}) + q^{k-t-x}(q^{n-k}-1) + q^{t+x+2}(q^{k-t+1}-1) \\
& \qquad + (q^{k+2}-q^{x+2}-q^{k-t+1}) + q(q^{2k-2t-x}+q^{x}+1) > 0.
\end{aligned}$$

The last four terms are positive for $q \geq 2$ since $k > x \geq 2$. For the first term, we have that

$$q^{n-t+2}-q^{n-t+1} \geq q^{n-t+1} \geq 2q^{n-t} > q^{n-t-x+1}+q^{2k-t+3},$$

which is true since $x \geq 2$ and $n \geq 2k+t+3$. ■

Lemma 4.5.12. *Suppose $k \geq 2t+2$, $t \geq 1$, $q \geq 3$ and $n \geq 2k+t+3$, then*

$$\begin{aligned}
(\theta_{k-t})^{2}\begin{bmatrix} t+3 \\ t+1 \end{bmatrix}\begin{bmatrix} k-t-2 \\ j \end{bmatrix} q^{(k-t-j-2)^2}\begin{bmatrix} n-k \\ k-t-j-2 \end{bmatrix} & \\
< \begin{bmatrix} k-t+1 \\ j+1 \end{bmatrix} q^{(k-t-j)(k-t-j-1)}\begin{bmatrix} n-k-1 \\ k-t-j-1 \end{bmatrix}&,
\end{aligned}$$

for all $j \in \{0,\dots,k-t-2\}$.

Proof.

$$\begin{aligned}
&(\theta_{k-t})^{2}\begin{bmatrix} t+3 \\ t+1 \end{bmatrix}\begin{bmatrix} k-t-2 \\ j \end{bmatrix} q^{(k-t-j-2)^2}\begin{bmatrix} n-k \\ k-t-j-2 \end{bmatrix} \\
&\qquad < \begin{bmatrix} k-t+1 \\ j+1 \end{bmatrix} q^{(k-t-j)(k-t-j-1)}\begin{bmatrix} n-k-1 \\ k-t-j-1 \end{bmatrix} \\
\Leftrightarrow \quad & \frac{(q^{k-t+1}-1)^2}{(q-1)^2}\frac{(q^{t+3}-1)(q^{t+2}-1)}{(q^2-1)(q-1)}\begin{bmatrix} n-k \\ k-t-j-2 \end{bmatrix} \\
&\quad \cdot\left(\frac{(q^{j+1}-1)(q^{k-t-j}-1)(q^{k-t-j-1}-1)}{(q^{k-t+1}-1)(q^{k-t}-1)(q^{k-t-1}-1)}\begin{bmatrix} k-t+1 \\ j+1 \end{bmatrix}\right) \\
&\qquad < \begin{bmatrix} k-t+1 \\ j+1 \end{bmatrix}\left(\frac{(q^{n-2k+t+j+2}-1)(q^{n-2k+t+j+1}-1)}{(q^{n-k}-1)(q^{k-t-j-1}-1)}\begin{bmatrix} n-k \\ k-t-j-2 \end{bmatrix}\right) q^{3k-3t-3j-4} \\
\Leftrightarrow \quad & \frac{(q^{k-t+1}-1)}{(q-1)^2}\frac{(q^{t+3}-1)(q^{t+2}-1)}{(q^2-1)(q-1)}\frac{(q^{j+1}-1)(q^{k-t-j}-1)(q^{k-t-j-1}-1)}{(q^{k-t}-1)(q^{k-t-1}-1)} \\
&\qquad < \frac{(q^{n-2k+t+j+2}-1)(q^{n-2k+t+j+1}-1)}{(q^{n-k}-1)(q^{k-t-j-1}-1)} q^{3k-3t-3j-4} \\
\Leftrightarrow \quad & (q^{n-k}-1)(q^{k-t-j-1}-1)(q^{k-t+1}-1)\frac{(q^{k-t-j}-1)(q^{k-t-j-1}-1)}{(q^{k-t}-1)(q^{k-t-1}-1)} \\
&\qquad < (q-1)^3(q^2-1)\frac{q^{n-2k+t+j+1}-1}{q^{j+1}-1}\frac{q^{n-2k+t+j+2}-1}{(q^{t+3}-1)(q^{t+2}-1)} q^{3k-3t-3j-4}
\end{aligned}$$

It is true that $\frac{q^a-1}{q^b-1} \leq q^{a-b}$ if and only if $b \geq a$. We use this bound twice in the last fraction on the left side of the inequality. Moreover, since $\frac{q^{n-2k+t+j+2}-1}{(q^{t+3}-1)(q^{t+2}-1)} \geq q^{n-2k-t+j-3} \geq 1$ it is sufficient to prove that

$$\begin{aligned}
& q^{n+k-2t-3j} < (q-1)^3(q^2-1)\frac{q^{n-2k+t+j+1}-1}{q^{j+1}-1}q^{n+k-4t-2j-7} \\
\Leftrightarrow \quad & \frac{q^{2t-j+7}}{q^{n-2k+t+j+1}-1} < \frac{(q-1)^3(q^2-1)}{q^{j+1}-1} \\
\Leftarrow \quad & \frac{q^{2t-j+7}}{q^{2t+j+4}-1}(q^{j+1}-1) < (q-1)^3(q^2-1) \\
\Leftarrow \quad & \frac{q^{7-j}}{q^{j+4}-1}(q^{j+1}-1) < (q-1)^3(q^2-1) \\
\Leftarrow \quad & \frac{q^{7}}{q^{4}-1}(q-1) < (q-1)^3(q^2-1)
\end{aligned}$$

The third inequality follows since $f(n) = \frac{q^{2t-j+7}}{q^{n-2k+t+j+1}-1}$ is decreasing and $n \geq 2k+t+3$. The fourth inequality follows since $h(t) = \frac{q^{2t-j+7}}{q^{2t+j+4}-1}$ is decreasing and $t \geq 0$ while the last inequality follows as $g(j) = \frac{q^{7-j}}{q^{j+4}-1}(q^{j+1}-1)$ is decreasing and $j \geq 0$. The last inequality is true for all $q \geq 3$. ■

Lemma 4.5.13. *Suppose* $k \geq t+2$, $t \geq 1$, $q \geq 3$ *and* $n \geq 2k+t+3$, *then*

$$(\theta_{k-t})^x \begin{bmatrix} n-t-x \\ k-t-x \end{bmatrix} \begin{bmatrix} t+x+1 \\ t+1 \end{bmatrix} < f_p(q,n,k,t)$$

for all $2 \leq x \leq k-t$.

Proof. From Lemma 4.5.11, it follows that it is sufficient to prove the lemma for $x = 2$. Hence, we have to prove the following inequalities, for which we use (4.8) and (4.5):

$$(\theta_{k-t})^2 \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \begin{bmatrix} t+3 \\ t+1 \end{bmatrix} < \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \left(1 + \theta_{t+2}q^{k-t-1}\frac{q^{n-k}-1}{q^{k-t-1}-1}\right) \qquad \text{for } k \leq 2t+2; \tag{4.19}$$

$$\begin{aligned}
&(\theta_{k-t})^2 \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \begin{bmatrix} t+3 \\ t+1 \end{bmatrix} \\
&\qquad < \theta_{k+1} + \sum_{j=0}^{k-t-2} \begin{bmatrix} k-t+1 \\ j+1 \end{bmatrix} q^{(k-t-j)(k-t-j-1)} \begin{bmatrix} n-k-1 \\ k-t-j-1 \end{bmatrix} \qquad \text{for } k \geq 2t+3.
\end{aligned} \tag{4.20}$$

We start by proving inequality (4.19). Suppose to the contrary that this inequality does not hold. Then we have that

$$\begin{aligned}
& (\theta_{k-t})^2 \begin{bmatrix} t+3 \\ t+1 \end{bmatrix} \geq 1 + \theta_{t+2}q^{k-t-1}\frac{q^{n-k}-1}{q^{k-t-1}-1} \\
\xRightarrow{L.1.10.2} \quad & \frac{q^{2k-2t+2}}{(q-1)^2}2q^{2t+2} > \frac{q^{t+3}-1}{q-1}q^{k-t-1}\frac{q^{n-k}-1}{q^{k-t-1}-1} \\
\xRightarrow{n \geq 2k+t+3} \quad & 2q^{k+t+5}(q^{k-t-1}-1) > (q-1)(q^{t+3}-1)(q^{n-k}-1) \\
& \qquad \geq (q-1)(q^{t+3}-1)(q^{k+t+3}-1) \\
\Rightarrow \quad & 0 > q^{2k+4}(q^{2t-k+3} - q^{2t-k+2} - 2) + q^{t+4}(2q^{k+1} - q^k - 1) \\
& \qquad + (q-1) + q^{t+3} + q^{k+t+3}.
\end{aligned}$$

All terms in the right hand side of the last inequality are non-negative since $k \leq 2t+2$ and $q \geq 3$. Hence, we have a contradiction which proves (4.19).

Now we prove inequality (4.20). We use Lemma 1.10.4 for the factor $\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}$ with $c = k-t-2$, and so, we have to prove the following inequality

$$(\theta_{k-t})^2 \begin{bmatrix} t+3 \\ t+1 \end{bmatrix} \sum_{j=0}^{k-t-2} \begin{bmatrix} k-t-2 \\ j \end{bmatrix} q^{(k-t-j-2)^2} \begin{bmatrix} n-k \\ k-t-j-2 \end{bmatrix}$$
$$< \theta_{k+1} + \sum_{j=0}^{k-t-2} \begin{bmatrix} k-t+1 \\ j+1 \end{bmatrix} q^{(k-t-j)(k-t-j-1)} \begin{bmatrix} n-k-1 \\ k-t-j-1 \end{bmatrix}.$$

Hence, it is sufficient to prove that

$$(\theta_{k-t})^2 \begin{bmatrix} t+3 \\ t+1 \end{bmatrix} \begin{bmatrix} k-t-2 \\ j \end{bmatrix} q^{(k-t-j-2)^2} \begin{bmatrix} n-k \\ k-t-j-2 \end{bmatrix}$$
$$< \begin{bmatrix} k-t+1 \\ j+1 \end{bmatrix} q^{(k-t-j)(k-t-j-1)} \begin{bmatrix} n-k-1 \\ k-t-j-1 \end{bmatrix},$$

for all $j \in \{0,1,\ldots,k-t-2\}$. This follows from Lemma 4.5.12. ■

Lemma 4.5.14. *Suppose* $n \geq 2k+t+3, q \geq 2, k \geq t+2, t \geq 1$, *then*

$$(\theta_{k-t})^x \begin{bmatrix} n-t-x \\ k-t-x \end{bmatrix} q^x \begin{bmatrix} t+x \\ x \end{bmatrix} < (\theta_{k-t})^2 \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} q^2 \begin{bmatrix} t+2 \\ 2 \end{bmatrix}$$

for all $2 < x \leq k-t$.

Proof. It is sufficient to prove that

$$(\theta_{k-t})^{x+1} \begin{bmatrix} n-t-x-1 \\ k-t-x-1 \end{bmatrix} q^{x+1} \begin{bmatrix} t+x+1 \\ x+1 \end{bmatrix} < (\theta_{k-t})^x \begin{bmatrix} n-t-x \\ k-t-x \end{bmatrix} q^x \begin{bmatrix} t+x \\ x \end{bmatrix}.$$

Since $n \geq 2k+t+3, q \geq 2, k \geq t+2, t \geq 1, 2 \leq x < k$, we have from (4.18) that

$$\left((\theta_{k-t})^x \begin{bmatrix} n-t-x \\ k-t-x \end{bmatrix} \right) q^x \begin{bmatrix} t+x \\ x \end{bmatrix} > \left((\theta_{k-t})^{x+1} \begin{bmatrix} n-t-x-1 \\ k-t-x-1 \end{bmatrix} \frac{\begin{bmatrix} t+x+2 \\ t+1 \end{bmatrix}}{\begin{bmatrix} t+x+1 \\ t+1 \end{bmatrix}} \right) q^x \begin{bmatrix} t+x \\ x \end{bmatrix}$$
$$> (\theta_{k-t})^{x+1} \begin{bmatrix} n-t-x-1 \\ k-t-x-1 \end{bmatrix} q^x \frac{q^{t+x+2}-1}{q^{t+x+1}-1} \begin{bmatrix} t+x+1 \\ x+1 \end{bmatrix}$$
$$> (\theta_{k-t})^{x+1} \begin{bmatrix} n-t-x-1 \\ k-t-x-1 \end{bmatrix} q^{x+1} \begin{bmatrix} t+x+1 \\ x+1 \end{bmatrix}.$$

This proves the lemma. ■

Lemma 4.5.15. *Suppose* $k \geq t+2, t \geq 1, q \geq 3$, *and* $n \geq 2k+t+3$, *then*

$$(\theta_{k-t})^x \begin{bmatrix} n-t-x \\ k-t-x \end{bmatrix} q^x \begin{bmatrix} t+x \\ x \end{bmatrix} < f_a(q,n,k,t)$$

for all $2 \leq x \leq k-t$.

Proof. From Lemma 4.5.14 it follows that it is sufficient to prove the lemma for $x = 2$. Hence we have to prove the following inequalities, for which we use (4.13) and (4.10):

$$(\theta_{k-t})^2 \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} q^2 \begin{bmatrix} t+2 \\ 2 \end{bmatrix} < \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \left(1+\theta_{t+1}q^{k-t-1}\frac{q^{n-k}-1}{q^{k-t-1}-1}\right) \quad \text{for } k \leq 2t+1; \tag{4.21}$$

$$\begin{aligned}(\theta_{k-t})^2 &\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} q^2 \begin{bmatrix} t+2 \\ 2 \end{bmatrix} \\ &< \theta_k + \sum_{j=0}^{k-t-2} \begin{bmatrix} k-t+1 \\ j+1 \end{bmatrix} q^{(k-t-j)(k-t-j-1)} \begin{bmatrix} n-k-1 \\ k-t-j-1 \end{bmatrix} \quad \text{for } k \geq 2t+2.\end{aligned} \tag{4.22}$$

We start by proving inequality (4.21). Suppose to the contrary that this inequality doesn't hold. Then we have that

$$\begin{aligned}
& & (\theta_{k-t})^2 \begin{bmatrix} t+2 \\ 2 \end{bmatrix} q^2 &\geq 1+\theta_{t+1}q^{k-t-1}\frac{q^{n-k}-1}{q^{k-t-1}-1} \\
&\xRightarrow{\underline{L.1.10.2}} & \frac{q^{2k-2t+2}}{(q-1)^2}2q^{2t+2} &> \frac{q^{t+2}-1}{q-1}q^{k-t-1}\frac{q^{n-k}-1}{q^{k-t-1}-1} \\
&\xRightarrow{\underline{n\geq 2k+t+3}} & 2q^{k+t+5}(q^{k-t-1}-1) &> (q-1)(q^{t+2}-1)(q^{n-k}-1) \\
& & &> (q-1)(q^{t+2}-1)(q^{k+t+3}-1) \\
&\Rightarrow & 0 &> q^{2k+4}(q^{2t-k+2}-q^{2t-k+1}-2)+q^{t+3}(2q^{k+2}-q^{k+1}-1) \\
& & &\quad + q^{t+2}+q^{k+t+3}+(q-1).
\end{aligned}$$

All terms in the right hand side of the last inequality are non-negative since $k \leq 2t+1$ and $q \geq 3$. Hence we have a contradiction which proves (4.21).

Now we prove inequality (4.22). We use Lemma 1.10.4 for the factor $\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}$ with $c = k-t-2$, and so, we have to prove the following inequality.

$$\begin{aligned}(\theta_{k-t})^2 q^2 \begin{bmatrix} t+2 \\ 2 \end{bmatrix} &\sum_{j=0}^{k-t-2} \begin{bmatrix} k-t-2 \\ j \end{bmatrix} q^{(k-t-j-2)^2} \begin{bmatrix} n-k \\ k-t-j-2 \end{bmatrix} \\ &< \theta_k + \sum_{j=0}^{k-t-2} \begin{bmatrix} k-t+1 \\ j+1 \end{bmatrix} q^{(k-t-j)(k-t-j-1)} \begin{bmatrix} n-k-1 \\ k-t-j-1 \end{bmatrix}\end{aligned}$$

Note that it is sufficient to prove the inequality below for all $j \in \{0, \ldots, k-t-2\}$.

$$\begin{aligned}(\theta_{k-t})^2 q^2 \begin{bmatrix} t+2 \\ 2 \end{bmatrix} &\begin{bmatrix} k-t-2 \\ j \end{bmatrix} q^{(k-t-j-2)^2} \begin{bmatrix} n-k \\ k-t-j-2 \end{bmatrix} \\ &< \begin{bmatrix} k-t+1 \\ j+1 \end{bmatrix} q^{(k-t-j)(k-t-j-1)} \begin{bmatrix} n-k-1 \\ k-t-j-1 \end{bmatrix},\end{aligned}$$

This inequality follows from Lemma 4.5.12, since $q^2 \begin{bmatrix} t+2 \\ 2 \end{bmatrix} < \begin{bmatrix} t+3 \\ 2 \end{bmatrix}$. ■

Lemma 4.5.16. *Suppose* $n \geq 2k+t+3, q \geq 3, k \geq t+2, t \geq 1$, *then*

$$2\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + (\theta_{t+1}\theta_{k-t} - \theta_{t+1} - 1)\theta_{k-t} \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} < f_p(q,n,k,t).$$

Proof. We have to prove the following inequalities:

$$\begin{aligned}
&2\begin{bmatrix}n-t-1\\k-t-1\end{bmatrix} + (\theta_{t+1}\theta_{k-t} - \theta_{t+1} - 1)\theta_{k-t}\begin{bmatrix}n-t-2\\k-t-2\end{bmatrix}\\
&\qquad < \begin{bmatrix}n-t-2\\k-t-2\end{bmatrix}\left(1+\theta_{t+2}q^{k-t-1}\frac{q^{n-k}-1}{q^{k-t-1}-1}\right) \qquad \text{for } k \leq 2t+2;
\end{aligned} \tag{4.23}$$

$$\begin{aligned}
&2\begin{bmatrix}n-t-1\\k-t-1\end{bmatrix} + (\theta_{t+1}\theta_{k-t} - \theta_{t+1} - 1)\theta_{k-t}\begin{bmatrix}n-t-2\\k-t-2\end{bmatrix}\\
&\qquad < \theta_{k+1} + \sum_{j=0}^{k-t-2}\begin{bmatrix}k-t+1\\j+1\end{bmatrix}q^{(k-t-j)(k-t-j-1)}\begin{bmatrix}n-k-1\\k-t-j-1\end{bmatrix} \text{ for } k \geq 2t+3.
\end{aligned} \tag{4.24}$$

We start by proving inequality (4.23). Suppose to the contrary that this inequality doesn't hold. Then we have that

$$\begin{aligned}
& 2\begin{bmatrix}n-t-1\\k-t-1\end{bmatrix} + (\theta_{t+1}\theta_{k-t} - \theta_{t+1} - 1)\theta_{k-t}\begin{bmatrix}n-t-2\\k-t-2\end{bmatrix}\\
& \qquad\qquad \geq \begin{bmatrix}n-t-2\\k-t-2\end{bmatrix}\left(1+\theta_{t+2}q^{k-t-1}\frac{q^{n-k}-1}{q^{k-t-1}-1}\right)\\
\Leftrightarrow\quad & 2\frac{q^{n-t-1}-1}{q^{k-t-1}-1} + (\theta_{t+1}\theta_{k-t} - \theta_{t+1} - 1)\theta_{k-t} \geq 1 + \theta_{t+2}q^{k-t-1}\frac{q^{n-k}-1}{q^{k-t-1}-1}\\
\Rightarrow\quad & 2(q^{n-t-1}-1) + (q^{k-t-1}-1)(\theta_{t+1}\theta_{k-t} - \theta_{t+1} - 1)\theta_{k-t} \geq \theta_{t+2}q^{k-t-1}(q^{n-k}-1)\\
\xRightarrow{L.1.10.2}\quad & 2(q^{n-t-1}-1)(q-1) + (q^{k-t-1}-1)(q^{k-t+1}-1)\frac{q^{k+3}}{(q-1)^2}\\
& \qquad\qquad > (q^{t+3}-1)q^{k-t-1}(q^{n-k}-1)\\
\Rightarrow\quad & 2q^{n-t} - 2q^{n-t-1} - 2q + 2 + \frac{q^{3k-2t+3}}{(q-1)^2} > q^{n+2} - q^{n-t-1} - q^{k+2} + q^{k-t-1}\\
\Rightarrow\quad & 0 > \left(q^{n+2} - \frac{q^{3k-2t+3}}{(q-1)^2} - 2q^{n-t}\right) + \left(q^{n-t-1} - q^{k+2} - 2\right) + q^{k-t-1} + 2q\\
\Rightarrow\quad & 0 > \left(q^{2k+3}(q^{t+2}-2) - \frac{q^{2k+5}}{(q-1)^2}\right) + \left(q^{2k+2} - q^{k+2} - 2\right) + q^{k-t-1} + 2q.
\end{aligned}$$

The last implication follows since $n \geq 2k+t+3$ and $k \leq 2t+2$. For $q \geq 3$, we have that all terms on the right hand side of the last inequality are non-negative. Hence we find a contradiction, which proves (4.23).

Now we prove inequality (4.24) for $k \geq 2t+3$. Suppose again to the contrary that this inequality doesn't hold. Then we have that

$$\begin{aligned}
& 2\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + (\theta_{t+1}\theta_{k-t} - \theta_{t+1} - 1)\theta_{k-t}\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \\
& \qquad \geq \theta_{k+1} + \sum_{j=0}^{k-t-2} \begin{bmatrix} k-t+1 \\ j+1 \end{bmatrix} q^{(k-t-j)(k-t-j-1)} \begin{bmatrix} n-k-1 \\ k-t-j-1 \end{bmatrix} \\
\xRightarrow{j=0} \quad & 2\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + (\theta_{t+1}\theta_{k-t} - \theta_{t+1} - 1)\theta_{k-t}\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \\
& \qquad \geq \theta_{k-t} q^{(k-t)(k-t-1)} \begin{bmatrix} n-k-1 \\ k-t-1 \end{bmatrix} \\
\xRightarrow{L.1.10.2} \quad & 4q^{(n-k)(k-t-1)} + \frac{q^{2k-t+4}}{(q-1)^3} 2q^{(n-k)(k-t-2)} \\
& \qquad \geq \theta_{k-t}\left(1+\frac{1}{q}\right) q^{(k-t)(k-t-1)+(n-2k+t)(k-t-1)} \\
\Rightarrow \quad & 4 + \frac{2}{q^{n-3k+t-4}(q-1)^3} \geq \theta_{k-t}\left(1+\frac{1}{q}\right) > \theta_{k-t} + 4 \\
\xRightarrow{n\geq 2k+t+3} \quad & \frac{2q^{k-2t+1}}{(q-1)^2} > q^{k-t+1} - 1 \\
\xRightarrow{q\geq 3} \quad & q^{k-2t+1} > q^{k-t+1} - 1.
\end{aligned}$$

The last inequality gives a contradiction for $q \geq 3, t \geq 1$. ■

Lemma 4.5.17. *Suppose* $n \geq 2k+t+3, q \geq 3, k \geq t+2, t \geq 1$, *then*

$$2\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + (\theta_{t+1}\theta_{k-t} - \theta_{t+1} - \theta_{k-t})\theta_{k-t}\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} < f_a(q,n,k,t).$$

Proof. We have to prove the following inequalities:

$$\begin{aligned}
& 2\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + (\theta_{t+1}\theta_{k-t} - \theta_{t+1} - \theta_{k-t})\theta_{k-t}\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \\
& \qquad < \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}\left(1 + \theta_{t+1}q^{k-t-1}\frac{q^{n-k}-1}{q^{k-t-1}-1}\right) \qquad \text{for } k \leq 2t+1; \quad (4.25) \\
& 2\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + (\theta_{t+1}\theta_{k-t} - \theta_{t+1} - \theta_{k-t})\theta_{k-t}\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \\
& \qquad < \theta_k + \sum_{j=0}^{k-t-2} \begin{bmatrix} k-t+1 \\ j+1 \end{bmatrix} q^{(k-t-j)(k-t-j-1)} \begin{bmatrix} n-k-1 \\ k-t-j-1 \end{bmatrix} \text{ for } k \geq 2t+2. \quad (4.26)
\end{aligned}$$

We start by proving inequality (4.25). Suppose to the contrary that this inequality doesn't hold. Then we have that

$$\begin{aligned}
& 2\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + (\theta_{t+1}\theta_{k-t} - \theta_{t+1} - \theta_{k-t})\theta_{k-t}\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \\
& \qquad \geq \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}\left(1 + \theta_{t+1}q^{k-t-1}\frac{q^{n-k}-1}{q^{k-t-1}-1}\right)
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow \quad & 2\frac{q^{n-t-1}-1}{q^{k-t-1}-1} + (\theta_{t+1}\theta_{k-t} - \theta_{t+1} - \theta_{k-t})\theta_{k-t} \geq 1 + \theta_{t+1}q^{k-t-1}\frac{q^{n-k}-1}{q^{k-t-1}-1} \\
\Rightarrow \quad & 2(q^{n-t-1}-1) + (q^{k-t-1}-1)(\theta_{t+1}\theta_{k-t} - \theta_{t+1} - \theta_{k-t})\theta_{k-t} \\
& \qquad > \theta_{t+1}q^{k-t-1}(q^{n-k}-1) \\
\xRightarrow{L.1.10.2} \quad & 2(q^{n-t-1}-1)(q-1) + (q^{k-t-1}-1)(q^{k-t+1}-1)\frac{q^{k+3}}{(q-1)^2} \\
& \qquad > (q^{t+2}-1)q^{k-t-1}(q^{n-k}-1) \\
\Rightarrow \quad & 2q^{n-t} - 2q^{n-t-1} - 2q + 2 + \frac{q^{3k-2t+3}}{(q-1)^2} > q^{n+1} - q^{n-t-1} - q^{k+1} + q^{k-t-1} \\
\Rightarrow \quad & 0 > \left(q^{n+1} - \frac{q^{3k-2t+3}}{(q-1)^2} - 2q^{n-t}\right) + \left(q^{n-t-1} - q^{k+1} - 2\right) + q^{k-t-1} + 2q \\
\xRightarrow[k\leq 2t+1]{n\geq 2k+t+3} \quad & 0 > \left(q^{2k+3}(q^{t+1}-2) - \frac{q^{2k+4}}{(q-1)^2}\right) + \left(q^{2k+2} - q^{k+1} - 2\right) + q^{k-t-1} + 2q.
\end{aligned}$$

For $q \geq 3$, we have that all terms on the right hand side of the last inequality are non-negative. Hence we find a contradiction, which proves (4.25).

Now we prove inequality (4.26). Suppose again to the contrary that this inequality doesn't hold. Then we have that

$$\begin{aligned}
& 2\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + (\theta_{t+1}\theta_{k-t} - \theta_{t+1} - \theta_{k-t})\theta_{k-t}\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \\
& \qquad \geq \theta_k + \sum_{j=0}^{k-t-2}\begin{bmatrix} k-t+1 \\ j+1 \end{bmatrix} q^{(k-t-j)(k-t-j-1)}\begin{bmatrix} n-k-1 \\ k-t-j-1 \end{bmatrix} \\
\xRightarrow{j=0} \quad & 2\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + (\theta_{t+1}\theta_{k-t} - \theta_{t+1} - \theta_{k-t})\theta_{k-t}\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \\
& \qquad \geq \theta_{k-t}q^{(k-t)(k-t-1)}\begin{bmatrix} n-k-1 \\ k-t-1 \end{bmatrix} \\
\xRightarrow{L.1.10.2} \quad & 4q^{(n-k)(k-t-1)} + \frac{q^{2k-t+4}}{(q-1)^3}2q^{(n-k)(k-t-2)} \\
& \qquad \geq \theta_{k-t}q^{(k-t)(k-t-1)}\left(1+\frac{1}{q}\right)q^{(n-2k+t)(k-t-1)} \\
\Rightarrow \quad & 4 + \frac{2}{q^{n-3k+t-4}(q-1)^3} \geq \theta_{k-t}\left(1+\frac{1}{q}\right) > \theta_{k-t} + 4 \\
\xRightarrow{n\geq 2k+t+3} \quad & \frac{2q^{k-2t+1}}{(q-1)^2} > q^{k-t+1} - 1 \\
\xRightarrow{q\geq 3} \quad & q^{k-2t+1} > q^{k-t+1} - 1.
\end{aligned}$$

The last inequality gives a contradiction for $q \geq 3$, since $t \geq 1$. ■

Lemma 4.5.18. *For* $2 \leq x < k-t+1, q \geq 3$ *and* $n \geq k+2$, *we have that*

$$\theta_{t+x+1}\begin{bmatrix} n-t-x \\ k-t-x \end{bmatrix} < \theta_{t+x}\begin{bmatrix} n-t-x+1 \\ k-t-x+1 \end{bmatrix}$$

Proof.

$$\begin{aligned}
& \theta_{t+x+1}\begin{bmatrix} n-t-x \\ k-t-x \end{bmatrix} < \theta_{t+x}\begin{bmatrix} n-t-x+1 \\ k-t-x+1 \end{bmatrix} \\
\Leftrightarrow \quad & q^{t+x+2}-1 < (q^{t+x+1}-1)\frac{q^{n-t-x+1}-1}{q^{k-t-x+1}-1} \\
\Leftrightarrow \quad & q^{k+3}-q^{t+x+2}-q^{k-t-x+1} < q^{n+2}-q^{n-t-x+1}-q^{t+x+1} \\
\Leftrightarrow \quad & -q^{k-t-x+1} < \left(q^{n+2}-q^{n-t-x+1}-q^{k+3}\right)+q^{t+x+1}(q-1).
\end{aligned}$$

Note that the right hand side of the last inequality is positive for $q \geq 3$, which proves the inequality. ■

Corollary 4.5.19. *For* $x < k-t+1, q \geq 3$ *and* $n \geq k+2$*, we have that*

$$\begin{aligned}
\theta_{t+x}\begin{bmatrix} n-t-x+1 \\ k-t-x+1 \end{bmatrix} &\leq \theta_{t+2}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} \text{ if } x \geq 2 \\
\theta_{t+x}\begin{bmatrix} n-t-x+1 \\ k-t-x+1 \end{bmatrix} &\leq \theta_{t+3}\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \text{ if } x \geq 3
\end{aligned}$$

Lemma 4.5.20. *Suppose that* $n \geq 2k+t+3, k \geq 2t+3, 2 \leq x \leq k-t+1, t \geq 1$*. Then we have that*

$$\begin{aligned}
\theta_{k+1} + \sum_{j=0}^{k-t-2} & \begin{bmatrix} k-t+1 \\ j+1 \end{bmatrix} q^{(k-t-j)(k-t-j-1)} \begin{bmatrix} n-k-1 \\ k-t-j-1 \end{bmatrix} \\
& > \theta_{t+x}\begin{bmatrix} n-t-x+1 \\ k-t-x+1 \end{bmatrix} + \theta_{k-t}^2\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} + \theta_{k-t-1}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix}.
\end{aligned}$$

Proof. Suppose to the contrary that the inequality in the statement of the lemma doesn't hold. Then we have that

$$\begin{aligned}
& \theta_{k+1} + \sum_{j=0}^{k-t-2} \begin{bmatrix} k-t+1 \\ j+1 \end{bmatrix} q^{(k-t-j)(k-t-j-1)} \begin{bmatrix} n-k-1 \\ k-t-j-1 \end{bmatrix} \\
& \qquad \leq \theta_{t+x}\begin{bmatrix} n-t-x+1 \\ k-t-x+1 \end{bmatrix} + \theta_{k-t}^2\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} + \theta_{k-t-1}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} \\
\xRightarrow[C.4.5.19]{x\geq 2, j=0} \quad & \theta_{k-t}q^{(k-t)(k-t-1)}\begin{bmatrix} n-k-1 \\ k-t-1 \end{bmatrix} \\
& \qquad < \theta_{t+2}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + \theta_{k-t}^2\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} + \theta_{k-t-1}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} \\
\xRightarrow{L.1.10.2} \quad & \frac{q^{k-t+1}-1}{q-1}q^{(k-t)(k-t-1)}\left(1+\frac{1}{q}\right)q^{(n-2k+t)(k-t-1)} \\
& \qquad < \frac{q^{t+3}-1}{q-1}2q^{(n-k)(k-t-1)} + \frac{(q^{k-t+1}-1)^2}{(q-1)^2}2q^{(n-k)(k-t-2)} \\
& \qquad\quad + \frac{q^{k-t}-1}{q-1}2q^{(n-k)(k-t-1)}
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow\qquad (q^{k-t+1}-1)\left(1+\frac{1}{q}\right) < 2(q^{t+3}-1) + 2\frac{(q^{k-t+1}-1)^2}{q^{n-k}(q-1)} + 2(q^{k-t}-1)\\
&\Rightarrow\qquad q^{t+3}\left(q^{k-2t-2}-2q^{k-2t-3}-2\right) + \left(3-\frac{1}{q}\right)\\
&\qquad\qquad + \left(\frac{q^{n-t}(q-1)-2(q^{k-t+1}-1)^2}{q^{n-k}(q-1)}\right) < 0
\end{aligned}$$

For $k \geq 2t+4, q \geq 3$ and $n \geq 2k+t+3$ all terms in the left hand side of the last inequality are non-negative, which gives a contradiction. For $k = 2t+3$ we have

$$\begin{aligned}
&\qquad (q^{t+4}-3q^{t+3}) + \left(3-\frac{1}{q}-2\frac{(q^{t+4}-1)^2}{q^{n-2t-3}(q-1)}\right) < 0\\
&\xRightarrow{n\geq 5t+9}\qquad (q^{t+4}-3q^{t+3}) + \left(3-\frac{1}{q}-2\frac{(q^{t+4}-1)^2}{q^{3t+7}(q-1)}\right) < 0\\
&\xRightarrow{t\geq 1}\qquad (q^{t+4}-3q^{t+3}) + \left(1-\frac{1}{q}\right) < 0,
\end{aligned}$$

which also gives a contradiction for $q \geq 3$ and $t \geq 1$. ■

Lemma 4.5.21. *Suppose that* $n \geq 2k+t+3, k \geq 2t+2, 3 \leq x \leq k-t+1, t \geq 1, q \geq 3$. *Then we have that*

$$\begin{aligned}
&\theta_k + \sum_{j=0}^{k-t-2} \begin{bmatrix} k-t+1 \\ j+1 \end{bmatrix} q^{(k-t-j)(k-t-j-1)} \begin{bmatrix} n-k-1 \\ k-t-j-1 \end{bmatrix}\\
&\qquad > \theta_{t+x}\begin{bmatrix} n-t-x+1 \\ k-t-x+1 \end{bmatrix} + \theta_{k-t}^2\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} + \theta_{k-t-1}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix}.
\end{aligned}$$

Proof. Suppose to the contrary that the inequality in the statement of the lemma doesn't hold. Then we have that

$$\begin{aligned}
&\qquad \theta_k + \sum_{j=0}^{k-t-2} \begin{bmatrix} k-t+1 \\ j+1 \end{bmatrix} q^{(k-t-j)(k-t-j-1)} \begin{bmatrix} n-k-1 \\ k-t-j-1 \end{bmatrix}\\
&\qquad\qquad \leq \theta_{t+x}\begin{bmatrix} n-t-x+1 \\ k-t-x+1 \end{bmatrix} + \theta_{k-t}^2\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} + \theta_{k-t-1}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix}\\
&\xRightarrow[C.4.5.19]{x\geq 3, j=0}\qquad \theta_{k-t}q^{(k-t)(k-t-1)}\begin{bmatrix} n-k-1 \\ k-t-1 \end{bmatrix}\\
&\qquad\qquad < \theta_{t+3}\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} + \theta_{k-t}^2\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} + \theta_{k-t-1}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix}\\
&\xRightarrow{L.1.10.2}\qquad \frac{q^{k-t+1}-1}{q-1}q^{(k-t)(k-t-1)}\left(1+\frac{1}{q}\right)q^{(n-2k+t)(k-t-1)}\\
&\qquad\qquad < \frac{q^{t+4}-1}{q-1}2q^{(n-k)(k-t-2)} + \frac{(q^{k-t+1}-1)^2}{(q-1)^2}2q^{(n-k)(k-t-2)} + \frac{q^{k-t}-1}{q-1}2q^{(n-k)(k-t-1)}\\
&\Rightarrow\qquad (q^{k-t+1}-1)\left(1+\frac{1}{q}\right) < 2\frac{q^{t+4}-1}{q^{n-k}} + 2\frac{(q^{k-t+1}-1)^2}{q^{n-k}(q-1)} + 2(q^{k-t}-1)\\
&\Rightarrow\qquad \left(q^{k-t+1}-2q^{k-t}-2\frac{q^{t+4}-1}{q^{n-k}}\right) + \left(1-\frac{1}{q}\right) + \left(q^{k-t}-2\frac{(q^{k-t+1}-1)^2}{q^{n-k}(q-1)}\right) < 0\\
&\xRightarrow{q\geq 3}\qquad \left(q^{k-t}-2\frac{q^{t+4}-1}{q^{n-k}}\right) + \left(1-\frac{1}{q}\right) + 2\left(\frac{q^{n-t}-(q^{k-t+1}-1)^2}{q^{n-k}(q-1)}\right) < 0.
\end{aligned}$$

For $n \geq 2k+t+3, q \geq 3$ the terms in the left hand side of the last inequality are non-negative, which gives a contradiction. ■

Lemma 4.5.22. *Suppose that* $n \geq 2k+t+3, k \geq 2t+2$ *and* $q \geq 3$*. Then we have that*

$$\begin{aligned}\theta_k + \sum_{j=0}^{k-t-2} \begin{bmatrix} k-t+1 \\ j+1 \end{bmatrix} q^{(k-t-j)(k-t-j-1)} \begin{bmatrix} n-k-1 \\ k-t-j-1 \end{bmatrix} \\ > q^2\theta_{t-1}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + \theta_{k-t}^2 \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} + \theta_{k-t-1}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix}\end{aligned}$$

Proof. Suppose to the contrary that the inequality in the statement of the lemma doesn't hold. Then we have that

$$\begin{aligned}
& \theta_k + \sum_{j=0}^{k-t-2} \begin{bmatrix} k-t+1 \\ j+1 \end{bmatrix} q^{(k-t-j)(k-t-j-1)} \begin{bmatrix} n-k-1 \\ k-t-j-1 \end{bmatrix} \\
& \qquad \leq q^2\theta_{t-1}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + \theta_{k-t}^2 \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} + \theta_{k-t-1}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} \\
\xRightarrow{j=0} \quad & \theta_k + \theta_{k-t} q^{(k-t)(k-t-1)} \begin{bmatrix} n-k-1 \\ k-t-1 \end{bmatrix} \\
& \qquad < q^2\theta_{t-1}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + \theta_{k-t}^2 \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} + \theta_{k-t-1}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} \\
\xRightarrow{L.1.10.2} \quad & \frac{q^{k-t+1}-1}{q-1} q^{(k-t)(k-t-1)} \left(1+\frac{1}{q}\right) q^{(n-2k+t)(k-t-1)} \\
& \qquad < \frac{q^t-1}{q-1} 2q^{(n-k)(k-t-1)+2} + \frac{(q^{k-t+1}-1)^2}{(q-1)^2} 2q^{(n-k)(k-t-2)} \\
& \qquad\quad + \frac{q^{k-t}-1}{q-1} 2q^{(n-k)(k-t-1)} \\
\Rightarrow \quad & (q^{k-t+1}-1)\left(1+\frac{1}{q}\right) < 2(q^{t+2}-q^2) + 2\frac{(q^{k-t+1}-1)^2}{q^{n-k}(q-1)} + 2(q^{k-t}-1) \\
\Rightarrow \quad & \left(q^{k-t+1} - 2q^{k-t} - 2q^{t+2}\right) + \left(1-\frac{1}{q}\right) + \left(q^{k-t} + 2q^2 - 2\frac{(q^{k-t+1}-1)^2}{q^{n-k}(q-1)}\right) < 0
\end{aligned} \tag{4.27}$$

$$\begin{aligned}
\xRightarrow{q\geq 3} \quad & q^{t+2}\left(q^{k-2t-2}-2\right) + \left(1-\frac{1}{q}\right) + 2\left(\frac{q^{n-t}+2q^{n-k+2}-(q^{k-t+1}-1)^2}{q^{n-k}(q-1)}\right) < 0 \\
\Rightarrow \quad & q^{t+2}\left(q^{k-2t-2}-2\right) + \left(1-\frac{1}{q}\right) + 2q^{2k-2t+2}\left(\frac{q^{n-2k+t-2}+2q^{n-3k+2t}-1}{q^{n-k}(q-1)}\right) < 0.
\end{aligned}$$

For $k \geq 2t+3, q \geq 3$ and $n \geq 2k+t+3$ all terms in the left hand side of the last inequality are non-negative, which gives a contradiction. For $k = 2t+2$ we have that $n > 2k+t+2 = 5t+6$, and by using (4.27) we have that

$$\begin{aligned}
& \left(q^{t+3}-3q^{t+2}\right) + \left(2q^2+1-\frac{1}{q}-2\frac{(q^{t+3}-1)^2}{q^{n-2t-2}(q-1)}\right) < 0 \\
\xRightarrow{n\geq 5t+7} \quad & \left(q^{t+3}-3q^{t+2}\right) + \left(2q^2+1-\frac{1}{q}-2\frac{(q^{t+3}-1)^2}{q^{3t+5}(q-1)}\right) < 0 \\
\xRightarrow{t\geq 1} \quad & \left(q^{t+3}-3q^{t+2}\right) + \left(2q^2-1-\frac{1}{q}\right) < 0
\end{aligned}$$

which also gives a contradiction for $q \geq 3$. ■

5 The Sunflower bound

> *It never hurts to keep looking for sunshine.*
>
> —Eeyore

The results in this chapter are joint work with prof. Aart Blokhuis and dr. Maarten De Boeck, and will appear in [15].

5.1 Introduction

A $(k+1,t+1)$-*SCID* is a set of k-dimensional subspaces in $\mathrm{PG}(n,q)$, that pairwise intersect in *precisely* a t-dimensional space (SCID stands for: *Subspaces with Constant Intersection Dimension*). Note that this set corresponds to a set of $(k+1)$-dimensional vector spaces, pairwise intersecting in a $(t+1)$-dimensional vector space. This indicates why we use the $(k+1,t+1)$-notation.
A $(k+1,t+1)$-SCID is also called a *t-intersecting constant dimension subspace code*, where the code words have projective dimension k. Note that $(k+1,0)$-SCIDs correspond with partial k-spreads in $\mathrm{PG}(n,q)$.

Investigating SCIDs is interesting for the link with coding theory. *Network coding* is a segment of information theory dealing with data transmission over lossy and noisy networks. In such networks, information travels from a set of sources to a set of receivers through several intermediate nodes. An optimal information rate can be achieved by performing linear combinations during transmissions in the intermediate nodes. This approach is called *random network coding*, and utilizes *subspace codes* [81]. In a subspace code, the code words are subspaces in a projective space, and the subspace distance $d(U,V)$ between two code words U and V is defined as follows: $d(U,V) = \dim(U) + \dim(V) - \dim(U \cap V)$. *Constant dimension subspace codes* are subspace codes whose elements all have the same dimension. They are the q-analogues of the classical codes. SCIDs are *equidistant constant dimension subspace codes* since the pairwise distances between the code words are equal.

An example of a $(k+1,t+1)$-SCID is a *sunflower*, which is a set of k-spaces, passing through the same t-space and having no points in common outside of this t-space. It can be shown that a t-intersecting constant dimension subspace code is a sunflower if the code has many code words.

Theorem 5.1.1 ([56, Theorem 1]). *A $(k+1,t+1)$-SCID C is a sunflower if*

$$|C| > \left(\frac{q^{k+1}-q^{t+1}}{q-1}\right)^2 + \left(\frac{q^{k+1}-q^{t+1}}{q-1}\right) + 1.$$

It is believed that the Sunflower bound is in general not tight. In [6], the Sunflower bound for $(k+1,1)$-SCIDs was studied.

Theorem 5.1.2 ([6, Theorem 2.1]). *Let C be a $(k+1,1)$-SCID, with $k \geq 4$. If*

$$|C| \geq \left(\frac{q^{k+1}-q}{q-1}\right)^2 + \left(\frac{q^{k+1}-q}{q-1}\right) - q^k,$$

then C is a sunflower.

In this chapter, we will give a better result for $(k+1,1)$-SCIDs, see Theorem 5.3.6. In this result, we improve the Sunflower bound with a factor $\frac{2}{\sqrt[6]{q}} + \frac{4}{\sqrt[3]{q}} - \frac{5}{\sqrt{q}}$, while in [6], the authors improve the bound with a lower order term q^k.

We suppose that $k \geq 3$ as for $(2,1)$-SCIDs we, more generally, known that every $(k+1,k)$-SCID is a sunflower or consists of k-spaces in a fixed $(k+1)$-space, see Theorem 2.0.6. For $(3,1)$-SCIDs, an almost complete classification is known, see [9].

Result 5.1.3 ([9]). *Let C be a set of planes in $\mathrm{PG}(n,q), q \geq 3$, pairwise intersecting in exactly a point. If $|C| \geq 3(q^2+q+1)$, then C is contained in a Klein quadric in $\mathrm{PG}(5,q)$, or C is a dual partial spread in $\mathrm{PG}(4,q)$, or all elements of C pass through a common point.*

In Section 5.2, we give some definitions and general lemmas. In Section 5.3, we start with the Main Lemma that gives an important inequality. Using this inequality, we continue with Theorem 5.3.6 that gives an improvement on the Sunflower bound if $k \geq 3$ and $q \geq 9$ (and if $q \geq 7$ and $k \geq 5$).

5.2 Preliminaries

From now on, we consider a fixed $(k+1,1)$-SCID $\mathcal{S}$ that is *not* a sunflower, of size $|\mathcal{S}| = (1-s)\theta_k^2$, $0 < s < 1$. Note that the size of $|\mathcal{S}|$ is smaller than the Sunflower bound for $s > \frac{1}{\theta_k} - \frac{1}{\theta_k^2}$. We will derive, for a fixed value of k and field size q, an upper bound on $1-s$.

Definition 5.2.1. Consider the SCID $\mathcal{S}$. The sets of points and lines that are contained in an element of $\mathcal{S}$ are denoted by $\mathcal{P}_{\mathcal{S}}$ and $\mathcal{L}_{\mathcal{S}}$ respectively.

Lemma 5.2.2. *Suppose $P \in \mathcal{P}_{\mathcal{S}}$, then P lies in at most θ_k elements of $\mathcal{S}$ and on at most $\theta_k \cdot \theta_{k-1}$ lines of $\mathcal{L}_{\mathcal{S}}$.*

Proof. There exists an element $S_0 \in \mathcal{S}$ not through P, since $\mathcal{S}$ is not a sunflower. Every k-space of $\mathcal{S}$ through P contains a point Q of S_0 and every line PQ with $Q \in S_0$ is contained in at most one k-space. In this way we find at most θ_k elements of $\mathcal{S}$ that contain P. The lemma follows since the number of lines through a point in a k-space is θ_{k-1}. ■

From now on, we distinguish 'rich' and 'poor' points and lines in $\mathcal{P}_{\mathcal{S}}$ and $\mathcal{L}_{\mathcal{S}}$. First we give the definition, then we continue with some counting arguments.

Definition 5.2.3. Suppose c, d are constants between s and 1. A point $P \in \mathcal{P}_{\mathcal{S}}$ is *c-rich* if it is included in more than $(1-c)\theta_k$ elements of $\mathcal{S}$. A point is *c-poor* if it is not c-rich. A line $l \in \mathcal{L}_{\mathcal{S}}$ is *(c,d)-rich* if it contains more than $(1-d)(q+1)$ c-rich points.

We will call c-rich and c-poor points, and (c,d)-rich lines *rich* and *poor* points, and *rich* lines respectively, if the constants c and d are clear from the context.

Lemma 5.2.4. *For the number r of c-rich points in an element of $\mathcal{S}$, we find:*

$$r \geq r_0 = \left(1 - \frac{s}{c}\right)\theta_k,$$

Proof. Fix $S_0 \in \mathcal{S}$, and count the number of elements in $\mathcal{S}$ that intersect S_0 in a point. By Lemma 5.2.2, we have that through every rich point P of S_0, there are at most $\theta_k - 1$ elements of $\mathcal{S}$ different from S_0. Through every line spanned by P and a point of such a k-space, there is at most one element of $\mathcal{S}$.
Every poor point of S_0 lies in at most $(1-c)\theta_k - 1$ other elements of $\mathcal{S}$ by definition. We double-count pairs (P, Z), with $P \in Z$, $Z \in \mathcal{S}$ where $P \in S_0$ and $Z \neq S_0$, to obtain the following inequality:

$$\begin{array}{rrcl} & r(\theta_k - 1) + (\theta_k - r)\left((1-c)\theta_k - 1\right) & \geq & |\mathcal{S}| - 1 \\ \Leftrightarrow & r(\theta_k - 1 - (1-c)\theta_k + 1) & \geq & (1-s)\theta_k^2 - 1 - (1-c)\theta_k^2 + \theta_k \\ \Rightarrow & rc\theta_k & \geq & (c-s)\theta_k^2 \\ \Leftrightarrow & r & \geq & (1 - \frac{s}{c})\theta_k. \end{array}$$

■

Lemma 5.2.5. *An element of $\mathcal{S}$ contains at least*

$$\alpha = \frac{\theta_k \theta_{k-1}}{q+1} \cdot \left(1 - \frac{s}{cd}\right)$$

(c, d)-rich lines and the total number of (c, d)-rich lines is at least $\alpha(1-s)\theta_k^2$.

Proof. Consider a k-space $S_0 \in \mathcal{S}$ and let β denote the number of poor lines in S_0. By counting pairs (P, l), with P a rich point in S_0, l a line in S_0 and $P \in l$, we find:

$$\left(\begin{bmatrix} k+1 \\ 2 \end{bmatrix} - \beta\right)(q+1) + \beta(1-d)(q+1) \geq r_0\theta_{k-1} = \left(1 - \frac{s}{c}\right)\theta_k\theta_{k-1},$$

which gives

$$\beta \leq \frac{s\theta_k\theta_{k-1}}{cd(q+1)}.$$

Hence, an element of $\mathcal{S}$ contains at least $\begin{bmatrix} k+1 \\ 2 \end{bmatrix} - \beta = \frac{\theta_k\theta_{k-1}}{q+1} - \frac{s\theta_k\theta_{k-1}}{cd(q+1)}$ (c, d)-rich lines. ■

Remark 5.2.6. In order to get a useful bound in the previous lemma, we need values of s, c and d such that $1 - \frac{s}{cd} \geq 0$ or $s \leq cd$. Later we will see that the values that we use for c and d satisfy these inequalities.

We continue with a lemma that will be useful to prove the Main Lemma and the theorems in the following section.

Lemma 5.2.7. *Let $\rho(s)$ be the average number of (c, d)-rich lines meeting two distinct elements S_1, S_2 of $\mathcal{S}$ in a c-rich point different from $S_1 \cap S_2$ (in the case the latter is c-rich). Then $\rho(s)$ is at least*

$$f(s) = \theta_k\theta_{k-1}q\frac{1-d}{1-s}\left(1 - \frac{s}{cd}\right)\left(1 - c - \frac{1}{\theta_k}\right)^2\left(1 - d - \frac{d}{q}\right).$$

Proof. We count triples (S_1, S_2, r) where r is a rich line connecting a rich point in $S_1 \setminus S_2$ with a rich point in $S_2 \setminus S_1$. Let $\rho_{\{S_1, S_2\}}$, $S_1, S_2 \in \mathcal{S}$, $S_1 \neq S_2$, be the number of rich lines meeting both $S_1 \setminus S_2$ and $S_2 \setminus S_1$ in a rich point. We define $\rho(s)$ as the average of the values $\rho_{\{S_1, S_2\}}$ with $S_1, S_2 \in \mathcal{S}$ and $S_1 \neq S_2$. On the one hand, the number of triples equals

$$(1-s)\theta_k^2\left((1-s)\theta_k^2 - 1\right)\rho(s) \leq (1-s)^2\theta_k^4\rho(s).$$

On the other hand, the number of triples is at least

$$(1-s)\theta_k^2\frac{\theta_k\theta_{k-1}}{q+1}\left(1-\frac{s}{cd}\right)\cdot(1-d)(q+1)((1-d)q-d)\cdot((1-c)\theta_k-1)^2,$$

as by Lemma 5.2.5, there are at least $(1-s)\theta_k^2\frac{\theta_k\theta_{k-1}}{q+1}\left(1-\frac{s}{cd}\right)$ rich lines, and on a rich line there are at least $(1-d)(q+1)((1-d)q-d)$ possibilities for an ordered pair of two distinct rich points P_1, P_2. Through those points, we find at least $((1-c)\theta_k-1)^2$ possibilities for the k-spaces $S_1, S_2 \in \mathcal{S}$ (not containing the line P_1P_2). This gives that the average $\rho(s)$ is at least $f(s)$. ■

5.3 Main Lemma and results

Using the combinatorial lemmas in the previous section, the main goal in this section is to find a an upper bound on $(1-s)$, as a function of the field size q. We start with the Main Lemma, that will be the basis of the theorems at the end of this section.

Main Lemma 5.3.1. Let $\mathcal{S}$ be a $(k+1,1)$-SCID in $\mathrm{PG}(n,q)$, with $|\mathcal{S}|=(1-s)\theta_k^2$, $k\geq 3$, that is not a sunflower. For all values $0<s<c,d<1$, we have the following inequality:

$$\left(1-\frac{s}{cd}\right)(1-d)(1-c)\left(1-c-\frac{1}{q^3}\right)^2\left(1-d-\frac{d}{q}\right)\left(1-d-\frac{1+d}{q}\right)q$$
$$\leq(1-s)^2+\frac{1-s}{q}. \quad (5.1)$$

Proof. Consider a pair of different k-spaces $S_1, S_2 \in \mathcal{S}$ having at least $f(s)$ connecting rich lines, then the $2k$-space $T=\langle S_1,S_2\rangle$ contains at least

$$f(s)\cdot\frac{(1-d)(q+1)-2}{q}=\left(1-d-\frac{1+d}{q}\right)f(s)$$

rich points: every rich line contains at least $(1-d)(q+1)-2$ rich points, not contained in $S_1\cup S_2$. Furthermore, every point P in the $2k$-space T, not in the union $S_1\cup S_2$, lies on at most q such connecting lines. That there are indeed at most q such lines, follows since $\langle P,S_1\rangle$ meets S_2 in a line ℓ through $S_1\cap S_2$. Hence, the lines through P, meeting both S_1 and S_2, are precisely the lines through P in the plane $\langle P,\ell\rangle$. In this plane there are q lines through P that do not contain $\mathcal{S}_1\cap\mathcal{S}_2$. Hence, each such point P is counted at most q times.

Since the dual of a $(k+1,1)$-SCID in a $2k$-space is a partial $(k-1)$-spread in this $2k$-space, we have that a $2k$-space contains at most $\lfloor\theta_{2k}/\theta_{k-1}\rfloor=q^{k+1}+q$ elements of $\mathcal{S}$. On the other hand, this $2k$-space contains at most θ_{k-1} points from each element of $\mathcal{S}$ not contained in T. Hence, the number of pairs (P,S_0), with $P\in\langle S_1,S_2\rangle$ a rich point in the k-space S_0, is at least $\left(1-d-\frac{1+d}{q}\right)f(s)(1-c)\theta_k$ and at most $(q^{k+1}+q)\theta_k+\left((1-s)\theta_k^2-(q^{k+1}+q)\right)\theta_{k-1}$. Hence,

$$\left(1-d-\frac{1+d}{q}\right)(1-c)f(s)\theta_k\leq(q^{k+1}+q)\theta_k+\left((1-s)\theta_k^2-(q^{k+1}+q)\right)\theta_{k-1}$$
$$\Rightarrow\left(1-d-\frac{1+d}{q}\right)(1-c)\frac{f(s)}{\theta_k\theta_{k-1}}\leq 1-s+\frac{q^{k+1}+q}{\theta_k^2\theta_{k-1}}q^k\leq 1-s+\frac{1}{q^{k-2}}.$$

The last inequality follows since $q^k(q^{k+1}+q) \le q^{2-k}\theta_k^2\theta_{k-1}$. This implies that

$$\left(1-\frac{s}{cd}\right)(1-d)(1-c)\left(1-c-\frac{1}{\theta_k}\right)^2\left(1-d-\frac{d}{q}\right)\left(1-d-\frac{1+d}{q}\right)q \le (1-s)^2+\frac{1-s}{q^{k-2}}, \tag{5.2}$$

which proves the lemma since $k \ge 3$. ■

Corollary 5.3.2. *Let $\mathcal{S}$ be a $(k+1,1)$-SCID in $\mathrm{PG}(n,q)$, with $|\mathcal{S}| = (1-s)\theta_k^2$, $k \ge 3$, that is not a sunflower. Suppose that*

$$\left(\frac{1}{q}-\frac{B(q,c,d)}{cd}\right)^2 - 4B(q,c,d)\left(\frac{1}{cd}-1\right) \ge 0.$$

Then we have, for all values $0 < s < c, d < 1$, that

$$(1-s) \le F(q,c,d) = \frac{1}{2}\left(\frac{B(q,c,d)}{cd}-\frac{1}{q}-\sqrt{\left(\frac{1}{q}-\frac{B(q,c,d)}{cd}\right)^2-4B(q,c,d)\left(\frac{1}{cd}-1\right)}\right)$$

$$\text{or } (1-s) \ge G(q,c,d) = \frac{1}{2}\left(\frac{B(q,c,d)}{cd}-\frac{1}{q}+\sqrt{\left(\frac{1}{q}-\frac{B(q,c,d)}{cd}\right)^2-4B(q,c,d)\left(\frac{1}{cd}-1\right)}\right)$$

with $B(q,c,d) = (1-d)(1-c)\left(1-c-\frac{1}{q^3}\right)^2\left(1-d-\frac{d}{q}\right)\left(1-d-\frac{1+d}{q}\right)q$.

Proof. Using inequality (5.1) from the Main Lemma, we immediately find the following quadratic inequality

$$(1-s)^2+\left(\frac{1}{q}-\frac{B(q,c,d)}{cd}\right)\cdot(1-s)+B(q,c,d)\left(\frac{1}{cd}-1\right) \ge 0,$$

which proves the corollary. ■

From now on, we put $c(q) = d(q) = 1-\frac{1}{\sqrt[6]{q}}-\frac{1}{2\sqrt[3]{q}}$. Since c and d must be non-negative by definition, we have to assume that $q \ge 7$. We denote $c(q), F(q,c(q),c(q)), G(q,c(q),c(q))$ and $B(q,c(q),c(q))$ by c_q, F_q, G_q and B_q respectively. We first give a lower bound on B_q.

Lemma 5.3.3. *Let $t = \sqrt[6]{q}$, $q \ge 7$, then*

$$B_q > \left(1+\frac{1}{2t}\right)^2\left(1+\frac{1}{2t}-\frac{1}{t^4}\right)^2\left(1+\frac{1}{2t}-\frac{1}{t^5}\right)\left(1+\frac{1}{2t}-\frac{2}{t^5}\right), \text{ and} \tag{5.3}$$

$$B_q > \left(1+\frac{1}{2t}\right)^2\left(1+\frac{1}{3t}\right)^2. \tag{5.4}$$

Proof. By using the equality $c_q = c_{t^6} = 1-\frac{1}{t}-\frac{1}{2t^2}$ and $t = \sqrt[6]{q} \ge \sqrt[6]{7}$, we have

$$\begin{aligned}
B_q &= (1-c_q)^2\left(1-c_q-\frac{1}{q^3}\right)^2\left(1-c_q-\frac{c_q}{q}\right)\left(1-c_q-\frac{1+c_q}{q}\right)q\\
&= \left(\frac{1}{t}+\frac{1}{2t^2}\right)^2\left(\frac{1}{t}+\frac{1}{2t^2}-\frac{1}{t^{18}}\right)^2\left(\frac{1}{t}+\frac{1}{2t^2}-\frac{1}{t^6}+\frac{1}{t^7}+\frac{1}{2t^8}\right)\\
&\quad\cdot\left(\frac{1}{t}+\frac{1}{2t^2}-\frac{2}{t^6}+\frac{1}{t^7}+\frac{1}{2t^8}\right)t^6\\
&= \left(1+\frac{1}{2t}\right)^2\left(1+\frac{1}{2t}-\frac{1}{t^{17}}\right)^2\left(1+\frac{1}{2t}-\frac{1}{t^5}+\frac{1}{t^6}+\frac{1}{2t^7}\right)\left(1+\frac{1}{2t}-\frac{2}{t^5}+\frac{1}{t^6}+\frac{1}{2t^7}\right)
\end{aligned}$$

Using this expression for B_q, we can check that the following two inequalities are true for all $t \geq \sqrt[6]{7}$, and so, for all $q \geq 7$.

$$B_q > \left(1+\frac{1}{2t}\right)^2\left(1+\frac{1}{2t}-\frac{1}{t^4}\right)^2\left(1+\frac{1}{2t}-\frac{1}{t^5}\right)\left(1+\frac{1}{2t}-\frac{2}{t^5}\right), \text{ and}$$
$$B_q > \left(1+\frac{1}{2t}\right)^2\left(1+\frac{1}{3t}\right)^2.$$

■

We continue by investigating for which values of $q \geq 7$ the condition $\left(\frac{1}{q}-\frac{B_q}{c_q^2}\right)^2 - 4B_q\left(\frac{1}{c_q^2}-1\right) \geq 0$, in Corollary 5.3.2, is true. Or equivalently, for which values of q, the argument of the square root in F_q and G_q is non-negative.

Lemma 5.3.4. *For $q \geq 7$, it is true that $\left(\frac{1}{q}-\frac{B_q}{c_q^2}\right)^2 - 4B_q\left(\frac{1}{c_q^2}-1\right) \geq 0$, with $B_q = (1-c_q)^2\left(1-c_q-\frac{1}{q^3}\right)^2\left(1-c_q-\frac{c_q}{q}\right)\left(1-c_q-\frac{1+c_q}{q}\right)q$ and $c_q = 1-\frac{1}{\sqrt[6]{q}}-\frac{1}{2\sqrt[3]{q}}$.*

Proof. Note that it follows from Lemma 5.3.3 that $B_q > 0$ if $q \geq 7$. Suppose that the inequality in the statement of the lemma does not hold. Then we have

$$\begin{aligned}
& \frac{B_q^2}{c_q^4} - \frac{2B_q}{qc_q^2} + \frac{1}{q^2} < 4B_q\left(\frac{1}{c_q^2}-1\right) \\
\Rightarrow\quad & \frac{B_q^2}{c_q^4} < 2B_q\left(\frac{2}{c_q^2}-2+\frac{1}{qc_q^2}\right) \\
\overset{B_q>0}{\Longleftrightarrow}\quad & B_q < 2c_q^2\left(2(1-c_q^2)+\frac{1}{q}\right) \\
\overset{t=\sqrt[6]{q}}{\Longleftrightarrow}\quad & B_{t^6} < 2\left(1-\frac{1}{t}-\frac{1}{2t^2}\right)^2\left(\frac{4}{t}-\frac{2}{t^3}-\frac{1}{2t^4}+\frac{1}{t^6}\right) \\
\overset{(5.4)}{\Longrightarrow}\quad & \left(1+\frac{1}{2t}\right)^2\left(1+\frac{1}{3t}\right)^2 < 2\left(1-\frac{1}{t}\right)^2\left(\frac{4}{t}-\frac{1}{t^3}\right) \\
\Leftrightarrow\quad & \left(t+\frac{1}{2}\right)^2\left(t+\frac{1}{3}\right)^2 < 2\,(t-1)^2\left(4t-\frac{1}{t}\right) \\
\Leftrightarrow\quad & t^4+\frac{5}{3}t^3+\frac{37}{36}t^2+\frac{5}{18}t+\frac{1}{36} < 8t^3-16t^2+6t+4-\frac{2}{t} \\
\Leftrightarrow\quad & t^4-\frac{19}{3}t^3+\frac{613}{36}t^2-\frac{103}{18}t-\frac{143}{36}+\frac{2}{t} < 0\,.
\end{aligned}$$

The last inequality gives a contradiction for all values of $t \geq \sqrt[6]{7}$, and so for all $q \geq 7$, which proves the lemma. ■

Now we prove that $G_q > 1$. This implies that the first bound in Corollary 5.3.2 holds, since $0 < s < 1$.

Lemma 5.3.5. *For $q \geq 7$, it is true that*

$$G_q = \frac{1}{2}\left(\frac{B_q}{c_q^2}-\frac{1}{q}+\sqrt{\left(\frac{1}{q}-\frac{B_q}{c_q^2}\right)^2-4B_q\left(\frac{1}{c_q^2}-1\right)}\right) > 1$$

with

$$B_q = (1-c_q)^2\left(1-c_q-\frac{1}{q^3}\right)^2\left(1-c_q-\frac{c_q}{q}\right)\left(1-c_q-\frac{1+c_q}{q}\right)q,$$
$$c_q = 1-\frac{1}{\sqrt[6]{q}}-\frac{1}{2\sqrt[3]{q}}.$$

Proof. We have to prove that

$$\frac{B_q}{c_q^2}-\frac{1}{q}+\sqrt{\left(\frac{1}{q}-\frac{B_q}{c_q^2}\right)^2-4B_q\left(\frac{1}{c_q^2}-1\right)}>2.$$

For all values of $q \geq 7$ such that $2-\frac{B_q}{c_q^2}+\frac{1}{q}<0$, the previous inequality is true. If $2-\frac{B_q}{c_q^2}+\frac{1}{q}\geq 0$, then it is equivalent to proving that

$$\begin{aligned}
&\left(\frac{1}{q}-\frac{B_q}{c_q^2}\right)^2-4B_q\left(\frac{1}{c_q^2}-1\right)>4+4\left(\frac{1}{q}-\frac{B_q}{c_q^2}\right)+\left(\frac{1}{q}-\frac{B_q}{c_q^2}\right)^2\\
\Leftrightarrow\quad & -\frac{B_q}{c_q^2}+B_q>1+\frac{1}{q}-\frac{B_q}{c_q^2}\\
\Leftrightarrow\quad & B_q>1+\frac{1}{q}.
\end{aligned}$$

Set $t=\sqrt[6]{q}$. From Lemma 5.3.3(5.4), we know that it is sufficient to prove the following inequality.

$$\begin{aligned}
&\left(1+\frac{1}{2t}\right)^2\left(1+\frac{1}{3t}\right)^2>1+\frac{1}{t^6}\\
\Leftrightarrow\quad & t^4+\frac{5}{3}t^3+\frac{37}{36}t^2+\frac{5}{18}t+\frac{1}{36}>t^4+\frac{1}{t^2}\\
\Leftrightarrow\quad & \frac{5}{3}t^3+\frac{37}{36}t^2+\frac{5}{18}t+\frac{1}{36}-\frac{1}{t^2}>0.
\end{aligned}$$

This last inequality is true for $t=\sqrt[6]{q}\geq\sqrt[6]{7}$, and so for $q\geq 7$, which proves the lemma. ■

Theorem 5.3.6. *A $(k+1,1)$-SCID in $\mathrm{PG}(n,q)$, $k\geq 3, q\geq 7$, that has more than $F_q\theta_k^2$ elements, is a sunflower. Here, we use*

$$F_q=\frac{1}{2}\left(\frac{B_q}{c_q^2}-\frac{1}{q}-\sqrt{\left(\frac{1}{q}-\frac{B_q}{c_q^2}\right)^2-4B_q\left(\frac{1}{c_q^2}-1\right)}\right)$$

and

$$B_q = (1-c_q)^2\left(1-c_q-\frac{1}{q^3}\right)^2\left(1-c_q-\frac{c_q}{q}\right)\left(1-c_q-\frac{1+c_q}{q}\right)q,$$
$$c_q = 1-\frac{1}{\sqrt[6]{q}}-\frac{1}{2\sqrt[3]{q}}.$$

In particular, we have that a $(k+1,1)$-SCID in $\mathrm{PG}(n,q)$, with more than $\left(\frac{2}{\sqrt[6]{q}}+\frac{4}{\sqrt[3]{q}}-\frac{5}{\sqrt{q}}\right)\theta_k^2$ elements is a sunflower.

Proof. From Corollary 5.3.2, Lemma 5.3.4 and Lemma 5.3.5, we know that $F_q\theta_k^2$ gives an upper bound on the size $|\mathcal{S}| = (1-s)\theta_k^2$ of a $(k+1,1)$-SCID, with $\mathcal{S}$ not a sunflower. Hence, a $(k+1,1)$-SCID with more than $F_q\theta_k^2$ elements is a sunflower.

We have to prove that

$$\begin{aligned}
&F_q \leq \frac{2}{\sqrt[6]{q}} + \frac{4}{\sqrt[3]{q}} - \frac{5}{\sqrt{q}} \\
\Leftrightarrow\ &\frac{B_q}{c_q^2} - \frac{1}{q} - \sqrt{\left(\frac{1}{q} - \frac{B_q}{c_q^2}\right)^2 - 4B_q\left(\frac{1}{c_q^2} - 1\right)} \leq \frac{4}{\sqrt[6]{q}} + \frac{8}{\sqrt[3]{q}} - \frac{10}{\sqrt{q}}.
\end{aligned}$$

If $\frac{B_q}{c_q^2} - \frac{1}{q} - \frac{4}{\sqrt[6]{q}} - \frac{8}{\sqrt[3]{q}} + \frac{10}{\sqrt{q}} \leq 0$, then this is true for all values of $q \geq 7$. If $\frac{B_q}{c_q^2} - \frac{1}{q} - \frac{4}{\sqrt[6]{q}} - \frac{8}{\sqrt[3]{q}} + \frac{10}{\sqrt{q}} > 0$, then it is equivalent to proving that

$$\begin{aligned}
&\left(\frac{1}{q} - \frac{B_q}{c_q^2}\right)^2 - 4B_q\left(\frac{1}{c_q^2} - 1\right) \\
&\qquad \geq \left(\frac{4}{\sqrt[6]{q}} + \frac{8}{\sqrt[3]{q}} - \frac{10}{\sqrt{q}}\right)^2 + 2\left(\frac{4}{\sqrt[6]{q}} + \frac{8}{\sqrt[3]{q}} - \frac{10}{\sqrt{q}}\right)\left(\frac{1}{q} - \frac{B_q}{c_q^2}\right) + \left(\frac{1}{q} - \frac{B_q}{c_q^2}\right)^2 \\
\Leftrightarrow\quad &B_q\left(-\frac{1}{c_q^2} + 1 + \frac{1}{c_q^2}\left(\frac{2}{\sqrt[6]{q}} + \frac{4}{\sqrt[3]{q}} - \frac{5}{\sqrt{q}}\right)\right) \\
&\qquad \geq \left(\frac{2}{\sqrt[6]{q}} + \frac{4}{\sqrt[3]{q}} - \frac{5}{\sqrt{q}}\right)^2 + \frac{1}{q}\left(\frac{2}{\sqrt[6]{q}} + \frac{4}{\sqrt[3]{q}} - \frac{5}{\sqrt{q}}\right) \\
\Leftrightarrow\quad &B_q\left(c_q^2 - 1 + \frac{2}{\sqrt[6]{q}} + \frac{4}{\sqrt[3]{q}} - \frac{5}{\sqrt{q}}\right) \\
&\qquad \geq c_q^2\left(\left(\frac{2}{\sqrt[6]{q}} + \frac{4}{\sqrt[3]{q}} - \frac{5}{\sqrt{q}}\right)^2 + \frac{1}{q}\left(\frac{2}{\sqrt[6]{q}} + \frac{4}{\sqrt[3]{q}} - \frac{5}{\sqrt{q}}\right)\right) \\
\overset{t=\sqrt[6]{q}}{\Longleftrightarrow}\quad &B_{t^6}\left(\frac{1}{4t^4} + \frac{4}{t^2} - \frac{4}{t^3}\right) \geq \left(1 - \frac{1}{t} - \frac{1}{2t^2}\right)^2\left(\left(\frac{2}{t} + \frac{4}{t^2} - \frac{5}{t^3}\right)^2 + \frac{1}{t^6}\left(\frac{2}{t} + \frac{4}{t^2} - \frac{5}{t^3}\right)\right).
\end{aligned}$$

In view of equation (5.3) in Lemma 5.3.3, it is sufficient to prove that

$$\begin{aligned}
&\left(1 + \frac{1}{2t}\right)^2\left(1 + \frac{1}{2t} - \frac{1}{t^4}\right)^2\left(1 + \frac{1}{2t} - \frac{1}{t^5}\right)\left(1 + \frac{1}{2t} - \frac{2}{t^5}\right)\left(\frac{1}{4t^4} + \frac{4}{t^2} - \frac{4}{t^3}\right) \\
&\qquad \geq \left(1 - \frac{1}{t} - \frac{1}{2t^2}\right)^2\left(\left(\frac{2}{t} + \frac{4}{t^2} - \frac{5}{t^3}\right)^2 + \frac{1}{t^6}\left(\frac{2}{t} + \frac{4}{t^2} - \frac{5}{t^3}\right)\right), \\
\Leftrightarrow\quad &\frac{157}{4t^4} + \frac{95}{4t^5} - \frac{2165}{16t^6} + \frac{173}{8t^7} + \frac{1411}{64t^8} + \frac{383}{64t^9} + \frac{1313}{256t^{10}} + \frac{69}{2t^{11}} + \frac{1177}{32t^{12}} - \frac{37}{8t^{13}} \\
&\quad - \frac{3315}{128t^{14}} - \frac{219}{8t^{15}} - \frac{1631}{64t^{16}} + \frac{3}{32t^{17}} + \frac{557}{32t^{18}} + \frac{151}{16t^{19}} + \frac{293}{32t^{20}} - \frac{1}{8t^{21}} \\
&\quad - \frac{11}{2t^{22}} - \frac{3}{2t^{23}} + \frac{1}{8t^{24}} \geq 0.
\end{aligned}$$

This inequality is true for all $t = \sqrt[6]{q} \geq \sqrt[6]{7}$, and so for $q \geq 7$. So, a $(k+1,1)$-SCID in $\mathrm{PG}(n,q)$, with at least $\left(\frac{2}{\sqrt[6]{q}} + \frac{4}{\sqrt[3]{q}} - \frac{5}{\sqrt{q}}\right)\theta_k^2$ elements, has more than $F_q\theta_k^2$ elements. This implies that this SCID is a sunflower, which proves the theorem. ■

Note that the bound $1-s \leq \frac{2}{\sqrt[6]{q}}+\frac{4}{\sqrt[3]{q}}-\frac{5}{\sqrt{q}}$ only gives an improvement for the Sunflower bound for large values of q. It is possible to show that for $q \geq 473$, this bound is an improvement on the bound in Theorem 5.1.2. For fixed, smaller values of q, an improved Sunflower bound can be found by investigating the bound $1-s \leq F_q$. This bound gives an improvement on the Sunflower bound if $F_q < 1-\frac{1}{\theta_k}+\frac{1}{\theta_k^2}$. For $k=3$ and $k=4$, this is the case for $q \geq 9$ and $q \geq 8$ respectively. For $k>4$, we have that $F_q < 1-\frac{1}{\theta_k}+\frac{1}{\theta_k^2}$, if $F_q < 1-\frac{1}{\theta_5}$, which is the case for $q \geq 7$. For these values of q and k, we also found that the bound $1-s \leq F_q$ improves the bound in Theorem 5.1.2.

q	F_q	$\frac{2}{\sqrt[6]{q}}+\frac{4}{\sqrt[3]{q}}-\frac{5}{\sqrt{q}}$	Bound Theorem 5.1.2
2^4	0.97698136	1.59732210	0.99975770
2^6	0.89046942	1.37500000	0.99999619
2^8	0.78319928	1.11116105	0.99999999
2^{10}	0.67282525	0.87056078	0.99999999
2^{12}	0.56493296	0.67187500	1.00000000
2^{14}	0.46301281	0.51527789	1.00000000
2^{16}	0.37118406	0.39466158	1.00000000
2^{18}	0.29280283	0.30273438	1.00000000
2^{20}	0.22886576	0.23291485	1.00000000

Table 5.1: Upper bound F_q and $\frac{2}{\sqrt[6]{q}}+\frac{4}{\sqrt[3]{q}}-\frac{5}{\sqrt{q}}$ on $1-s=\frac{|\mathcal{S}|}{\theta_k^2}$ in column 1 and 2. Upper bound from Theorem 5.1.2, for $k=3$ on $\frac{|\mathcal{S}|}{\theta_k^2}$ in column 3.

In Table 5.1, we give the values of the upper bound F_q and $\frac{2}{\sqrt[6]{q}}+\frac{4}{\sqrt[3]{q}}-\frac{5}{\sqrt{q}}$ on $1-s=\frac{|\mathcal{S}|}{\theta_k^2}$, for some specific values of q. The values in this table confirm that the bound $\frac{2}{\sqrt[6]{q}}+\frac{4}{\sqrt[3]{q}}-\frac{5}{\sqrt{q}}$ is a good approximation for F_q for large values of q. In the third column, the upper bound from Theorem 5.1.2, for $k=3$ on $\frac{|\mathcal{S}|}{\theta_k^2}$ is given.

Note that for fixed values of k and q, there is a possibility to find a slightly better bound than the bound F_q, by using our techniques. Given the fixed values for k and q in inequality (5.2), we can choose the values of c and d such that we get the best bound for $(1-s)$. We describe this technique in the example below.

Example 5.3.7. *Suppose that $q=2^8=256$ and $k=5$, then we find from (5.2), that*

$$\left(1-\frac{s}{cd}\right)(1-d)(1-c)\left(1-c-\frac{1}{\theta_5(2^8)}\right)^2\left(1-d-\frac{d}{2^8}\right)\left(1-d-\frac{1+d}{2^8}\right)2^8 \leq (1-s)^2+\frac{1-s}{2^{24}}$$

$$\begin{aligned}
&\Leftrightarrow \left(1-\frac{s}{cd}\right)B(c,d) \leq (1-s)^2+\frac{1-s}{2^{24}}\\
&\Leftrightarrow (1-s)^2+(1-s)\left(\frac{1}{2^{24}}-\frac{B(c,d)}{cd}\right)-\left(1-\frac{1}{cd}\right)B(c,d) \geq 0\\
&\Leftrightarrow 1-s \leq \frac{1}{2}\left(\frac{B(c,d)}{cd}-\frac{1}{2^{24}}-\sqrt{\left(\frac{1}{2^{24}}-\frac{B(c,d)}{cd}\right)^2-4\left(\frac{1}{cd}-1\right)B(c,d)}\right),
\end{aligned}$$

with $B(c,d)=(1-d)(1-c)\left(1-c-\frac{1}{\theta_5(2^8)}\right)^2\left(1-d-\frac{d}{2^8}\right)\left(1-d-\frac{1+d}{2^8}\right)2^8$. By using a computer algebra package, we find a very good bound on $1-s$ for $c=0.53152285$ and $d=0.5294$. For these *values, we find the bound* $1 - s \leq 0.7825095$. *Hence, this gives a small improvement on the bound* $1 - s \leq F_q = 0.78319928$, *for which we used* $c(2^8) = d(2^8) = 0.5244047$. *Note that the bound, given by the Sunflower Theorem 5.1.1, and the bound given in [6] are both larger than* $0.999999999\theta_k^2$ *for* $q = 2^8 = 256$ *and* $k = 5$. *This indicates that our new bound is a clear improvement.*

6 The chromatic number of some Kneser graphs

> *All colors are the friends of their neighbors and the lovers of their opposites.*
>
> —Marc Chagall

The results in this chapter have been obtained in a collaboration with prof. Klaus Metsch and dr. Daniel Werner, and will appear in [48] and [47].

6.1 Introduction

A *flag* in $\mathrm{PG}(n,q)$ is a set F of non-trivial subspaces of $\mathrm{PG}(n,q)$ (that is, different from $\emptyset$ and $\mathrm{PG}(n,q)$) such that for all $\alpha, \beta \in F$ one has $\alpha \subset \beta$ or $\beta \subset \alpha$. The subset $\{\dim(\alpha)+1 \mid \alpha \in F\}$, in which we use the projective dimension, is called the *type* of F and it is a subset of $\{1,2,\ldots,n\}$. Note that the number of elements in a flag is equal to the size of its type, since every two elements in a flag have a different dimension. Two flags F and G are in *general position* if $\alpha \cap \beta = \emptyset$ or $\langle \alpha, \beta \rangle = \mathrm{PG}(n,q)$ for all $\alpha \in F$ and $\beta \in G$.

Notation 6.1.1. *Although a flag is a set, we will write flags $\{\alpha, \beta\}$ of cardinality two of projective spaces as ordered pairs (α, β) where $\dim(\alpha) < \dim(\beta)$.*

For $\Omega \subseteq \{1,2,\ldots,n\}$, we define the q-Kneser graph $qK_{n+1;\Omega}$ to be the graph whose vertices are all the flags of type Ω of $\mathrm{PG}(n,q)$ with two vertices adjacent when the corresponding flags are in general position. For $k \in \{1,\ldots,n\}$, we put $qK_{n+1;k} = qK_{n+1;\{k\}}$, and this q-Kneser graph is the graph in the Grassmann scheme corresponding to the relation $\mathcal{R}_k$, see Example 1.9.5.

We are interested in the chromatic number of these graphs and hence in their independence number α. An independent set of the Kneser graph is a set of flags that are mutually not in general position. An independent set of flags in this graph, will also be called an *Erdős-Ko-Rado set* of flags, in short, *EKR set*. Thus, the chromatic number of a Kneser graph is the smallest number of EKR sets whose union comprises all flags.

An example of an EKR set of flags of type $\Omega \subseteq \{2,3,\ldots,n\}$ is a *point-pencil* $\mathcal{F}_\Omega(P)$ with base point $P \in \mathrm{PG}(n,q)$. This is the set of all flags F of type Ω and for which $F \cup \{P\}$ is a flag. We use the notation $\mathcal{F}(P)$ if the type of the flags is clear from the context. Note that a point-pencil $\mathcal{F}_\Omega(P)$ for $|\Omega| = 1$, is equal to a point-pencil of subspaces in a projective space, which is defined in Section 1.6.

We now describe a strategy that — in some cases — is sufficient to determine the independence number and that we will apply in this chapter. Recall that χ and α are the chromatic and independence number of a graph, and let V be its vertex set. Let $\Gamma = qK_{n+1;\Omega}$ be the q-Kneser graph with $\Omega \subseteq \{1,2,\ldots,n\}$. We assume that we have constructed a coloring of Γ of size χ, and we suppose that C is a coloring with $|C| \leq \chi$. Furthermore, we assume that $\alpha'(\Gamma)$ is an integer, smaller than $\alpha(\Gamma)$, such that one has structural information on all cocliques with more than $\alpha'(\Gamma)$ vertices. Hence, this last assumption asks for a Hilton-Milner type theorem on the flags. Now, if $\alpha'(\Gamma) \cdot |C| < |V|$, then at least $(|V| - \alpha'(\Gamma)|C|)/(\alpha(\Gamma) - \alpha'(\Gamma))$ color classes of g have cardinality larger than $\alpha'(\Gamma)$ and hence one has structural information on these color classes. This structural information is sometimes enough to provide a lower bound on $|C|$ and sometimes even suffices to show that $|C| = \chi$.

This approach was successfully applied for many Kneser graphs $qK_{n,\Omega}$ with $|\Omega| = 1$ in [12, 13]. One of the most important results for $|\Omega| = 1$ is the following one.

Theorem 6.1.2 ([12, Theorem 1.5]). *If $k \geq 2$, and either $q \geq 3$ and $n \geq 2k+2$, or $q = 2$ and $n \geq 2k+3$, then the chromatic number of the q-Kneser graph is $\chi(qK_{n+1;k+1}) = \begin{bmatrix} n-k+1 \\ 1 \end{bmatrix}$. Moreover, each color class of a minimum coloring is contained in a point-pencil and the base points of these point-pencils are the points of a fixed subspace of dimension $n+1-k$.*

For $|\Omega| \geq 2$, much less is known. Even the independence number of these graphs is only known in a few cases. One recent result is the following.

Theorem 6.1.3 ([32, Theorem 3.1]). *If S is an independent set of the q-Kneser graph $qK_{n+1,\Omega}$, with $\Omega = \{1, 2, \dots, n\}$, then*

$$|S| \leq \frac{\theta_n \theta_{n-1} \theta_{n-2} \dots \theta_2 \theta_1}{q^{(n+1)/2} + 1}.$$

The proof of this result uses algebraic arguments and thus does not produce structural information on cocliques that have fewer than this number of vertices. So this result only gives a lower bound for the chromatic number. In contrast to this, the independence number as well as structural information on large cocliques of $qK_{5;\{2,4\}}$ has been given in [14]. For $qK_{2d+1,\{d,d+1\}}$, it has been given for $d = 2$ in [11] and for $d = 3$ in [94].

This chapter is organized as follows. In Section 6.2, we determine the optimal colorings of the Kneser graph $qK_{5;\{2,4\}}$. In Section 6.3, we investigate the Kneser graph $qK_{5;\{2,3\}}$. In Section 6.3.1, we provide several examples for optimal colorings of this graph. In Section 6.3.2, we consider three points P_1, P_2, P_3 and a set M of points in $\mathrm{PG}(4,q)$, q large, with $M \cap \langle P_1, P_2, P_3 \rangle = \emptyset$ and $|M| = cq^3$ for some positive constant $c < 1$. We prove that, if for each of the three points P_i, the number of lines through this point meeting M is small, then there exists a solid S that contains at least mq^2 points of M, where m is a constant. This will be a crucial tool in Section 6.3.3, where we determine the chromatic number of the Kneser graph $qK_{5;\{2,3\}}$ for large values of q. Recently, also the chromatic number of the Kneser graph $qK_{2d+1;\{d,d+1\}}$, for $d \geq 3$, was investigated [48]. In Section 6.4, we give an overview of the main results.

6.2 The chromatic number of the Kneser graph $qK_{5;\{2,4\}}$ of line-solid flags in $\mathrm{PG}(4,q)$

Recall that a point-pencil $\mathcal{F}(P) = \mathcal{F}_{\{2,4\}}(P)$ is the set of all line-solid flags in $\mathrm{PG}(4,q)$, whose line (and so solid) contains the point P. Note that $|\mathcal{F}(P)| = \theta_3\theta_2$.

Example 6.2.1. *If S is a solid of $\mathrm{PG}(4,q)$, then $\{\mathcal{F}(P) \mid P \in S\}$ is a covering of $qK_{5;\{2,4\}}$ with θ_3 independent sets.*

This example shows that there exists a coloring of $qK_{5;\{2,4\}}$ with θ_3 color classes where each color class is a subset of a point-pencil. Theorem 6.2.3 below implies that every coloring with at most θ_3 color classes has the same structure as Example 6.2.1. For the proof of Theorem 6.2.3, we use the following result.

Theorem 6.2.2 ([14, Theorem 1]). *The independence number of* $qK_{5;\{2,4\}}$ *is* $a_0 = \theta_3\theta_2$ *and every independent set of* $qK_{5;\{2,4\}}$ *that is not contained in a point-pencil has at most* $a_1 = 2q^4 + 3q^3 + 4q^2 + 2q + 1$ *elements.*

Theorem 6.2.3. *Let* $q \geq 3$*. Suppose that* $\mathcal{C}$ *is a covering of the vertices of* $qK_{5;\{2,4\}}$ *consisting of* $q^3 + q^2 + q + 1$ *maximal independent sets. Then* $\mathcal{C}$ *consists of all point-pencils with base point contained in a given solid.*

Proof. From Theorem 6.2.2, and using its notation, we have $|F| = a_0$ or $|F| \leq a_1$ for each $F \in \mathcal{C}$. Moreover, $|F| = a_0$ implies $F = \mathcal{F}(P)$ for some point P. Let M be the set of points P with $\mathcal{F}(P) \in \mathcal{C}$. Let $\mathcal{L}$ be the set of lines that contain at least one point of M. For $L \in \mathcal{L}$, we denote by c_L the number of points in M that are contained in L. By double counting the pairs (P, L), with $P \in M$ and $L \in \mathcal{L}$, we find

$$\sum_{L\in\mathcal{L}} c_L = |M|\theta_3,$$

since every point is contained in θ_3 lines. Next, we double count all triples $(P, P', L) \in M \times M \times \mathcal{L}$ with $L = \langle P, P' \rangle$. Since any two distinct points of M span a line, we find

$$\sum_{L\in\mathcal{L}} c_L(c_L - 1) = |M|(|M| - 1).$$

For $L \in \mathcal{L}$, we have $1 \leq c_L \leq q + 1$, and $c_L = q + 1$ if all points of L belong to M. It follows that

$$(q + 1)\sum_{L\in\mathcal{L}}(c_L - 1) \geq |M|(|M| - 1),$$

and so

$$|\mathcal{L}| = \sum_{L\in\mathcal{L}} c_L - \sum_{L\in\mathcal{L}}(c_L - 1) \leq |M|\theta_3 - \frac{|M|(|M| - 1)}{q + 1} \tag{6.1}$$

with equality if and only if $c_L \in \{1, q + 1\}$ for all $L \in \mathcal{L}$. Since the number of solids through a line is θ_2, the union of all sets $\mathcal{F}(P)$, with $P \in M$, contains $|\mathcal{L}|\theta_2$ flags of type $\{2, 4\}$. If we put $x = \theta_3 - |M|$, then $\mathcal{C}$ contains x independent sets of cardinality at most a_1 and, hence, we have

$$\left|\bigcup_{F\in\mathcal{C}} F\right| \leq \left(|M|\theta_3 - \frac{|M|(|M| - 1)}{q + 1}\right)\theta_2 + xa_1. \tag{6.2}$$

Since the union of all independent sets in $\mathcal{C}$ is the set of all flags of type $\{2, 4\}$ and thus has cardinality $\begin{bmatrix}5\\2\end{bmatrix}\theta_2$, it follows that (use $|M| = \theta_3 - x$ and $a_1 = (2q^2 + q + 1)\theta_2$)

$$\begin{aligned}
& \frac{\theta_4\theta_3}{q+1}\theta_2 - \frac{(\theta_3 - x)\theta_3(q + 1) - (\theta_3 - x)(\theta_3 - x - 1)}{q + 1}\theta_2 \leq xa_1 \\
\Leftrightarrow \quad & \frac{\theta_4\theta_3 - (\theta_3 - x)(\theta_4 + x)}{q + 1}\theta_2 \leq x(2q^2 + q + 1)\theta_2 \\
\Leftrightarrow \quad & \frac{x^2 + xq^4}{q + 1} \leq x(2q^2 + q + 1) \\
\xLeftrightarrow{q+1>0} \quad & x\left(x + q^4 - \left(2q^2 + q + 1\right)(q + 1)\right) \leq 0 \\
\Leftrightarrow \quad & x\left(x + q^4 - 2q^3 - 3q^2 - 2q - 1\right) \leq 0.
\end{aligned} \tag{6.3}$$

First, consider the case $q \geq 4$. Then $q^4 - 2q^3 - 3q^2 - 2q - 1 > 0$, and so (6.3) implies $x = 0$ and we have equality in (6.2), and so as well in (6.1). Hence, $c_L \in \{1, q+1\}$ for all $L \in \mathcal{L}$. That is, each $L \in \mathcal{L}$ has the property that either one or all of its points belong to M. This implies that the union of all points of M is itself a subspace. Since it contains $|M| = q^3 + q^2 + q + 1$ points, this subspace has dimension 3 and we are done.

Now, suppose that $q = 3$. Then (6.3) gives $x(x-7) \leq 0$, which shows that $x \leq 7$ and thus $|M| \geq 33$. If $c_L \leq q$ holds for all $L \in \mathcal{L}$, then we could improve the bound (6.2) by replacing $q+1$ in the denominator by q:

$$\begin{aligned} & \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q \theta_2 \leq \left(|M|\theta_3 - \frac{|M|(|M|-1)}{q} \right) \theta_2 + xa_1 \\ \overset{q=3}{\Longleftrightarrow} \quad & \frac{-13}{3}x^2 + \frac{325}{3}x - 1690 \geq 0, \end{aligned}$$

which gives a contradiction for $x \geq 0$. Hence, there exists some $L \in \mathcal{L}$ with $c_L = q + 1 = 4$. Each of the remaining $|M| - 4 \geq 29$ points of M spans a plane with L. Since the number of planes through L is 13, it follows that there exists a plane π (through L) that contains at least $4 + 3 = 7$ elements of M. Similarly, since $|M| \geq 33 = 26 + 7$, one of the four solids through π contains at least $7 + \lceil \frac{26}{4} \rceil = 14$ elements of M. Let τ be a solid through π which contains at least $t \geq 14$ elements of M. Then the number of lines, that contain one of these t points is at most $130 + 27t$. The first term is the total number of lines in τ, and the second term is the product of the number t of points of M in τ and the number of lines through such a point not in τ. We have equality only if all 130 lines of τ belong to $\mathcal{L}$. If $P \in M$, with $P \notin \tau$, then t of the 40 lines through P contain an element of M that is contained in τ. It follows that

$$|\mathcal{L}| \leq 130 + 27t + (|M| - t)(40 - t).$$

The union of the independent sets $\mathcal{F}(P)$, with $P \in M$, has size $|\mathcal{L}|\theta_2$. Since the remaining x independent sets of $\mathcal{C}$ each contain at most a_1 flags, and since the total number of $\{2,4\}$-flags is $\begin{bmatrix} 5 \\ 2 \end{bmatrix}_3 \theta_2$, it follows that

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix}_3 \theta_2(3) \leq |\mathcal{L}|\theta_2(3) + xa_1 \leq (130 + 27t + (40 - x - t)(40 - t))\theta_2(3) + xa_1.$$

Since $a_1 = 22 \cdot \theta_2(3)$, we can divide by $\theta_2(3)$ and find

$$0 \leq (t - 14)(t + x - 39) - 4x - 26. \tag{6.4}$$

Since $14 \leq t \leq |M| = 40 - x$, it follows first that $t > 39 - x$, that is $t = 40 - x = |M|$. Then (6.4) gives $0 \leq -5x$ and, hence, $x = 0$, $t = 40$ and $|M| = 40$. This implies that $\mathcal{C}$ consists of the sets $\mathcal{F}(P)$ for the 40 points P of τ. ■

Remark 6.2.4. From Theorem 6.2.3 and duality, it follows that the chromatic number of the Kneser graph $qK_{5;\{1,3\}}$ is θ_3. Moreover, for every color class C of a minimum coloring, it holds that all planes of the flags in C are contained in a solid S_C, and all these solids S_C contain the same fixed point P.

6.3 The chromatic number of the Kneser graph $qK_{5;\{2,3\}}$ of line-plane flags in $\mathrm{PG}(4, q)$

In this section, we will prove that, for large q, the chromatic number of the Kneser Graph $qK_{5,\{2,3\}}$ is $\theta_3 - q$. More specifically, we will prove the following result.

Theorem 6.3.1. *For $q > 160 \cdot 36^5$, the chromatic number of the Kneser graph $qK_{5;\{2,3\}}$ is $q^3 + q^2 + 1$. Up to duality, for each color class C of a minimum coloring there is a unique point-pencil F such that $F \cup C$ is independent, and the base points of these point-pencils are $q^3 + q^2 + 1$ distinct points of a solid.*

6.3.1 Colorings of the Kneser graph $qK_{5;\{2,3\}}$

Recall that a flag of type $\{2, 3\}$ corresponds to a *line-plane flag* of $\mathrm{PG}(4, q)$. Hence, it is a set $\{\ell, \pi\}$ of a line ℓ and a plane π, with ℓ contained in π. Two flags (ℓ, π) and (ℓ', π') are adjacent in $qK_{5;\{2,3\}}$ if and only if the flags are in general position in $\mathrm{PG}(4, q)$. This means $l \cap \pi' = \emptyset = l' \cap \pi$ and also implies that $\pi \cap \pi'$ is a point. Recall that an independent set of the Kneser graph is a set of line-plane flags pairwise not in general position, or in short, an EKR set of line-plane flags. Thus, the chromatic number of the Kneser graph $qK_{5;\{2,3\}}$ is the smallest number of EKR sets whose union comprises all line-plane flags.

Point-pencils of line-plane flags are EKR sets. However, these are not maximal and are contained in more than one maximal EKR set, as we shall see below. Note that the flags of type $\{d, d + 1\}$ in $\mathrm{PG}(2d, q)$ are self-dual, and that the dual of two flags in general position are flags that are in general position too. Hence, there are maximal EKR sets that arise as the dual of the maximal EKR sets that contain a point-pencil.

Example 6.3.2 (EKR sets). *Let $\mathcal{M}$ be the set of all line-plane flags of $\mathrm{PG}(4, q)$. For point-line flags (P, ℓ), point-solid flags (P, S), and plane-solid flags (τ, S), we define the EKR sets*

$$\begin{aligned}
\mathcal{F}(P, \ell) &= \{(h, \pi) \in \mathcal{M} \mid P \in h \textit{ or } \ell \subset \pi\},\\
\mathcal{F}(P, S) &= \{(h, \pi) \in \mathcal{M} \mid P \in h \textit{ or } P \in \pi \subset S\},\\
\mathcal{F}(S, P) &= \{(h, \pi) \in \mathcal{M} \mid \pi \subset S \textit{ or } P \in h \subset S\},\\
\mathcal{F}(S, \tau) &= \{(h, \pi) \in \mathcal{M} \mid \pi \subset S \textit{ or } h \subset \tau\}.
\end{aligned}$$

Let F be one of the examples above. In the first two cases we call $\mathcal{F}(P) = \{(h, \pi) \in \mathcal{M} \mid P \in h\}$ the generic part *and $F \setminus \mathcal{F}(P)$ the* special part *of F. In the remaining two cases, we call $\mathcal{F}(S) = \{(h, \pi) \in \mathcal{M} \mid \pi \subset S\}$ the* generic part *and $F \setminus \mathcal{F}(S)$ the* special part *of F.*

Note that examples 1 and 4 as well as 2 and 3 are each other's dual. Also, all four examples have cardinality

$$e_0 = \theta_2(\theta_3 + q^2),$$

and their special parts have cardinality $q^2\theta_2$. It was shown in [11] that these examples are the largest EKR sets of line-plane flags in $\mathrm{PG}(4, q)$. We reformulate their result as follows.

Theorem 6.3.3 ([11, Proposition 2.1]). *Let $\mathcal{F}$ be an EKR set of line-plane flags of $\mathrm{PG}(4, q)$. Then $|\mathcal{F}| \le e_0$ and equality occurs if and only if $\mathcal{F}$ is one of the sets defined in Example 6.3.2.*

We will explain in the appendix (Section 6.3.4) how the following stability result can be derived from [11].

Result 6.3.4. *Every EKR set of line-plane flags of $\mathrm{PG}(4, q)$, which is not a subset of one of the sets defined in Example 6.3.2, has cardinality at most*

$$e_1 = 4q^4 + 9q^3 + 4q^2 + q + 1.$$

Example 6.3.5 (Coverings of $qK_{5;\{2,3\}}$). *Let S be a solid of $\mathrm{PG}(4, q)$.*

1) *Consider a set W of q points of S and suppose that there is a map ν from the set of points in $S \setminus W$ to the set of lines of S such that $P \in \nu(P)$ for all $P \in S \setminus W$ and such that every line of S that meets W lies in the image of ν. Then $\mathfrak{F} = \{\mathcal{F}(P, \nu(P)) \mid P \in S \setminus W\}$ is a set of EKR sets whose union is the set of all line-plane flags of* $\mathrm{PG}(4, q)$.

   Proof. We show that every line-plane flag (l, π) in $\mathrm{PG}(4, q)$ is covered by the set $\mathfrak{F}$. If (l, π) is a flag such that $l \cap S$ contains a point P of $S \setminus W$, then $(l, \pi) \in \mathcal{F}(P, \nu(P))$. If (l, π) is a flag such that $l \cap S$ contains no point of $S \setminus W$, then $l \cap S$ is a point Q contained in W. The line $l_0 = \pi \cap S$ contains the point $Q \in W$, and so this line is the image of ν of a point P'. Hence, $\nu(P') = l_0$, and so (l, π) is contained in the flag $\mathcal{F}(P', \nu(P'))$. This proves that every line-plane flag is contained in an element of $\mathfrak{F}$. ■

   We provide examples of a set W and a map ν satisfying these conditions:

   (a) *Suppose that W is a set of q points $P_1, \dots, P_q$ which are contained in a common line ℓ and let P_0 be the last remaining point of ℓ. For each plane π of S through ℓ, fix a numbering $\ell_1(\pi), \dots, \ell_q(\pi)$ of the lines of π through P_0, different from ℓ. Define the map ν from the set $S \setminus W$ to the line-set of S by $\nu(P_0) = \ell$ and $\nu(P) = PP_i$, if $P \notin \ell$ and $P \in \ell_i(\langle P, \ell \rangle)$.*

   (b) *Suppose that W is a set of q points $P_1, \dots, P_q$ in a plane π. Furthermore, suppose that there is a map ν from $\pi \setminus W$ to the set of lines in π, such that every line in π through a point of W is contained in the image of ν. Then one can extend this map to $S \setminus W$ as follows: the q points in W meet at most $q(q + 1)$ lines of π and thus there is at least one line $g \subseteq \pi$ which does not meet the set W. Let $\pi_1, \dots, \pi_q$ be the planes through g in S different from π and, for all $i \in \{1, \dots, q\}$ and all $P \in \pi_i \setminus \pi$, set $\nu(P) = PP_i$.*

   Obviously, one can define such a map ν on a plane $\pi \setminus W$ if W only spans a line therein, because then the construction in (a) can be used. However, one can also find such a map ν if W spans the plane π and we give a simple construction in the case where $q - 1$ points $P_1, \dots, P_{q-1}$ of W are contained in a common line ℓ_0 and the last point P_q of W satisfies $\pi = \langle P_q, \ell_0 \rangle$. We let Q_0 and Q_1 be the two remaining points of ℓ_0 and we fix a numbering $\ell_1, \dots, \ell_q$ of the lines different from ℓ_0 of π through Q_0, such that $\ell_q = Q_0P_q$. Then, for all $i \in \{1, \dots, q - 1\}$ and all $P \in \ell_i \setminus \{Q_0\}$, we set $\nu(P) = PP_i$. Furthermore, we set $\nu(Q_0) = \ell_0$, $\nu(Q_1) = Q_1P_q$ and for all $P \in \ell_q \setminus \{Q_0, P_q\}$, we set $\nu(P) = \ell_q$.

2) *Finally, we give an example which uses both EKR sets with special part coming from a solid and EKR sets with special part coming from a line, that is, EKR sets $\mathcal{F}(P, Z)$ and $\mathcal{F}(Q, l)$ for a point-solid flag (P, Z) and a point-line flag (Q, l), respectively.*

   Here, let W be again a set of q points $P_1, \dots, P_q$ of S, and suppose that these points only span a line ℓ of S. Let P_0 be the last remaining point of ℓ. For any plane π with $\ell \subseteq \pi \subseteq S$, fix a numbering $\ell_1(\pi), \dots, \ell_q(\pi)$ of the lines of π through P_0 different from ℓ as well as a numbering $S_1(\pi), \dots, S_q(\pi)$ of the solids containing π, and different from S. Put

$$\mathfrak{F}_1(\pi) = \bigcup_{i=1}^{q} \{\mathcal{F}(P, PP_i) \mid P_0 \neq P \in \ell_i(\pi)\},$$

$$\mathfrak{F}_2(\pi) = \bigcup_{i=1}^{q} \{\mathcal{F}(P, S_i(\pi)) \mid P_0 \neq P \in \ell_i(\pi)\}.$$

   Now, let Π be the set consisting of all planes of S that contain ℓ and for every subset R of Π, put

$$\mathfrak{F}(R) = \{\mathcal{F}(P_0, \ell)\} \cup \bigcup_{\pi \in R} \mathfrak{F}_1(\pi) \cup \bigcup_{\pi \in \Pi \setminus R} \mathfrak{F}_2(\pi).$$

Then, for all $R \subseteq \Pi$, the set $\mathfrak{F}(R)$ consists of $\theta_3 - q$ EKR sets whose union is the set of all line-plane flags. Note that for $R = \Pi$, this example $\mathfrak{F}(\Pi)$ coincides with the example described above in 1(a).

Proof. We show that every line-plane flag (l, α) in $\mathrm{PG}(4, q)$ is covered by the set $\mathfrak{F}(R)$. If (l, α) is a flag such that $l \cap S$ contains a point P of $S \setminus W$, then (l, α) is contained in the point-pencil $\mathcal{F}(P)$. This point-pencil is contained in $\mathcal{F}(P_0, \ell)$ if $P = P_0$. If $P \neq P_0$, then $\mathcal{F}(P)$ is contained in an element of $\mathfrak{F}_1(\langle P, \ell\rangle)$ or in $\mathfrak{F}_2(\langle P, \ell\rangle)$ depending on whether $\langle P, \ell\rangle$ is contained in R or not. If (l, α) is a flag such that $l \cap S$ contains no point of $S \setminus W$, then $l \cap S$ is a point P_i contained in W, and the line $l_0 = \alpha \cap S$ contains this point. Now there are two cases, depending on whether $\pi = \langle \ell, l_0\rangle$ is contained in R or not. If $\pi \in R$, then $(l, \alpha) \in \mathcal{F}(l_0 \cap \ell_i, \nu(l_0 \cap \ell_i))$, which is contained in $\mathfrak{F}_1(\pi)$. Suppose now that $\pi \notin R$, and let $S_j(\pi)$ be the solid through π spanned by π and α. Then $(l, \alpha) \in \mathcal{F}(l_0 \cap \ell_j, S_j(\pi))$, which is contained in $\mathfrak{F}_2(\pi)$. This proves that every line-plane flag is contained in an element of $\mathfrak{F}$. ■

This list of examples is not a complete list of all colorings with $\theta_3 - q$ colors. For example, one can also find colorings by replacing all EKR sets in a coloring described above by their dual structure. However, since there are examples of colorings with $\theta_3 - q$ colors, we know that the chromatic number of Γ is at most $\theta_3 - q$ and the list above provides several examples of colorings of this size. We will prove in Section 6.3.3 that the chromatic number is in fact equal to $\theta_3 - q$, provided q is large enough.

6.3.2 A lemma on point sets

Lemma 6.3.6. *Suppose that M is a set of points in $\mathrm{PG}(4, q)$, and that P_1, P_2, P_3 are three non-collinear points such that the plane $\pi = \langle P_1, P_2, P_3\rangle$ has no points in M. Let m, n and d be positive real numbers such that the following hold:*

- *Each of the points P_1, P_2, P_3 lies on at most nq^2 lines that meet M,*
- *$|M| = dq^3$,*
- *$q > 32\frac{n^5 m}{d^5}$.*

Then there exists a solid S through π with $|S \cap M| \geq mq^2$.

Proof. Let π_j, $1 \leq j \leq q^2 + q$, be the planes through the line P_1P_2 different from π, and, for $i \in \{1, 2\}$ and $j \in \{1, \ldots, q^2 + q\}$, let a_{ij} be the number of lines of π_j through P_i that meet M. Then $x_j = |\pi_j \cap M| \leq a_{1j}a_{2j}$. This implies that $\sqrt{x_j} \leq \frac{1}{2}(a_{1j} + a_{2j})$. Since each of P_1 and P_2 lies on at most nq^2 lines that meet M, it follows that

$$nq^2 \geq \frac{1}{2}\sum_j (a_{1j} + a_{2j}) \geq \sum_j \sqrt{x_j}.$$

Put $R = \{j \mid x_j \geq cq^2\}$ with $c = \frac{d^2}{4n^2}$. Then

$$nq^2 \geq \sum_{j \notin R} \sqrt{x_j} \geq \frac{1}{\sqrt{cq}} \sum_{j \notin R} x_j \geq \frac{1}{\sqrt{cq}}(|M| - |R|q^2),$$

since the sum of x_j over all j is $|M|$ and since each plane π_j, with $j \in R$, meets M in at most q^2 points. It follows that

$$|R|q^2 \geq |M| - nq^2\sqrt{c}q.$$

Assume to the contrary that every solid through π meets M in at most mq^2 points. Then every solid through π contains at most $\frac{mq^2}{cq^2}$ planes π_j, with $j \in R$. Hence, the number of solids through π that contain a plane π_j, with $j \in R$, is at least $\frac{|R|c}{m}$. This implies that P_3 lies on at least

$$\frac{|R|c}{m} \cdot cq^2$$

lines that meet M. Hence,

$$\frac{|R|}{m}c^2q^2 \leq nq^2.$$

Comparing this to the lower bound for $|R|$, we find

$$\begin{aligned}(|M| - nq^2\sqrt{c}q)\frac{c^2}{m} \leq nq^2 &\Rightarrow\ |M| \leq \sqrt{c}nq^3 + \frac{mnq^2}{c^2}\\ &\Rightarrow\ dq^3 \leq \frac{d}{2}q^3 + 16\frac{mn^5q^2}{d^4}\\ &\Rightarrow\ q \leq 32\frac{mn^5}{d^5},\end{aligned}$$

in which the second implication follows since $c = \frac{d^2}{4n^2}$. This contradicts the hypothesis in the statement of the lemma. $\blacksquare$

Remark 6.3.7. The restriction on q, imposed by this lemma, is the main reason why we can prove Theorem 6.3.1 only for very large values of q. The remaining arguments in the next section are valid for smaller values of q.

6.3.3 The chromatic number of $qK_{5;\{2,3\}}$

In this section we prove, for large values of q, that the chromatic number of $qK_{5;\{2,3\}}$ is $\theta_3 - q$. Note that from Example 6.3.5, we already know a coloring with this many colors, so we only have to show that one cannot do better.

Theorem 6.3.8. *Let $\mathfrak{F} = \{F_1, \dots, F_{\theta_3-q}\}$ be a multiset (so we allow $F_i = F_j$ for $i \neq j$) of $\theta_3 - q$ EKR sets of line-plane flags of* $\mathrm{PG}(4, q)$*, $q > 160 \cdot 36^5$, whose union consists of all line-plane flags of* $\mathrm{PG}(4, q)$*. We put $J = \{1 \leq j \leq \theta_3 - q : |F_j| > e_1\}$ and $I \subseteq J$ is the set of indices i such that the generic part of F_i is based on a point P_i. We suppose the following:*

1. *For $j \in J$, the set F_j is one of the EKR sets defined in Example 6.3.2, which implies that $|F_j| = e_0$.*
2. *For distinct $i, j \in J$, the EKR sets F_i and F_j have distinct generic parts.*
3. *For at least $\frac{1}{2}|J|$ indices $j \in J$, the generic part of F_j is based on a point. Hence, $|I| \geq \frac{1}{2}|J|$.*

Then each $F \in \mathfrak{F}$ has e_0 elements and is based on a point P_F and the points P_F, $F \in \mathfrak{F}$, are $\theta_3 - q$ mutually distinct points of a solid.

Note that in this theorem, we suppose that the sets F_j, with $j \in J$, are maximal EKR sets. The proof of this theorem is carried out in Lemmas 6.3.9-6.3.20. In all these lemmas, we suppose that $\mathfrak{F}$ is as in the theorem and that $q > 160 \cdot 36^5$. We note that Lemma 6.3.9 is valid for all q and Lemma 6.3.10 requires only $q \geq 41$.

Lemma 6.3.9. *The number of all line-plane flags of* $\mathrm{PG}(4, q)$ *is equal to*

$$\begin{bmatrix}5\\3\end{bmatrix} \cdot \begin{bmatrix}3\\2\end{bmatrix} = |\mathfrak{F}|e_0 - q^2\theta_2(2q^3 + q^2 + q + 1).$$

Lemma 6.3.10. *Let S be a solid and let $q \geq 41$. Denote by c_1 the number of indices $i \in I$ with $P_i \notin S$ and by c_3 the number of EKR sets $F \in \mathfrak{F}$ with $|F| \leq e_1$. Then $(|I| - c_1) + c_3 < 5q^2$ or $c_1 + c_3 \leq 4q^2$.*

Proof. We have $|I| \geq \frac{1}{2}|J| = \frac{1}{2}(\theta_3 - q - c_3)$. We know that for all $i \in I$ the set F_i is based on a point P_i and we set $A = \{a \in I \mid P_a \in S\}$. For $a \in A$, the set F_a contains θ_2^2 flags (ℓ, π) with $P \in \ell \subseteq S$. Since there are $(q^2 + 1)\theta_2$ lines in S, there are at most $(q^2 + 1)\theta_2^2$ flags (ℓ, π) with $\ell \subseteq S$. It follows that

$$\left|\bigcup_{a\in A} F_a\right| \leq |A|(e_0 - \theta_2^2) + (q^2 + 1)\theta_2^2. \tag{6.5}$$

If $i \in I \setminus A$, then for each $a \in A$, the sets F_i and F_a share the θ_2 line-plane flags (P_iP_a, π). Different values of a in A correspond to disjoint sets of θ_2 flags, and, hence, F_i contains at least $|A|\theta_2$ flags that are contained in $\cup_{a\in A}F_a$. It follows that

$$\left|\bigcup_{i\in I} F_i \setminus \bigcup_{a\in A} F_a\right| \leq |I \setminus A|e_0 - |A||I \setminus A|\theta_2. \tag{6.6}$$

Therefore, we have that

$$\begin{aligned}
\begin{bmatrix}5\\3\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} &\leq \left|\bigcup_{a\in A} F_a\right| + \left|\bigcup_{i\in I} F_i \setminus \bigcup_{a\in A} F_a\right| + \left|\bigcup_{i\in J\setminus I} F_i\right| + \left|\bigcup_{i\notin J} F_i\right| \\
\Rightarrow |\mathfrak{F}|e_0 - q^2\theta_2(2q^3 + q^2 + q + 1) & \\
&\leq |A|(e_0 - \theta_2^2) + (q^2 + 1)\theta_2^2 + |I \setminus A|e_0 - |A||I \setminus A|\theta_2 + |J \setminus I|e_0 + c_3e_1 \\
\Rightarrow |A|\theta_2^2 - (q^2 + 1)\theta_2^2 + |A|(|I| - |A|)\theta_2 + c_3(e_0 - e_1) & \\
&\leq q^2\theta_2(2q^3 + q^2 + q + 1).
\end{aligned}$$

The first implication follows by Lemma 6.3.9, and the inequalities (6.5) and (6.6). The second implication follows since $|\mathfrak{F}| = \theta_3 - q = |A| + |I \setminus A| + |J \setminus I| + c_3$.

We use that $|A| = |I| - c_1$, and $e_0 - e_1 \geq \theta_2(q^3 - 2q^2 - 4q + 5)$ for $q \geq 3$. If we divide both sides by θ_2, then we have that

$$(|I| - c_1)\theta_2 + c_1(|I| - c_1) + c_3(q^3 - 2q^2 - 4q + 5) \leq 2q^5 + (2q^2 + 1)\theta_2. \tag{6.7}$$

Assume the statement of the lemma is not true. Then

$$0 \leq (c_1 + c_3 - 4q^2)(|I| - c_1 + c_3 - 5q^2). \tag{6.8}$$

If we add the right hand side of (6.8) to the right hand side of (6.7), we find the following inequality.

$$\begin{aligned}&(|I|-c_1)\theta_2+c_1(|I|-c_1)+c_3(q^3-2q^2-4q+5)\\&\quad\leq 2q^5+(2q^2+1)\theta_2+(c_1+c_3-4q^2)(|I|-c_1+c_3-5q^2).\end{aligned}$$

If we replace in this inequality $|I|$ by $\frac{1}{2}(\theta_3-q-c_3)+z$ with $z=|I|-\frac{1}{2}(\theta_3-q-c_3)$, and multiply both sides with 2, we find that

$$\begin{aligned}&(\theta_3-q-c_3+2z)\theta_2-2c_1\theta_2+c_1(\theta_3-q-c_3+2z)-2c_1^2+2c_3(q^3-2q^2-4q+5)\\&\quad\leq 4q^5+2(2q^2+1)\theta_2+(c_1+c_3-4q^2)(\theta_3-q+c_3+2z-2c_1-10q^2)\\&\Leftrightarrow 2(5q^2+q+1-c_3)z+(q^3+8q^2-9q+8-c_3)c_3+q^5\\&\quad\leq 38q^4+2q^3+q+1+(2q+2)c_1. \qquad (6.9)\end{aligned}$$

Since $|I|\geq\frac{1}{2}(\theta_3-q-c_3)$, we have that $z\geq 0$. Furthermore, from (6.7), we have that

$$c_3(q^3-2q^2-4q+5)\leq 2q^5+(2q^2+1)\theta_2,$$

which implies that $c_3\leq 3q^2$ for $q\geq 10$. Hence, $(q^3+8q^2-9q+8-c_3)c_3\geq 0$ as well as $2(5q^2+q+1-c_3)z\geq 0$, so, for $q\geq 10$, (6.9) implies that

$$q^5\leq 38q^4+2q^3+q+1+(2q+2)c_1.$$

As $c_1\leq|I|\leq|\mathfrak{F}|=\theta_3-q$, this is a contradiction for $q\geq 41$. ■

Lemma 6.3.11. *There exists a solid S such that*

$$|\{F\in\mathfrak{F}:|F|\leq e_1\}|+|\{i\in I:P_i\notin S\}|\leq 4q^2.$$

Proof. Let c_3 be the number of $F\in\mathfrak{F}$ with $|F|\neq e_0$ and thus $|F|\leq e_1$. Then $\mathfrak{F}$ contains $|I|\geq\frac{1}{2}(\theta_3-q-c_3)$ EKR sets that are maximal EKR sets based on a point. Let these be G_i, $i=1,\dots,|I|$, let R_i be the base point of G_i and put

$$g_i=\left|G_i\cap\bigcup_{j=1}^{i-1}G_j\right|.$$

Then we have that

$$\left|\bigcup_{i\in I}G_i\right|=|I|e_0-\sum_{i\in I}g_i. \qquad (6.10)$$

We may assume that the sequence $g_1,\dots,g_{|I|}$ is monotone increasing. We want to show that g_j for $j=\frac{1}{4}q^3+q^2+2q+1$ is less than $9q^2\theta_2$. Suppose that this is not the case, then we would have that $\sum_{i=j}^{|I|}g_i\geq(|I|-j+1)9q^2\theta_2$. We know that

$$\begin{aligned}&\begin{bmatrix}5\\3\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}\leq\left|\bigcup_{i\in I}G_i\right|+\left|\bigcup_{i\in J\setminus I}F_i\right|+\left|\bigcup_{i\notin J}F_i\right|\\&\Rightarrow|\mathfrak{F}|e_0-q^2\theta_2(2q^3+q^2+q+1)\\&\qquad\leq|I|e_0-\sum_{i\in I}g_i+|J\setminus I|e_0+c_3e_1\\&\Rightarrow\sum_{i\in I}g_i+c_3(e_0-e_1)\leq q^2\theta_2(2q^3+q^2+q+1)\\&\Rightarrow(|I|-j+1)9q^2\theta_2+c_3(e_0-e_1)\leq q^2\theta_2(2q^3+q^2+q+1).\end{aligned}$$

The first implication follows again by Lemma 6.3.9 and (6.10). The second implication follows since $|\mathfrak{F}| = \theta_3 - q = |I| + |J \setminus I| + c_3$, and the third implication follows by the assumption that $\sum_{i=j}^{|I|} g_i \geq (|I| - j + 1)9q^2\theta_2$.

Using the lower bound for $|I| \geq \frac{1}{2}(\theta_3 - q - c_3)$, as well as $e_0 - e_1 \geq \theta_2 \left(q^3 - 2q^2 - 4q + 5\right)$ for $q \geq 3$, and $j = \frac{1}{4}q^3 + q^2 + 2q + 1$, we find that

$$\begin{aligned}
&\left(\frac{1}{4}q^3 - \frac{1}{2}q^2 - 2q + \frac{1}{2} - \frac{1}{2}c_3\right) 9q^2\theta_2 + c_3\theta_2 \left(q^3 - 2q^2 - 4q + 5\right) \leq q^2\theta_2(2q^3 + q^2 + q + 1) \\
\Leftrightarrow\ & c_3 \left(2q^3 - 13q^2 - 8q + 10\right) \leq -\frac{1}{2}q^5 + 11q^4 + 38q^3 - 7q^2 \\
\Rightarrow\ & c_3 < 0.
\end{aligned}$$

The last implication is true for $q \geq 26$. Since $c_3 \geq 0$, we find a contradiction, and so our assumption was false. Hence, we have that $g_j < 9q^2\theta_2$ and, therefore, $g_i < 9q^2\theta_2$ for all $i \leq j$. Now, let Q_1, Q_2 and Q_3 be three non-collinear points in $\{R_i : i \in \{j - q - 1, \ldots, j\}\}$ and let $\mathcal{P}$ be the set of all points R_i, with $i \leq j - q - 2$, that do not lie in the plane $\pi = \langle Q_1, Q_2, Q_3\rangle$. Recall that $j = \frac{1}{4}q^3 + q^2 + 2q + 1$. Then $|\mathcal{P}| \geq j - q - 2 - (\theta_2 - 3) > \frac{1}{4}q^3$. Also, each of the points Q_i is contained in less than $9q^2$ lines that meet $\mathcal{P}$, since every such line lies in θ_2 flags that are contained in the union of the G_i, with $i \leq j - q - 2$. Then we use Lemma 6.3.6 with $M = \mathcal{P}, n = 9, d = \frac{1}{4}$ and $m = 5$. Hence, since $q \geq 32\frac{n^5 m}{d^5} = 160 \cdot 36^5$, we find a solid that contains at least $5q^2$ points of $\mathcal{P}$. The statement follows now from Lemma 6.3.10. ■

Remark 6.3.12. Note that there is precisely one solid that contains all but at most $4q^2$ points $P_i, i \in I$: if there would be two such solids S_1, S_2, then the number of points $P_i, i \in I$, in $S_1 \cup S_2$ would be at least $2(\theta_3 - q - 4q^2) - \theta_2$. For $q \geq 9$, this number of points is larger than the total number $\theta_3 - q$ of EKR sets F_i in $\mathfrak{F}$, which gives a contradiction.

Notation 6.3.13. *From now on, we denote by S the unique solid that contains all but at most $4q^2$ of the points P_i, with $i \in I$, and we use the following notation:*

- $C_0 = \{F_i \mid i \in I,\ P_i \in S\}$.
- $C_1 = \{F_i \mid i \in I,\ P_i \notin S\}$.
- $C_2 = \{F_i \mid i \in J \setminus I\}$.
- $C_3 = \{F_i \mid i \in \{1, \ldots, \theta_3 - q\} \setminus J\}$.
- $c_i = |C_i|$ *for* $i \in \{0, \ldots, 3\}$.
- $W = \{P \in S \mid P \neq P_i,\ \forall i \in I\}$.
- *Let M be the set of all line-plane flags (l, π) for which $l \cap S$ is a point which lies in W.*

Lemma 6.3.14. *We have*

(a) $C_0 \cup C_1 \cup C_2 \cup C_3$ *is a partition of* $\mathfrak{F}$.

(b) $c_1 + c_3 \leq 4q^2$.

(c) $|W| = \theta_3 - c_0$.

(d) Every point of W lies in the plane of exactly $q^3\theta_2$ flags of M.

(e) $|M| = |W|q^3\theta_2$.

(f) $c_3 \leq 2q^2 + 6q$ *for* $q \geq 22$.

Proof. Statement (a) is obvious from the notation introduced above. The choice of S implies statement (b). Since no two members of $\mathfrak{F}$ of size e_0 have the same generic part, we have $|W| = |S \backslash C_0| = |S| - |C_0| = \theta_3 - c_0$ and thus statement (c). Furthermore, each point $P \in W$ is contained in q^3 lines that meet S only in P and each such line lies in θ_2 planes. Hence, for every point $P \in W$, exactly $q^3\theta_2$ flags (ℓ, π) of M satisfy $\ell \cap S = P$, which proves statements (d) and (e). Finally, statement (f) follows from Lemma 6.3.9 and $q \geq 22$:

$$\begin{aligned} & |\mathfrak{F}|e_0 - q^2\theta_2(2q^3 + q^2 + q + 1) \leq |J|e_0 + c_3e_1 \\ \Leftrightarrow \quad & c_3(e_0 - e_1) \leq q^2\theta_2(2q^3 + q^2 + q + 1) \\ \Rightarrow \quad & c_3 \leq 2q^2 + 6q. \end{aligned}$$

■

Lemma 6.3.15.

(a) Suppose that $F \in C_0$. Then the generic part of F does not contain a flag of M.

(b) Suppose that $F \in C_1$. Then $|F \cap M| \leq |W|\theta_2 + q^2\theta_2$.

(c) Suppose that $F \in C_2$, with base solid H. If $H = S$, then we have that $|F \cap M| \leq q^2\theta_2$. If $H \neq S$, then $|F \cap M| \leq |H \cap W|q^2(q + 1) + q^2\theta_2$.

Proof. (a) The flags of the generic part of F either have a line that is contained in S or that meets S in the base point of F, which is not in W. Therefore these flags do not belong to M.

(b) We know that F is based on a point P. The generic part of F consists of all flags whose line contains P. As $P \notin S$, we see that the generic part of F has exactly $|W|\theta_2$ flags in M. The special part of F has $q^2\theta_2$ flags and thus at most this many flags of M.

(c) We know that F is based on a solid H. The generic part of H consists of all flags whose plane lies in H. Hence, if $H = S$, the generic part contains no flag of M, and if $H \neq S$, it contains exactly $|H \cap W|q^2(q + 1)$ flags of M. The special part of F has $q^2\theta_2$ flags and thus at most this many flags of M. ■

Lemma 6.3.16. *Suppose that z is an integer such that all except at most one plane of S have at most z points in W. Then*

$$|W|q^3\theta_2 \leq c_1\left(|W| + q^2\right)\theta_2 + c_2(zq^2(q + 1) + q^2\theta_2) + c_3e_1 + s + q^3(q + 1)\theta_2,$$

where s is the number of flags of M that are contained in the special part of F for some EKR set F of C_0. If every plane of S has at most z points in W, then

$$|W|q^3\theta_2 \leq c_1(|W| + q^2)\theta_2 + c_2(zq^2(q + 1) + q^2\theta_2) + c_3e_1 + s.$$

Proof. Each of the $|M| = |W|q^3\theta_2$ flags of M is contained in some member of $\mathfrak{F} = C_0 \cup C_1 \cup C_2 \cup C_3$. Hence, $|W|q^3\theta_2 \leq \sum_{i=0}^{3} |(\cup_{F \in C_i} F) \cap M|$. If there exists a plane of S with more than z points in W, then denote by z' its number of points in W. Otherwise put $z' = z$. Since a plane of S lies in q solids other than S, the preceding lemma shows that $\cup_{F \in C_2} F$ and M share at most

$$\begin{aligned} & (c_2 - q)(zq^2(q + 1) + q^2\theta_2) + q(z'q^2(q + 1) + q^2\theta_2) \\ & \quad = c_2(zq^2(q + 1) + q^2\theta_2) + (z' - z)q^3(q + 1) \end{aligned}$$

flags. Using this, together with the previous lemma and the fact that $|F| \le e_1$ for $F \in C_3$, we find that

$$\begin{aligned}|W|q^3\theta_2 &\le \left|\left(\bigcup_{F\in C_0} F\right)\cap M\right| + \left|\left(\bigcup_{F\in C_1} F\right)\cap M\right| + \left|\left(\bigcup_{F\in C_2} F\right)\cap M\right| + \left|\left(\bigcup_{F\in C_3} F\right)\cap M\right| \\ &\le \sum_{F\in C_0}|F\cap M| + \sum_{F\in C_1}|F\cap M| + \sum_{F\in C_3}|F\cap M| \\ &\qquad + c_2(zq^2(q+1)+q^2\theta_2) + (z'-z)q^3(q+1) \\ &\le s + c_1(|W|+q^2)\theta_2 + c_2(zq^2(q+1)+q^2\theta_2) + (z'-z)q^3(q+1) + c_3e_1.\end{aligned}$$

Now we use $z' - z \le \theta_2$ to find the first assertion and $z' - z = 0$ to find the second assertion in the statement of the lemma. ■

Lemma 6.3.17. *Let π_1 and π_2 be distinct planes of S. Then*

$$|(\pi_1\cup\pi_2)\cap W|\, q^2(q+1) \le 6q^3(q+4) + 3q(|W|-q)(q+1). \tag{6.11}$$

Proof. Put $W' = (\pi_1\cup\pi_2)\cap W$, and let M' be the subset of M that consists of all flags of M whose line meets S in a point of W'. Lemma 6.3.14 (d) shows that $|M'| = |W'|q^3\theta_2$. Each flag of M' lies in at least one of the EKR sets of $\mathfrak{F} = C_0\cup C_1\cup C_2\cup C_3$. Hence, $|M'| \le d_0+d_1+d_2+d_3$, where d_i is the number of elements of M' that lie in some member of C_i.

For $F \in C_3$, we have $|F\cap M'| \le |F| \le e_1$. Hence, $d_3 \le c_3e_1$.

If $F \in C_1$, then $|F| = e_0$ and F is based on a point $P \notin S$, so the flags of M' that lie in the generic part of F are precisely the $|W'|\theta_2$ flags whose line contains P and a point of W'. Since the special part of F has $q^2\theta_2$ flags, it follows that $d_1 \le c_1(|W'|+q^2)\theta_2$.

Consider $F \in C_2$. Then $|F| = e_0$ and F is based on a solid H. If $H = S$, then the lines of all flags of the generic part of F are contained in S and hence $F\cap M' = \emptyset$. Now we consider the case when $H \ne S$. Then the number of flags of M' in the generic part of F is $|H\cap W'|q^2(q+1)$. This number is at most $(2q+1)q^2(q+1)$, if the plane $H\cap S$ is different from π_1 and from π_2, and it is $|W\cap\pi_i|q^2(q+1)$, if $H\cap S = \pi_i$. Since there are exactly q solids that meet S in π_1 and as many that meet S in π_2, it follows that the number of flags of M' that lie in the generic part of at least one EKR set of C_2 is at most

$$\begin{aligned}&q(|W\cap\pi_1|+|W\cap\pi_2|)q^2(q+1) + (c_2-2q)(2q+1)q^2(q+1) \\ &\quad \le q(|W\cap\pi_1|+|W\cap\pi_2|)q^2(q+1) + c_2(2q+1)q^2(q+1).\end{aligned}$$

The special part of each EKR set of C_2 has $q^2\theta_2$ flags and thus at most this many flags of M'. Using $|W\cap\pi_1|+|W\cap\pi_2| \le |W'|+q+1$, it follows that

$$d_2 \le q(|W'|+q+1)q^2(q+1) + c_2(2q+1)q^2(q+1) + c_2q^2\theta_2.$$

Finally, we consider an EKR set F of C_0. Then $|F| = e_0$ and F is based on a point P. We know from Lemma 6.3.15 (a) that only the special part T of F can contribute to M'. For T, there are the following possibilities:

- There exists a line ℓ with $P \in \ell$ and T consists of all flags whose plane contains ℓ and whose line does not contain P. If ℓ meets S only in P, then $|T\cap M'| = |W'|q$. If ℓ is contained in S, then $|T\cap M'| = |\ell\cap W'|q^3$ which is at most $2q^3$ if $P \notin \pi_1\cup\pi_2$, and at most q^4 if $P \in \pi_1\cup\pi_2$. Since $|W'| \le 2q^2+q+1$, it follows that $|T\cap M'| \le q^4$ if $P \in \pi_1\cup\pi_2$, and $|T\cap M'| \le q(2q^2+q+1)$ otherwise.

- There exists a solid H with $P \in H$ and T consists of all line-plane flags (h, τ) with $P \in \tau \subseteq H$ and $P \notin h$. Then $T \cap M' = \emptyset$ if $H = S$, and $|T \cap M'| = |H \cap W'|q^2$ if $H \neq S$. In the second case, this number is $|W' \cap \pi_i|q^2$ if $H \cap S = \pi_i$ for some $i \in \{1, 2\}$, and it is at most $(2q+1)q^2$ if $H \cap S \notin \{\pi_1, \pi_2\}$. Note that $H \cap S = \pi_i$ implies $P \in \pi_i$, so that $|W' \cap \pi_i| \leq q^2 + q$ and hence $|W' \cap \pi_i|q^2 \leq q^3(q+1)$.

Summarizing, we see that $|T \cap M'| \leq q(2q^2+q+1)$ if $P \notin \pi_1 \cup \pi_2$, and $|T \cap M'| \leq q^3(q+1)$ if $P \in \pi_1 \cup \pi_2$, which proves

$$\begin{aligned} d_0 &\leq (c_0 - 2q^2 - q - 1 + |W'|)q(2q^2+q+1) + (2q^2+q+1-|W'|)q^3(q+1) \\ &= c_0 q(2q^2+q+1) + 2q^6 - q^5 - 2q^4 - 4q^3 - 2q^2 - q - |W'|(q^4 - q^3 - q^2 - q) \\ &\leq c_0 q(2q^2+q+1) + 2q^6 - q^5 - |W'|(q^4 - q^3 - q^2 - q). \end{aligned}$$

It follows that

$$\begin{aligned} |W'|q^3\theta_2 = |M'| &\leq d_0 + d_1 + d_2 + d_3 \\ &\leq c_0 q(2q^2+q+1) + 2q^6 - q^5 - |W'|(q^4 - q^3 - q^2 - q) \\ &\quad + c_1(|W'| + q^2)\theta_2 + q(|W'| + q + 1)q^2(q+1) \\ &\quad + c_2(2q+1)q^2(q+1) + c_2 q^2\theta_2 + c_3 e_1 \end{aligned}$$

and simplifications show that

$$\begin{aligned} |W'|q^4\theta_1 \leq |W'|q\theta_2 &+ q^3(2q^3+2q+1) + c_0 q(q^2+\theta_2) \\ &+ \underbrace{c_1(|W'| + q^2)\theta_2 + c_2 q^2(\theta_2 + 2q^2 + 3q + 1) + c_3 e_1}_{=\xi}. \end{aligned} \tag{6.12}$$

We put $\delta = c_1 + c_2 + c_3$, which also implies that $c_0 = \theta_3 - q - \delta$. Since $|W'| \leq 2q^2 + q + 1$, we have that

$$\begin{aligned} \xi &\leq c_1(3q^2+q+1)\theta_2 + c_2 q^2(\theta_2 + 2q^2 + 3q + 1) + c_3 e_1 \\ &= \delta(3q^2+q+1)\theta_2 - c_2(3q^2+2q+1) + c_3(e_1 - (3q^2+q+1)\theta_2) \\ &\leq \delta(3q^2+q+1)\theta_2 + (2q^2+6q)(q^4+5q^3-q^2-q). \end{aligned}$$

The last inequality follows from Lemma 6.3.14 (f).
Using this bound on ξ, as well as $|W'| \leq q^2 + \theta_2$ and $c_0 = \theta_3 - q - \delta$ on the right hand side of inequality (6.12), we find that

$$\begin{aligned} |W'|q^4\theta_1 &\leq 6q^6 + 21q^5 + 35q^4 - 3q^2 + 2q + \delta(3q^4 + 2q^3 + 4q^2 + q + 1) \\ &\leq 6q^6 + 24q^5 + \delta(3q^4 + 3q^3) \\ &= 6q^5(q+4) + 3q^3\delta(q+1). \end{aligned}$$

Substituting $\delta = |W| - q$ in the last expression implies the statement. ■

Lemma 6.3.18. *We have* $c_0 \geq q^3 - 18q + 1$ *and thus* $|W| \leq q^2 + 19q$.

Proof. Let π_1 and π_2 be planes of S such that $|\pi_1 \cap W| \geq |\pi_2 \cap W| \geq |\pi \cap W|$ for every plane π of S other than π_1 and π_2. Put $z = |\pi_2 \cap W|$. The number s occurring in the assertion of Lemma 6.3.16 is at most $c_0 q^2\theta_2$, since the special part of each EKR set of C_0 has cardinality $q^2\theta_2$. Therefore, Lemma 6.3.16 shows that

$$|W|(q^3 - c_1)\theta_2 \leq c_0 q^2\theta_2 + c_1 q^2\theta_2 + c_2(zq^2(q+1) + q^2\theta_2) + c_3 e_1 + q^3(q+1)\theta_2.$$

Since $c_0 + c_1 + c_2 + c_3 = \theta_3 - q$, the right hand side is equal to

$$(\theta_3 - q)q^2\theta_2 + c_2zq^2(q+1) + c_3(e_1 - q^2\theta_2) + q^3(q+1)\theta_2.$$

Using $c_3 \leq 2q^2 + 6q$ from Lemma 6.3.14 (f) and the definition of e_1 implies

$$\begin{aligned} |W|(q^3 - c_1)\theta_2 &\leq q^7 + 9q^6 + 38q^5 + 58q^4 + 22q^3 + 9q^2 + 6q + c_2zq^2(q+1) \\ &\leq q^7 + 10q^6 + c_2zq^2(q+1). \end{aligned}$$

The last inequality follows since $q \geq 40$. We put $\delta = c_1 + c_2 + c_3$, such that $|W| = \theta_3 - c_0 = \delta + q$ and thus

$$(\delta + q)(q^3 - c_1)\theta_2 \leq q^7 + 10q^6 + \delta zq^2(q+1). \tag{6.13}$$

Now, Lemma 6.3.17 states

$$|(\pi_1 \cup \pi_2) \cap W|q^2(q+1) \leq 6q^4 + 24q^3 + 3\delta(q^2 + q)$$

and, since $|(\pi_1 \cup \pi_2) \cap W| \geq |\pi_1 \cap W| + |\pi_2 \cap W| - (q+1) \geq 2z - q - 1$, this implies

$$2zq^2(q+1) \leq 7q^4 + 26q^3 + q^2 + 3\delta(q^2+q) \leq 8q^4 + 3\delta(q^2+q). \tag{6.14}$$

The last inequality uses $q \geq 27$. Combining (6.13) with (6.14) and using $c_1 \leq 4q^2$ results in

$$\begin{aligned} (\delta+q)(q^3 - 4q^2)\theta_2 \leq{}& \; q^7 + 10q^6 + \delta\left(4q^4 + \frac{3}{2}\delta(q^2+q)\right) \\ \Leftrightarrow \quad & \delta^2\frac{3}{2}(q+1) + \delta(4q^3 - q\theta_2(q-4)) + q^6 + 10q^5 - q^2\theta_2(q-4) \geq 0. \end{aligned} \tag{6.15}$$

It is easy to verify that this inequality is not satisfied for $\delta = q^2 + 18q$ nor for $\delta = \frac{2}{3}q^3 - 7q^2$. Since (6.15) is a quadratic inequality in δ, it follows that δ does not lie in the interval $[q^2 + 18q, \frac{2}{3}q^3 - 7q^2]$. However, we have $\delta = \theta_3 - q - c_0$ as well as $c_0 + c_1 = |I| \geq \frac{1}{2}(\theta_3 - q - c_3)$. Furthermore, since $c_1 + c_3 \leq 4q^2$ by Lemma 6.3.14(b), this implies $\delta < \frac{2}{3}q^3 - 7q^2$ for $q \geq 70$. We conclude that $\delta \leq q^2 + 18q$, and hence $|W| \leq q^2 + 19q$. ■

Lemma 6.3.19. *Every plane of S has at most $10q$ points in W.*

Proof. From Lemma 6.3.17 and Lemma 6.3.18, it follows, for $q \geq 72$, that

$$\begin{aligned} & |(\pi_1 \cup \pi_2) \cap W|\, q^2(q+1) \leq 6q^3(q+4) + 3q(q^2+18q)(q+1) \\ \Rightarrow \quad & |(\pi_1 \cup \pi_2) \cap W| \leq 9q + 72 \leq 10q \\ \Rightarrow \quad & |\pi_1 \cap W| \leq 10q, \end{aligned}$$

for all planes π_1 (and $\pi_2 \neq \pi_1$) in $\mathcal{S}$. ■

Lemma 6.3.20. *We have $\mathfrak{F} = C_0$.*

Proof. As in the previous proofs, we put $\delta = c_1 + c_2 + c_3$, which again implies $|W| = q + \delta$ as well as $\delta = \theta_3 - q - c_0$. From Lemmas 6.3.18 and 6.3.19, we have $|W| \leq q^2 + 19q$ and $|\pi \cap W| \leq 10q$ for all planes π of S. Therefore, Lemma 6.3.15 shows that $|F \cap M| \leq (2q^2 + 19q)\theta_2$ for $F \in C_1$, and $|F \cap M| \leq 11q^4 + 11q^3 + q^2$ for $F \in C_2$. Hence, each of the EKR sets $F \in C_1 \cup C_2$ satisfies $|F \cap M| \leq 12q^4$ for $q \geq 12$. Since $e_1 < 12q^4$, the same holds for $F \in C_3$. Therefore, the total contribution of all EKR sets in $C_1 \cup C_2 \cup C_3$ to M is at most $12\delta q^4 = 12(|W| - q)q^4$. Furthermore, the generic part of every EKR set in C_0 is disjoint from M and thus it remains to consider the special parts $T(F)$ of the EKR sets $F \in C_0$. In view of that we define

$$\begin{aligned}\alpha &= |\{F \in C_0 : T(F) \text{ is based on a line } \ell \subset S\}|,\\ \beta &= |\{F \in C_0 : T(F) \text{ is based on a solid } H\}|,\\ \gamma &= |\{F \in C_0 : T(F) \text{ is based on a line } \ell \not\subseteq S\}|.\end{aligned}$$

Moreover, we let A be the set of lines ℓ of S such that $\mathcal{F}(P,\ell) \in C_0$ for some point P of ℓ and we let B be the set of all point-solid pairs (P,H) with $\mathcal{F}(P,H) \in C_0$ and $H \neq S$. Then $\alpha + \beta + \gamma = c_0$, $|A| \leq \alpha$ and $|B| \leq \beta$. Recall that if $F \in C_0$ is such that $T(F)$ is solid based with solid S, then $T(F)$ does not contribute to M. Therefore, we find an upper bound on the number $|M| = |W|q^3\theta_2$ of flags of M:

$$|W|q^3\theta_2 \leq 12(|W|-q)q^4 + \sum_{\ell \in A} |\ell \cap W|q^3 + \sum_{(P,H)\in B} |H \cap W|q^2 + \gamma|W|q. \tag{6.16}$$

Furthermore, since the product of two consecutive integers is non-negative we have

$$\begin{aligned}0 &\leq \sum_{\ell \in A}(|\ell \cap W| - 1)(|\ell \cap W| - 2)\\ &= \sum_{\ell \in A} |\ell \cap W|(|\ell \cap W| - 1) - 2\sum_{\ell \in A} |\ell \cap W| + 2|A|\\ &\leq |W|(|W|-1) - 2\sum_{\ell \in A} |\ell \cap W| + 2|A|.\end{aligned}$$

The last inequality follows from counting the triples (P_1, P_2, l), with $P_1, P_2 \in W \cap l$, $P_1 \neq P_2$ and $l \in A$, in two ways. Since $\alpha + \beta + \gamma = \theta_3 - |W|$ and $|A| \leq \alpha$, we have $|A| \leq \theta_3 - |W| - \beta - \gamma$ and thus this equation implies

$$\sum_{\ell \in A} |\ell \cap W| \leq \frac{1}{2}|W|(|W|-3) + \theta_3 - \beta - \gamma.$$

Using this and $|B| \leq \beta$ in (6.16), we find

$$\begin{aligned}L = |W|q^3\left(\theta_2 - \frac{1}{2}(|W|-3)\right) &\leq 12(|W|-q)q^4 + (\theta_3 - \gamma)q^3\\ &\quad + \gamma|W|q + \sum_{(P,H)\in B}(|H \cap W| - q)q^2.\end{aligned} \tag{6.17}$$

Now, we first show that the coefficient of γ in this inequality is negative, so that we may omit the term in γ therein. Since $|W| \leq q^2 + 19q$, we have $L \geq \frac{1}{3}|W|q^5$ for $q \geq 52$. Furthermore, Lemma 6.3.19 shows $|H \cap W| \leq 10q$ for all $(P,H) \in B$ and, since $|B| + \gamma \leq \beta + \gamma \leq \theta_3 - |W|$, we find that

$$\gamma|W|q + \sum_{(P,H)\in B}(|H \cap W| - q)q^2 \leq \gamma(q^3 + 19q^2) + 9q^3|B| \leq (\theta_3 - |W|)9q^3.$$

Using this as well as $|W| \leq q^2 + 19q$ and $L \geq \frac{1}{3}|W|q^5$ in Equation (6.17), we have that

$$\begin{aligned}&\frac{1}{3}|W|q^5 \leq 12(q^2 + 18q)q^4 + \theta_3 q^3 + (\theta_3 - |W|)9q^3\\ \Leftrightarrow\quad &|W|\left(\frac{q^2}{3} + 9\right) \leq 12(q^2 + 18q)q + 10\,\theta_3\\ \Rightarrow\quad &|W| \leq 66q + 678.\end{aligned}$$

Hence, the coefficient $|W|q - q^3$ of γ in (6.17) is negative for $q \geq 76$ and therefore the term in γ can be omitted in the inequality. Doing that, replacing $|W|$ by $q + \delta$ and simplifying we find that

$$(q+\delta)q\left(q^2 + \frac{1}{2}(q - \delta + 5)\right) \leq 12\delta q^2 + \theta_3 q + \sum_{(P,H)\in B}(|H \cap W| - q). \tag{6.18}$$

If π is a plane of S, then the number of $(P, H) \in B$, with $H \cap S = \pi$, is at most $\theta_2 - |\pi \cap W|$. Also, if π_1 and π_2 are distinct planes of S, then

$$|\pi_1 \cap W| + |\pi_2 \cap W| \leq |W| + |\pi_1 \cap \pi_2 \cap W| \leq 2q + 1 + \delta. \tag{6.19}$$

We claim that

$$\sum_{(P,H)\in B}(|H \cap W| - q) \leq \frac{1}{2}(\theta_3 - q - \delta)(\delta + 1). \tag{6.20}$$

Since $|B| \leq \theta_3 - q - \delta$, this is clear if $|H \cap W| - q \leq \frac{1}{2}(\delta + 1)$ for all $H \in B$. Hence, we may assume that there exists a flag $(P_0, H_0) \in B$ with $x = |H_0 \cap W| - q \geq \frac{1}{2}(\delta + 1)$. From $q + x = |H_0 \cap W| \leq |W| = q + \delta$, we find $x \leq \delta$. If $(P, H) \in B$, with $H \cap S = H_0 \cap S$, then $P \in H_0 \cap S$ and $P \notin W$ and hence there are at most $\theta_2 - q - x = q^2 + 1 - x$ such points. If $(P, H) \in B$, with $H \cap S \neq H_0 \cap S$, then (6.19) implies $|H \cap W| - q \leq \delta + 1 - x$.
Now, if $|B| \geq 2(q^2+1-x)$, then for at most half of the elements of B, it holds that $|H \cap W| - q = x$, while for the other elements of B, we have that $|H \cap W| - q \leq \delta + 1 - x$. Hence, the average value of $|H \cap W| - q$ taken over all $(P, H) \in B$ is less than $\frac{1}{2}(x + (\delta + 1 - x)) = \frac{1}{2}(\delta + 1)$ and then (6.20) follows from $|B| \leq \theta_3 - q - \delta$.
If, on the other hand, $|B| \leq 2(q^2 + 1 - x)$, then $|B| \leq 2q^2$ and since $|H \cap W| - q \leq x$ for all $(P, H) \in B$ we find, using $q > 160 \cdot 36^5$ and $\delta = |W| - q \leq q^2 + 18q$ from Lemma 6.3.18, that

$$\sum_{(P,H)\in B}(|H \cap W| - q) \leq 2q^2 x \leq 2q^2\delta \leq \frac{1}{2}(\theta_3 - q - \delta)\delta \leq \frac{1}{2}(\theta_3 - q - \delta)(\delta + 1).$$

We have handled all cases and thus (6.20) is verified. Now, we may use the bound (6.20) in Equation (6.18) to find

$$(q+\delta)q\left(q^2 + \frac{1}{2}(q - \delta + 5)\right) \leq 12\delta q^2 + \theta_3 q + \frac{1}{2}(\theta_3 - q - \delta)(\delta + 1),$$

which is equivalent to

$$\begin{aligned} & \frac{1}{2}\delta q(q^2 - 25q - \delta + 5) + \frac{1}{2}\delta^2 \leq (q^2 - q + 1)q + \frac{1}{2} \\ \Leftrightarrow \quad & \delta^2(q-1) - \delta q(q^2 - 25q + 5) + 2q(q^2 - q + 1) + 1 \geq 0. \end{aligned} \tag{6.21}$$

For $q \geq 73$, this inequality is false for $\delta = 3$ and $\delta = \frac{1}{2}q^2$. Hence, this inequality is false for all values of δ between 3 and $\frac{1}{2}q^2$. Using $\delta = |W| - q < \frac{1}{2}q^2$, this implies $\delta < 3$ and thus $\delta \leq 2$, that is, it remains to show $\delta \notin \{1, 2\}$.

First consider $\delta = 2$. Then Equation (6.18) shows that

$$\frac{1}{2}(3q^3 - 45q^2 + 4q) \leq \sum_{(P,H)\in B}(|H \cap W| - q).$$

Since $|B| \leq c_0 = \theta_3 - q - \delta$ and since $|H \cap W| \leq |W| = q + 2$ for all $(P, H) \in B$, this implies $|H \cap W| > q + 1$ and thus $|H \cap W| = q + 2 = |W|$ for at least $\frac{1}{2}(q^3 - 47q^2 + 4q + 2)$ elements $(P, H) \in B$. Note that $|H \cap W| = q + 2$ implies $W \subseteq H$, that is, $W \subseteq H \cap S$. Therefore, W spans a plane σ of S. However, $(P, H) \in B$ with $W \subseteq H$ implies $P \in H \cap S = \sigma$ and this may happen at most $\theta_2 - |W| = q^2 - 1$ times, a contradiction for $q \geq 49$.

Now, suppose that $\delta = 1$. Then Equation (6.18) shows that

$$\frac{1}{2}(q^3 - 21q^2 + 2q) \leq \sum_{(P,H) \in B}(|H \cap W| - q)$$

and, since $|H \cap W| \leq |W| = q + 1$ for all $(P, H) \in B$, this implies that there are $\frac{1}{2}(q^3 - 21q^2 + 2q)$ elements $(P, H) \in B$ with $|H \cap W| > q$ and thus $|H \cap W| = |W| = q + 1$. Now, if W spans a plane σ, then we have seen above that there are at most $\theta_2 - |W| = q^2$ elements $(P, H) \in B$ with $W \leq H$, a contradiction for $q \geq 24$. Therefore, we may assume that W spans a line ℓ, only. Hence, finally, there exists only one EKR set F in $\mathfrak{F} \setminus C_0$. Now, the special parts of the EKR sets of C_0 do not contain any flag (h, π) with $\pi \cap S = \ell$ and therefore these $q^2\theta_2$ flags must lie in F. This implies that F may not be a subset of a solid-based EKR set, nor may it be a subset of a point based EKR set with point outside of S. Hence, we have $|F| \leq e_1$.

Now, reconsider the set M of all $|W|q^3\theta_2 = (q + 1)q^3\theta_2$ flags (h, τ) such that $h \cap S$ is a point of W. Each point $P \in S \setminus W$ is the base point of exactly one EKR set of C_0 and we let $S(P)$ be its special part. Then M is a subset of the union of F and the sets $S(P)$ with $P \in S \setminus W$. The $q^2(q+1)$ points of $S \setminus W$ are distributed in the $q + 1$ planes of S through ℓ. Consider such a plane π and let

- γ_π be the number of points $P \in \pi \setminus \ell$ for which $S(P)$ is based on a line that meets S only in P,
- let α_π be the number of points $P \in \pi \setminus \ell$ for which $S(P)$ is based on a line that is contained in S, and
- let β_π be the number of pairs $(P, H) \in B$ with $P \in \pi$.

Then there are at most $\gamma_\pi(q + 1)q + \alpha_\pi q^3$ flags in M that lie in $S(P)$ for some point $P \in \pi \setminus \ell$ such that $S(P)$ is based on a line. Now, consider the β_π pairs $(P, H) \in B$ with $P \in \pi$. The special part $S(P)$ of every such pair contains $|H \cap \ell|q^2$ pairs of M. If $\ell \not\subseteq H$, then this is q^2 and otherwise it is $q^2(q + 1)$. For distinct $(P_1, H_1), (P_2, H_2) \in B$ with $P_1, P_2 \in \pi$ and $\pi \subseteq H_1 = H_2$, the q^2 flags $(g, \tau) \in M$ for which $\tau \cap S = P_1P_2$ (and hence $g \cap S = P_1P_2 \cap g$) lie in both $S(P_1, H_1)$ and $S(P_2, H_2)$, so that the number of flags of M that lie in $S(P_2)$ but not in $S(P_1)$ is at most q^3. Since there are q solids through π different from S, these arguments show that the union of the special parts $S(P)$ for the β_π points is at most $q \cdot (q + 1)q^2 + (\beta_\pi - q)q^3 = (\beta_\pi + 1)q^3$. Therefore, since $\alpha_\pi + \beta_\pi + \gamma_\pi$ equals the number q^2 of points of $\pi \setminus \ell$, we have that the union of the special parts $S(P)$ for all points $P \in \pi \setminus \ell$ contains at most

$$\gamma_\pi(q + 1)q + \alpha_\pi q^3 + (\beta_\pi + 1)q^3 \leq (\gamma_\pi + \alpha_\pi + \beta_\pi)q^3 + q^3 = (q^2 + 1)q^3.$$

Since there are $q + 1$ planes of S through ℓ, it follows that

$$(q + 1)q^3\theta_2 \leq |F| + (q + 1)(q^2 + 1)q^3$$

which shows that $|F| \geq (q + 1)q^4$. This is a contradiction to $|F| \leq e_1$ for $q \geq 5$. ■

The previous lemma concludes the proof of Theorem 6.3.8.

Proof of Theorem 6.3.1: Consider a coloring of the Kneser graph $qK_{5;\{2,3\}}$, $q > 160 \cdot 36^5$, with $t \leq \theta_3 - q$ color classes $C_1, \dots, C_t$. Define $C_i = \emptyset$ for $t < i \leq \theta_3 - q$. Each set C_i is an EKR set of line-plane flags of $\mathrm{PG}(4, q)$. If $|C_i| > e_1$, then let $\bar{C}_i$ be a maximal EKR set containing C_i; it follows from Theorem 6.3.3 and the appendix below that $|\bar{C}_i| = e_0$ and $\bar{C}_i$ is one of the sets defined in Example 6.3.2. For each i, we now define a set F_i. For each i, with $|C_i| \leq e_1$, define $F_i = C_i$. Now consider an index i with $|C_i| > e_1$. If there exists an index $j < i$ with $|C_j| > e_1$ and such that $\bar{C}_i$ and $\bar{C}_j$ have the same generic part, then let F_i be the special part of $\bar{C}_i$ (this implies $|F_i| = q^2\theta_2 < e_1$), and otherwise put $F_i = \bar{C}_i$. Let J be the set of indices i with $|F_i| = e_0$. Consider the multiset $\mathfrak{F} = \{F_i \mid 1 \leq i \leq \theta_3 - q\}$. Then each F_i is an EKR set and the union of the F_i is the set of all line-plane flags.

Case 1. For at least $\frac{1}{2}|J|$ indices $i \in J$, the generic part of F_i is based on a point. Then $\mathfrak{F}$ satisfies the hypotheses of Theorem 6.3.8. The conclusion of this theorem implies that $J = \{1, 2, \dots, \theta_3 - q\}$, that the generic part of all F_i is based on a point, and that the base points are $\theta_3 - q$ distinct points of a solid. This implies that $t = \theta_3 - q$, that $|C_i| > e_1$ and $F_i = \bar{C}_i$ for all i. Note that $F_i = \bar{C}_i$ might not be uniquely determined by C_i, however its base point is. This follows from the fact that two maximal EKR sets based on distinct points (are easily seen to) have less than e_1 elements in common and, hence, C_i can not be contained in both. This proves Theorem 6.3.1 in this case.

Case 2. For less than $\frac{1}{2}|J|$, indices $i \in J$ the generic part of F_i is based on a point. Then for more than $\frac{1}{2}|J|$ indices i, the generic part is based on a solid and we can apply the first case in the dual space. This proves Theorem 6.3.1 in this case.

6.3.4 Appendix

In [11], the authors investigate EKR sets of line-plane flags in $\mathrm{PG}(4, q)$. We adapt their notation in this appendix and suppose that $q \geq 3$. In the proof of their classification result, they consider EKR sets $\mathcal{C}$ of line-plane flags in $\mathrm{PG}(4, q)$ which are not contained in one of the sets given in Example 6.3.2. For this, the authors distinguish several cases for the structure of such a set $\mathcal{C}$, depending on the number of *red lines.*

1. If there are θ_3 red lines, then the EKR set $\mathcal{C}$ must be one of the sets in Example 6.3.2, see Case F in [11, Section 4.1].
2. If there are θ_2 red lines through a point in a solid, then $|\mathcal{C}| \leq \theta_2^2 + q^2(q^2-1) + 2q^2(q+1)^2 < e_1$, see Case E in [11, Section 4.1].
3. If there are θ_2 red lines in a plane A_0, then there the authors do not provide an upper bound, but only show that in this case, the sets cannot be contained in a set of Example 6.3.2, see Case D in [11, Section 4.1]. In order to derive Result 6.3.4, we first have to provide an upper bound for that case, too, and we shall do so below.

   We are in the situation that there is one red plane A_0 and all of its lines are red as well. If there are more than $q + 1$ red planes, then the arguments in the second paragraph of [11, Section D] show that the number of elements in the EKR set is at most $\theta_2^2 + q^2(q+1)^2 + q^4 + q^3$, which is smaller than e_1. So here we consider the case that there are at most q red planes apart from A_0.

   Note first that if A is a yellow plane, then $A \cap A_0$ is a line (so $\langle A_0, A \rangle$ is a solid) and A has a unique point $p(A)$ which lies in A_0 and such that a flag (L, A) is in $\mathcal{C}$ if and only if $p(A) \in L$.

The following holds and will be used several times below: if A_1 and A_2 are yellow planes, then

$$p(A_1) \in A_2 \text{ or } p(A_2) \in A_1 \text{ or } \langle A_0, A_1\rangle = \langle A_0, A_2\rangle. \tag{6.22}$$

Now there are two possibilities.

- Suppose that for any two yellow planes A_1 and A_2 with $p(A_1) = p(A_2)$ we have $A_0 \cap A_1 = A_0 \cap A_2$. Then each point $P = p(B)$, with B a yellow plane, corresponds to a unique line $l_B = B \cap A_0$. If there is a line $l_B \subset A_0$ such that l_B is contained in more than q yellow planes, different from A_0, then for every other yellow plane C with $p(C) \neq p(B)$, it holds that $p(C) \in l_B$ or $p(B) \in l_C$, see (6.22). Hence, there are at most $(2q + 1)(q^2 + q)$ yellow planes. If there is no line $l \subset A_0$ contained in more than q yellow planes, then there are at most $q\theta_2$ yellow planes.
- Suppose that there is a point P and two yellow planes A_1 and A_2 with $A_0 \cap A_1 \neq A_0 \cap A_2$ and $p(A_1) = p(A_2) = P$. Then each yellow plane A must satisfy $P \in A$ or $A \subseteq \langle A_0, A_1\rangle$ or $A \subseteq \langle A_0, A_2\rangle$. The number of yellow planes is thus at most $2(q^3 + q^2 + q) + (q + 1)(q^2 - q)$. Note that equality can occur only when the solids $\langle A_0, A_1\rangle$ and $\langle A_0, A_2\rangle$ are distinct.

In any case, the number of yellow planes is at most $y = 3q^3 + 2q^2 + q$. If A_0 is the only red plane, it follows that $|\mathcal{C}| \leq \theta_2^2 + yq \leq 4q^4 + 4q^3 + 4q^2 + 2q + 1$. If A_0 is not the only red plane and there are q other red planes A, then we treat these as the yellow planes above by choosing for $p(A)$ any point of $A \cap A_0$. Then the bound for $|\mathcal{C}|$ is almost the same except that we have to add $q \cdot q^2$, namely q^2 more flags for each of the q red planes. Hence, $|\mathcal{C}| \leq e_1$.

4. If there are at most $q + 1$ red lines, then we use the proofs of Lemmas 4.1, 4.2 and 4.3 in [11] to find that $|\mathcal{C}| < 4q^4 + 9q^3 + 4q^2 + q + 1$.

Hence, we find that the weakest of these upper bounds is the number $e_1 = 4q^4 + 9q^3 + 4q^2 + q + 1$ and it is given in the general case of the proof of [11, Lemma 4.3].

6.4 The chromatic number of the Kneser graph $qK_{2d+1;\{d,d+1\}}$, $d \geq 3$

In this section, we give an overview of the methods and results proven in [48]. The details and proofs appeared in the PhD thesis of dr. Daniel Werner [112]. The results in this part are joint work with prof. Klaus Metsch and dr. Daniel Werner.

In this section, we investigate the Kneser graph whose vertices are flags in $\mathrm{PG}(2d, q)$, such that each flag contains a projective $(d - 1)$-space π and a projective d-space τ, with $\pi \subseteq \tau$.

For this generalized chromatic number problem, we again used the strategy mentioned in Section 6.1. For this, we assume that we have constructed a coloring of size the chromatic number χ and we used a stability result (and conjecture) on the cocliques. The coclique number as well as structural information on large cocliques of $qK_{2d+1,\{d,d+1\}}$ has been given for $d = 2$ in [11] and for $d = 3$ in [94]. We used the results in [11] in the previous section, to show that $\chi(qK_{5,\{2,3\}}) = q^3 + q^2 + 1$ for $q > 160 \cdot 36^5$. The first aim in this project was to determine the chromatic number of $qK_{7,\{3,4\}}$ for large q using the results of [94]. However, our approach in this project was able to deal with the general case of the graphs $qK_{2d+1,\{d,d+1\}}$, for all $d \geq 3$.

Recall that the set $\mathcal{F}(P)$ is a point-pencil of flags of $\mathrm{PG}(2d,q)$ of type $\{d,d+1\}$. Dually, for every hyperplane H in $\mathrm{PG}(2d,q)$, we denote by $F(H)$ the set of all flags of type $\{d,d+1\}$ whose d-space is contained in H and call this set a *dual point-pencil.* Note that point-pencils and dual point-pencils are cocliques of cardinality $\approx q^{d^2-d-1}$ but they are not maximal cocliques. For $d=2$, every maximal coclique containing a point-pencil or a dual point-pencil has cardinality $\theta_2(\theta_3+q^2)$. For $d\geq 3$ there are different maximal cocliques, and they do not all have the same size. However, the structure of the large maximal cocliques can still be described quite precisely.

Example 6.4.1 (EKR sets).

1. *For a point P and a set $\mathcal{U}$ of d-dimensional subspaces through P, such that for all $\tau,\tau'\in\mathcal{U}$ we have $\dim(\tau\cap\tau')\geq 1$, we define*

$$\mathcal{F}(P,\mathcal{U})=\{(\pi,\tau)\in V(\Gamma)\mid P\in\pi \text{ or } \tau\in\mathcal{U}\}.$$

We again call $\{(\pi,\tau)\in\mathcal{F}(P,\mathcal{U})\mid P\in\pi\}$ the generic part *and $\{(\pi,\tau)\in\mathcal{F}(P,\mathcal{U})\mid P\notin\pi\}$ the* special part *of $\mathcal{F}(P,\mathcal{U})$. We also say that $\mathcal{F}(P,\mathcal{U})$ is* based on the point P *and call P the* base point *of $\mathcal{F}(P,\mathcal{U})$.*

2. *Dually, for a hyperplane H and a set $\mathcal{E}$ of subspaces of dimension $d-1$ in H with pairwise non-empty intersection, we define*

$$\mathcal{F}(H,\mathcal{E})=\{(\pi,\tau)\in V(\Gamma)\mid \tau\subseteq H \text{ or } \pi\in\mathcal{E}\}.$$

We call $\{(\pi,\tau)\in\mathcal{F}(H,\mathcal{E})\mid\tau\subseteq H\}$ the generic part *and $\{(\pi,\tau)\in\mathcal{F}(H,\mathcal{E})\mid\tau\not\subseteq H\}$ the* special part *of $\mathcal{F}(P,\mathcal{U})$. We also say that $\mathcal{F}(H,\mathcal{E})$ is* based on *the hyperplane H.*

We continue with some examples of colorings.

Example 6.4.2 (coloring of $qK_{2d+1,\{d,d+1\}}$). *Let $U\subseteq\mathrm{PG}(2d,q)$ be a subspace of dimension $d+1$, consider a set W of q points of U and let L be the set of lines of U that meet W. Furthermore, suppose there exists an injective map ν from L to the point set $\{P\in U|P\notin W\}$, such that $\nu(l)\in l$ for all $l\in L$. Let S_l be the set of all d-spaces through the line l. Then*

$$\{\mathcal{F}(\nu(l),S_l)\mid l\in L\}\cup\{\mathcal{F}(P,\emptyset)\mid P\in U\setminus(\nu(L)\cup W)\}$$

is a set of cocliques of $qK_{2d+1,\{d,d+1\}}$ whose union contains all vertices of $qK_{2d+1,\{d,d+1\}}$.

Remark 6.4.3. (a) Since there are $\theta_{d+1}-q$ cocliques in the given coverings, we find

$$\chi(qK_{2d+1,\{d,d+1\}})\leq\theta_{d+1}-q.$$

(b) There are different possibilities for (W,ν) satisfying the required condition in Example 6.4.2. We describe an explicit example. Let $P_0,\dots,P_q$ be the points of a line $\ell\subseteq U$ and set $W=\{P_1,\dots,P_q\}$. For each plane π of U through ℓ, fix a numbering $h_P(\pi)$, $P\in W$, of the lines different from ℓ of π, containing P_0. Define ν by $\nu(\ell)=P_0$ and $\nu(l)=l\cap h_{l\cap\ell}(\langle\ell,l\rangle)$ for $l\in L\setminus\{\ell\}$. This map ν has the property that $U=\nu(L)\cup W$.

It is also possible to construct maps ν satisfying $U\neq\nu(L)\cup W$, for example for odd $q\geq 5$, when W consists of q points of a conic in a plane of U, but we omit the details.

(c) We can find different coverings in cocliques by replacing all cocliques of the coverings described in Example 6.4.2 by their dual structure.

Recall that our strategy uses a stability result on the cocliques in the Kneser graph. Hence, we make the following conjecture.

Conjecture 6.4.4. *For every integer $d \geq 2$, there is an integer $\rho(d)$ such that every maximal coclique of the Kneser graph $qK_{2d+1,\{d,d+1\}}$ contains a point-pencil, a dual point-pencil, or has at most $\rho(d) \cdot q^{d^2+d-2}$ elements.*

This conjecture is true for $d = 2$, which was implicitly proven in [11], see Section 6.3.4, and it is true for $d = 3$, as is shown in [94].

Our main result is the following.

Theorem 6.4.5. *If Conjecture 6.4.4 is true for some integer $d \geq 3$, then*

$$\chi(qK_{2d+1,\{d,d+1\}}) = \frac{q^{d+2}-1}{q-1} - q$$

for sufficiently large q, depending on d and $\rho(d)$. Moreover, if $\mathfrak{F}$ is a family of this many maximal cocliques that cover the vertex set, then — up to duality — there exists a $(d+1)$-dimensional subspace U in $\mathrm{PG}(2d, q)$ and an injective map μ from $\mathfrak{F}$ to the set of points of U such that $\mathcal{F}(\mu(C)) \subseteq C$ for all $C \in \mathfrak{F}$.

Since the conjecture is true for $d = 3$, we find the following corollary.

Corollary 6.4.6. *For $q > 3 \cdot 7^{15} \cdot 2^{56}$, we have $\chi(qK_{7,\{3,4\}}) = q^4 + q^3 + q^2 + 1$.*

Part II

Cameron-Liebler sets

7 Introduction

> *The main application of Pure Mathematics is to make you happy.*
>
> —Hendrik Lenstra

In the first part of the thesis, we investigated intersection problems. In this part, we continue with the research on Cameron-Liebler sets in different contexts. It will become clear that results on intersection problems can be applied.

7.1 Definition

In [28], Cameron and Liebler introduced specific line classes in $\mathrm{PG}(3,q)$ when investigating the orbits of the subgroups of the collineation group of $\mathrm{PG}(3,q)$. It is well known, by Block's Lemma [76, Section 1.6], that a collineation group of a finite projective space $\mathrm{PG}(n,q)$ has at least as many orbits on lines as on points. Cameron and Liebler tried to determine which collineation groups have equally many point and line orbits. From Lemma 1.8.3, we know that these point and line orbits form a tactical decomposition. More specifically, a symmetrical tactical decomposition, since the number of point and line classes is the same.

We continue with some trivial examples of subgroups of $G = \mathrm{P\Gamma L}(4,q)$, with equally many orbits on the lines and points of $\mathrm{PG}(3,q)$.

Example 7.1.1. *Consider a point P and a plane π in $\mathrm{PG}(3,q)$, with $P \notin \pi$.*

1. *$Stab_G(P)$ has two orbits on the points; namely P and $\mathrm{PG}(3,q) \setminus P$, and has two orbits on the lines, namely the lines containing P and the lines not containing P.*
2. *$Stab_G(\pi)$ has two orbits on the points; namely the points in π and the points not in π, and has two orbits on the lines, namely the lines contained in π and the lines not contained in π.*
3. *$Stab_G(\{P,\pi\})$, with $P \notin \pi$, has three orbits on the points; the point P, the points in π, the points in $\mathrm{PG}(3,q) \setminus (\{P\} \cup \pi)$, and has three orbits on the lines; the lines through P, the lines in π, the lines not in π and not through P.*

Cameron and Liebler found that the line orbits of the subgroups with equally many orbits on lines and points, fulfill the following (equivalent) combinatorial and algebraic properties.

Result 7.1.2 ([28, Proposition 3.1]). *Let $\mathcal{L}$ be a set of lines in $\mathrm{PG}(3,q)$, with characteristic vector χ and let A be the point-line incidence matrix of $\mathrm{PG}(3,q)$. Then the following properties are equivalent.*

1. $\chi \in \mathrm{im}(A^T)$,
2. $\chi \in \ker(A)^\perp$,
3. *for every regulus $\mathcal{R}$, we have that $|\mathcal{L} \cap \mathcal{R}| = |\mathcal{L} \cap \mathcal{R}'|$, with $\mathcal{R}'$ the opposite regulus of $\mathcal{R}$,*
4. *there is a number x such that $|\mathcal{L} \cap S| = x$ for every spread S,*

5. *there is a number x such that $|\mathcal{L} \cap S| = x$ for every Desarguesian spread S.*

A set of lines which satisfies one of these properties (and so all of them) was first called a *special line class* by Cameron and Liebler, and was later called a *Cameron-Liebler set of lines* by other researchers. The number x in the result above, is called the *parameter* of the Cameron-Liebler line set.

Hence, the line orbits of a collineation group of $\mathrm{PG}(3, q)$ which has the property that it has the same number of orbits on the points as on the lines, are Cameron-Liebler line sets, see [28].

We will see later that the converse is not true, see Example 8.3.2.4, and Remark 8.3.3. The original aim was to classify the Cameron-Liebler sets, in order to find information on the collineation groups with the 'orbit'-property. Up to now, the Cameron-Liebler line sets in $\mathrm{PG}(3, q)$ are not yet fully classified. On the other hand, the original group theoretic question, in $\mathrm{PG}(n, q)$, is solved by Cameron, Bamberg and Penttila [27, 3].

Theorem 7.1.3. *A subgroup G of $\mathrm{P\Gamma L}(n, q)$, having equally many orbits on points and lines*

1. *stabilizes a hyperplane π and acts line-transitively on it, or (dually)*
2. *fixes a point P and acts line-transitively on the quotient space, or*
3. *is line-transitive. In this case, there are three possibilities.*
    - *G contains $\mathrm{PSL}(n + 1, q)$,*
    - *$G = A_7 \leq \mathrm{PGL}(4, 2)$,*
    - *G is the normalizer in $\mathrm{PGL}(5, 2)$ of a Singer cyclic group of $\mathrm{PG}(4, 2)$.*

The link between the group theoretical question and Cameron-Liebler sets, can be generalized to other contexts. The lemma below follows from the ideas in Block's Lemma [10], and was given in [110, Lemma 3.3.11].

Lemma 7.1.4. *Let G be a group acting on two finite sets X and X' with orbits $O_1, O_2, \ldots, O_m$ in X and orbits $O'_1, O'_2, \ldots, O'_{m'}$ in X'. Suppose $R \subseteq X \times X'$ is a G-invariant relation with corresponding $(|X| \times |X'|)$-matrix A, defined over $\mathbb{R}$.*

1. *The images $A^T \chi_{O_i}$ are linear combinations of the vectors $\chi_{O'_j}$.*
2. *If A has full row rank, then $m \leq m'$, and if $m = m'$, then all characteristic vectors $\chi_{O'_j}$ are linear combinations of the vectors $A^T \chi_{O_i}$, and so, $\chi_{O'_j} \in \mathrm{im}(A^T)$.*

Remark 7.1.5. The set of points and lines in $\mathrm{PG}(3, q)$ forms a 2-design, and hence, its incidence matrix A has full row rank, see Result 1.1.5. So, the above lemma states that if the number of orbits on the lines equals the number of orbits on the points, then for each line orbit $\chi_{O'}$ it follows that $\chi_{O'} \in \mathrm{im}(A^T)$. From Theorem 7.1.2(1), we know that this last property, $\chi \in \mathrm{im}(A^T)$, defines Cameron-Liebler line sets in $\mathrm{PG}(3, q)$.

Note that Lemma 7.1.4 gives a way to define and investigate Cameron-Liebler sets in other settings.

Penttila further investigated the Cameron-Liebler line sets in $\mathrm{PG}(3, q)$, and found more equivalent definitions for them [99]. After a large number of results regarding these Cameron-Liebler sets of lines in the projective space $\mathrm{PG}(3, q)$, Cameron-Liebler sets of k-spaces in $\mathrm{PG}(2k + 1, q)$ [104], and Cameron-Liebler line sets in $\mathrm{PG}(n, q)$ [51] were defined. Drudge generalized the concept of Cameron-Liebler line sets in $\mathrm{PG}(3, q)$ to Cameron-Liebler line sets in $\mathrm{PG}(n, q)$. These line sets can also be defined by many equivalent definitions, see Definition 7.1.8.

Definition 7.1.6. A *switching k-set* in $\mathrm{PG}(n,q)$ is a partial k-spread $\mathcal{R}$ for which there exists a partial k-spread $\mathcal{R}'$ such that $\mathcal{R}\cap\mathcal{R}'=\emptyset$, and $\cup_{P\in\mathcal{R}}P=\cup_{P\in\mathcal{R}'}P$, in other words, $\mathcal{R}$ and $\mathcal{R}'$ have no common members and cover the same set of points in $\mathrm{PG}(n,q)$. We say that $\mathcal{R}$ and $\mathcal{R}'$ form a *pair of conjugate switching k-sets.*

Theorem 7.1.7 ([51, Theorem 3.2]). *Let A be the point-line incidence matrix of $\mathrm{PG}(n,q)$. Let $\mathcal{L}$ be a set of lines in $\mathrm{PG}(n,q)$, $n\geq 3$, with characteristic vector χ, and x so that $|\mathcal{L}|=x\theta_{n-1}$. Then the following properties are equivalent.*

1. $\chi\in\mathrm{im}(A^T)$,
2. $\chi\in\ker(A)^{\perp}$,
3. *for every pair of conjugate switching 1-sets $\mathcal{R}$ and $\mathcal{R}'$, we have that $|\mathcal{L}\cap\mathcal{R}|=|\mathcal{L}\cap\mathcal{R}'|$,*
4. *for every line ℓ, the number of lines of $\mathcal{L}$ disjoint from ℓ is $(x-\chi(\ell))q^2\theta_{n-3}$,*
5. *for every line ℓ, the number of lines of $\mathcal{L}$, different from ℓ, that intersect ℓ is $x(q+1)+\chi(\ell)(q^2\theta_{n-3}-1)$,*
6. *for every point P and k-space π, with $P\in\pi$, it holds that*

$$|star(P)\cap\mathcal{L}|+\frac{\theta_{n-2}}{\theta_{k-1}\theta_{k-2}}|line(\pi)\cap\mathcal{L}|=x+\frac{\theta_{n-2}}{\theta_{k-2}}|pencil(P,\pi)\cap\mathcal{L}|.$$

In addition, if n is odd, then the following conditions are also equivalent.

7. $|\mathcal{L}\cap\mathcal{S}|=x$ *for every line spread $\mathcal{S}$ in $\mathrm{PG}(n,q)$,*
8. $|\mathcal{L}\cap\mathcal{S}|=x$ *for every Desarguesian line spread $\mathcal{S}$ in $\mathrm{PG}(n,q)$.*

If $n=3$, then the above conditions are also equivalent to:

9. *for every pair of disjoint lines ℓ_1 and ℓ_2, there are $x+q(\chi(\ell_1)+\chi(\ell_2))$ lines meeting both.*

Definition 7.1.8. A set $\mathcal{L}$ of lines in $\mathrm{PG}(n,q)$ that fulfills one of the statements in Theorem 7.1.7 (and consequently all of them) is called a *Cameron-Liebler set of lines* in $\mathrm{PG}(n,q)$ with parameter x.

Remark 7.1.9. Cameron-Liebler line sets in $\mathrm{PG}(n,q)$ correspond to tight sets of type 1, in the Grassmann graph $J_q(n+1,2)$, see Definition 1.7.7. Recall that in this graph, the vertices are the lines in $\mathrm{PG}(n,q)$ and two vertices are adjacent if the corresponding lines meet in a point. From statement 5. in Theorem 7.1.7, it follows that a Cameron-Liebler line set $\mathcal{L}$ in $\mathrm{PG}(n,q)$ is an intriguing set with values $y=x(q+1)$ and $y'=x(q+1)+q^2\theta_{n-3}-1$. By investigating the eigenvalues of the Grassmann graph, it follows that $y'-y=\lambda$, with λ the largest eigenvalue of the graph. Hence, $\mathcal{L}$ is also a tight set of type 1.

The examination of Cameron-Liebler sets in projective spaces started the motivation for defining and investigating Cameron-Liebler sets of generators in polar spaces [36], Cameron-Liebler classes in finite sets [39] and Cameron-Liebler sets of k-spaces in $\mathrm{PG}(n,q)$ and in $\mathrm{AG}(n,q)$. Furthermore, Cameron-Liebler sets can be introduced for any distance-regular graph. This has been done in the past under various names: Boolean degree 1 functions [59], completely regular codes of strength 0 and covering radius 1 [95], ... We refer to the introduction of [59] for an overview. Note that the definitions do not always coincide, e.g. for polar spaces, see Chapter 10 and [35, 36].

We have seen some algebraic, combinatorial and geometrical definitions for Cameron-Liebler sets. The main question, independent of the context where Cameron-Liebler sets are investigated, is always the same: for which values of the parameter x do there exist Cameron-Liebler sets and which examples correspond to a given parameter x? We will partially solve this question for Cameron-Liebler sets of k-spaces in $\mathrm{PG}(n, q)$, see Chapter 8, and for Cameron-Liebler sets of generators in polar spaces, see Chapter 10. In Chapter 9, we mention the definition and several results of Cameron-Liebler sets in $\mathrm{AG}(n, q)$.

8

Cameron-Liebler sets of k-spaces in $\mathrm{PG}(n, q)$

> *La géométrie est l'art du raisonnement correct à partir de figures mal dessinées.*
>
> —Henri Poincaré

In this chapter, we investigate Cameron-Liebler sets of k-spaces in $\mathrm{PG}(n, q)$. The results in this chapter are joint work with prof. Aart Blokhuis and dr. Maarten De Boeck, and appeared in [16]. In Section 8.1, we list several equivalent definitions for these Cameron-Liebler sets, by generalizing the known results about Cameron-Liebler line sets in $\mathrm{PG}(n, q)$, see [51], and Cameron-Liebler sets of k-spaces in $\mathrm{PG}(2k + 1, q)$, see [104]. In Section 8.2, we make the link between these Cameron-Liebler sets and Boolean degree one functions. Several properties of Cameron-Liebler sets are given in Section 8.3. In the last section, we use these properties to prove the following classification result: there is no Cameron-Liebler set of k-spaces in $\mathrm{PG}(n, q)$, $n > 3k + 1$, with parameter x such that $2 \leq x \leq \frac{1}{\sqrt[8]{2}} q^{\frac{n}{2}-\frac{k^2}{4}-\frac{3k}{4}-\frac{3}{2}} (q-1)^{\frac{k^2}{4}-\frac{k}{4}+\frac{1}{2}} \sqrt{q^2+q+1}$, (see Theorem 8.4.13).

8.1 The characterization theorem

Let Δ_k be the collection of k-spaces in $\mathrm{PG}(n, q)$, for $0 \leq k \leq n$, and let A be the incidence matrix of the points and the k-spaces of $\mathrm{PG}(n, q)$: the rows of A are indexed by the points and the columns by the k-spaces.

In this chapter, we will use the Grassmann scheme $J_q(n + 1, k + 1)$, see Example 1.9.5. Recall that there is an orthogonal decomposition $V_0 \perp V_1 \perp \cdots \perp V_{k+1}$ of $\mathbb{R}^{\Delta_k}$ in maximal common eigenspaces of $A_0, A_1, \ldots, A_{k+1}$, see Result 1.9.3. Consider the distance *one* relation $\mathcal{R}_1$ and let V_j be the eigenspace corresponding to the eigenvalue P_{j1} from Lemma 8.1.2. Using this (classical) ordering, we find the following lemma.

Lemma 8.1.1. *For the Grassmann scheme $J_q(n + 1, k + 1)$, we have that $\mathrm{im}(A^T) = V_0 \perp V_1$ and $V_0 = \langle \boldsymbol{j} \rangle$.*

Hence, this is well defined, with respect to the assumption on V_0 and V_1 in Section 1.9. In the following lemmas and theorems, we denote the disjointness matrix A_{k+1} by K since the corresponding graph is the q-Kneser graph $qK_{n+1:k+1}$. Kneser graphs also appeared in Chapter 6, where we investigated the chromatic number of some generalized Kneser graphs.

Before we start with proving some equivalent definitions for a Cameron-Liebler set of k-spaces, we give some lemmas and definitions that we will need in the characterization Theorem 8.1.6.

Lemma 8.1.2 ([52]). *Consider the Grassmann scheme $J_q(n + 1, k + 1)$. The eigenvalue P_{ji} of the distance-i relation for V_j is given by:*

$$P_{ji} = \sum_{s=\max\{0,j-i\}}^{\min\{j,k+1-i\}} (-1)^{j+s} \begin{bmatrix} j \\ s \end{bmatrix} \begin{bmatrix} n-k+s-j \\ n-k-i \end{bmatrix} \begin{bmatrix} k+1-s \\ i \end{bmatrix} q^{i(i+s-j)+\binom{j-s}{2}}.$$

Lemma 8.1.3. *If $P_{1i}, i \geq 1$, is the eigenvalue of A_i corresponding to V_j, then $j = 1$.*

Proof. We need to prove that $P_{1i} \neq P_{ji}$ for q a prime power and $j > 1$. We will first introduce $\phi_i(j) = \max\{a \mid q^a | P_{ji}\}$, which is the exponent of q in the factorization of P_{ji}. Note that $\begin{bmatrix} a \\ b \end{bmatrix}$ equals 1 modulo q and note that it is sufficient to show that $\phi_i(j), j > 1$, is different from $\phi_i(1)$ for all i. By Lemma 8.1.2, we see that

$$\phi_i(j) = \min\left\{ i(i+s-j) + \binom{j-s}{2} \mid \max\{0, j-i\} \leq s \leq \min\{j, k+1-i\} \right\}$$

unless there are two or more terms with a power of q with minimal exponent as factor and that have zero as their sum. If s is the integer in $\{\max\{0, j-i\}, \dots, \min\{j, k+1-i\}\}$ closest to $j - i - \frac{1}{2}$, then $f_{ij}(s) = i(i+s-j) + \binom{j-s}{2}$ is minimal.

- If $j \leq i$, we see that $f_{ij}(s)$ is minimal for $s = 0$. Then we find $\phi_i(j) = \frac{1}{2}j^2 - (i + \frac{1}{2})j + i^2$. We see that for a fixed i, $\phi_i(k-1) > \phi_i(k), k \leq i$. Note that the minimal value for $f_{ij}(s)$ is reached for only one s.
- If $j \geq i$, we see that $f_{ij}(s)$ is minimal for $s = j - i$. Then we find $\phi_i(j) = \binom{i}{2}$. Again we note that the minimal value for $f_{ij}(s)$ is reached for only one s.

We can conclude the following inequality for a given $i \geq 1$:

$$\phi_i(1) > \phi_i(2) > \cdots > \phi_i(i) = \phi_i(i+1) = \cdots = \phi_i(k+1)\,.$$

This implies the statement for $i \neq 1$.

For $i = 1$, we have that

$$\begin{aligned}
&P_{11} = P_{j1} \\
&\Leftrightarrow -\begin{bmatrix} k+1 \\ 1 \end{bmatrix} + \begin{bmatrix} n-k \\ 1 \end{bmatrix}\begin{bmatrix} k \\ 1 \end{bmatrix} q = -\begin{bmatrix} j \\ 1 \end{bmatrix}\begin{bmatrix} k-j+2 \\ 1 \end{bmatrix} + \begin{bmatrix} n-k \\ 1 \end{bmatrix}\begin{bmatrix} k+1-j \\ 1 \end{bmatrix} q \\
&\Leftrightarrow -(q^{k+1}-1)(q-1) + (q^{n-k}-1)(q^k-1)q \\
&\qquad\quad = -(q^j-1)(q^{k-j+2}-1) + (q^{n-k}-1)(q^{k-j+1}-1)q \\
&\Leftrightarrow q^{n+1} + q = q^{n-j+2} + q^j \\
&\Leftrightarrow j = 1 \vee j = n+1.
\end{aligned}$$

So, we can see that they are different if $j \neq n+1$. This is always true since $j \in \{1, \dots, k+1\}$ and $k < n$. ■

Note that for $j \geq 1$, it was already known that $|P_{ji}| \leq |P_{1i}|$. This result was shown in [22, Proposition 5.4(ii)].

Lemma 8.1.4. *Let π be a k-dimensional subspace in $\mathrm{PG}(n,q)$ with χ_π the characteristic vector of the set $\{\pi\}$. If $\mathcal{Z}$ is the set of all k-spaces in $\mathrm{PG}(n,q)$ disjoint from π with characteristic vector $\chi_{\mathcal{Z}}$, then*

$$\chi_{\mathcal{Z}} - q^{k^2+k}\begin{bmatrix} n-k-1 \\ k \end{bmatrix}\left(\begin{bmatrix} n \\ k \end{bmatrix}^{-1}\boldsymbol{j} - \chi_\pi\right) \in \ker(A).$$

Proof. Let v_π be the incidence vector of π with its positions corresponding to the points of $\mathrm{PG}(n,q)$. Note that $A\chi_\pi = v_\pi$. We have that $A\chi_{\mathcal{Z}} = q^{k^2+k}\begin{bmatrix} n-k-1 \\ k \end{bmatrix}(\boldsymbol{j} - v_\pi)$ since $\mathcal{Z}$ is the set of all k-spaces disjoint from π and every point not in π is contained in $q^{k^2+k}\begin{bmatrix} n-k-1 \\ k \end{bmatrix}$ k-spaces skew to π (see Lemma 1.10.1). The lemma now follows from

$$\begin{aligned} &\chi_{\mathcal{Z}} - q^{k^2+k}\begin{bmatrix} n-k-1 \\ k \end{bmatrix}\left(\begin{bmatrix} n \\ k \end{bmatrix}^{-1}\boldsymbol{j} - \chi_\pi\right) \in \ker(A) \\ \Leftrightarrow \quad & A\chi_{\mathcal{Z}} = q^{k^2+k}\begin{bmatrix} n-k-1 \\ k \end{bmatrix}\left(\begin{bmatrix} n \\ k \end{bmatrix}^{-1}A\boldsymbol{j} - A\chi_\pi\right) . \end{aligned}$$

■

Definition 8.1.5. An *m-cover* $\mathcal{S}_m$ of k-spaces in $\mathrm{PG}(n,q)$ is a (multi-)set of k-spaces such that every point in $\mathrm{PG}(n,q)$ is contained in precisely m elements of $\mathcal{S}_m$.

Note that the 1-covers of k-spaces in $\mathrm{PG}(n,q)$ are the k-spreads in $\mathrm{PG}(n,q)$. Hence, 1-covers only exist for $(k+1)|(n+1)$. For $m>1$, there are some examples of m-covers known with $(k+1)\nmid(n+1)$. A trivial example is the set of all lines in $\mathrm{PG}(4,q)$. It is easy to see that this is a θ_3-cover of lines, with $k+1=2\nmid 5=n+1$.

We want to make a combination of a generalization of Theorem 3.2 in [51] and Theorem 3.7 in [104] to give several equivalent definitions for a Cameron-Liebler set of k-spaces in $\mathrm{PG}(n,q)$.

Theorem 8.1.6. *Let $\mathcal{L}$ be a non-empty set of k-spaces in* $\mathrm{PG}(n,q), n\geq 2k+1$, *with characteristic vector χ, and x so that $|\mathcal{L}|=x\begin{bmatrix} n \\ k \end{bmatrix}$. Then the following properties are equivalent.*

1. $\chi\in\mathrm{im}(A^T)$.
2. $\chi\in\ker(A)^\perp$.
3. *For every k-space π, the number of elements of $\mathcal{L}$ disjoint from π is* $(x-\chi(\pi))\begin{bmatrix} n-k-1 \\ k \end{bmatrix}q^{k^2+k}$.
4. *The vector* $\chi - x\frac{q^{k+1}-1}{q^{n+1}-1}\boldsymbol{j}$ *is a vector in* V_1.
5. $\chi\in V_0\perp V_1$.
6. *For a given $i\in\{1,\dots,k+1\}$ and any k-space π, the number of elements of $\mathcal{L}$, meeting π in a $(k-i)$-space is given by:*

$$\begin{cases} \left((x-1)\frac{q^{k+1}-1}{q^{k-i+1}-1}+q^i\frac{q^{n-k}-1}{q^i-1}\right)q^{i(i-1)}\begin{bmatrix} n-k-1 \\ i-1 \end{bmatrix}\begin{bmatrix} k \\ i \end{bmatrix} & \text{if } \pi\in\mathcal{L} \\ x\begin{bmatrix} n-k-1 \\ i-1 \end{bmatrix}\begin{bmatrix} k+1 \\ i \end{bmatrix}q^{i(i-1)} & \text{if } \pi\notin\mathcal{L} \end{cases} .$$

7. *for every pair of conjugate switching k-sets $\mathcal{R}$ and $\mathcal{R}'$, we have that* $|\mathcal{L}\cap\mathcal{R}|=|\mathcal{L}\cap\mathcal{R}'|$.

If $\mathrm{PG}(n,q)$ *admits a k-spread, then the following properties are equivalent to the previous ones.*

8. $|\mathcal{L}\cap\mathcal{S}|=x$ *for every k-spread $\mathcal{S}$ in* $\mathrm{PG}(n,q)$.
9. $|\mathcal{L}\cap\mathcal{S}|=x$ *for every Desarguesian k-spread $\mathcal{S}$ in* $\mathrm{PG}(n,q)$.
10. *For every $m\in\mathbb{N}$, it holds that $|\mathcal{L}\cap\mathcal{S}_m|=mx$ for every m-cover of k-spaces $\mathcal{S}_m$ in* $\mathrm{PG}(n,q)$.

Proof. We first prove that properties $1,2,3,4,5,6$ are equivalent by proving the following implications:

- $1\Leftrightarrow 2$: This follows since $\mathrm{im}(B^T)=\ker(B)^\perp$ for every matrix B.

- 2 ⇒ 3: We assume that $\chi \in \ker(A)^{\perp}$. Let $\pi \in \Delta_k$ and $\mathcal{Z}$ the set of k-spaces disjoint from π. By Lemma 8.1.4, we know that
$$\chi_{\mathcal{Z}} - q^{k^2+k}\begin{bmatrix} n-k-1 \\ k \end{bmatrix}\left(\begin{bmatrix} n \\ k \end{bmatrix}^{-1}\boldsymbol{j} - \chi_{\pi}\right) \in \ker(A).$$
Since $\chi \in \ker(A)^{\perp}$, this implies
$$\begin{aligned} &\chi_{\mathcal{Z}}\cdot\chi - q^{k^2+k}\begin{bmatrix} n-k-1 \\ k \end{bmatrix}\left(\begin{bmatrix} n \\ k \end{bmatrix}^{-1}\boldsymbol{j}\cdot\chi - \chi_{\pi}\cdot\chi\right) = 0 \\ \Leftrightarrow\ &|\mathcal{Z}\cap\mathcal{L}| - q^{k^2+k}\begin{bmatrix} n-k-1 \\ k \end{bmatrix}\left(\begin{bmatrix} n \\ k \end{bmatrix}^{-1}|\mathcal{L}| - \chi(\pi)\right) = 0 \\ \Leftrightarrow\ &|\mathcal{Z}\cap\mathcal{L}| = (x-\chi(\pi))q^{k^2+k}\begin{bmatrix} n-k-1 \\ k \end{bmatrix}\ . \end{aligned}$$
Hence, this last equality proves that the number of elements of $\mathcal{L}$, disjoint from π is $(x-\chi(\pi))q^{k^2+k}\begin{bmatrix} n-k-1 \\ k \end{bmatrix}$.

- 3 ⇒ 4: By expressing property 3 in vector notation, we find that $K\chi = (x\boldsymbol{j}-\chi)\begin{bmatrix} n-k-1 \\ k \end{bmatrix}q^{k^2+k}$ and, since by Lemma 1.10.1, we have $K\boldsymbol{j} = q^{(k+1)^2}\begin{bmatrix} n-k \\ k+1 \end{bmatrix}$, we see that $v = \chi - x\frac{q^{k+1}-1}{q^{n+1}-1}\boldsymbol{j}$ is an eigenvector of K:
$$\begin{aligned} Kv &= K\left(\chi - x\frac{q^{k+1}-1}{q^{n+1}-1}\boldsymbol{j}\right) \\ &= (x\boldsymbol{j}-\chi)\begin{bmatrix} n-k-1 \\ k \end{bmatrix}q^{k^2+k} - x\frac{q^{k+1}-1}{q^{n+1}-1}q^{(k+1)^2}\begin{bmatrix} n-k \\ k+1 \end{bmatrix}\boldsymbol{j} \\ &= \begin{bmatrix} n-k-1 \\ k \end{bmatrix}q^{k^2+k}\left(x\boldsymbol{j} - \chi - x\frac{q^{n+1}-q^{k+1}}{q^{n+1}-1}\boldsymbol{j}\right) \\ &= -\begin{bmatrix} n-k-1 \\ k \end{bmatrix}q^{k^2+k}\left(\chi - x\frac{q^{k+1}-1}{q^{n+1}-1}\boldsymbol{j}\right) \\ &= P_{1,k+1}v\ . \end{aligned}$$
By Lemma 8.1.3, for $i = k+1$, we know that $v \in V_1$.

- 4 ⇒ 5: This follows since $V_0 = \langle \boldsymbol{j} \rangle$, see Lemma 8.1.1.

- 5 ⇒ 1: This follows again from Lemma 8.1.1.

- 4 ⇒ 6: Denote $\chi - x\frac{q^{k+1}-1}{q^{n+1}-1}\boldsymbol{j}$ by v. The matrix A_i corresponds to the relation $\mathcal{R}_i$. This implies that $(A_i\chi)_{\pi}$ gives the number of k-spaces in $\mathcal{L}$ that intersect π in a $(k-i)$-space.

$$\begin{aligned} A_i\chi &= A_iv + x\frac{q^{k+1}-1}{q^{n+1}-1}A_i\boldsymbol{j} = P_{1i}v + x\frac{q^{k+1}-1}{q^{n+1}-1}P_{0i}\boldsymbol{j} \\ &= \left(-\begin{bmatrix} n-k-1 \\ i-1 \end{bmatrix}\begin{bmatrix} k+1 \\ i \end{bmatrix}q^{i(i-1)} + \begin{bmatrix} n-k \\ i \end{bmatrix}\begin{bmatrix} k \\ i \end{bmatrix}q^{i^2}\right)\left(\chi - x\frac{q^{k+1}-1}{q^{n+1}-1}\boldsymbol{j}\right) \\ &\quad + x\frac{q^{k+1}-1}{q^{n+1}-1}\begin{bmatrix} n-k \\ i \end{bmatrix}\begin{bmatrix} k+1 \\ i \end{bmatrix}q^{i^2}\boldsymbol{j} \\ &= \left(\begin{bmatrix} n-k \\ i \end{bmatrix}\begin{bmatrix} k \\ i \end{bmatrix}q^{i^2} - \begin{bmatrix} k+1 \\ i \end{bmatrix}\begin{bmatrix} n-k-1 \\ i-1 \end{bmatrix}q^{i(i-1)}\right)\chi \\ &\quad + x\frac{q^{k+1}-1}{q^{n+1}-1}q^{i(i-1)}\left(\begin{bmatrix} n-k-1 \\ i-1 \end{bmatrix}\begin{bmatrix} k+1 \\ i \end{bmatrix} - \begin{bmatrix} n-k \\ i \end{bmatrix}\begin{bmatrix} k \\ i \end{bmatrix}q^{i} + \begin{bmatrix} n-k \\ i \end{bmatrix}\begin{bmatrix} k+1 \\ i \end{bmatrix}q^{i}\right)\boldsymbol{j} \end{aligned}$$

$$\begin{aligned}
&= \left(\begin{bmatrix} n-k \\ i \end{bmatrix}\begin{bmatrix} k \\ i \end{bmatrix} q^{i^2} - \begin{bmatrix} k+1 \\ i \end{bmatrix}\begin{bmatrix} n-k-1 \\ i-1 \end{bmatrix} q^{i(i-1)}\right)\chi \\
&\quad + x\frac{q^{k+1}-1}{q^{n+1}-1} q^{i(i-1)} \begin{bmatrix} n-k-1 \\ i-1 \end{bmatrix}\begin{bmatrix} k \\ i \end{bmatrix}\left(\frac{q^{k+1}-1}{q^{k-i+1}-1} - \frac{q^{n-k}-1}{q^{i}-1}q^{i}\left(1-\frac{q^{k+1}-1}{q^{k-i+1}-1}\right)\right)\boldsymbol{j} \\
&= \left(\begin{bmatrix} n-k \\ i \end{bmatrix}\begin{bmatrix} k \\ i \end{bmatrix} q^{i^2} - \begin{bmatrix} k+1 \\ i \end{bmatrix}\begin{bmatrix} n-k-1 \\ i-1 \end{bmatrix} q^{i(i-1)}\right)\chi + x\begin{bmatrix} n-k-1 \\ i-1 \end{bmatrix}\begin{bmatrix} k+1 \\ i \end{bmatrix} q^{i(i-1)}\boldsymbol{j}
\end{aligned}$$

This proves the implication for every $i \in \{1, \dots, k+1\}$.

- $6 \Rightarrow 4$: We follow the approach of [104, Lemma 3.5] where we look for an eigenvalue of A_i and we define $\beta_i = x\begin{bmatrix} k+1 \\ i \end{bmatrix}\begin{bmatrix} n-k-1 \\ i-1 \end{bmatrix}q^{i(i-1)}$.
  From property 6, we know that
  $$\begin{aligned}
  A_i\chi &= x\begin{bmatrix} k+1 \\ i \end{bmatrix}\begin{bmatrix} n-k-1 \\ i-1 \end{bmatrix} q^{i(i-1)}(\boldsymbol{j}-\chi) \\
  &\quad + \left((x-1)\frac{q^{k+1}-1}{q^{k-i+1}-1} + q^{i}\frac{q^{n-k}-1}{q^{i}-1}\right) q^{i(i-1)} \begin{bmatrix} n-k-1 \\ i-1 \end{bmatrix}\begin{bmatrix} k \\ i \end{bmatrix}\chi \\
  &= \left(\begin{bmatrix} n-k \\ i \end{bmatrix}\begin{bmatrix} k \\ i \end{bmatrix} q^{i^2} - \begin{bmatrix} k+1 \\ i \end{bmatrix}\begin{bmatrix} n-k-1 \\ i-1 \end{bmatrix} q^{i(i-1)}\right)\chi + x\begin{bmatrix} n-k-1 \\ i-1 \end{bmatrix}\begin{bmatrix} k+1 \\ i \end{bmatrix} q^{i(i-1)}\boldsymbol{j} \\
  &= P_{1i}\chi + \beta_i\boldsymbol{j}\ .
  \end{aligned}$$
  Then we can see that $v_i = \chi + \frac{\beta_i}{P_{1i}-P_{0i}}\boldsymbol{j}$ is an eigenvector for A_i with eigenvalue P_{1i}:
  $$\begin{aligned}
  A_i\left(\chi + \frac{\beta_i}{P_{1i}-P_{0i}}\boldsymbol{j}\right) &= P_{1i}\chi + \beta_i\boldsymbol{j} + \frac{\beta_i}{P_{1i}-P_{0i}}P_{0i}\boldsymbol{j} \\
  &= P_{1i}\left(\chi + \frac{\beta_i}{P_{1i}-P_{0i}}\boldsymbol{j}\right).
  \end{aligned}$$
  By Lemma 8.1.3, we know that $\chi + \frac{\beta_i}{P_{1i}-P_{0i}}\boldsymbol{j} = \chi - x\frac{q^{k+1}-1}{q^{n+1}-1}\boldsymbol{j} \in V_1$.

We show that properties 8, 9 and 10 are equivalent with the previous properties if $\mathrm{PG}(n,q)$ admits a k-spread.

- $2 \Rightarrow 10$: Let $\mathcal{S}_m$ be an m-cover of k-spaces in $\mathrm{PG}(n,q)$ and let χ_m be its characteristic vector. Note that $\chi_m(i) = j$ if the i'th element is contained j times in $\mathcal{S}_m$. Hence, χ_m doesn't have to be a $\{0,1\}-$vector. Then we know that $\chi_m - m\begin{bmatrix} n \\ k \end{bmatrix}^{-1}\boldsymbol{j} \in \ker(A)$. Since $\chi \in \ker(A)^{\perp}$, we have that
  $$0 = \chi \cdot \left(\chi_m - m\begin{bmatrix} n \\ k \end{bmatrix}^{-1}\boldsymbol{j}\right) = |\mathcal{L} \cap \mathcal{S}_m| - m|\mathcal{L}|\begin{bmatrix} n \\ k \end{bmatrix}^{-1},$$
  so $|\mathcal{L} \cap \mathcal{S}_m| = m|\mathcal{L}|\begin{bmatrix} n \\ k \end{bmatrix}^{-1} = mx$.
- $10 \Rightarrow 8$: A k-spread in $\mathrm{PG}(n,q)$ is an m-cover for $m = 1$.
- $8 \Rightarrow 9$: Trivial.
- $9 \Rightarrow 3$: Suppose that $\mathrm{PG}(n,q)$ contains k-spreads, hence also Desarguesian k-spreads. We know that the group $\mathrm{PGL}(n+1,q)$ acts transitively on the pairs of pairwise disjoint k-spaces. Let n_i, for $i = 1, 2$, be the number of Desarguesian k-spreads that contain i fixed pairwise disjoint k-spaces. This number only depends on i, and not on the chosen k-spaces, by the above transitivity property.
  Let π be a fixed k-space. The number of pairs $(\pi', \mathcal{S})$, with $\mathcal{S}$ a Desarguesian k-spread that

contains π and π' is equal to $q^{(k+1)^2}\begin{bmatrix}n-k\\k+1\end{bmatrix}\cdot n_2 = n_1\cdot\left(\frac{q^{n+1}-1}{q^{k+1}-1}-1\right)$, so $\frac{n_1}{n_2}=q^{k(k+1)}\begin{bmatrix}n-k-1\\k\end{bmatrix}$. By counting the number of pairs $(\pi',\mathcal{S})$, with $\pi'\in\mathcal{L}$ and $\mathcal{S}$ a Desarguesian k-spread that contains π and π', we find that the number of k-spaces in $\mathcal{L}$, disjoint from a fixed k-space π, is given by $(x-\chi(\pi))\frac{n_1}{n_2}=(x-\chi(\pi))q^{k(k+1)}\begin{bmatrix}n-k-1\\k\end{bmatrix}$.

To end this proof, we show that property 7 is equivalent with the other properties.

- $2\Rightarrow 7$: Let $\chi_{\mathcal{R}}$ and $\chi_{\mathcal{R}'}$ be the characteristic vectors of the pair of conjugate switching k-sets $\mathcal{R}$ and $\mathcal{R}'$ respectively. As $\mathcal{R}$ and $\mathcal{R}'$ cover the same set of points, we find: $\chi_{\mathcal{R}}-\chi_{\mathcal{R}'}\in\ker(A)$. This implies $0=\chi\cdot(\chi_{\mathcal{R}}-\chi_{\mathcal{R}'})=\chi\cdot\chi_{\mathcal{R}}-\chi\cdot\chi_{\mathcal{R}'}$, so that $\chi\cdot\chi_{\mathcal{R}}=|\mathcal{L}\cap\mathcal{R}|=|\mathcal{L}\cap\mathcal{R}'|=\chi\cdot\chi_{\mathcal{R}'}$.

- $7\Rightarrow 1$: We first show that property 7 implies the other properties if $n=2k+1$. For any two k-spreads $\mathcal{S}_1,\mathcal{S}_2$, the sets $\mathcal{S}_1\setminus\mathcal{S}_2$ and $\mathcal{S}_2\setminus\mathcal{S}_1$ form a pair of conjugate switching k-sets. So $|\mathcal{L}\cap(\mathcal{S}_1\setminus\mathcal{S}_2)|=|\mathcal{L}\cap(\mathcal{S}_2\setminus\mathcal{S}_1)|$, which implies that $|\mathcal{L}\cap\mathcal{S}_1|=|\mathcal{L}\cap\mathcal{S}_2|=c$.

  Now we prove that this constant c equals $x=|\mathcal{L}|\begin{bmatrix}2k+1\\k\end{bmatrix}^{-1}$. Let n_i, for $i=0,1$, be the number of k-spreads containing i fixed pairwise disjoint k-spaces. This number only depends on i, and not on the chosen k-spaces. The number of pairs $(\pi,\mathcal{S})$, with $\mathcal{S}$ a k-spread that contains π, is equal to $\begin{bmatrix}2k+2\\k+1\end{bmatrix}\cdot n_1=n_0\cdot\frac{q^{2k+2}-1}{q^{k+1}-1}$, which implies that $\frac{n_0}{n_1}=\begin{bmatrix}2k+1\\k\end{bmatrix}$.

  By counting the number of pairs $(\pi,\mathcal{S})$, with $\mathcal{S}$ a k-spread that contains π, and where $\pi\in\mathcal{L}$, we find, that the number of k-spaces in $\mathcal{L}\cap\mathcal{S}$ equals $|\mathcal{L}|\frac{n_1}{n_0}=|\mathcal{L}|\begin{bmatrix}2k+1\\k\end{bmatrix}^{-1}=x$. This implies property 8, and hence, property 1.

  Now we prove that implication $7\Rightarrow 1$ also holds if $n>2k+1$. Given a subspace τ in $\mathrm{PG}(n,q)$, we will use the notation $A_{|\tau}$ for the submatrix of A, where we only have the rows, corresponding with the points of τ, and the columns corresponding with the k-spaces in τ. We know that the matrix $A_{|\tau}$ has full rank by Result 1.1.5.
  Let Π be a $(2k+1)$-dimensional subspace in $\mathrm{PG}(n,q)$. By property 7, we know that for every two k-spreads $\mathcal{R},\mathcal{R}'$ in Π, we have $|\mathcal{L}\cap\mathcal{R}|=|\mathcal{L}\cap\mathcal{R}'|$ since $\mathcal{R}\setminus\mathcal{R}'$ and $\mathcal{R}'\setminus\mathcal{R}$ are conjugate switching k-sets. This implies that $\chi_{\mathcal{L}|\Pi}\in\operatorname{im}\left(A^T_{|\Pi}\right)$ by the arguments above applied for the $(2k+1)$-space Π. So, there is a linear combination of the rows of $A_{|\Pi}$ equal to $\chi_{\mathcal{L}|\Pi}$. This linear combination is unique since $A_{|\Pi}$ has full row rank. Now we will show that the linear combination of $\chi_{\mathcal{L}}$ is uniquely defined by the vectors $\chi_{\mathcal{L}|\Pi}$, with Π varying over all $(2k+1)$-spaces in $\mathrm{PG}(n,q)$.

  We show, for every two $(2k+1)$-spaces Π,Π', that the coefficients of the row corresponding to a point in $\Pi\cap\Pi'$ in the linear combination of $\chi_{\mathcal{L}|\Pi}$ and in the linear combination of $\chi_{\mathcal{L}|\Pi'}$ are equal.

  Suppose $\chi_{\mathcal{L}|\Pi}=a_1r_1+a_2r_2+\cdots+a_lr_l+a_{l+1}r_{l+1}+\cdots+a_mr_m$ and $\chi_{\mathcal{L}|\Pi'}=b_{l+1}r_{l+1}+\cdots+b_mr_m+b_{m+1}r_{m+1}+\cdots+b_sr_s$, where $r_1,\ldots,r_l,\ldots,r_m$ and $r_{l+1},\ldots,r_m,\ldots,r_s$ are the rows corresponding with the points of Π and Π', respectively. Note that we only look at the columns corresponding with the k-spaces in Π and Π', respectively.

  We now look at the space $\Pi\cap\Pi'$, and at the corresponding columns in A. Recall that $A_{|\Pi\cap\Pi'}$ also has full row rank, so the linear combination that gives $\chi_{\mathcal{L}|(\Pi\cap\Pi')}$ is unique, and equal to the ones corresponding with Π and Π', restricted to $\Pi\cap\Pi'$. This proves that $a_i=b_i$ for $l+1\leq i\leq m$. Here we also used the fact that the entry in A corresponding with a point of $\Pi\setminus\Pi'$ or $\Pi'\setminus\Pi$ and a k-space in $\Pi\cap\Pi'$ is zero.

By using all $(2k+1)$-spaces, we see that $\chi_{\mathcal{L}}$ is uniquely defined, and by construction we have $\chi_{\mathcal{L}} \in \mathrm{im}(A^T)$. Note that we only used that property 7 holds for conjugate switching k-sets inside a $(2k+1)$-dimensional subspace. ■

Definition 8.1.7. A set $\mathcal{L}$ of k-spaces in $\mathrm{PG}(n,q)$ that fulfills one of the statements in Theorem 8.1.6 (and consequently all of them) is called a *Cameron-Liebler set of k-spaces* in $\mathrm{PG}(n,q)$ with parameter $x = |\mathcal{L}|\begin{bmatrix}n\\k\end{bmatrix}^{-1}$.

Similar to Remark 7.1.9, and by using statement 6. in Theorem 8.1.6, it can be seen that the Cameron-Liebler sets of k-spaces in $\mathrm{PG}(n,q)$ correspond to the tight sets of type 1 in the Grassmann graph $J_q(n+1,k+1)$.

From Theorem 8.1.6.8, we know that the parameter of a Cameron-Liebler set of k-spaces in $\mathrm{PG}(n,q)$ is always an integer if $\mathrm{PG}(n,q)$ admits a k-spread, and so, if $k+1$ is a divisor of $n+1$. For $k+1 \nmid n+1$, this is not always the case, while the parameter of Cameron-Liebler line sets in $\mathrm{PG}(3,q)$ and the parameter of Cameron-Liebler sets of generators in polar spaces are always integers (see [36, Theorem 4.8]).

Remark 8.1.8. The link between Cameron-Liebler sets of k-spaces in $\mathrm{PG}(n,q)$, and the original group theoretical question of Cameron and Liebler follows from Lemma 7.1.4. For this, we also use that the set of points and k-spaces in $\mathrm{PG}(n,q)$ forms a 2-design, and so, the incidence matrix A has full row rank, see Result 1.1.5. So, we find that the orbits of a collineation group, with the same number of orbits on the points and k-spaces, are Cameron-Liebler sets. The reverse statement is not true: not every Cameron-Liebler set is an orbit of a collineation group with the 'orbit'-property. An example of such a Cameron-Liebler set is the union of the set of all k-spaces through a point P and the set of all k-spaces in a hyperplane H, with $P \notin H$.

We end this section with showing an extra property of Cameron-Liebler sets of k-spaces in $\mathrm{PG}(n,q)$.

Proposition 8.1.9. *Let $\mathcal{L}$ be a Cameron-Liebler set of k-spaces in $\mathrm{PG}(n,q)$, then we find the following equality for every j-dimensional subspace α and every i-dimensional subspace τ, with $\alpha \subset \tau$ and $j < k < i$:*

$$|[k]_\alpha \cap \mathcal{L}| + \frac{\begin{bmatrix}n-j-1\\k-j\end{bmatrix}(q^{k-j}-1)}{\begin{bmatrix}i\\k\end{bmatrix}(q^{i-k}-1)}|[k]^\tau \cap \mathcal{L}| = \frac{\begin{bmatrix}n-j-1\\k-j\end{bmatrix}}{\begin{bmatrix}i-j-1\\k-j\end{bmatrix}}|[k]^\tau_\alpha \cap \mathcal{L}| + \frac{\begin{bmatrix}n-j-1\\k-j-1\end{bmatrix}}{\begin{bmatrix}n\\k\end{bmatrix}}|\mathcal{L}| \, .$$

Here $[k]_\alpha$, $[k]^\tau$ and $[k]^\tau_\alpha$ denote the set of all k-spaces through α, the set of all k-spaces in τ and the set of all k-spaces in τ through α, respectively.

Proof. Let $\chi_{[\alpha]}$, $\chi_{[\tau]}$ and $\chi_{[\alpha,\tau]}$ be the characteristic vectors of $[k]_\alpha$, $[k]^\tau$ and $[k]^\tau_\alpha$, respectively, and define

$$v = \chi_{[\alpha]} + \frac{\begin{bmatrix}n-j-1\\k-j\end{bmatrix}(q^{k-j}-1)}{\begin{bmatrix}i\\k\end{bmatrix}(q^{i-k}-1)}\chi_{[\tau]} - \frac{\begin{bmatrix}n-j-1\\k-j\end{bmatrix}}{\begin{bmatrix}i-j-1\\k-j\end{bmatrix}}\chi_{[\alpha,\tau]} - \frac{\begin{bmatrix}n-j-1\\k-j-1\end{bmatrix}}{\begin{bmatrix}n\\k\end{bmatrix}}\boldsymbol{j} \, .$$

Since

$$\left(A\chi_{[\alpha]}\right)_P = \begin{cases}\begin{bmatrix}n-j-1\\k-j-1\end{bmatrix} & \text{for } P \notin \alpha\\ \begin{bmatrix}n-j\\k-j\end{bmatrix} & \text{for } P \in \alpha,\end{cases} \qquad \left(A\chi_{[\tau]}\right)_P = \begin{cases}0 & \text{for } P \notin \tau\\ \begin{bmatrix}i\\k\end{bmatrix} & \text{for } P \in \tau,\end{cases}$$

$$\left(A\chi_{[\alpha,\tau]}\right)_P = \begin{cases}0 & \text{for } P \notin \tau\\ \begin{bmatrix}i-j-1\\k-j-1\end{bmatrix} & \text{for } P \in \tau \setminus \alpha\\ \begin{bmatrix}i-j\\k-j\end{bmatrix} & \text{for } P \in \alpha,\end{cases}$$

we can calculate $(Av)_{P'}$ for every point P', and see that $Av = 0$. This implies that $v \in \ker(A)$. Let χ be the characteristic vector of $\mathcal{L}$. By Definition 2 in Theorem 8.1.6, we know that $\chi \in \ker(A)^{\perp}$, so, by calculating $\chi \cdot v$, the lemma follows. ■

For $k = 1$, K. Drudge showed in [51] that the property in Proposition 8.1.9 is not only a necessary, but also a sufficient property for a Cameron-Liebler line set in $\mathrm{PG}(n,q)$. For $k > 1$ we pose it as an open problem to show that this property is also sufficient.

8.2 Boolean degree one functions

Another way to approach Cameron-Liebler sets of k-spaces in $\mathrm{PG}(n,q)$ is by the theory of Boolean degree one functions. Boolean functions are $\{0,1\}$-valued functions on a finite domain Ω. Each Boolean function f on $\Omega = \{\omega_1, \omega_2, \dots, \omega_n\}$ corresponds to an n-dimensional $\{0,1\}$-vector v, such that the i'th element of v is equal to $f(\omega_i)$. Furthermore, f also corresponds to a set $\mathcal{L}_f$, such that $\mathcal{L}_f = \{\omega \in \Omega | f(\omega) = 1\}$.

Boolean functions can be described for several classical association schemes, including the Johnson scheme, Grassmann scheme, and graphs from polar spaces, as well as for some other domains such as permutation groups. In this section, we give the link between these functions and Cameron-Liebler sets. For more information, we refer to [59].

In all settings, we have some form of coordinates: an element in $\{1, 2, \dots, n\}$ in the Johnson graph $J(n,k)$; a point in the Grassmann graph $J_q(n+1, k+1)$, or for most graphs related to polar spaces; and, a transposition (ij) for the graphs derived from permutation groups. For a coordinate x, we denote the characteristic function of x by x^+: $x^+(\pi) = 1$ if the element x is contained in the object π, and $x^+(\pi) = 0$ otherwise. Then, a Boolean degree one function is a $\{0,1\}$-valued function on the vertices that can be written as $f = c + \sum_i c_i x_i^+$.

We will go into more detail for the projective setting. Let Δ_k be the set of all k-spaces in $\mathrm{PG}(n,q)$. A point $P \in \mathrm{PG}(n,q)$ induces a characteristic function P^+ on Δ_k:

$$\forall \pi \in \Delta_k : P^+(\pi) = \begin{cases} 1 & \text{if } P \in \pi \\ 0 & \text{if } P \notin \pi. \end{cases}$$

Note that this function corresponds with the vector $A^T \chi_P$, with χ_P the characteristic vector of the point P, and A the point-k-space incidence matrix.

Definition 8.2.1. A *Boolean degree one function* on the set of k-spaces in $\mathrm{PG}(n,q)$ is a $\{0,1\}$-valued function of the form:

$$f : \Delta_k \to \mathbb{R} : \pi \mapsto c + \sum_{i=1}^{\theta_n} a_i P_i^+(\pi),$$

with $a_i, c \in \mathbb{R}$ and $\{P_i \mid 1 \leq i \leq \theta_n\}$ the set of points in $\mathrm{PG}(n,q)$.

Let $\mathcal{L}_f = \{\pi \in \Delta_k | f(\pi) = 1\}$ be the set, corresponding to the Boolean degree one function f on Δ_k. It is clear that the Boolean function $f = P^+$, with P a point in $\mathrm{PG}(n,q)$, is a Boolean degree one function. Note that the set $\mathcal{L}_f$, with $f = P^+$, is precisely the point-pencil with vertex P. In general, the sets $\mathcal{L}_f$, with f a Boolean degree one function on the set of k-spaces in $\mathrm{PG}(n,q)$, are precisely Cameron-Liebler sets of k-spaces in $\mathrm{PG}(n,q)$. For the proof of this theorem, we refer to [89, Theorem 2.3.2].

Theorem 8.2.2. *Consider the projective space* $\mathrm{PG}(n,q)$*, then a set* $\mathcal{L}$ *is a Cameron-Liebler set of* k*-spaces in* $\mathrm{PG}(n,q)$ *if and only if* $\mathcal{L}=\mathcal{L}_f$ *for some Boolean degree one function* f *on the set of* k*-spaces in* $\mathrm{PG}(n,q)$*.*

8.3 Properties of Cameron-Liebler sets of k-spaces in $\mathrm{PG}(n,q)$

We start with some properties of Cameron-Liebler sets of k-spaces in $\mathrm{PG}(n,q)$ that can easily be proved.

Lemma 8.3.1. *Let* $\mathcal{L}$ *and* $\mathcal{L}'$ *be two Cameron-Liebler sets of* k*-spaces in* $\mathrm{PG}(n,q)$ *with parameters* x *and* x' *respectively, then the following statements are valid.*

1. $0 \leq x \leq \frac{q^{n+1}-1}{q^{k+1}-1}$.
2. *The set of all* k*-spaces in* $\mathrm{PG}(n,q)$ *not in* $\mathcal{L}$ *is a Cameron-Liebler set of* k*-spaces with parameter* $\frac{q^{n+1}-1}{q^{k+1}-1}-x$.
3. *If* $\mathcal{L}\cap\mathcal{L}'=\emptyset$*, then* $\mathcal{L}\cup\mathcal{L}'$ *is a Cameron-Liebler set of* k*-spaces with parameter* $x+x'$.
4. *If* $\mathcal{L}'\subseteq\mathcal{L}$*, then* $\mathcal{L}\setminus\mathcal{L}'$ *is a Cameron-Liebler set of* k*-spaces with parameter* $x-x'$.

We continue with some examples of Cameron-Liebler sets of k-spaces in $\mathrm{PG}(n,q)$. We refer to these examples as the *trivial examples.*

Example 8.3.2. *Trivial examples of Cameron-Liebler sets of* k*-spaces in* $\mathrm{PG}(n,q)$.

1. *The empty set (parameter* 0*).*
2. *The set of all* k*-spaces through a point* P*, so the point-pencil with vertex* P *(parameter* 1*). This follows immediately from the theory of Boolean degree one functions.*
3. *The set of all* k*-spaces in a fixed hyperplane (parameter* $\frac{q^{n-k}-1}{q^{k+1}-1}$*). Note that this parameter is not an integer if* $k+1\nmid n+1$*, or equivalently, if* $\mathrm{PG}(n,q)$ *does not contain a* k*-spread.*
4. *The union of all* k*-spaces through a point* P*, together with the set of* k*-spaces in a fixed hyperplane* H*, with* $P\notin H$ *(parameter* $x=1+\frac{q^{n-k}-1}{q^{k+1}-1}$*).*
5. *The complement of these four examples: these are Cameron-Liebler sets with parameter* $x=\frac{q^{n+1}-1}{q^{k+1}-1}, x=\frac{q^{n+1}-1}{q^{k+1}-1}-1, x=q^{n-k}$ *and* $x=q^{n-k}-1$ *respectively.*

Remark 8.3.3. Example 4. is a Cameron-Liebler set $\mathcal{L}$ in $\mathrm{PG}(n,q)$, but is not an orbit of k-spaces of a symmetrical tactical decomposition in the collineation group. This was proven in [27, 96], and follows from the following observation. If $\mathcal{L}$ would arise from a symmetrical tactical decomposition $\mathcal{T}$, then, since P is the unique point of $\mathrm{PG}(n,q)$, such that through P there pass $\begin{bmatrix} n \\ k \end{bmatrix}$ k-spaces of $\mathcal{L}$, we have that $\{P\}$ must be a point class of $\mathcal{T}$. But, a k-space $\pi\in\mathcal{L}$ contains either one or no points of $\{P\}$, depending on whether $P\in\pi$ or not. Hence, $\mathcal{L}$ cannot be a class of k-spaces of $\mathcal{T}$.

In [93], several properties of Cameron-Liebler sets of k-spaces in $\mathrm{PG}(2k+1,q)$ were given. We will first generalize some of these results to use them in Section 8.4.

Lemma 8.3.4. *Let π and π' be two disjoint k-spaces in $\mathrm{PG}(n,q)$ with $\Sigma = \langle \pi, \pi' \rangle$, let P be a point in $\Sigma \setminus (\pi \cup \pi')$ and let P' be a point not in Σ. Then the number of k-spaces disjoint from π and π' equals $W(q,n,k)$, the number of k-spaces disjoint from π and π' through P equals $W_\Sigma(q,n,k)$ and the number of k-spaces disjoint from π and π' through P' equals $W_{\bar{\Sigma}}(q,n,k)$.*

Here, $W(q,n,k), W_\Sigma(q,n,k), W_{\bar{\Sigma}}(q,n,k)$ are given by:

$$\begin{aligned}
W(q,n,k) &= \sum_{i=-1}^{k} W_i(q,n,k)\\
W_\Sigma(q,n,k) &= \frac{1}{(q^{k+1}-1)^2}\sum_{i=0}^{k} W_i(q,n,k)(q^{i+1}-1)\\
W_{\bar{\Sigma}}(q,n,k) &= \frac{1}{q^{n+1}-q^{2k+2}}\sum_{i=-1}^{k-1} W_i(q,n,k)(q^{k+1}-q^{i+1})\\
W_i(q,n,k) &= \begin{cases} q^{2k^2+k+\frac{3i^2}{2}-\frac{i}{2}-3ik}\left[\begin{smallmatrix} n-2k-1\\ k-i\end{smallmatrix}\right]\left[\begin{smallmatrix} k+1\\ i+1\end{smallmatrix}\right]\prod_{j=0}^{i}(q^{k-j+1}-1) & \mathit{if}\, i \geq 0\\ q^{2(k+1)^2}\left[\begin{smallmatrix} n-2k-1\\ k+1\end{smallmatrix}\right] & \mathit{if}\, i=-1\end{cases}.
\end{aligned}$$

Proof. To count the number of k-spaces π'', that are disjoint from π and π', we first count the number of possible intersections $\pi'' \cap \Sigma$.

We count the number of i-spaces in Σ, disjoint from π and π', by counting $((P_0, P_1, \ldots, P_i), \sigma_i)$ in two ways. Here σ_i is an i-space in Σ, disjoint from π and π', and the points $P_0, P_1, \ldots, P_i$ form a basis of σ_i. For the ordered basis $(P_0, P_1, \ldots, P_i)$ we have $\prod_{j=0}^{i}\frac{q^{2j}(q^{k-j+1}-1)^2}{q-1}$ possibilities since there are $\left[\begin{smallmatrix} 2k+2\\ 1\end{smallmatrix}\right] - 2\left[\begin{smallmatrix} k+j+1\\ 1\end{smallmatrix}\right] + \left[\begin{smallmatrix} 2j\\ 1\end{smallmatrix}\right] = \frac{q^{2j}(q^{k-j+1}-1)^2}{q-1}$ possibilities for P_j if $P_0, P_1, \ldots, P_{j-1}$ are given. By a similar argument, we find that the number of ordered bases $(P_0, P_1, \ldots, P_i)$ for a given σ_i is $\prod_{j=0}^{i}\frac{q^{j}(q^{i-j+1}-1)}{q-1}$. In this way we find that the number of i-spaces in Σ, disjoint from π and π', is given by:

$$\frac{\prod_{j=0}^{i}\frac{q^{2j}(q^{k-j+1}-1)^2}{q-1}}{\prod_{j=0}^{i}\frac{q^{j}(q^{i-j+1}-1)}{q-1}} = \prod_{j=0}^{i}\frac{q^{j}(q^{k-j+1}-1)^2}{q^{i-j+1}-1} = q^{\binom{i+1}{2}}\begin{bmatrix} k+1\\ i+1\end{bmatrix}\prod_{j=0}^{i}(q^{k-j+1}-1).$$

Now we count, for a given i-space σ_i in Σ, the number of k-spaces π'' through σ_i such that $\pi'' \cap \Sigma = \sigma_i$. This equals the number of $(k-i-1)$-spaces in $\mathrm{PG}(n-i-1,q)$, disjoint from a $(2k-i)$-space, and is equal to $q^{(k-i)(2k-i+1)}\left[\begin{smallmatrix} n-2k-1\\ k-i\end{smallmatrix}\right]$ by Lemma 1.10.1. By this lemma, we also see that the number of k-spaces disjoint from Σ is given by $q^{(k+1)(2k+2)}\left[\begin{smallmatrix} n-2k-1\\ k+1\end{smallmatrix}\right]$. This implies that $W_i(q,n,k)$, $-1 \leq i \leq k$, is the number of k-spaces disjoint from π and π', and intersecting Σ in an i-space.

Now we have enough information to count the number of k-spaces disjoint from π and π':

$$W(q,n,k) = \sum_{i=-1}^{k} W_i(q,n,k)\ .$$

We use the same arguments to calculate $W_\Sigma(q,n,k)$ and $W_{\bar{\Sigma}}(q,n,k)$. By double counting (P, π''), with π'' a k-space through $P \in \Sigma$ disjoint from π and π', and double counting (P', π''), with π'' a

k-space through $P' \notin \Sigma$ disjoint from π and π', we find:

$$\left(\begin{bmatrix}2k+2\\1\end{bmatrix} - 2\begin{bmatrix}k+1\\1\end{bmatrix}\right) \cdot W_{\Sigma}(q,n,k) = \sum_{i=0}^{k} W_i(q,n,k) \cdot \begin{bmatrix}i+1\\1\end{bmatrix} \text{ and}$$

$$\left(\begin{bmatrix}n+1\\1\end{bmatrix} - \begin{bmatrix}2k+2\\1\end{bmatrix}\right) \cdot W_{\bar{\Sigma}}(q,n,k) = \sum_{i=-1}^{k-1} W_i(q,n,k) \cdot \left(\begin{bmatrix}k+1\\1\end{bmatrix} - \begin{bmatrix}i+1\\1\end{bmatrix}\right) .$$

This implies:

$$\begin{aligned} W_{\Sigma}(q,n,k) &= \frac{1}{(q^{k+1}-1)^2} \sum_{i=0}^{k} W_i(q,n,k)(q^{i+1}-1) \\ W_{\bar{\Sigma}}(q,n,k) &= \frac{1}{q^{n+1}-q^{2k+2}} \sum_{i=-1}^{k-1} W_i(q,n,k)(q^{k+1}-q^{i+1}) . \end{aligned}$$

■

From now on, we denote $W_i(q,n,k)$, $W_{\Sigma}(q,n,k)$ and $W_{\bar{\Sigma}}(q,n,k)$ by W_i, W_{Σ} and $W_{\bar{\Sigma}}$ if the dimensions n, k and the field size q are clear from the context.

Lemma 8.3.5. *Let $\mathcal{L}$ be a Cameron-Liebler set of k-spaces in $\mathrm{PG}(n,q)$ with parameter x.*

1. *For every $\pi \in \mathcal{L}$, there are s_1 elements of $\mathcal{L}$ meeting π.*

2. *For skew $\pi, \pi' \in \mathcal{L}$ and a k-spread $\mathcal{S}_0$ in $\Sigma = \langle \pi, \pi' \rangle$, there exist exactly d_2 subspaces in $\mathcal{L}$ that are skew to both π and π' and there exist s_2 subspaces in $\mathcal{L}$ that meet both π and π'.*

   Here, d_2, s_1 and s_2 are given by:

   $$\begin{aligned} d_2(q,n,k,x,\mathcal{S}_0) &= (W_{\Sigma} - W_{\bar{\Sigma}})|\mathcal{S}_0 \cap \mathcal{L}| - 2W_{\Sigma} + xW_{\bar{\Sigma}} \\ s_1(q,n,k,x) &= x\begin{bmatrix}n\\k\end{bmatrix} - (x-1)\begin{bmatrix}n-k-1\\k\end{bmatrix}q^{k^2+k} \\ s_2(q,n,k,x,\mathcal{S}_0) &= x\begin{bmatrix}n\\k\end{bmatrix} - 2(x-1)\begin{bmatrix}n-k-1\\k\end{bmatrix}q^{k^2+k} + d_2(q,n,k,x,\mathcal{S}_0) , \end{aligned}$$

   where W_{Σ} and $W_{\bar{\Sigma}}$ are given by Lemma 8.3.4.

3. *Define $d_2'(q,n,k,x) = (x-2)W_{\Sigma}$ and $s_2'(q,n,k,x) = x\begin{bmatrix}n\\k\end{bmatrix} - 2(x-1)\begin{bmatrix}n-k-1\\k\end{bmatrix}q^{k^2+k} + d_2'(q,n,k,x)$. If $n > 3k+1$, then $|\mathcal{S}_0 \cap \mathcal{L}| \leq x$ for every k-spread $\mathcal{S}_0$ in Σ. Moreover we have that $d_2(q,n,k,x,\mathcal{S}_0) \leq d_2'(q,n,k,x)$ and $s_2(q,n,k,x,\mathcal{S}_0) \leq s_2'(q,n,k,x)$.*

Proof.

1. This follows directly from Theorem 8.1.6(3) and $|\mathcal{L}| = x\begin{bmatrix}n\\k\end{bmatrix}$.

2. Let χ_{π} and $\chi_{\pi'}$ be the characteristic vectors of $\{\pi\}$ and $\{\pi'\}$, respectively, and let $\mathcal{Z}$ be the set of all k-spaces in $\mathrm{PG}(n,q)$ disjoint from π and π', and let $\chi_{\mathcal{Z}}$ be its characteristic vector. Furthermore, let v_{π} and $v_{\pi'}$ be the incidence vectors of π and π', respectively, with their positions corresponding to the points of $\mathrm{PG}(n,q)$. Note that $A\chi_{\pi} = v_{\pi}$ and $A\chi_{\pi'} = v_{\pi'}$. By Lemma 8.3.4, we know the numbers W_{Σ} and $W_{\bar{\Sigma}}$ of k-spaces disjoint from π and π', through a point P, if $P \in \Sigma$ and $P \notin \Sigma$ respectively. Let $\mathcal{S}_0$ be a k-spread in Σ and let v_{Σ} be the

incidence vector of Σ (as a point set). We find:

$$\begin{aligned} A\chi_{\mathcal{Z}} &= W_\Sigma(v_\Sigma - v_\pi - v_{\pi'}) + W_{\bar{\Sigma}}(\boldsymbol{j} - v_\Sigma) \\ &= W_\Sigma(A\chi_{\mathcal{S}_0} - A\chi_\pi - A\chi_{\pi'}) + W_{\bar{\Sigma}}\left(\begin{bmatrix} n \\ k \end{bmatrix}^{-1} A\boldsymbol{j} - A\chi_{\mathcal{S}_0}\right) \\ \Leftrightarrow \qquad & \chi_{\mathcal{Z}} - W_\Sigma(\chi_{\mathcal{S}_0} - \chi_\pi - \chi_{\pi'}) - W_{\bar{\Sigma}}\left(\begin{bmatrix} n \\ k \end{bmatrix}^{-1} \boldsymbol{j} - \chi_{\mathcal{S}_0}\right) \in \ker(A). \end{aligned}$$

We know that the characteristic vector χ of $\mathcal{L}$ is included in $\ker(A)^\perp$. This implies:

$$\begin{aligned} & \chi_{\mathcal{Z}} \cdot \chi = W_\Sigma(\chi_{\mathcal{S}_0} \cdot \chi - \chi(\pi) - \chi(\pi')) + W_{\bar{\Sigma}}(x - \chi_{\mathcal{S}_0} \cdot \chi) \\ \Leftrightarrow \qquad & |\mathcal{Z} \cap \mathcal{L}| = W_\Sigma(|\mathcal{S}_0 \cap \mathcal{L}| - 2) + W_{\bar{\Sigma}}(x - |\mathcal{S}_0 \cap \mathcal{L}|) \\ \Leftrightarrow \qquad & |\mathcal{Z} \cap \mathcal{L}| = (W_\Sigma - W_{\bar{\Sigma}})|\mathcal{S}_0 \cap \mathcal{L}| - 2W_\Sigma + xW_{\bar{\Sigma}}\ , \end{aligned}$$

which gives the formula for $d_2(q,n,k,x,\mathcal{S}_0)$. The formula for $s_2(q,n,k,x,\mathcal{S}_0)$ follows from the inclusion-exclusion principle.

3. Suppose Σ is a $(2k+1)$-space in $\mathrm{PG}(n,q)$, and suppose $\mathcal{S}_0$ is a k-spread in Σ such that $|\mathcal{S}_0 \cap \mathcal{L}| > x$. By property 1 in Theorem 8.1.6, we know that the characteristic vector χ of $\mathcal{L}$ can be written as $\sum_{P \in \mathrm{PG}(n,q)} x_P r_P^T$ for some $x_P \in \mathbb{R}$ where r_P is the row of A corresponding to the point P. Let χ_π be the characteristic vector of the set $\{\pi\}$ with π a k-space, then $\chi_\pi \cdot \chi = \sum_{P \in \pi} x_P$ equals 1 if $\pi \in \mathcal{L}$ and 0 if $\pi \notin \mathcal{L}$. As $\chi \cdot \boldsymbol{j} = |\mathcal{L}| = x\begin{bmatrix} n \\ k \end{bmatrix}$, we find that $\sum_{P \in \mathrm{PG}(n,q)} x_P = x$.
   If $|\mathcal{S}_0 \cap \mathcal{L}| > x$, then $\chi \cdot \chi_{S_0} = \sum_{P \in \Sigma} x_P > x$. From these observations, it follows that $\sum_{P \in \mathrm{PG}(n,q) \setminus \Sigma} x_P = \sum_{P \in \mathrm{PG}(n,q)} x_P - \sum_{P \in \Sigma} x_P$ is negative. As $n > 3k+1$, there exists a k-space τ in $\mathrm{PG}(n,q)$, disjoint from Σ, with $\chi_\tau \cdot \chi = \sum_{P \in \tau} x_P$ negative, which gives the contradiction.

   It follows that $|\mathcal{S}_0 \cap \mathcal{L}| \le x$. Since this is true for every k-spread $\mathcal{S}_0$ in every $(2k+1)$-space in $\mathrm{PG}(n,q)$, the statement holds. ■

In the remainder of this chapter, we will use the upper bound $d_2'(q,n,k,x)$ and $s_2'(q,n,k,x)$ instead of $d_2(q,n,k,x,\mathcal{S}_0)$ and $s_2(q,n,k,x,\mathcal{S}_0)$ respectively, since they are independent of the chosen k-spread $\mathcal{S}_0$.

The following lemma is a generalization of Lemma 2.4 in [93].

Lemma 8.3.6. *Let c, n, k be non-negative integers with $n > 3k+1$ and*

$$(c+1)s_1 - \binom{c+1}{2} s_2' > x\begin{bmatrix} n \\ k \end{bmatrix}\ ,$$

then no Cameron-Liebler set of k-spaces in $\mathrm{PG}(n,q)$ with parameter x contains $c+1$ mutually skew k-spaces.

Proof. Assume that $\mathrm{PG}(n,q)$ has a Cameron-Liebler set $\mathcal{L}$ of k-spaces with parameter x that contains $c+1$ mutually disjoint k-spaces $\pi_0, \pi_1, \dots, \pi_c$. Lemma 8.3.5 shows that π_i meets at least $s_1(q,n,k,x) - is_2(q,n,k,x)$ elements of $\mathcal{L}$ that are skew to $\pi_0, \pi_1, \dots, \pi_{i-1}$. This implies that $x\begin{bmatrix} n \\ k \end{bmatrix} = |\mathcal{L}| \ge (c+1)s_1 - \sum_{i=0}^{c} is_2 \ge (c+1)s_1 - \sum_{i=0}^{c} is_2'$ which contradicts the assumption. ■

8.4 Classification results

In this section, we will list some classification results for Cameron-Liebler sets of k-spaces in $\mathrm{PG}(n,q)$. We start with some known classification results for $k=1$. For $n=3$, Cameron and Liebler proved that the sets in Example 8.3.2 are the only examples of Cameron-Liebler line sets with parameter equal to $0,1,2,q^2-1,q^2$ and q^2+1 [28]. They also conjectured that the only Cameron-Liebler line sets in $\mathrm{PG}(3,q)$ are the trivial ones. This conjecture was disproven, and several non-trivial examples of Cameron-Liebler sets are known now. In [26, 30, 31, 51, 57, 58, 63], constructions of non-trivial Cameron-Liebler line sets with parameter $x=\frac{q^2+1}{2}$, $x=\frac{q^2-1}{2}$ and $x=\frac{(q+1)^2}{3}$, were given, and other classification results were discussed in [28, 62, 64, 65, 91, 92, 103].

The strongest classification results are given in [64, 92], the latter of which proves the following result.

Theorem 8.4.1 ([92, Theorem 1.1]). *There are no Cameron-Liebler line sets in* $\mathrm{PG}(3,q)$ *with parameter*

$$2<x\leq q\sqrt[3]{\frac{q}{2}}-\frac{2}{3}q.$$

In [64], Metsch and Gavrilyuk found a strong classification result, using a modular equality. This result rules out roughly at least one half of all possible parameters x.

Theorem 8.4.2 ([64, Theorem 1.1]). *Let $\mathcal{L}$ be a Cameron-Liebler line set with parameter x in* $\mathrm{PG}(3,q)$*. Then for every plane π and every point P of* $\mathrm{PG}(3,q)$ *it holds that*

$$\binom{x}{2}+n(n-x)\equiv 0 \mod (q+1).$$

Here, n is the number of lines of $\mathcal{L}$ in the plane π, and through the point P respectively.

Regarding the Cameron-Liebler sets of k-spaces in $\mathrm{PG}(2k+1,q)$, the most important classification result is described in [93].

Theorem 8.4.3 ([93]). *There does not exist a Cameron-Liebler set of planes in* $\mathrm{PG}(5,q)$ *with parameter x satisfying* $2<x<\frac{q}{3}$*. For $k\geq 3$, there exists a positive integer q_0 with the following properties. If q is a prime power satisfying $q\geq q_0$ and $k<q\log q-q-1$, then* $\mathrm{PG}(2k+1,q)$ *has no Cameron-Liebler sets of k-spaces with parameter x for* $2<x<\frac{q}{5}$.

Moreover, for $q\in\{2,3,4,5\}$, a complete classification is known for Cameron-Liebler sets of k-spaces in $\mathrm{PG}(n,q)$, see [59]. There, the authors show that the only Cameron-Liebler sets in this context are the trivial Cameron-Liebler sets, independent of the values of k and n.

Now we continue with several new classification results for Cameron-Liebler sets of k-spaces in $\mathrm{PG}(n,q)$. In the following lemma, we start with the classification for the parameters $x\in\,]0,1[\,\cup\,]1,2[$.

Lemma 8.4.4. *There are no Cameron-Liebler sets of k-spaces in* $\mathrm{PG}(n,q)$ *with parameter $x\in\,]0,1[$ and if $n\geq 3k+2$, then there are no Cameron-Liebler sets of k-spaces with parameter $x\in\,]1,2[$.*

Proof. Suppose there is a Cameron-Liebler set $\mathcal{L}$ of k-spaces with parameter $x\in\,]0,1[$. Then $\mathcal{L}$ is not the empty set, so suppose $\pi\in\mathcal{L}$. By property 3 in Theorem 8.1.6, we find that the number of k-spaces in $\mathcal{L}$ disjoint from π is negative, which gives the contradiction.

Suppose there is a Cameron-Liebler set $\mathcal{L}$ of k-spaces with parameter $x\in\,]1,2[$ in $\mathrm{PG}(n,q)$, $n\geq 3k+2$. By property 3 in Theorem 8.1.6, we know that there are at least two disjoint k-spaces

$\pi, \pi' \in \mathcal{L}$. By Lemma 8.3.5$(2, 3)$, we know that there are $d_2 \leq d_2'$ elements of $\mathcal{L}$ disjoint from π and π'. Since d_2' is negative for $x \in]1, 2[$, we find a contradiction. ■

We continue with a classification result for Cameron-Liebler sets of k-spaces with parameter $x = 1$, where we will use the Erdős-Ko-Rado result from Theorem 2.0.3, for $t = 0$.

Theorem 8.4.5. *Let $\mathcal{L}$ be a Cameron-Liebler set of k-spaces with parameter $x = 1$ in $\mathrm{PG}(n, q)$, $n \geq 2k + 1$. Then $\mathcal{L}$ is a point-pencil or $n = 2k + 1$ and $\mathcal{L}$ is the set of all k-spaces in a hyperplane of $\mathrm{PG}(2k + 1, q)$.*

Proof. The theorem follows immediately from Theorem 2.0.3 since, by Theorem 8.1.6(3), we know that $\mathcal{L}$ is a family of pairwise intersecting k-spaces of size $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$. ■

We continue this section by showing that there are no Cameron-Liebler sets of k-spaces in $\mathrm{PG}(n, q)$, $n \geq 3k + 2$, with parameter $2 \leq x \leq \frac{1}{\sqrt[8]{2}} q^{\frac{n}{2} - \frac{k^2}{4} - \frac{3k}{4} - \frac{3}{2}} (q - 1)^{\frac{k^2}{4} - \frac{k}{4} + \frac{1}{2}} \sqrt{q^2 + q + 1}$. For this classification result, we will use the Hilton-Milner theorem for projective spaces, see Theorem 2.0.5.

To simplify the notations, we denote $q^{\frac{n}{2} - \frac{k^2}{4} - \frac{3k}{4} - \frac{3}{2}} (q - 1)^{\frac{k^2}{4} - \frac{k}{4} + \frac{1}{2}} \sqrt{q^2 + q + 1}$ by $f(q, n, k)$. Recall that the set of all k-spaces in a hyperplane in $\mathrm{PG}(n, q)$ is a Cameron-Liebler set of k-spaces with parameter $x = \frac{q^{n-k} - 1}{q^{k+1} - 1}$ (see Example 8.3.2.3) and note that $f(q, n, k) \in \mathcal{O}(\sqrt{q^{n-2k}})$ while $\frac{q^{n-k} - 1}{q^{k+1} - 1} \in \mathcal{O}(q^{n-2k-1})$.

Lemma 8.4.6. *For $n \geq 2k + 2$, we have*

$$\begin{bmatrix} n \\ k \end{bmatrix} > \begin{bmatrix} n - k - 1 \\ k \end{bmatrix} q^{k^2 + k} > W_\Sigma \,.$$

If also $k \geq 2$, then

$$\begin{bmatrix} n - k - 1 \\ k \end{bmatrix} q^{k^2 + k} > q^{nk - k^2} + q^{nk - k^2 - 1} + q^{nk - k^2 - 2} \,.$$

Proof. The first inequality follows since $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ is the number of k-spaces through a fixed point in $\mathrm{PG}(n, q)$, $\left[\begin{smallmatrix} n-k-1 \\ k \end{smallmatrix}\right] q^{k^2 + k}$ is the number of k-spaces through a fixed point disjoint from a given k-space not through that point (see Lemma 1.10.1), and W_Σ is the number of k-spaces through a fixed point and disjoint from two given k-spaces not through that point.

The second inequality, for $k \geq 2, n \geq 2k + 2$, follows from the calculations below, in which we define $\prod_{i=0}^{k-3} g(i) = 1$, for $k = 2$.

$$\begin{aligned} \begin{bmatrix} n - k - 1 \\ k \end{bmatrix} q^{k^2 + k} &= \left(\prod_{i=0}^{k-3} \left(\frac{q^{n-k-1-i} - 1}{q^{k-i} - 1} \right) \right) \left(\frac{q^{n-2k+1} - 1}{q - 1} \cdot \frac{q^{n-2k} - 1}{q^2 - 1} \right) q^{k^2 + k} \\ &> q^{(n-2k-1)(k-2)} (q^{n-2k} + q^{n-2k-1} + q^{n-2k-2}) q^{n-2k-2} q^{k^2 + k} \\ &= q^{nk - k^2} + q^{nk - k^2 - 1} + q^{nk - k^2 - 2} \,. \end{aligned}$$ ■

Notation 8.4.7. *We denote $\Delta(q, n, k) = \left[\begin{smallmatrix} n-k-1 \\ k \end{smallmatrix}\right] q^{k^2 + k}$ and $C(q, n, k) = \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] - \left[\begin{smallmatrix} n-k-1 \\ k \end{smallmatrix}\right] q^{k^2 + k}$. Then, according to Lemma 8.3.5, we can write*

$$\begin{aligned} s_1(q, n, k, x) &= xC(q, n, k) + \Delta(q, n, k) \quad \textit{and} \\ s_2'(q, n, k, x) &= xC(q, n, k) + (2 - x)\Delta(q, n, k) + (x - 2)W_\Sigma. \end{aligned}$$

We denote $\Delta(q, n, k)$ and $C(q, n, k)$ by Δ and C if q, n and k are clear from the context.

Lemma 8.4.8. *If* $n \geq 2k+1$ *and* $q \geq 3$, *then*

$$W_\Sigma \leq \Delta - \frac{C}{2}.$$

Proof. First, using the definition of W_Σ as given in Lemma 8.3.4, we find

$$\begin{aligned} W_\Sigma &= \frac{1}{(q^{k+1}-1)^2} \sum_{i=0}^{k} (q^{i+1}-1) q^{2k^2+k+\frac{3i^2}{2}-\frac{i}{2}-3ik} \begin{bmatrix} n-2k-1 \\ k-i \end{bmatrix} \begin{bmatrix} k+1 \\ i+1 \end{bmatrix} \prod_{j=0}^{i} (q^{k-j+1}-1) \\ &= q^{2k^2+k} \sum_{i=0}^{k} q^{\frac{3i^2}{2}-\frac{i}{2}-3ik} \begin{bmatrix} n-2k-1 \\ k-i \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} \prod_{j=1}^{i} (q^{k-j+1}-1)\ . \end{aligned}$$

Here, the final product is considered 1 if $i=0$ (the 'empty' product). Now, using the definitions of Δ and C as in Notation 8.4.7, the inequality stated above can be written as:

$$q^{2k^2+k} \sum_{i=0}^{k} q^{\frac{3i^2}{2}-\frac{i}{2}-3ik} \begin{bmatrix} n-2k-1 \\ k-i \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} \prod_{j=1}^{i} (q^{k-j+1}-1) \leq \frac{3}{2} \begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k} - \frac{1}{2} \begin{bmatrix} n \\ k \end{bmatrix}\ . \tag{8.1}$$

For $k=1$, this reduces to

$$q^3 \begin{bmatrix} n-3 \\ 1 \end{bmatrix} + q(q-1) \leq \frac{3}{2} \begin{bmatrix} n-2 \\ 1 \end{bmatrix} q^2 - \frac{1}{2} \begin{bmatrix} n \\ 1 \end{bmatrix} \qquad \Leftrightarrow \qquad \frac{q-1}{2} \geq 0\ ,$$

which is true for all $q \geq 2$. So, we will from now on assume that $k \geq 2$.

Repeatedly applying the left equality in (1.3) from Result 1.10.3, we find that $\begin{bmatrix} n \\ k \end{bmatrix} = q^{k^2+k} \begin{bmatrix} n-k-1 \\ k \end{bmatrix} + \sum_{i=0}^{k} q^{ik} \begin{bmatrix} n-i-1 \\ k-1 \end{bmatrix}$, so inequality (8.1) can be rewritten as

$$\begin{aligned} q^{2k^2+k} \sum_{i=0}^{k} q^{\frac{3i^2}{2}-\frac{i}{2}-3ik} \begin{bmatrix} n-2k-1 \\ k-i \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} \prod_{j=1}^{i} (q^{k-j+1}-1) + \frac{1}{2} \sum_{i=0}^{k} q^{ik} \begin{bmatrix} n-i-1 \\ k-1 \end{bmatrix} & \\ \leq \begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k}\ . & \end{aligned}$$

We now apply Lemma 1.10.4 on the right hand side of this inequality and we see that it is equivalent with

$$\begin{aligned} q^{2k^2+k} \sum_{i=1}^{k} q^{\frac{3i^2}{2}-\frac{i}{2}-3ik} \begin{bmatrix} n-2k-1 \\ k-i \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} \prod_{j=1}^{i} (q^{k-j+1}-1) + \frac{1}{2} \sum_{i=0}^{k} q^{ik} \begin{bmatrix} n-i-1 \\ k-1 \end{bmatrix} & \\ \leq q^{k^2+k} \sum_{i=1}^{k} q^{(k-i)^2} \begin{bmatrix} n-2k-1 \\ k-i \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix}\ . & \end{aligned} \tag{8.2}$$

Now, we note that $\prod_{j=1}^{i} (q^{k-j+1}-1) \leq q^{(i-1)(k+1)-\frac{i(i-1)}{2}} (q^{k-i+1}-1)$ for $i \geq 1$. So, in order to prove (8.2), it is sufficient to show that the following inequality is valid:

$$\begin{aligned} \frac{1}{2} \sum_{i=0}^{k} q^{ik} \begin{bmatrix} n-i-1 \\ k-1 \end{bmatrix} &\leq q^{k^2+k} \sum_{i=1}^{k} \left(q^{(k-i)^2} - q^{(k-i)(k-i-1)-1} (q^{k-i+1}-1) \right) \begin{bmatrix} n-2k-1 \\ k-i \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} \\ &= q^{k^2+k} \sum_{i=1}^{k} q^{(k-i)(k-i-1)-1} \begin{bmatrix} n-2k-1 \\ k-i \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} \\ &= q^{2k^2-2k+1} \begin{bmatrix} n-2k-1 \\ k-1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix} + q^{k^2+k} \sum_{i=2}^{k} q^{(k-i)(k-i-1)-1} \begin{bmatrix} n-2k-1 \\ k-i \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix}\ . \end{aligned} \tag{8.3}$$

Applying Lemma 1.10.2.1 on the left hand side in (8.3), we find that

$$\frac{1}{2}\sum_{i=0}^{k} q^{ik}\begin{bmatrix} n-i-1 \\ k-1 \end{bmatrix} \le q^{(k-1)(n-k)}\sum_{i=0}^{k} q^{i} = q^{(k-1)(n-k)}\frac{q^{k+1}-1}{q-1} \,. \tag{8.4}$$

Now applying Lemma 1.10.2.4 on the first term of the right hand side in (8.3), we find that

$$q^{2k^2-2k+1}\begin{bmatrix} n-2k-1 \\ k-1 \end{bmatrix}\begin{bmatrix} k \\ 1 \end{bmatrix} \ge \left(1+\frac{1}{q}\right) q^{(k-1)(n-k)+1}\frac{q^{k}-1}{q-1} = (q+1)q^{(k-1)(n-k)}\frac{q^{k}-1}{q-1} \,. \tag{8.5}$$

From (8.4) and (8.5), it follows that in order to prove (8.3), it is sufficient to show that the following inequality is valid:

$$q^{k+1}-1 \le (q+1)\left(q^{k}-1\right) \quad \Leftrightarrow \quad q^{k} \ge q \,,$$

This statement is clearly true. ■

Lemma 8.4.9. *If* $x \le \frac{1}{\sqrt[8]{2}} f(q,n,k)$ *and* $n \ge 2k+2$*, then* $\frac{\Delta}{C} > \sqrt[4]{2}x^2$.

Proof. We want to prove that

$$\begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k} > \sqrt[4]{2}x^2\left(\begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k}\right).$$

We first look at the case $k \ge 2$. Given a k-space π in $\mathrm{PG}(n-1,q)$, the number of $(k-1)$-spaces meeting π equals $\begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k}$ by Lemma 1.10.1. We know that this number is smaller than the product of the number of points $Q \in \pi$ and the number of $(k-1)$-spaces through Q. This implies that

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k} &\le \begin{bmatrix} k+1 \\ 1 \end{bmatrix}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \\ &= \frac{q^{k+1}-1}{q-1}\cdot\frac{(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^{k-1}-1)\cdots(q-1)} \\ &< \frac{q^{nk-\frac{k^2}{2}-n+\frac{3k}{2}+1}}{(q-1)^{\frac{k^2}{2}-\frac{k}{2}+1}} \,. \end{aligned}$$

From this computation and the assumption on x, it follows that

$$\begin{aligned} \sqrt[4]{2}x^2\left(\begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k}\right) < (f(q,n,k))^2\frac{q^{nk-\frac{k^2}{2}-n+\frac{3k}{2}+1}}{(q-1)^{\frac{k^2}{2}-\frac{k}{2}+1}} &= q^{nk-k^2-2}(q^2+q+1) \\ &\le \begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k} \,, \end{aligned}$$

where the final inequality is given by Lemma 8.4.6 (which we can apply since $k \ge 2$).

Now we look at the case $k = 1$. We have to prove that

$$\begin{bmatrix} n-2 \\ 1 \end{bmatrix} q^2 > \sqrt[4]{2}x^2\left(\begin{bmatrix} n \\ 1 \end{bmatrix} - \begin{bmatrix} n-2 \\ 1 \end{bmatrix} q^2\right) \quad \Leftrightarrow \quad \frac{q^{n-2}-1}{q^2-1}q^2 > \sqrt[4]{2}x^2 \,.$$

By the assumption on x, it is sufficient to prove that

$$\frac{q^{n-2}-1}{q^2-1}q^2 > f(q,n,1)^2 = q^{n-5}(q^3-1) \quad \Leftrightarrow \quad q^{n-2}+q^{n-3}-q^{n-5}-q^2 > 0 \,,$$

which is clearly true since $n \ge 4$. ■

Lemma 8.4.10. *Let $\mathcal{L}$ be a Cameron-Liebler set of k-spaces in $\mathrm{PG}(n,q)$, $n \geq 3k+2$, with parameter $2 \leq x \leq \frac{1}{\sqrt[8]{2}} f(q,n,k)$, then $\mathcal{L}$ cannot contain $\left\lfloor \frac{3}{2}x \right\rfloor$ mutually disjoint k-spaces.*

Proof. We apply Lemma 8.3.6, with $c+1 = \left\lfloor \frac{3}{2}x \right\rfloor$ and have to show that

$$\left\lfloor \frac{3}{2}x \right\rfloor s_1 - \binom{\left\lfloor \frac{3}{2}x \right\rfloor}{2} s_2' > x \begin{bmatrix} n \\ k \end{bmatrix} .$$

Using Notation 8.4.7 and Lemma 8.4.8, we see that it is sufficient to prove that

$$\begin{aligned} &\left\lfloor \frac{3}{2}x \right\rfloor (xC+\Delta) - x(\Delta + C) \\ &\qquad - \frac{1}{2}\left\lfloor \frac{3}{2}x \right\rfloor \left(\left\lfloor \frac{3}{2}x \right\rfloor - 1 \right) \left(xC - (x-2)\Delta + (x-2)\left(\Delta - \frac{C}{2} \right) \right) > 0 \\ \Leftrightarrow \qquad & \Delta \left(\left\lfloor \frac{3}{2}x \right\rfloor - x \right) > C \left(x - \left\lfloor \frac{3}{2}x \right\rfloor x + \frac{1}{2}\left\lfloor \frac{3}{2}x \right\rfloor \left(\left\lfloor \frac{3}{2}x \right\rfloor - 1 \right) \left(\frac{x}{2} + 1 \right) \right) . \end{aligned}$$

From Lemma 8.4.9, we know that $\frac{\Delta}{C} > \sqrt[4]{2}x^2$. Hence, it is sufficient to prove that

$$\left(\left\lfloor \frac{3}{2}x \right\rfloor - x \right) \sqrt[4]{2}x^2 > x - \left\lfloor \frac{3}{2}x \right\rfloor x + \frac{1}{2}\left\lfloor \frac{3}{2}x \right\rfloor \left(\left\lfloor \frac{3}{2}x \right\rfloor - 1 \right) \left(\frac{x}{2} + 1 \right) \tag{8.6}$$

for all admissible x. We denote $\frac{3}{2}x - \left\lfloor \frac{3}{2}x \right\rfloor$ by ε. Then, $0 \leq \varepsilon < 1$. We rewrite (8.6) as

$$\begin{aligned} & \left(\frac{3}{2}x - \varepsilon - x \right) \sqrt[4]{2}x^2 > x - \left(\frac{3}{2}x - \varepsilon \right) x + \frac{1}{2}\left(\frac{3}{2}x - \varepsilon \right) \left(\frac{3}{2}x - \varepsilon - 1 \right) \left(\frac{x}{2} + 1 \right) \\ \Leftrightarrow \qquad & -\left(\frac{x+2}{4} \right) \varepsilon^2 + \left(\frac{(3-4\sqrt[4]{2})x^2 + x - 2}{4} \right) \varepsilon + \frac{(8\sqrt[4]{2}-9)x^3 + 12x^2 - 4x}{16} > 0 . \qquad (8.7) \end{aligned}$$

The nontrivial zero of the quadratic function $f(\varepsilon) = -\left(\frac{x+2}{4}\right)\varepsilon^2 + \left(\frac{(3-4\sqrt[4]{2})x^2+x-2}{4}\right)\varepsilon$ is smaller than 1 for any x, so $f(\varepsilon) > f(1)$ for any $\varepsilon \in [0,1[$ regardless of x. So, to prove (8.7), it is sufficient to prove

$$\begin{aligned} & \left(\frac{1}{2}\sqrt[4]{2} - \frac{9}{16} \right) x^3 + \left(\frac{3}{2} - \sqrt[4]{2} \right) x^2 - \frac{1}{4}x - 1 \geq 0 \\ \Leftrightarrow \qquad & (x-2)\left((8\sqrt[4]{2} - 9)x^2 + 6x + 8 \right) \geq 0 , \end{aligned}$$

which is clearly true for $x \geq 2$. ■

Lemma 8.4.11. *If $2 \leq x \leq \frac{1}{\sqrt[8]{2}} f(q,n,k)$ and $n \geq 2k+2$ and $q \geq 3$, then*

$$\begin{aligned} & \frac{x-1}{\frac{3}{2}x - 2} \begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k} - \left(\frac{3}{2}x - 3 \right) s_2' > x \begin{bmatrix} n \\ k \end{bmatrix} - x \begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k} \quad \textit{and} \\ & \frac{x-1}{\frac{3}{2}x - 2} \begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k} - \left(\frac{3}{2}x - 3 \right) s_2' > \begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k} + q^{k+1} . \end{aligned}$$

Proof. To prove the first inequality, we rewrite it using Notation 8.4.7.

$$\frac{x-1}{\frac{3}{2}x - 2}\Delta - \left(\frac{3}{2}x - 3 \right) (xC + (2-x)\Delta + (x-2)W_\Sigma) > xC .$$

Using Lemma 8.4.8, we see that it is sufficient to prove

$$\frac{x-1}{\frac{3}{2}x-2}\Delta > C\left(\frac{3}{4}x^2+x-3\right) .$$

From Lemma 8.4.9, we know that $\frac{\Delta}{C} > \sqrt[4]{2}x^2$. Hence, it is sufficient to prove that

$$\frac{x-1}{\frac{3}{2}x-2}\sqrt[4]{2}x^2 > \left(\frac{3}{4}x^2+x-3\right) \quad\Leftrightarrow\quad \left(\sqrt[4]{2}-\frac{9}{8}\right)x^3-\sqrt[4]{2}x^2+\frac{13}{2}x-6>0 .$$

Using a computer algebra package, we find that the last inequality is valid for all $x \geq 2$.

To prove the second inequality for $k \geq 2$, it is sufficient to prove that

$$\begin{aligned} & x\begin{bmatrix} n \\ k \end{bmatrix} - x\begin{bmatrix} n-k-1 \\ k \end{bmatrix}q^{k^2+k} > \begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n-k-1 \\ k \end{bmatrix}q^{k^2+k} + q^{k+1} \\ \Leftrightarrow\quad & q^{k+1} < (x-1)\left(\begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n-k-1 \\ k \end{bmatrix}q^{k^2+k}\right) = (x-1)\sum_{i=0}^{k} q^{ik}\begin{bmatrix} n-i-1 \\ k-1 \end{bmatrix} , \end{aligned}$$

whereby we applied repeatedly the left equality in (1.3) from Result 1.10.3. We immediately see that

$$(x-1)\sum_{i=0}^{k} q^{ik}\begin{bmatrix} n-i-1 \\ k-1 \end{bmatrix} > q^{k^2}\begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} > q^{(n-k)(k-1)+k} > q^{2k+2} > q^{k+1} .$$

For $k=1$, we prove the second inequality directly. Note that $s'_2 = x+2q$. The inequality reduces to

$$\begin{aligned} & \frac{x-1}{\frac{3}{2}x-2}\cdot\frac{q^{n-2}-1}{q-1}q^2 - \left(\frac{3}{2}x-3\right)(x+2q) > q^2+q+1 \\ \Leftrightarrow\quad & \frac{x-1}{\frac{3}{2}x-2}\cdot\frac{q^{n-2}-1}{q-1}q^2 > \frac{3}{2}x^2+3(q-1)x+q^2-5q+1 . \end{aligned} \tag{8.8}$$

Recall that $2 \leq x \leq \frac{1}{\sqrt[8]{2}}f(q,n,1) = \frac{1}{\sqrt[8]{2}}q^{\frac{n-5}{2}}\sqrt{q^3-1} < q^{\frac{n-2}{2}}$. We look at the left hand side of (8.8) and find

$$\begin{aligned} \frac{x-1}{\frac{3}{2}x-2}\cdot\frac{q^{n-2}-1}{q-1}q^2 &= \left(\frac{2}{3}+\frac{2}{3(3x-4)}\right)\frac{q^{n-2}-1}{q-1}q^2 \\ &> \left(\frac{2}{3}+\frac{2}{9(x-1)}\right)\frac{q^{n-2}-1}{q-1}q^2 \\ &> \frac{2}{3}\frac{q^{n-2}-1}{q-1}q^2 + \frac{2}{9\left(q^{\frac{n-2}{2}}-1\right)}\frac{q^{n-2}-1}{q-1}(q^2-1) \\ &= \frac{2}{3}\frac{q^{n-2}-1}{q-1}q^2 + \frac{2}{9}\left(q^{\frac{n-2}{2}}+1\right)(q+1) . \end{aligned}$$

For the right hand side of (8.8), we find that

$$\begin{aligned} \frac{3}{2}x^2+3(q-1)x+q^2-5q+1 &< \frac{3}{2\sqrt[4]{2}}q^{n-5}\left(q^3-1\right)+3(q-1)q^{\frac{n-2}{2}}+q^2-5q+1 \\ &< \frac{3}{2}q^{n-5}\left(q^3-1\right)+3(q-1)q^{\frac{n-2}{2}}+q^2-5q+1 . \end{aligned}$$

So, to prove (8.8), it is sufficient to prove that

$$\begin{aligned} &\frac{2}{3}\frac{q^{n-2}-1}{q-1}q^2+\frac{2}{9}\left(q^{\frac{n-2}{2}}+1\right)(q+1)\geq\frac{3}{2}q^{n-5}\left(q^3-1\right)+3(q-1)q^{\frac{n-2}{2}}+q^2-5q+1\\ \Leftrightarrow\quad &\frac{2}{3}q^{n-1}-\frac{5}{6}q^{n-2}+\frac{2}{3}\frac{q^{n-4}-1}{q-1}q^2+\frac{3}{2}q^{n-5}-q^{\frac{n-2}{2}}\left(\frac{25}{9}q-\frac{29}{9}\right)-q^2+\frac{47}{9}q-\frac{7}{9}\geq 0\,. \end{aligned} \tag{8.9}$$

For $n=4,5$, we can check this to be true for all $q\geq 3$ using computer algebra software. For $n\geq 6$, we rewrite (8.9) as follows:

$$\begin{aligned} \frac{5}{18}(q-3)q^{n-2}+\frac{q^{\frac{n}{2}}}{18}\left(7q^{\frac{n-2}{2}}-50\right)+\frac{2}{3}\frac{q^{n-4}-1}{q-1}q^2&\\ +\left(\frac{29}{9}q^{\frac{n-2}{2}}-q^2\right)+\frac{47}{9}q+\left(\frac{3}{2}q^{n-5}-\frac{7}{9}\right)\geq 0\,.& \end{aligned}$$

Here each of the terms in the left hand side is positive for $q\geq 3$ since $n\geq 6$, which proves the second inequality in the statement for $k=1$. ■

Lemma 8.4.12. *If $\mathcal{L}$ is a Cameron-Liebler set of k-spaces in* $\mathrm{PG}(n,q)$*, $n\geq 3k+2$ and $q\geq 3$, with parameter $2\leq x\leq\frac{1}{\sqrt[8]{2}}f(q,n,k)$, then $\mathcal{L}$ contains a point-pencil.*

Proof. Let π be a k-space in $\mathcal{L}$ and let c be the maximal number of elements of $\mathcal{L}$ that are pairwise disjoint. By Theorem 8.1.6(3), there are $(x-1)\left[{n-k-1 \atop k}\right]q^{k^2+k}$ k-spaces in $\mathcal{L}$ disjoint from π. Within this collection of k-spaces, we find at most $c-1$ spaces $\sigma_1,\sigma_2,\ldots,\sigma_{c-1}$ that are pairwise disjoint. By Lemma 8.4.10, $c-1\leq\left\lfloor\frac{3}{2}x\right\rfloor-2$. By the pigeonhole principle, we find an index i so that σ_i meets at least $\frac{x-1}{c-1}\left[{n-k-1 \atop k}\right]q^{k^2+k}\geq\frac{x-1}{\left\lfloor\frac{3}{2}x\right\rfloor-2}\left[{n-k-1 \atop k}\right]q^{k^2+k}$ elements of $\mathcal{L}$ that are skew to π. We denote this collection of k-spaces disjoint from π and meeting σ_i in at least a point by $\mathcal{F}_i$.

Now we want to show that $\mathcal{F}_i$ contains a family of pairwise intersecting subspaces. For any σ_j with $j\neq i$, we find at most s'_2 elements that meet σ_i and σ_j. In this way, we find that there are at least $\frac{x-1}{\left\lfloor\frac{3}{2}x\right\rfloor-2}\left[{n-k-1 \atop k}\right]q^{k^2+k}-(c-2)s'_2\geq\frac{x-1}{\frac{3}{2}x-2}\left[{n-k-1 \atop k}\right]q^{k^2+k}-\left(\frac{3}{2}x-3\right)s'_2$ elements of $\mathcal{L}$ that meet σ_i, are disjoint from π and that are disjoint from σ_j for all $j\neq i$. We denote this subset of $\mathcal{F}_i\subseteq\mathcal{L}$ by $\mathcal{F}'_i$. This collection $\mathcal{F}'_i$ of k-spaces is a set of pairwise intersecting k-spaces: if two elements α,β in $\mathcal{F}'_i$ would be disjoint, then $(\{\sigma_1,\ldots,\sigma_{c-1}\}\setminus\{\sigma_i\})\cup\{\alpha,\beta,\pi\}$ would be a collection of $c+1$ pairwise disjoint elements of $\mathcal{L}$, which is impossible since we supposed that c is the size of a maximal set of pairwise disjoint k-spaces in $\mathcal{L}$. By Lemma 8.4.11, we have $\frac{x-1}{\frac{3}{2}x-2}\left[{n-k-1 \atop k}\right]q^{k^2+k}-\left(\frac{3}{2}x-3\right)s'_2>\left[{n \atop k}\right]-\left[{n-k-1 \atop k}\right]q^{k^2+k}+q^{k+1}$ since $2\leq x\leq\frac{1}{\sqrt[8]{2}}f(q,n,k)$. This implies that $\cap_{F\in\mathcal{F}'_i}F$ is not empty by Theorem 2.0.5; let P be a point contained in $\cap_{F\in\mathcal{F}'_i}F$. We conclude that $\mathcal{F}'_i$ is a part of the point-pencil through P.

We conclude by showing that $\mathcal{L}$ contains the whole point-pencil through P. If $\gamma\notin\mathcal{L}$ is a k-space through P, then γ meets at least $\frac{x-1}{\frac{3}{2}x-2}\left[{n-k-1 \atop k}\right]q^{k^2+k}-\left(\frac{3}{2}x-3\right)s'_2>x\left[{n \atop k}\right]-x\left[{n-k-1 \atop k}\right]q^{k^2+k}$ elements of $\mathcal{F}'_i\subseteq\mathcal{L}$, where the inequality follows from Lemma 8.4.11. This contradicts Theorem 8.1.6.3. ■

Theorem 8.4.13. *There are no Cameron-Liebler sets of k-spaces in* $\mathrm{PG}(n,q)$*, $n\geq 3k+2$ and $q\geq 3$, with parameter $2\leq x\leq\frac{1}{\sqrt[8]{2}}q^{\frac{n}{2}-\frac{k^2}{4}-\frac{3k}{4}-\frac{3}{2}}(q-1)^{\frac{k^2}{4}-\frac{k}{4}+\frac{1}{2}}\sqrt{q^2+q+1}$.*

Proof. We prove this result using induction on x. By Lemma 8.4.12, we know that $\mathcal{L}$ contains the point-pencil $[P]_k$ through a point P. By Lemma 8.3.1(4), $\mathcal{L} \setminus [P]_k$ is a Cameron-Liebler set of k-spaces with parameter $(x-1)$, which by the induction hypothesis (in case $x-1 \geq 2$) or by Lemma 8.4.4 (in case $1 < x-1 < 2$) does not exist, or which is a point-pencil (in case $x-1 = 1$) by Theorem 8.4.5. In the former case, there is an immediate contradiction; in the latter case, $\mathcal{L}$ contains two disjoint point-pencils of k-spaces, a contradiction. ■

Remark 8.4.14. We cannot compare this classification result with classification results already known, for Cameron-Liebler sets of k-spaces in $\mathrm{PG}(2k+1, q)$, $k \geq 1$, since the parameters n and k of these spaces do not fulfill the condition "$n \geq 3k+2$" in Theorem 8.4.13. For $q \in \{2, 3, 4, 5\}$, a complete classification is known for Cameron-Liebler sets of k-spaces in $\mathrm{PG}(n, q)$, see [59]. There, the authors show that the only Cameron-Liebler sets in this context are the trivial Cameron-Liebler sets, independent of the values of k and n. Hence, for small values of q this result is stronger than the classification result in the previous theorem.

9 Cameron-Liebler sets of k-spaces in $\mathrm{AG}(n, q)$

> *You know, people think mathematics is complicated.*
> *Mathematics is the simple bit. It's the stuff we can understand.*
> *It's cats that are complicated. I mean, what is it in those little molecules and*
> *stuff that make one cat behave differently than another, or that make a cat?*
> *And how do you define a cat?*
> *I have no idea.*
>
> —John Conway

In this section, we give a short overview of the results proven in [46] and [44]. The results in this part are joint work with dr. Ferdinand Ihringer, Jonathan Mannaert, prof. Leo Storme and prof. Andrea Švob.

Similar to the definition of Cameron-Liebler sets of k-spaces in $\mathrm{PG}(n, q)$, we have the following definition in the affine context.

Definition 9.0.1. A set $\mathcal{L}$ of k-spaces in $\mathrm{AG}(n, q)$ is a Cameron-Liebler set of k-spaces of parameter x in $\mathrm{AG}(n, q)$ if and only if every k-spread in $\mathrm{AG}(n, q)$ has x elements in common with $\mathcal{L}$.

In contrast to k-spreads in $\mathrm{PG}(n, q)$, we note that there exist k-spreads in $\mathrm{AG}(n, q)$, for every $k \leq n$, which implies that the definition above is well defined. An example of an affine k-spread in $\mathrm{AG}(n, q)$ is the following. Embed $\mathrm{AG}(n, q)$ in the projective space $\mathrm{PG}(n, q)$, and let H be the hyperplane at infinity. Consider a $(k-1)$-space π in H and let S_p be the set of all k-spaces through π. The set of all affine k-spaces corresponding to the elements of S_p, restricted to $\mathrm{AG}(n, q)$, is a k-spread in this affine space.

There is a strong link between Cameron-Liebler sets of k-spaces in $\mathrm{PG}(n, q)$ and $\mathrm{AG}(n, q)$.

Theorem 9.0.2. *Let $\mathcal{L}$ be a Cameron-Liebler set of k-spaces with parameter x in $\mathrm{PG}(n, q)$ which does not contain k-spaces in some hyperplane H. Then $\mathcal{L}$ is a Cameron-Liebler set of k-spaces with parameter x of $\mathrm{AG}(n, q) \cong \mathrm{PG}(n, q) \setminus H$.*
If $\mathcal{L}$ is a Cameron-Liebler set of k-spaces of $\mathrm{AG}(n, q)$ with parameter x, then $\mathcal{L}$ is a Cameron-Liebler set of k-spaces of $\mathrm{PG}(n, q)$ with parameter x in the projective closure $\mathrm{PG}(n, q)$ of $\mathrm{AG}(n, q)$.

Using the link between $\mathrm{PG}(n, q)$ and $\mathrm{AG}(n, q)$, it was possible to give several equivalent definitions for Cameron-Liebler sets of k-spaces in $\mathrm{AG}(n, q)$. A second consequence of this link, is that the classification result for Cameron-Liebler sets of k-spaces in $\mathrm{PG}(n, q)$ (Theorem 8.4.13) implies the following result.

Theorem 9.0.3. *There are no Cameron-Liebler sets of k-spaces in $\mathrm{AG}(n, q)$, $n \geq 3k + 2$ and $q \geq 3$, with parameter $2 \leq x \leq \frac{1}{\sqrt[8]{2}} q^{\frac{n}{2} - \frac{k^2}{4} - \frac{3k}{4} - \frac{3}{2}} (q-1)^{\frac{k^2}{4} - \frac{k}{4} + \frac{1}{2}} \sqrt{q^2 + q + 1}$.*

For $k = 1, n = 3$, we also find a classification result, using a modular equality in the affine context, similar to Theorem 8.4.2.

Theorem 9.0.4. *Let $\mathcal{L}$ be a Cameron-Liebler line set in* $\mathrm{AG}(3,q)$ *with parameter x, then the following equation holds:*

$$x(x-1)\equiv 0 \qquad \mod 2(q+1).$$

We also found a non-trivial Cameron-Liebler line example $\mathcal{L}_a$ in $\mathrm{AG}(3,q)$ with parameter $x=\frac{q^2-1}{2}$. This example could be derived from a non-trivial Cameron-Liebler line example $\mathcal{L}_p$ in $\mathrm{PG}(3,q)$ [31, 58], since in this example, there is a (hyper)plane that contains no elements of $\mathcal{L}_p$.

10

Cameron-Liebler sets of generators in finite classical polar spaces

> *Isaac Newton zag een appel, en dacht 'zie ik dat goed?'*
> *Het is niet wat het is, het is wat je d'r mee doet.*
>
> —Bart Peeters, *Het is niet wat het is*

The results in this chapter are joint work with dr. Maarten De Boeck and appeared in [35].

10.1 Introduction

We investigate Cameron-Liebler sets in finite classical polar spaces. The finite classical polar spaces are the hyperbolic quadrics $Q^+(2d-1,q)$, the parabolic quadrics $Q(2d,q)$, the elliptic quadrics $Q^-(2d+1,q)$, the Hermitian polar spaces $H(2d-1,q^2)$ and $H(2d,q^2)$, and the symplectic polar spaces $W(2d-1,q)$, with q a prime power. For more information on these polar spaces, we refer to Section 1.5.

Here we study the sets of generators defined by the following definition, with A the incidence matrix of points and generators. We call these sets *degree one Cameron-Liebler sets.*

Definition 10.1.1. A degree one Cameron-Liebler set of generators in a finite classical polar space $\mathcal{P}$ is a set of generators in $\mathcal{P}$, with characteristic vector χ such that $\chi \in \mathrm{im}(A^T)$.

This definition corresponds with the definition of Boolean degree one functions for generators in polar spaces. In Section 8.2, we introduced Boolean degree one functions in projective spaces. Analogously, they can be defined in polar spaces, by replacing the set Δ_k of k-spaces in $\mathrm{PG}(n,q)$, by the set of generators in a polar space $\mathcal{P}$. Similarly, for generators, their definition corresponds to the fact that the corresponding characteristic vector lies in $V_0 \perp V_1$, which are eigenspaces of the related association scheme. In [36], M. De Boeck, M. Rodgers, L. Storme and A. Švob introduced Cameron-Liebler sets of generators in the finite classical polar spaces. In this article, Cameron-Liebler sets of generators in the polar spaces are defined by the *disjointness-definition* and the authors give several equivalent definitions for these Cameron-Liebler sets. Note that this definition is the polar-space-equivalent for the disjointness-definition in the projective context, see Theorem 8.1.6.3. Furthermore, this definition for polar spaces does not require that the parameter x is an integer, but it is proved in [36, Theorem 4.8] that $x \in \mathbb{N}$.

Definition 10.1.2 ([36]). Let $\mathcal{P}$ be a finite classical polar space with parameter e and rank d. A set $\mathcal{L}$ of generators in $\mathcal{P}$ is a Cameron-Liebler set of generators in $\mathcal{P}$, with parameter x, if and only if for every generator π in $\mathcal{P}$, the number of elements of $\mathcal{L}$, disjoint from π, equals $(x-\chi(\pi))q^{\binom{d-1}{2}+e(d-1)}$.

Using association scheme notation we can interpret the previous definition as follows. The characteristic vector of a Cameron-Liebler set is contained in $V_0 \perp W$, with W the eigenspace of the disjointness matrix A_d corresponding to a specific eigenvalue. It can be seen that W always contains V_1, but it does not necessarily coincide with V_1. Hence, for some polar spaces, Cameron-Liebler sets and degree one Cameron-Liebler sets will coincide, but for others not.

Type I	Type II	Type III
$Q^-(2d+1,q)$ $Q(2d,q)$, d even $Q^+(2d-1,q)$, d odd $W(2d-1,q)$, d even $H(2d-1,q)$, q square $H(2d,q)$, q square	$Q^+(2d-1,q)$, d even	$Q(4n+2,q)$ $W(4n+1,q)$

Table 10.1: Three types of polar spaces

In this chapter, we consider three different types of polar spaces, see Table 10.1. Type I and II correspond with type I and II respectively, defined in [36], while type III corresponds with the union of type III and IV in [36], as we handle the symplectic polar spaces $W(4n+1,q)$, for both q odd and q even, in the same way. Definition 10.1.2 and Definition 10.1.1 are equivalent for the polar spaces of type I by [36, Theorem 3.7, Theorem 3.15]. For the polar spaces of type II, we can consider the (degree one) Cameron-Liebler sets of one class of generators; we see that Cameron-Liebler sets and degree one Cameron-Liebler sets coincide when we only consider one class (see [36, Theorem 3.16]). For the polar spaces of type III, this equivalence no longer applies and for these polar spaces, any degree one Cameron-Liebler set is also a regular Cameron-Liebler set, but not vice versa.

In Table 10.2, we give an overview of properties that we will prove throughout this chapter. For this, we distinguish between sufficient properties, necessary properties and characteristic properties or definitions, for Cameron-Liebler sets and for degree one Cameron-Liebler sets for polar spaces of type III. Note that a characteristic property is both necessary and sufficient. In the last column, also the reference to the corresponding result is given.

Suppose in this table that $\mathcal{L}$ is a set of generators in the polar space $\mathcal{P}$ of type III, with characteristic vector χ. Suppose also that π is a generator in $\mathcal{P}$, not necessarily in $\mathcal{L}$.

Property	CL	degree one CL	
$\chi \in V_0 \perp V_1$.	S	C	(Theorem 10.1.5)
$\forall \pi \in \mathcal{P}, \lvert\{\tau \in \mathcal{L} \mid \dim(\tau \cap \pi) = d-i-1\}\rvert =$ (10.1), for $0 \leq i < d$	S	C	(Theorem 10.2.1)
$\forall \pi \in \mathcal{P}$: $\lvert\{\tau \in \mathcal{L} \mid \tau \cap \pi = \emptyset\}\rvert = (x-\chi(\pi))q^{\binom{d}{2}}$.	C	N	(Theorem 10.2.1)
$\chi - \frac{x}{q^d+1}\boldsymbol{j}$ is an eigenvector of A_d with eigenvalue $-q^{\binom{d}{2}}$.	C	N	(Lemma 10.2.3.2)
If $\mathcal{P}$ admits a spread, then $\lvert\mathcal{L} \cap S\rvert = x, \forall$ spread S of $\mathcal{P}$.	C	N	(Lemma 10.2.3.3)

Table 10.2: Overview of the sufficient (S), necessary (N) and characterising (C) properties.

Recall that Cameron-Liebler sets were originally introduced by a group-theoretical argument, see Section 7.1. Note that for a polar space $\mathcal{P}$, we cannot use Lemma 7.1.4 to find a group-theoretical definition for degree one Cameron-Liebler sets of generators in $\mathcal{P}$. This follows from the fact that the incidence matrix A does not have full row rank, see [23, Theorem 9.4.3].

In Section 10.1.1, we discuss several properties of the eigenvalues of the association scheme for generators of finite classical polar spaces. In Section 10.2, we give an overview of the equivalent definitions and several properties of degree one Cameron-Liebler sets in polar spaces. In Section 10.3, we give an equivalent definition for Cameron-Liebler sets in the hyperbolic quadrics $Q^+(2d-$

$1, q)$, d even. In Section 10.4, we prove some classification results for degree one Cameron-Liebler sets, in particular in the polar spaces $W(5, q)$ and $Q(6, q)$. We end this chapter with a new, non-trivial example of a Cameron-Liebler set of planes in $Q^+(5, q)$, described in Section 10.5.

10.1.1 The association scheme for generators in polar spaces

Let $\mathcal{P}$ be a finite classical polar space of rank d and let Ω be its set of generators. The relations $\mathcal{R}_i$ on Ω are defined as follows: $(\pi, \pi') \in \mathcal{R}_i$ if and only if $\dim(\pi \cap \pi') = d - i - 1$, for generators $\pi, \pi' \in \Omega$, with $i = 0, ..., d$. We define A_i as the adjacency matrix of the relation $\mathcal{R}_i$. By the theory of association schemes, we know that there is an orthogonal decomposition $V_0 \perp V_1 \perp \cdots \perp V_d$ of $\mathbb{R}^\Omega$ in common eigenspaces of $A_0, A_1, ..., A_d$. Consider the distance *one* relation $\mathcal{R}_1$ and let V_j be the eigenspace corresponding to the eigenvalue P_{j1} from Lemma 10.1.3. Although there are several association schemes linked to a polar space, in this chapter, we will refer to the association scheme defined above as *the* association scheme of a polar space.

Lemma 10.1.3 ([110, Theorem 4.3.6]). *In the association scheme of a polar space over $\mathbb{F}_q$ of rank d and parameter e, the eigenvalue P_{ji} of the relation $\mathcal{R}_i$ corresponding to the subspace V_j is given by:*

$$P_{ji} = \sum_{s=\max\{0,j-i\}}^{\min\{j,d-i\}} (-1)^{j+s} \begin{bmatrix} j \\ s \end{bmatrix} \begin{bmatrix} d-j \\ d-i-s \end{bmatrix} q^{e(i+s-j)+\binom{j-s}{2}+\binom{i+s-j}{2}}.$$

.

Before we start with investigating the Cameron-Liebler sets of generators in finite classical polar spaces, we give an important lemma about the eigenvalues P_{ji}.

Lemma 10.1.4. *In the association scheme of polar spaces, the eigenvalue P_{1i} of A_i corresponds only with the eigenspace V_1 for $i \neq 0$, that is, $P_{1i} \neq P_{ji}, \forall j \neq 1$, except in the following cases.*

1. *The hyperbolic quadrics $Q^+(2d-1, q)$. Here $P_{1i} = P_{d-1,i}$ for i even, so P_{1i} also corresponds with V_{d-1}, for every relation $\mathcal{R}_i$, i even.*
2. *The parabolic quadrics $Q(4n+2, q)$ and the symplectic spaces $W(4n+1, q)$. Here $P_{1d} = P_{dd}$, so P_{1d} also corresponds with V_d for the disjointness relation $\mathcal{R}_d$.*

Proof. We need to prove, given a fixed $i \neq 0$ and $j \neq 1$, that $P_{1i} \neq P_{ji}$, except for the two cases described in the statement of the lemma. For $j = 0$ and for all $i \neq 0$, it is easy to calculate that $P_{1i} \neq P_{0i}$, so we may suppose that $j > 1$.

For $i = 1$, we can directly compare the eigenvalues P_{11} and P_{j1}.

$$\begin{aligned} P_{11} = P_{j1} &\Leftrightarrow \begin{bmatrix} d-1 \\ 1 \end{bmatrix} q^e - 1 = \begin{bmatrix} d-j \\ 1 \end{bmatrix} q^e - \begin{bmatrix} j \\ 1 \end{bmatrix} \\ &\Leftrightarrow \frac{-q+1+(q^{d-1}-1)q^e}{q-1} = \frac{-q^j+1+(q^{d-j}-1)q^e}{q-1} \\ &\Leftrightarrow (q^{d-j+e-1}+1)(q^{j-1}-1) = 0. \end{aligned}$$

Since $j > 1$, the last equation gives a contradiction for any q.

For $i \geq 2$, we introduce $\phi_i(j) = \max\{k \mid\mid q^k | P_{ji}\}$, the exponent of q in P_{ji}. If $P_{ji} = 0$, we put $\phi_i(j) = \infty$. We will show that $\phi_i(j)$ is different from $\phi_i(1)$ for most values of i and j. For $j = 1$, we find that

$$P_{1i} = -\begin{bmatrix} d-1 \\ d-i \end{bmatrix} q^{\binom{i-1}{2}+e(i-1)} + \begin{bmatrix} d-1 \\ i \end{bmatrix} q^{\binom{i}{2}+ei} = q^{\binom{i-1}{2}+e(i-1)} \left(\begin{bmatrix} d-1 \\ i \end{bmatrix} q^{i-1+e} - \begin{bmatrix} d-1 \\ i-1 \end{bmatrix} \right).$$

We can see that $\phi_i(1) = \binom{i-1}{2} + e(i-1)$, since $i-1+e \geq 1$ and $\begin{bmatrix} a \\ b \end{bmatrix} = 1 \pmod{q}$ for all $0 \leq b \leq a$.

In Lemma 10.1.3, we see that $\phi_i(j)$ depends on the last factor of every term in the sum. To find $\phi_i(j)$, we first need to find all integer values z such that $q^{e(i+z-j)+\binom{j-z}{2}+\binom{i+z-j}{2}}$ is a factor of every term in the sum, or equivalently, such that $f_{ji} : \mathbb{Z} \to \mathbb{Z} : s \mapsto e(i+s-j) + \binom{j-s}{2} + \binom{i+s-j}{2}$ reaches its minimum for such a value z. So for most cases, we have that $\phi_i(j) = f_{ij}(z)$, but in some cases it occurs that two values of z correspond with opposite terms with factor $q^{\phi_i(j)}$. These cases, we have to investigate separately.

We can check that z is the unique integer or one of two integers in $[\max\{0, j-i\}, \ldots, \min\{j, d-i\}]$ closest to $j - \frac{i}{2} - \frac{e}{2}$. Since $i \geq 2$, we have three possibilities for the value of z, as we always have $j - i \leq j - \frac{i}{2} - \frac{e}{2} < j$:

- $z = 0$ if $j - \frac{i}{2} - \frac{e}{2} < 0$,
- $z \in \{j - \frac{i}{2} - \frac{e}{2}, j - \frac{i}{2} - \frac{e}{2} \pm \frac{1}{2}\}$ if $0 \leq j - \frac{i}{2} - \frac{e}{2} \leq d - i$,
- $z = d - i$ if $j - \frac{i}{2} - \frac{e}{2} > d - i$.

Now we handle these three cases.

- If $j - \frac{i}{2} - \frac{e}{2} < 0$, we see that f_{ji} is minimal for the integer $z = 0$.

  We note that in this case there is only 1 value of s, namely 0, for which the corresponding term is divisible by $q^{\phi_i(j)}$ but not by $q^{\phi_i(j)+1}$. This is important to exclude the case where 2 terms with factor $q^{\phi_i(j)}$ would be each others opposite.

  We find that $\phi_i(j) = f_{ji}(0) = \binom{i}{2} + (j-i)(j-e)$, and since $\phi_i(1) = \binom{i-1}{2} + e(i-1)$, the values $\phi_i(j)$ and $\phi_i(1)$ are equal if and only if $j = 1 \vee j = i + e - 1$. We only have to check the latter case, and recall that $j - \frac{i}{2} - \frac{e}{2} < 0$. It follows that $i + e < 2$, a contradiction since we supposed $i \geq 2$.

- If $0 \leq j - \frac{i}{2} - \frac{e}{2} \leq d - i$, we see that f_{ji} is minimal for the integer z closest to $j - \frac{i}{2} - \frac{e}{2}$.

  In Table 10.3, we list the different cases depending on e and the parity of i. Note that we have to check, for $e = 0, i$ odd, for $e = 1, i$ even, and for $e = 2, i$ odd, that the two values of z do not correspond with two opposite terms with factor $q^{\phi_i(j)}$. By calculating and taking into account the conditions $0 \leq j - \frac{i}{2} - \frac{e}{2} \leq d - i$, we find out that those cases do not correspond with two opposite terms, except in the following cases:

  - $e = 0, j = \frac{d}{2}$ and i odd,
  - $e = 1, j = \frac{d}{2} + 1, i = \frac{d}{2}$ and i even,
  - $e = 2, j = \frac{d}{2} + 2, i = \frac{d}{2}$ and i odd.

  In these cases, $P_{ij} = 0$, so $\phi_i(j) = \infty \neq \phi_i(1)$.

  Moreover, for every e, i and $j > 1$, $\phi_i(j) = f_{ij}(z)$ is independent of j, see the fifth column in Table 10.3. In the last column, we give the values of i for which $\phi_i(j) = \phi_i(1)$. As we supposed $i \geq 2$, we see that we have to check the eigenvalues for $i = 2$ if $e \in \{0, \frac{1}{2}, 1\}$ and for $i = 3$ if $e = 0$ in detail.

e	i	z	$\phi_i(j) = f_{ji}(z)$	$\phi_i(1)$	S
$Q^+(2d-1,q)$					
0	even	$j - \frac{i}{2}$	$\frac{i(i-2)}{4}$	$\frac{(i-1)(i-2)}{2}$	$\{2\}$
	odd	$j - \frac{i}{2} \pm \frac{1}{2}$	$\begin{cases} \frac{(i-1)^2}{4} & \text{if } j \neq \frac{d}{2} \\ \infty & \text{if } j = \frac{d}{2} \end{cases}$	$\frac{(i-1)(i-2)}{2}$	$\{3\}$
$\mathcal{H}(2d-1,q)$, with q square					
$\frac{1}{2}$	even	$j - \frac{i}{2}$	$\frac{i(i-1)}{4}$	$\frac{(i-1)^2}{2}$	$\{2\}$
	odd	$j - \frac{i}{2} - \frac{1}{2}$	$\frac{i(i-1)}{4}$	$\frac{(i-1)^2}{2}$	$\emptyset$
$Q(2d,q)$, $W(2d-1,q)$, with $d \not\equiv 0 \mod 4$					
1	even	$j - \frac{i}{2} - \frac{1}{2} \pm \frac{1}{2}$	$\frac{i^2}{4}$	$\frac{i(i-1)}{2}$	$\{2\}$
	odd	$j - \frac{i}{2} - \frac{1}{2}$	$\frac{i^2-1}{4}$	$\frac{i(i-1)}{2}$	$\emptyset$
$Q(2d,q)$, $W(2d-1,q)$, with $d \equiv 0 \mod 4$					
1	even, $i \neq \frac{d}{2}$	$j - \frac{i}{2} - \frac{1}{2} \pm \frac{1}{2}$	$\frac{i^2}{4}$	$\frac{i(i-1)}{2}$	$\{2\}$
	$i = \frac{d}{2}$	$j - \frac{i}{2} - \frac{1}{2} \pm \frac{1}{2}$	$\begin{cases} \infty & \text{if } j = \frac{d}{2} + 1 \\ \frac{i^2}{4} & \text{else} \end{cases}$	$\frac{i(i-1)}{2}$	$\{2\}$
	odd	$j - \frac{i}{2} - \frac{1}{2}$	$\frac{i^2-1}{4}$	$\frac{i(i-1)}{2}$	$\emptyset$
$\mathcal{H}(2d,q)$, with q square					
$\frac{3}{2}$	even	$j - \frac{i}{2} - 1$	$\frac{(i-1)(i+2)}{4}$	$\frac{i^2-1}{2}$	$\emptyset$
	odd	$j - \frac{i}{2} - \frac{1}{2}$	$\frac{(i-1)(i+2)}{4}$	$\frac{i^2-1}{2}$	$\emptyset$
$Q^-(2d+1,q)$, with $d \not\equiv 2 \mod 4$					
2	even	$j - \frac{i}{2} - 1$	$\frac{i^2}{4} + \frac{i}{2} - 1$	$\frac{(i-1)(i+2)}{2}$	$\emptyset$
	odd	$j - \frac{i}{2} - 1 \pm \frac{1}{2}$	$\frac{(i-1)(i+3)}{4}$	$\frac{(i-1)(i+2)}{2}$	$\emptyset$
$Q^-(2d+1,q)$, with $d \equiv 2 \mod 4$					
2	even	$j - \frac{i}{2} - 1$	$\frac{i^2}{4} + \frac{i}{2} - 1$	$\frac{(i-1)(i+2)}{2}$	$\emptyset$
	odd, $i \neq \frac{d}{2}$	$j - \frac{i}{2} - 1 \pm \frac{1}{2}$	$\frac{(i-1)(i+3)}{4}$	$\frac{(i-1)(i+2)}{2}$	$\emptyset$
	$i = \frac{d}{2}$	$j - \frac{i}{2} - 1 \pm \frac{1}{2}$	$\begin{cases} \infty & \text{if } j = \frac{d}{2} + 2 \\ \frac{(i-1)(i+3)}{4} & \text{else} \end{cases}$	$\frac{(i-1)(i+2)}{2}$	$\emptyset$

Table 10.3: For $0 \leq j - \frac{i}{2} - \frac{e}{2} \leq d - i$, with $S = \{i \geq 2 \mid \phi_i(j) = \phi_i(1)\}$.

  - Case $i = 2$ and $e \in \{0, \frac{1}{2}, 1\}$:

$$\begin{aligned} & P_{12} = P_{j2} \\ \Leftrightarrow \quad & -\begin{bmatrix} d-1 \\ 1 \end{bmatrix} q^e + \begin{bmatrix} d-1 \\ 2 \end{bmatrix} q^{1+2e} = \begin{bmatrix} j \\ 2 \end{bmatrix} q - \begin{bmatrix} d-j \\ 1 \end{bmatrix} \begin{bmatrix} j \\ 1 \end{bmatrix} q^e + \begin{bmatrix} d-j \\ 2 \end{bmatrix} q^{1+2e} \\ \Leftrightarrow \quad & \left(\begin{bmatrix} d-1 \\ 2 \end{bmatrix} - \begin{bmatrix} d-j \\ 2 \end{bmatrix} \right) q^{2e} + \begin{bmatrix} d-j-1 \\ 1 \end{bmatrix} \begin{bmatrix} j-1 \\ 1 \end{bmatrix} q^e = \begin{bmatrix} j \\ 2 \end{bmatrix}. \end{aligned}$$

    For $e = \frac{1}{2}$ and $e = 1$, we see that the right and left hand side of the last equation are different modulo q, since $j > 1$. So we can assume $e = 0$.

$$\begin{aligned} & P_{12} = P_{j2} \\ \Leftrightarrow \quad & \frac{(q^{d-1}-1)(q^{d-2}-1)}{(q^2-1)(q-1)} - \frac{(q^{d-j}-1)(q^{d-j-1}-1)}{(q^2-1)(q-1)} + \frac{(q^{d-j-1}-1)(q^{j-1}-1)}{(q-1)(q-1)} \\ & \qquad = \frac{(q^j-1)(q^{j-1}-1)}{(q^2-1)(q-1)} \\ \Leftrightarrow \quad & q^{2d-3} - q^{2d-2j-1} + q - q^{2j-1} = 0 \\ \Leftrightarrow \quad & q(q^{2j-2}-1)(q^{2(d-j-1)}-1) = 0. \end{aligned}$$

    Since $j > 1$, we see that $P_{12} = P_{j2}$ if and only if $j = d - 1$. This corresponds with the first exception in the lemma with $i = 2$.

  - Case $i = 3$ and $e = 0$.

$$\begin{aligned} & P_{13} = P_{j3} \\ \Leftrightarrow \quad & -\begin{bmatrix} d-1 \\ 2 \end{bmatrix} q + \begin{bmatrix} d-1 \\ 3 \end{bmatrix} q^3 = -\begin{bmatrix} j \\ 3 \end{bmatrix} q^3 + \begin{bmatrix} j \\ 2 \end{bmatrix} \begin{bmatrix} d-j \\ 1 \end{bmatrix} q - \begin{bmatrix} j \\ 1 \end{bmatrix} \begin{bmatrix} d-j \\ 2 \end{bmatrix} q + \begin{bmatrix} d-j \\ 3 \end{bmatrix} q^3 \\ \Leftrightarrow \quad & -\begin{bmatrix} d-1 \\ 2 \end{bmatrix} + \begin{bmatrix} d-1 \\ 3 \end{bmatrix} q^2 = -\begin{bmatrix} j \\ 3 \end{bmatrix} q^2 + \begin{bmatrix} j \\ 2 \end{bmatrix} \begin{bmatrix} d-j \\ 1 \end{bmatrix} - \begin{bmatrix} j \\ 1 \end{bmatrix} \begin{bmatrix} d-j \\ 2 \end{bmatrix} + \begin{bmatrix} d-j \\ 3 \end{bmatrix} q^2. \end{aligned}$$

    Since the right and left hand side of the last equation are different modulo q, we see that $P_{13} \neq P_{j3}$ for $j > 1$. Recall that $\begin{bmatrix} a \\ b \end{bmatrix} = 1 \pmod{q}$.

- If $j - \frac{i}{2} - \frac{e}{2} > d - i$, we see that f_{ji} is minimal for the integer $z = d - i$. Remark again that there is only one value of s for which the corresponding term is divisible by $q^{\phi_i(j)}$ but not by $q^{\phi_i(j)+1}$. This excludes the case where 2 terms with factor $q^{\phi_i(j)}$ would be each others opposite.

  We find that $\phi_i(j) = f_{ji}(d-i) = (j-e-d+1)(j-d+i-1) + \binom{i-1}{2} + e(i-1)$, and we know that $\phi_i(1) = \binom{i-1}{2} + e(i-1)$. These two values $\phi_i(j)$ and $\phi_i(1)$ are equal if and only if $j = e + d - 1$ or $j = d - i + 1$.

  - Suppose $j = d + e - 1$. As $j, d \in \mathbb{Z}$, we know that $e \in \mathbb{Z}$. If $e = 2$, then $j = d + 1 > d$, a contradiction. For $e = 1$, we find that $P_{1i} = P_{di}$ if and only if $i = d$ and d odd. This corresponds to the polar spaces $Q(4n+2, q)$ and $W(4n+1, q)$. For $e = 0$ and $j = d-1$, we find that $P_{1i} = P_{d-1,i}$ for i even. This corresponds to the exception for the polar spaces $Q^+(2d-1, q)$ and i even.
  - Suppose $j = d - i + 1$. Since $j - \frac{i}{2} - \frac{e}{2} > d - i$, we know that $i + e < 2$, which gives a contradiction as we supposed $i \geq 2$. ■

We continue with well-known theorems, linked to the Bose-Mesner algebra of the association scheme, that will be useful in the following sections (see Result 1.9.3). The first theorem follows from [36, Theorem 2.14].

Theorem 10.1.5. *Let $\mathcal{P}$ be a finite classical polar space of rank d and parameter e, and let Ω be the set of all generators of $\mathcal{P}$. Consider the eigenspace decomposition $\mathbb{R}^{\Omega} = V_0 \perp V_1 \perp \cdots \perp V_d$ related to the association scheme, and using the classical order. Let A be the point-generator incidence matrix of $\mathcal{P}$, then $\operatorname{im}(A^T) = V_0 \perp V_1$ and $V_0 = \langle \boldsymbol{j} \rangle$.*

The following theorem was already proved in [40, Proposition 3.7] from a different point of view. The ideas are already present in [2, Lemma 2] and [110, Lemma 2.1.3]. For the sake of completeness, we add a proof below.

Theorem 10.1.6. *Let $\mathcal{R}_i$ be a relation of an association scheme on the set Ω with adjacency matrix A_i and let $\mathcal{L} \subseteq \Omega$ be a set, with characteristic vector χ, such that for any $\pi \in \Omega$, we have that*

$$|\{x \in \mathcal{L} | (x, \pi) \in \mathcal{R}_i\}| = \begin{cases} \alpha_i \text{ if } \pi \in \mathcal{L} \\ \beta_i \text{ if } \pi \notin \mathcal{L}. \end{cases}$$

Then $\alpha_i - \beta_i = P$ is an eigenvalue of A_i and $v_i = \chi + \frac{\beta_i}{P - P_{0i}} \boldsymbol{j} \in V$ with V the eigenspace of A_i for the eigenvalue P.

The eigenspace V in the previous theorem can be seen as the direct sum of several eigenspaces of the association scheme. Note that an association scheme is not necessary in this theorem, a regular relation suffices. Furthermore, the set $\mathcal{L}$, described in this theorem, is an intriguing set in the graph $\Gamma = (\Omega, \mathcal{R}_i)$, see Definition 1.7.7.

Proof. We show that $v_i = \chi + \frac{\beta_i}{P - P_{0i}} \boldsymbol{j}$, with $P = \alpha_i - \beta_i$ is an eigenvector for the matrix A_i with eigenvalue P:

$$\begin{aligned} A_i \left(\chi + \frac{\beta_i}{P - P_{0i}} \boldsymbol{j} \right) &= \alpha_i \chi + \beta_i (\boldsymbol{j} - \chi) + \frac{\beta_i}{P - P_{0i}} P_{0i} \boldsymbol{j} \\ &= P \left(\chi + \frac{\beta_i}{P - P_{0i}} \boldsymbol{j} \right). \end{aligned}$$

So we find that $\chi + \frac{\beta_i}{P - P_{0i}} \boldsymbol{j} \in V$. ■

10.2 Degree one Cameron-Liebler sets

In this section, we investigate the degree one Cameron-Liebler sets and give an equivalent definition. Every degree one Cameron-Liebler set $\mathcal{L}$ has a parameter x, which can be defined as

$$x = \frac{|\mathcal{L}|}{\prod_{i=0}^{d-2} (q^{e+i} + 1)}.$$

For now it is clear that $x \in \mathbb{Q}$, but, in Lemma 10.4.1 we will prove that $x \in \mathbb{N}$.

Using Lemma 10.1.4 and Theorem 10.1.6, we can give a new equivalent definition for these degree one Cameron-Liebler sets of generators in polar spaces. The following theorem is an extension of Lemma 4.9 in [36].

Theorem 10.2.1. *Let $\mathcal{P}$ be a finite classical polar space, of rank d with parameter e, let $\mathcal{L}$ be a set of generators of $\mathcal{P}$ and i be an integer with $1 \leq i \leq d$. If $\mathcal{L}$ is a degree one Cameron-Liebler set* *of generators in $\mathcal{P}$, with parameter x, then the number of elements of $\mathcal{L}$ meeting a generator π in a $(d-i-1)$-space equals*

$$\begin{cases} \left((x-1)\begin{bmatrix} d-1\\ i-1\end{bmatrix} + q^{i+e-1}\begin{bmatrix} d-1\\ i\end{bmatrix}\right) q^{\binom{i-1}{2}+(i-1)e} & \text{if } \pi\in\mathcal{L}\\ x\begin{bmatrix} d-1\\ i-1\end{bmatrix} q^{\binom{i-1}{2}+(i-1)e} & \text{if } \pi\notin\mathcal{L}.\end{cases} \tag{10.1}$$

Moreover, if this property holds for a polar space $\mathcal{P}$ and an integer i such that

- *i is odd for $\mathcal{P} = Q^+(2d-1,q)$,*
- *$i \neq d$ for $\mathcal{P} = Q(2d,q)$ or $\mathcal{P} = W(2d-1,q)$ both with d odd or*
- *i is arbitrary otherwise,*

then $\mathcal{L}$ is a degree one Cameron-Liebler set with parameter x.

Proof. Consider first a degree one Cameron-Liebler set $\mathcal{L}$ of generators in the polar space $\mathcal{P}$ with characteristic vector χ. As $\chi \in V_0 \perp V_1$, we have $\chi = v + a\boldsymbol{j}$ for some $v \in V_1$ and some $a \in \mathbb{R}$. Since $|\mathcal{L}| = \langle j, \chi\rangle = x\prod_{i=0}^{d-2}(q^{i+e}+1)$, we find that $a = \frac{x}{q^{d+e-1}+1}$, hence $\chi = \frac{x}{q^{d+e-1}+1}\boldsymbol{j} + v$. Recall that the matrix A_i is the incidence matrix of the relation $\mathcal{R}_i$, which describes whether the dimension of the intersection of two generators equals $d-i-1$ or not. This implies that the vector $A_i\chi$, on the position corresponding to a generator π, gives the number of generators in $\mathcal{L}$, meeting π in a $(d-i-1)$-space. We have

$$\begin{aligned}
A_i\chi &= A_i v + \frac{x}{q^{d+e-1}+1}A_i\boldsymbol{j} = P_{1i}v + \frac{x}{q^{d+e-1}+1}P_{0i}\boldsymbol{j}\\
&= \left(\begin{bmatrix} d-1\\ i\end{bmatrix} q^{\binom{i}{2}+ei} - \begin{bmatrix} d-1\\ i-1\end{bmatrix} q^{\binom{i-1}{2}+e(i-1)}\right) v + \frac{x}{q^{d+e-1}+1}\begin{bmatrix} d\\ i\end{bmatrix} q^{\binom{i}{2}+ei}\boldsymbol{j}\\
&= \left(\begin{bmatrix} d-1\\ i\end{bmatrix} q^{\binom{i}{2}+ei} - \begin{bmatrix} d-1\\ i-1\end{bmatrix} q^{\binom{i-1}{2}+e(i-1)}\right)\left(\chi - \frac{x}{q^{d+e-1}+1}\boldsymbol{j}\right)\\
&\qquad + \frac{x}{q^{d+e-1}+1}\begin{bmatrix} d\\ i\end{bmatrix} q^{\binom{i}{2}+ei}\boldsymbol{j}\\
&= \frac{xq^{\binom{i-1}{2}+e(i-1)}}{q^{d+e-1}+1}\left(\begin{bmatrix} d-1\\ i-1\end{bmatrix} - \begin{bmatrix} d-1\\ i\end{bmatrix} q^{i+e-1} + \begin{bmatrix} d\\ i\end{bmatrix} q^{i+e-1}\right)\boldsymbol{j}\\
&\qquad + q^{\binom{i-1}{2}+e(i-1)}\left(\begin{bmatrix} d-1\\ i\end{bmatrix} q^{i+e-1} - \begin{bmatrix} d-1\\ i-1\end{bmatrix}\right)\chi\\
&= q^{\binom{i-1}{2}+e(i-1)}\left(x\begin{bmatrix} d-1\\ i-1\end{bmatrix}\boldsymbol{j} + \left(\begin{bmatrix} d-1\\ i\end{bmatrix} q^{i+e-1} - \begin{bmatrix} d-1\\ i-1\end{bmatrix}\right)\chi\right),
\end{aligned}$$

which proves the first implication.

For the proof of the other implication, suppose that $\mathcal{L}$ is a set of generators in $\mathcal{P}$ with the property described in the statement of the theorem. We apply Theorem 10.1.6 with Ω the set of all generators in $\mathcal{P}$, $\mathcal{R}_i$ the relation $\{(\pi,\pi') \mid \dim(\pi\cap\pi') = d-i-1\}$, and

$$\begin{aligned}
\alpha_i &= \left((x-1)\begin{bmatrix} d-1\\ i-1\end{bmatrix} + q^{i+e-1}\begin{bmatrix} d-1\\ i\end{bmatrix}\right) q^{\binom{i-1}{2}+(i-1)e},\\
\beta_i &= x\begin{bmatrix} d-1\\ i-1\end{bmatrix} q^{\binom{i-1}{2}+(i-1)e}.
\end{aligned}$$

As $\alpha_i - \beta_i = P_{1i}$, we find that $v_i = \chi + \frac{\beta_i}{P_{1i}-P_{0i}}\boldsymbol{j} \in V_1$, for the admissible values of i, by Lemma 10.1.4. Hence, by Definition 10.1.1, $\mathcal{L}$ is a degree one Cameron-Liebler set in $\mathcal{P}$. ■

Remark 10.2.2. This definition is also a new equivalent definition for Cameron-Liebler sets of generators in polar spaces of type I, as for these polar spaces, degree one Cameron-Liebler sets and Cameron-Liebler sets coincide.

In the following lemma, we give some properties of degree one Cameron-Liebler sets in a polar space.

Lemma 10.2.3. *Let $\mathcal{L}$ be a degree one Cameron-Liebler set of generators in a polar space $\mathcal{P}$ and let χ be the characteristic vector of $\mathcal{L}$. Denote $\frac{|\mathcal{L}|}{\prod_{i=0}^{d-2}(q^{e+i}+1)}$ again by x. Then $\mathcal{L}$ has the following properties:*

1. *$\chi = \frac{x}{q^{d+e-1}+1}\boldsymbol{j} + v$ with $v \in V_1$,*
2. *$\chi - \frac{x}{q^{d+e-1}+1}\boldsymbol{j}$ is an eigenvector with eigenvalue P_{1i} for all adjacency matrices A_i in the association scheme,*
3. *if $\mathcal{P}$ admits a spread, then $|\mathcal{L} \cap S| = x$ for every spread $\mathcal{S}$ of $\mathcal{P}$.*

Proof. The first property follows from the first part of the proof of Theorem 10.2.1. The second property follows from the first property since $\chi - \frac{x}{q^{d+e-1}+1}\boldsymbol{j} \in V_1$.

Consider now a spread S in $\mathcal{P}$ with characteristic vector χ_S and let A be the point-generator incidence matrix of $\mathcal{P}$. Since $\chi \in \mathrm{im}(A^T) = \ker(A)^\perp$ and by [36, Lemma 3.6(i), $m = 1$], which gives that $u = \chi_S - \frac{1}{\prod_{i=0}^{d-2}(q^{e+i}+1)}\boldsymbol{j} \in \ker(A)$, we find, by taking the inner product of u and χ, that

$$|\mathcal{L} \cap S| = \langle \chi_S, \chi \rangle = \frac{1}{\prod_{i=0}^{d-2}(q^{e+i}+1)}\langle \boldsymbol{j}, \chi \rangle = \frac{1}{\prod_{i=0}^{d-2}(q^{e+i}+1)}|\mathcal{L}| = x.$$

■

We also give some properties of degree one Cameron-Liebler sets of generators in polar spaces that can easily be proved. They are similar to the properties for Cameron-Liebler sets of k-spaces in $\mathrm{PG}(n, q)$, see Lemma 8.3.1.

Lemma 10.2.4. *Let $\mathcal{L}$ and $\mathcal{L}'$ be two degree one Cameron-Liebler sets of generators in a polar space $\mathcal{P}$ with parameters x and x' respectively, then the following statements are valid.*

1. *$0 \leq x, x' \leq q^{d-1+e} + 1$.*
2. *$|\mathcal{L}| = x \prod_{i=0}^{d-2}(q^{i+e} + 1)$.*
3. *The set of all generators in the polar space $\mathcal{P}$ not in $\mathcal{L}$ is a degree one Cameron-Liebler set of generators in $\mathcal{P}$ with parameter $q^{d-1+e} + 1 - x$.*
4. *If $\mathcal{L} \cap \mathcal{L}' = \emptyset$, then $\mathcal{L} \cup \mathcal{L}'$ is a degree one Cameron-Liebler set of generators in $\mathcal{P}$ with parameter $x + x'$.*
5. *If $\mathcal{L} \subseteq \mathcal{L}'$, then $\mathcal{L} \setminus \mathcal{L}'$ is a degree one Cameron-Liebler set of generators in $\mathcal{P}$ with parameter $x - x'$.*

Lemma 10.2.5 ([59, Lemma 2.3]). *Let $\mathcal{P}$ be a polar space of rank d and let $\mathcal{P}'$ be a polar space, embedded in $\mathcal{P}$ with the same rank d. If $\mathcal{L}$ is a degree one Cameron-Liebler set in $\mathcal{P}$, then the restriction of $\mathcal{L}$ to $\mathcal{P}'$ is again a degree one Cameron-Liebler set.*

Note that Theorem 10.2.1 does not hold for some values of i, dependent on the polar space $\mathcal{P}$, since for these cases, we cannot apply Lemma 10.1.4. We will now show that there are examples of generator sets that admit the property of Theorem 10.2.1 for the non-admitted values of i, but that are not degree one Cameron-Liebler sets. These are however Cameron-Liebler sets in the sense of [36].

Example 10.2.6. *By investigating [36, Example 4.6], we find an example of a Cameron-Liebler set in a polar space of type III with $d = 3$, that is not a degree one Cameron-Liebler set: a base-plane. A* base-plane *in a polar space $\mathcal{P}$ of rank 3 with base the plane π is the set of all planes in $\mathcal{P}$, intersecting π in at least a line.*

Let $\mathcal{P}$ be a polar space of type III of rank 3, so $\mathcal{P} = W(5, q)$ or $\mathcal{P} = Q(6, q)$. Let π be a plane and let $\mathcal{L}$ be the base-plane with base π. This set $\mathcal{L}$ is a Cameron-Liebler set in $\mathcal{P}$, but not a degree one Cameron-Liebler set. This follows from Theorem 10.2.1 with $i = 1$: The number of generators of $\mathcal{L}$, meeting a plane α of $\mathcal{L}$ in a line, depends on whether α equals π or not. As those two numbers, for $\alpha = \pi$ and $\alpha \neq \pi$ are different, the property in Theorem 10.2.1 does not hold. This implies that the set $\mathcal{L}$ is not a degree one Cameron-Liebler set. By similar arguments, we can also use Theorem 10.2.1 with $i = 2$, to show that a base-plane is not a degree one Cameron-Liebler set. However, the equalities for $i = 3$ in Theorem 10.2.1 hold.

Example 10.2.7. *A hyperbolic class is the set of all generators of one class of a hyperbolic quadric $Q^+(4n+1, q)$ embedded in a polar space $\mathcal{P}$ with $\mathcal{P} = Q(4n+2, q)$ or $\mathcal{P} = W(4n+1, q)$, q even. We know that this set is a Cameron-Liebler set, see [36, Remark 3.25], but we can prove that this set is not a degree one Cameron-Liebler set, by considering $\mathrm{im}(B^T)$, where B is the incidence matrix of hyperbolic classes and generators. Every hyperbolic class corresponds to a row in the matrix B. If the characteristic vectors of all hyperbolic classes would lie in $V_0 \perp V_1$, then $\mathrm{im}(B^T) \subseteq V_0 \perp V_1$. This gives a contradiction since $\mathrm{im}(B^T) = V_0 \perp V_1 \perp V_d$ by [36, Lemma 3.26].*
Note that for the polar spaces $W(4n+1, q)$, q odd, we do not have Example 10.2.7, as there is no hyperbolic quadric $Q^+(4n+1, q)$ embedded in these symplectic polar spaces.

In the previous remark, we found that one class of a hyperbolic quadric $Q^+(4n+1, q)$ embedded in a $Q(4n+2, q)$ or $W(4n+1, q)$, q even, is not a degree one Cameron-Liebler set. In the next example, we show that an embedded hyperbolic quadric, that is, taking both hyperbolic classes, is a degree one Cameron-Liebler set in the polar spaces $Q(4n+2, q)$ and $W(4n+1, q)$, q even.

Example 10.2.8 ([36, Example 4.4]). *Consider a polar space $\mathcal{P}$, with $\mathcal{P} = Q(4n+2, q)$ or $\mathcal{P} = W(4n+1, q)$, q even. By Lemma 10.2.5, we know that the set of generators in an embedded hyperbolic quadric $Q^+(4n+1, q)$ is a degree one Cameron-Liebler set, and hence, also a Cameron-Liebler set.*

Example	CL	degree one CL
All generators of $\mathcal{P}$.	×	×
Point-pencil.	×	×
Base-plane for $d = 3$ (defined in Example 10.2.6).	×	
Hyperbolic class (defined in Example 10.2.7).	×	
Embedded hyperbolic quadric (defined in Example 10.2.8).	×	×

Table 10.4: Examples of Cameron-Liebler and degree one Cameron-Liebler sets.

10.3 Polar spaces $Q^+(2d-1,q)$, d even

In the previous section, we introduced degree one Cameron-Liebler sets while in this section we consider Cameron-Liebler sets defined with the 'disjointness-definition' (Definition 10.1.2). We focus on Cameron-Liebler sets contained in one class of generators in the polar spaces $Q^+(2d-1,q)$, d even. These Cameron-Liebler sets were introduced in [36, Section 3] and are defined in only one class of generators, in contrast to the (degree one) Cameron-Liebler sets in other polar spaces.

Recall, from Example 1.5.6, that the generators of a hyperbolic quadric $Q^+(2d-1,q)$ can be divided in two classes such that for any two generators π and π' we have $\dim(\pi \cap \pi') \equiv 1 \pmod 2$ if and only if π and π' belong to the same class. By restricting the classical association scheme of the hyperbolic quadric $Q^+(2d-1,q)$ to the even relations, we define an association scheme for one class of generators. For more information, see [36, Remark 2.18 and Lemma 3.12]. Let $\mathcal{R}'_i$ and A'_i be $\mathcal{R}_{2i}$ and A_{2i} respectively, restricted to the rows and columns corresponding to the generators of this class. Let V'_j be $V_j \perp V_{d-j}$, also restricted to the subspace corresponding to these generators.

For the polar spaces $Q^+(2d-1,q)$, d even, we thus have the relations $\mathcal{R}'_i$, $i = 0,\dots,\frac{d}{2}$, and the eigenspaces V'_j, $j = 0,\dots,\frac{d}{2}$. For this association scheme on one class of generators, we give the analogue of Lemma 10.1.4.

Lemma 10.3.1. *The eigenvalue $P_{1,2i}$ of $A'_i = A_{2i}$ corresponds only with the eigenspace $V'_1 = V_1 \perp V_{d-1}$ for the classical polar spaces $Q^+(2d-1,q)$, d even.*

Proof. This lemma follows from Lemma 10.1.4 as for the hyperbolic quadrics $Q^+(2d-1,q)$ we found that $P_{1k} = P_{d-1,k}$ for k even. This implies that the eigenvalue $P_{1,2i}$ corresponds with $V_1 \perp V_{d-1}$. ■

Here again, we find a new equivalent definition.

Theorem 10.3.2. *Let $\mathcal{G}$ be a class of generators of the hyperbolic quadric $Q^+(2d-1,q)$ of even rank d and let $\mathcal{L}$ be a set of generators of $\mathcal{G}$. The set $\mathcal{L}$ is a Cameron-Liebler set of generators in $\mathcal{G}$ if and only if for every generator π in $\mathcal{G}$, the number of elements of $\mathcal{L}$ meeting π in a $(d-2i-1)$-space equals*

$$\begin{cases} \left((x-1)\begin{bmatrix} d-1 \\ 2i-1 \end{bmatrix} + q^{2i-1}\begin{bmatrix} d-1 \\ 2i \end{bmatrix}\right) q^{(2i-1)(i-1)} & \text{if } \pi \in \mathcal{L} \\ x\begin{bmatrix} d-1 \\ 2i-1 \end{bmatrix} q^{(2i-1)(i-1)} & \text{if } \pi \notin \mathcal{L}. \end{cases}$$

Proof. Let $\mathcal{L}$ be a set of generators in $\mathcal{G}$ with the property described in the theorem, then the first implication is a direct application of Theorem 10.1.6 with Ω the set of all generators in $\mathcal{G}$, $\mathcal{R}_i$ the relation $R'_i = \{(\pi,\pi') | \dim(\pi \cap \pi') = d-2i-1\}$, and

$$\alpha_i = \left((x-1)\begin{bmatrix} d-1 \\ 2i-1 \end{bmatrix} + q^{2i-1}\begin{bmatrix} d-1 \\ 2i \end{bmatrix}\right) q^{(2i-1)(i-1)},$$
$$\beta_i = x\begin{bmatrix} d-1 \\ 2i-1 \end{bmatrix} q^{(2i-1)(i-1)}.$$

As $\alpha_i - \beta_i = P_{1,2i}$, we find that $v_i = \chi + \frac{\beta_i}{P_{1,2i}-P_{0,2i}}\boldsymbol{j} \in V'_1$, hence $\chi \in V'_0 \perp V'_1$ and, by [36, Lemma 3.15], we know that $\chi \in \mathrm{im}(A^T)$. Now it follows from [36, Definition 3.16(iv)] that $\mathcal{L}$ is a (degree one) Cameron-Liebler set of $\mathcal{G}$. The other implication is [36, Lemma 4.10]. ■

10.4 Classification results

We try to use the ideas from the classification results for Cameron-Liebler sets of polar spaces of type I and the polar spaces $Q^+(2d-1,q)$, d even, in [36, Section 6], to find classification results for degree one Cameron-Liebler sets in polar spaces.
We start with a lemma that proves that the parameter x is always an integer.

Recall from the first part of this thesis that an Erdős-Ko-Rado (EKR) set of k-spaces is a set of k-spaces which are pairwise not disjoint (see Chapter 2).

Lemma 10.4.1. *If $\mathcal{L}$ is a degree one Cameron-Liebler set in a polar space $\mathcal{P}$ with parameter x, then $x \in \mathbb{N}$.*

Proof. For all polar spaces, except the hyperbolic quadrics $Q^+(2d-1,q)$, d even, we refer to [36, Lemma 4.8].

Suppose that $\mathcal{L}$ is a degree one Cameron-Liebler set in $\mathcal{P} = Q^+(2d-1,q)$, d even, with parameter x. Then $\mathcal{L}$ is also a Cameron-Liebler set in $\mathcal{P}$ with parameter x. If Ω_1 and Ω_2 are the two classes of generators in $\mathcal{P}$, then $\mathcal{L}\cap\Omega_1$ and $\mathcal{L}\cap\Omega_2$ are Cameron-Liebler sets of Ω_1 and Ω_2 with parameter x, by [36, Theorem 3.20]. Hence, x is the parameter of a Cameron-Liebler set in one class of generators of $Q^+(2d-1,q)$, d even. This implies, by [36, Lemma 4.8], that $x \in \mathbb{N}$. ■

Now we continue with a classification result for degree one Cameron-Liebler sets with parameter 1 in all polar spaces.

Theorem 10.4.2. *A degree one Cameron-Liebler set in a polar space $\mathcal{P}$ of rank d with parameter 1 is a point-pencil.*

Proof. For the polar spaces of type I and III, the theorem follows from [36, Theorem 6.4] as any degree one Cameron-Liebler set is a Cameron-Liebler set and since a base-plane and a hyperbolic class, are no degree one Cameron-Liebler sets (see Remark 10.2.6 and Remark 10.2.7).

Let $\mathcal{L}$ be a degree one Cameron-Liebler set with parameter 1 in a polar space $\mathcal{P}$ of type II. Then, $\mathcal{P}$ is the hyperbolic quadric $Q^+(4n-1,q)$ with Ω_1 and Ω_2 the two classes of generators. By [36, Theorem 3.20], we know that $\mathcal{L}\cap\Omega_1$ and $\mathcal{L}\cap\Omega_2$ are Cameron-Liebler sets in Ω_1,Ω_2 respectively, with parameter 1. Using [36, Theorem 6.4], we see that $\mathcal{L}\cap\Omega_i$ is a point-pencil or a base-solid if $n=2$ for $i=1,2$. A *base-solid* is the set of all 3-spaces intersecting a fixed 3-space (the base) in precisely a plane. Note that all elements of the base-solid belong to a different class of the hyperbolic quadric than the base itself.

If $n=2$, so $d=4$, and $\mathcal{L}\cap\Omega_1$ or $\mathcal{L}\cap\Omega_2$ is a base-solid with base π, then there are at least $(q+1)(q^2+1)$ elements of $\mathcal{L}$ meeting π in a plane. This contradicts Theorem 10.2.1, whether $\pi\in\mathcal{L}$ or not. So we find, for all $n\geq 1$, that $\mathcal{L}\cap\Omega_1$ and $\mathcal{L}\cap\Omega_2$ are both point-pencils with vertex v_1 and v_2 respectively. Now we show that $v_1=v_2$. Suppose $v_1\neq v_2$. Consider a generator $\alpha\in\Omega_2\setminus\mathcal{L}$ through v_1. Then α intersects θ_{d-2} generators of $\mathcal{L}\cap\Omega_1$ in a $(d-2)$-space through v_1. This gives a contradiction with Theorem 10.2.1, which proves that $v_1=v_2$. Hence, $\mathcal{L}$ is a point-pencil through $v_1=v_2$. ■

The classification result in [36, Theorem 6.7] for polar spaces of type I is also valid for degree one Cameron-Liebler sets in all polar spaces.

Theorem 10.4.3. *Let $\mathcal{P}$ be a finite classical polar space of rank d and parameter e, and let $\mathcal{L}$ be a degree one Cameron-Liebler set of $\mathcal{P}$ with parameter x. If $x \leq q^{e-1}+1$, then $\mathcal{L}$ is the union of x point-pencils whose vertices are pairwise non-collinear or $x = q^{e-1}+1$ and $\mathcal{L}$ is the set of generators in an embedded polar space of rank d and with parameter $e-1$.*

Proof. In Lemma 6.5, Theorem 6.6 and Theorem 6.7 of [36], the authors use [36, Lemma 4.9] to prove the classification result. We can use the same proof since we can apply Theorem 10.2.1 instead of [36, Lemma 4.9]. ■

Note that the last possibility corresponds to an embedded hyperbolic quadric $Q^+(2d-1,q)$ if $\mathcal{P} = Q(2d,q)$ or $\mathcal{P} = W(2d-1,q)$ with q even. For $\mathcal{P} = H(2d,q)$, the Hermitian variety $H(2d-1,q)$ can be embedded, and for $\mathcal{P} = Q^-(2d+1,q)$, the parabolic quadric $Q(2d,q)$ and, for q even $W(2d-1,q)$, can be embedded. If $\mathcal{P} = W(4n+1,q)$ with q odd, then $\mathcal{P}$ admits no embedded polar space with rank n and parameter $e-1=0$.

For the symplectic polar space $W(5,q)$ and the parabolic quadric $Q(6,q)$, we give a stronger classification result. Recall that the polar spaces $W(5,q)$ and $Q(6,q)$ are isomorphic for q even, see Remark 1.5.7. We start with some lemmas.

Lemma 10.4.4. *Let $\mathcal{L}$ be a degree one Cameron-Liebler set of generators (planes) in $W(5,q)$ or $Q(6,q)$ with parameter x.*

1. *For every $\pi \in \mathcal{L}$, there are s_1 elements of $\mathcal{L}$ meeting π (including π).*
2. *For skew $\pi, \pi' \in \mathcal{L}$, there exist exactly d_2 subspaces in $\mathcal{L}$ that are skew to both π and π' and there exist s_2 subspaces in $\mathcal{L}$ that meet both π and π'.*

   Here, d_2, s_1 and s_2 are given by:

$$\begin{aligned} d_2(q,x) &= (x-2)q^2(q-1) \\ s_1(q,x) &= x(q^2+1)(q+1)-(x-1)q^3 = q^3 + x(q^2+q+1) \\ s_2(q,x) &= x(q^2+1)(q+1) - 2(x-1)q^3 + d_2(q,x). \end{aligned}$$

Proof. Let $\mathcal{P}$ be the polar space $W(5,q)$ or $Q(6,q)$, hence $d=3$ and $e=1$.

1. This follows directly from Theorem 10.2.1, for $i=d$ and $|\mathcal{L}| = x(q^2+1)(q+1)$.
2. Let χ_π and $\chi_{\pi'}$ be the characteristic vectors of $\{\pi\}$ and $\{\pi'\}$, respectively. Let $\mathcal{Z}$ be the set of all planes in $\mathcal{P}$ disjoint from π and π', and let $\chi_{\mathcal{Z}}$ be its characteristic vector. Furthermore, let v_π and $v_{\pi'}$ be the incidence vectors of π and π', respectively, with their positions corresponding to the points of $\mathcal{P}$. Note that $A\chi_\pi = v_\pi$ and $A\chi_{\pi'} = v_{\pi'}$.
   The number of planes through a point $P \notin \pi \cup \pi'$ and disjoint from π and π' is the number of lines in $P^\perp$, disjoint from the lines corresponding to π and π'. By [80, Corollary 19], this number equals $q^2(q-1)$, and we find:

$$\begin{aligned} A\chi_{\mathcal{Z}} &= q^2(q-1)(\boldsymbol{j} - v_\pi - v_{\pi'}) \\ &= q^2(q-1)\left(A\frac{\boldsymbol{j}}{(q^2+1)(q+1)} - A\chi_\pi - A\chi_{\pi'}\right) \\ \Leftrightarrow \quad \chi_{\mathcal{Z}} - q^2(q-1)&\left(\frac{\boldsymbol{j}}{(q^2+1)(q+1)} - \chi_\pi - \chi_{\pi'}\right) \in \ker(A). \end{aligned}$$

We know that the characteristic vector χ of $\mathcal{L}$ is included in $\ker(A)^{\perp}$. This implies:

$$\begin{aligned} & \chi_{\mathcal{Z}} \cdot \chi = q^2(q-1)\left(\frac{\boldsymbol{j} \cdot \chi}{(q^2+1)(q+1)} - \chi(\pi) - \chi(\pi')\right) \\ \Leftrightarrow \quad & |\mathcal{Z} \cap \mathcal{L}| = (x-2)q^2(q-1) \end{aligned}$$

which gives the formula for $d_2(q,x)$. The formula for $s_2(q,x)$ follows from the inclusion-exclusion principle. ■

In the following lemma, corollary and theorem, we will use s_1, s_2, d_2 to denote the values $s_1(q,x)$, $s_2(q,x)$, $d_2(q,x)$ if the field size q and the parameter x are clear from the context. For the definition of these values, we refer to the previous lemma.

The following lemma is a generalization of Lemma 2.4 in [93]. Note that we used a similar lemma to find classification results in the projective context, see Lemma 8.3.6.

Lemma 10.4.5. *If c is a non-negative integer such that*

$$(c+1)s_1 - \binom{c+1}{2} s_2 > x(q^2+1)(q+1) \ ,$$

then no degree one Cameron-Liebler set of generators in $W(5,q)$ or $Q(6,q)$ with parameter x contains $c+1$ mutually skew generators.

Proof. Let $\mathcal{P}$ be the polar space $W(5,q)$ or $Q(6,q)$ and assume that $\mathcal{P}$ has a degree one Cameron-Liebler set $\mathcal{L}$ of generators with parameter x that contains $c+1$ mutually disjoint subspaces $\pi_0, \pi_1, \dots, \pi_c$. Lemma 10.4.4 shows that π_i, meets at least $s_1(q,x) - i \cdot s_2(q,x)$ elements of $\mathcal{L}$ that are skew to $\pi_0, \pi_1, \dots, \pi_{i-1}$. Hence, $x(q^2+1)(q+1) = |\mathcal{L}| \geq (c+1)s_1 - \sum_{i=0}^{c} is_2$ which contradicts the assumption. ■

Corollary 10.4.6. *A degree one Cameron-Liebler set of generators in $W(5,q)$ or $Q(6,q)$ with parameter $2 \leq x \leq \sqrt[3]{2q^2} - \frac{\sqrt[3]{4q}}{3} + \frac{1}{6}$ contains at most x pairwise disjoint generators.*

Proof. Let $\mathcal{L}$ be a degree one Cameron-Liebler set of generators in $W(5,q)$ or $Q(6,q)$ with parameter x. Using Lemma 10.4.5 for $e=1, d=3, c=x$, we find that if $q^3 - q^2x + \frac{q+1}{2}x^2 - \frac{q+1}{2}x^3 > 0$, then $\mathcal{L}$ contains at most x pairwise disjoint generators. Since $f_q(x) = q^3 - q^2x - \frac{q+1}{2}x^2(x-1)$ is decreasing on $[1, +\infty[$, we find that it is sufficient that $f_q\left(\sqrt[3]{2q^2} - \frac{\sqrt[3]{4q}}{3} + \frac{1}{6}\right) > 0$, as we only consider the values of x in $\left[2, \dots, \sqrt[3]{2q^2} - \frac{\sqrt[3]{4q}}{3} + \frac{1}{6}\right]$. It can be checked that $f_q\left(\sqrt[3]{2q^2} - \frac{\sqrt[3]{4q}}{3} + \frac{1}{6}\right) > 0$ for all $q \geq 2$. ■

Theorem 10.4.7. *A degree one Cameron-Liebler set $\mathcal{L}$ of generators in $W(5,q)$ or $Q(6,q)$ with parameter $2 \leq x \leq \sqrt[3]{2q^2} - \frac{\sqrt[3]{4q}}{3} + \frac{1}{6}$ is the union of α embedded hyperbolic quadrics $Q^+(5,q)$, that pairwise have no plane in common, and $x - 2\alpha$ point-pencils whose vertices are pairwise non-collinear and not contained in the α hyperbolic quadrics $Q^+(5,q)$. For the polar space $Q(6,q)$ or $W(5,q)$ with q even, $\alpha \in \{0, ..., \lfloor \frac{x}{2} \rfloor\}$, for the polar space $W(5,q)$ with q odd, $\alpha = 0$.*

Proof. Let $\mathcal{P}$ be the polar space $W(5,q)$ or $Q(6,q)$ and $\mathcal{L}$ be a degree one Cameron-Liebler set in $\mathcal{P}$. Note that the generators in these polar spaces are planes. By Corollary 10.4.6, there are c pairwise disjoint planes $\pi_1, \pi_2, \dots, \pi_c$, with $c \leq x$, in $\mathcal{L}$. Let S_i be the set of planes in $\mathcal{L}$ intersecting π_i and not intersecting π_j for all $j \neq i$. By Lemma 10.4.4, there are, for a fixed i, at least $s_1 - (c-1)s_2 \geq s_1 - (x-1)s_2 = q^3 - (x-2)q^2 - (x^2-2x)(q+1)$ planes in S_i. As S_i is an EKR set by Corollary 10.4.6, S_i has to be a part of a point-pencil (PP), a base plane (BP) or one class of an embedded hyperbolic quadric $Q^+(5,q)$ (CEHQ). Note that if $\mathcal{P}$ is $W(5,q)$, with q odd, then $\mathcal{P}$ cannot contain a CEHQ, so for this polar space, the only possibilities are a PP or BP, by [33, Theorem 4.9 and 4.17]. Using Theorem 10.2.1, we can prove that if the set S_i is a part of a PP, BP or CEHQ, then $\mathcal{L}$ has to contain all planes of this PP, BP or CEHQ. We show this for the case where the set of planes forms a part of a PP. So assume S_i is a subset of the point-pencil with vertex P, and there is a plane $\gamma \notin \mathcal{L}$ through P. This would imply that γ meets at least $q^3 - (x-2)q^2 - (x^2-2x)(q+1)$ planes in $\mathcal{L}$ non-trivially. This gives a contradiction by Theorem 10.2.1 for $i = 1$ and $i = 2$, as $\gamma \notin \mathcal{L}$ intersects precisely $x(q^2+q+1) < q^3 - (x-2)q^2 - (x^2-2x)(q+1)$ planes of $\mathcal{L}$ in a point or in a line. This argument also works for the BP and CEHQ, so we can conclude that if $\mathcal{L}$ contains an S_i which is a part of a PP, BP or CEHQ, then $\mathcal{L}$ has to contain the whole PP, BP or CEHQ respectively, which we will call $\mathcal{L}_i$.

Remark first that $\mathcal{L}$ cannot contain a BP with base π as then $\pi \in \mathcal{L}$ intersects $q^3 + q^2 + q > q^2 + q + x - 1$ planes of $\mathcal{L}$ in a line, which gives a contradiction with Theorem 10.2.1. This implies that all sets $\mathcal{L}_i$ are PP's or CEHQ's. Now we show that every two sets of planes $\mathcal{L}_i$ and $\mathcal{L}_j$ are disjoint. Suppose first that $\mathcal{L}_i$ and $\mathcal{L}_j$ are two PP's with vertices P_i and P_j respectively, that have at least a plane in common. Then there are at most $q+1$ planes in $\mathcal{L}_i \cap \mathcal{L}_j$ and let β be one of them. Now we see that β meets at least $2(q^3+q^2+q+1)-(q+1)$ elements of $\mathcal{L}$ non-trivially, contradicting Theorem 10.2.1. If $\mathcal{L}_i$ and $\mathcal{L}_j$ are two CEHQ's or a CEHQ and a PP that have at least a plane in common, then we can use the same arguments as above: In both cases, there are at most $q+1$ planes in $\mathcal{L}_i \cap \mathcal{L}_j$, which implies that a plane $\beta \in \mathcal{L}_i \cap \mathcal{L}_j$ meets at least $2(q^3+q^2+q+1)-(q+1)$ elements of $\mathcal{L}$ non-trivially, contradicting Theorem 10.2.1.

Now we know that $\mathcal{L}$ contains the disjoint union of $c \leq x$ sets $\mathcal{L}_i$ of planes, where every set is a PP or CEHQ. As the number of planes in a PP or CEHQ equals $(q^2+1)(q+1)$, and the total number of planes in $\mathcal{L}$ equals $x(q^2+1)(q+1)$ (see Lemma 10.2.4(2)), we see that $\mathcal{L}$ equals the union of x sets $\mathcal{L}_i$ such that any two sets have no plane in common.

To finish this proof, we want to show that the only possible composition of $\mathcal{L}$ consists of PP's and embedded hyperbolic quadrics. If $\mathcal{L}$ contains one class of an embedded hyperbolic quadric, then $\mathcal{L}$ also contains the other class of this hyperbolic quadric. This also follows from Theorem 10.2.1: suppose $\mathcal{L}$ contains only one class of an embedded hyperbolic quadric and let π be a plane of the other class of this embedded hyperbolic quadric. Then we can show that π is also a plane of $\mathcal{L}$: we know that π meets q^2+q+1 planes of the hyperbolic quadric in a line, so at least so many planes of $\mathcal{L}$, in a line. But if $\pi \notin \mathcal{L}$, then, by Theorem 10.2.1, π can only meet $x < \sqrt[3]{2q^2}$ planes of $\mathcal{L}$ in a line, a contradiction.

This implies that $\mathcal{L}$ has to be the union of point-pencils and embedded hyperbolic quadrics that pairwise have no plane in common. Note that two point-pencils have no plane in common if the corresponding vertices are non-collinear. As there exists a partial ovoid of size $q+1$ in $\mathcal{P}$, we can find x pairwise disjoint point-pencils. Note that for q odd and $\mathcal{P} = W(5,q)$, there are no embedded hyperbolic quadrics, so in this case $\mathcal{L}$ is the union of x point-pencils with non-collinear vertices. We end the proof by showing that, for $\mathcal{P} = Q(6,q)$ or $\mathcal{P} = W(5,q)$ and q even, there exist embedded hyperbolic quadrics in $\mathcal{P}$ that have no plane in common. It suffices to show this only for $\mathcal{P} = Q(6,q)$, by the connection between $Q(6,q)$ and $W(5,q)$ for q even. Consider two embedded hyperbolic quadrics $Q^+(5,q)$ in $Q(6,q)$, that intersect in a parabolic quadric $Q(4,q)$. These two hyperbolic quadrics have no planes in common as the generators of $Q(4,q)$ are lines. Note that the union of embedded hyperbolic quadrics that pairwise have no plane in common, together with the union of point-pencils with non-collinear vertices not contained in the embedded hyperbolic quadrics, is a degree one Cameron-Liebler set by Lemma 10.2.4(4), as a point-pencil is a degree one Cameron-Liebler set and for $\mathcal{P} \neq W(5,q)$ or q even, an embedded hyperbolic quadric of the same rank is also a degree one Cameron-Liebler set. ■

This theorem agrees with Conjecture 5.1.3 in [59], as this conjecture says that every degree one Cameron-Liebler set in a finite classical polar space, with rank d sufficiently large, is the union of non-degenerate hyperplane sections and point-pencils that pairwise have no generator in common.

Remark 10.4.8. Recall that the union of point-pencils and embedded hyperbolic quadrics, that pairwise have no plane in common, is also an example of a degree one Cameron-Liebler set of generators in the other polar spaces of type III (see Lemma 10.2.4 and Example 10.2.8).

We also note that we could not generalize this classification result to other classical polar spaces, as for these polar spaces, there is not enough information known about large EKR sets in these polar spaces. For the polar spaces $Q^+(4n+1,q)$, there are some EKR results in [34]. Since in this case, the large examples of EKR sets have much more elements than the largest known Cameron-Liebler sets, we cannot use these results.

10.5 New example of a degree one Cameron-Liebler set in $Q^+(5,q)$

In this section, we give an example of a degree one Cameron-Liebler set of generators in $Q^+(5,q)$, $q = p^h$ odd, found by dr. Maarten De Boeck, prof. Morgan Rodgers and myself. To explain the construction of the example, we use the Klein correspondence between the lines of $\mathrm{PG}(3,q)$ and the points of $Q^+(5,q)$, see Section 1.5. Recall that the generators of $Q^+(5,q)$ are planes which can be divided into two classes (see Remark 1.5.6), the Latin planes and the Greek planes. More precisely, by the Klein correspondence, the points of a Latin plane in $Q^+(5,q)$ correspond to the set of lines through a fixed point in $\mathrm{PG}(3,q)$, and the points of a Greek plane in $Q^+(5,q)$ correspond to the set of lines in a fixed plane in $\mathrm{PG}(3,q)$.

Consider the hyperbolic quadric $Q = Q^+(3,q)$ in $\mathrm{PG}(3,q)$, defined by the equation $x_0x_1+x_2x_3 = 0$. The lines of Q correspond to the set of points of two conics $C \cup C'$ in $Q^+(5,q)$, such that for the planes $\alpha = \langle C\rangle$ and $\alpha' = \langle C'\rangle$, it holds that α' is the image of α under the polarity of $Q^+(5,q)$.

Every point $P \in \mathrm{PG}(3,q)$ gives rise to a Latin plane π_l^P and a Greek plane π_g^P in $Q^+(5,q)$: the points of π_l^P correspond to all the lines through P in $\mathrm{PG}(3,q)$, and the points of π_g^P correspond to the all lines in the plane $P^\perp$. Here, $\perp$ is the polarity related to the quadric Q in $\mathrm{PG}(3,q)$, with corresponding matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Definition 10.5.1. A point $P(x_0,x_1,x_2,x_3) \in \mathrm{PG}(3,q)$ is a *square point* if $x_0x_1+x_2x_3$ is a square different from 0 in $\mathbb{F}_q$. A point $P(x_0,x_1,x_2,x_3) \in \mathrm{PG}(3,q)$ is a *non-square point* if $x_0x_1+x_2x_3$ is a non-square in $\mathbb{F}_q$.

Now we can partition the set of planes in $Q^+(5,q)$ into the following sets.

- $\mathcal{S}_l = \{\pi_l^P | P \text{ is a square point}\}$
- $\mathcal{NS}_l = \{\pi_l^P | P \text{ is a non-square point}\}$
- $\mathcal{O}_l = \{\pi_l^P | P \in Q\}$
- $\mathcal{S}_g = \{\pi_g^P | P \text{ is a square point}\}$
- $\mathcal{NS}_g = \{\pi_g^P | P \text{ is a non-square point}\}$
- $\mathcal{O}_g = \{\pi_g^P | P \in Q\}$

It is known that a 2-secant to Q in $\mathrm{PG}(3,q)$, q odd, contains $\frac{q-1}{2}$ square points and $\frac{q-1}{2}$ non-square points. A line disjoint from Q in $\mathrm{PG}(3,q)$ contains $\frac{q+1}{2}$ square points and $\frac{q+1}{2}$ non-square points. For a tangent line ℓ to Q, there are two possibilities; ℓ contains q square points, or ℓ contains q non-square points, see [72, Table 15.5(c)]. In the first case, ℓ is a *square tangent line*. In the latter case, ℓ is a *non-square tangent line*.

We partition the set of points in $Q^+(5,q)$ into the following sets.

- The set $\mathcal{X}_{1S}$ of points in $Q^+(5,q)$ corresponding to the square tangent lines to Q.
- The set $\mathcal{X}_{1NS}$ of points in $Q^+(5,q)$ corresponding to the non-square tangent lines to Q.
- The set $\mathcal{X}_2$ of points in $Q^+(5,q)$ corresponding to the 2-secants to Q.
- The set $\mathcal{X}_0$ of points in $Q^+(5,q)$ corresponding to the lines disjoint from Q.
- The set $\mathcal{X}_\infty = C \cup C'$ of points in $Q^+(5,q)$ corresponding to the lines of Q.

We present two lemmas that will be useful in the remainder of the construction.

Lemma 10.5.2. *If l is a square tangent line to Q in $\mathrm{PG}(3,q)$, then $l^\perp$ is a square tangent line if $q \equiv 1 \mod 4$, and $l^\perp$ is a non-square tangent line if $q \equiv 3 \mod 4$. If l is a non-square tangent line to Q in $\mathrm{PG}(3,q)$, then $l^\perp$ is a non-square tangent line if $q \equiv 1 \mod 4$, and $l^\perp$ is a square tangent line if $q \equiv 3 \mod 4$.*

Proof. Consider a tangent line l to Q in $\mathrm{PG}(3,q)$. Since the orthogonal group $PGO_+(4,q)$ of $Q^+(3,q)$ acts transitively on the points of $Q = Q^+(3,q)$ (see [74, Theorem 22.6.4]), we may suppose that l contains the point $(1,0,0,0)$ of Q, and so $l = \langle (1,0,0,0),(0,0,1,t)\rangle$, for a fixed $t \in \mathbb{F}_q \setminus \{0\}$. Note that l is a square tangent line if and only if t is a square in $\mathbb{F}_q$. By using the matrix A of the polarity $\perp$, we find that $T_{(1,0,0,0)}(Q)$ is the plane defined by $x_1 = 0$, while $T_{(0,0,1,t)}(Q)$ is the plane defined by $tx_2 + x_3 = 0$. The intersection of these two planes gives that $l^\perp = \langle (1,0,0,0),(0,0,1,-t)\rangle$. The lemma follows since $l^\perp$ is a square line if and only if $-t$ is a square in $\mathbb{F}_q$, and -1 is a square $\mathbb{F}_q$ if and only if $q \equiv 1 \mod 4$. ■

Lemma 10.5.3. *If l is a bisecant to Q in $\mathrm{PG}(3,q)$, then $l^\perp$ is also a bisecant to Q. Furthermore, if l is a line skew to Q in $\mathrm{PG}(3,q)$, then $l^\perp$ is also skew to Q.*

Proof. Note that for a bisecant l to Q, we have that $l \cap Q$ is a hyperbolic quadric $Q^+(1,q)$. For a line l skew to Q, we have that $l \cap Q$ is empty and is equal to $Q^-(1,q)$. The lemma follows now from [74, Theorem 22.7.2]. ■

In the following proposition, we prove that the partitions $\{\mathcal{X}_{1S}, \mathcal{X}_{1NS}, \mathcal{X}_2, \mathcal{X}_0, \mathcal{X}_\infty\}$ and $\{\mathcal{S}_l, \mathcal{S}_g, \mathcal{NS}_l, \mathcal{NS}_g, \mathcal{O}_l, \mathcal{O}_g\}$ form a point-tactical decomposition.

Proposition 10.5.4. *The partition of the points $\{\mathcal{X}_{1S}, \mathcal{X}_{1NS}, \mathcal{X}_2, \mathcal{X}_0, \mathcal{X}_\infty\}$ and the partition of the planes $\{\mathcal{S}_l, \mathcal{S}_g, \mathcal{NS}_l, \mathcal{NS}_g, \mathcal{O}_l, \mathcal{O}_g\}$ of $Q^+(5,q)$ give a point-tactical decomposition with matrix B_1 if $q \equiv 1 \mod 4$ and the matrix B_3 if $q \equiv 3 \mod 4$.*

$$B_1 = \begin{array}{cc} \begin{array}{cccccc} \mathcal{S}_l & \mathcal{S}_g & \mathcal{NS}_l & \mathcal{NS}_g & \mathcal{O}_l & \mathcal{O}_g \end{array} & \\ \begin{pmatrix} q & q & 0 & 0 & 1 & 1 \\ 0 & 0 & q & q & 1 & 1 \\ \frac{q-1}{2} & \frac{q-1}{2} & \frac{q-1}{2} & \frac{q-1}{2} & 2 & 2 \\ \frac{q+1}{2} & \frac{q+1}{2} & \frac{q+1}{2} & \frac{q+1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & q+1 & q+1 \end{pmatrix} & \begin{array}{c} \mathcal{X}_{1S} \\ \mathcal{X}_{1NS} \\ \mathcal{X}_2 \\ \mathcal{X}_0 \\ \mathcal{X}_\infty \end{array} \end{array}$$

$$B_3 = \begin{array}{c@{}c} \begin{array}{cccccc} \mathcal{S}_l & \mathcal{S}_g & \mathcal{NS}_l & \mathcal{NS}_g & \mathcal{O}_l & \mathcal{O}_g \end{array} & \\ \left(\begin{array}{cccccc} q & 0 & 0 & q & 1 & 1 \\ 0 & q & q & 0 & 1 & 1 \\ \frac{q-1}{2} & \frac{q-1}{2} & \frac{q-1}{2} & \frac{q-1}{2} & 2 & 2 \\ \frac{q+1}{2} & \frac{q+1}{2} & \frac{q+1}{2} & \frac{q+1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & q+1 & q+1 \end{array}\right) & \begin{array}{c} \mathcal{X}_{1S} \\ \mathcal{X}_{1NS} \\ \mathcal{X}_2 \\ \mathcal{X}_0 \\ \mathcal{X}_\infty \end{array} \end{array}$$

Proof. We find these matrices by using the Klein correspondence and so, we will prove the lemma using the lines of $\mathrm{PG}(3,q)$ instead of the points of $Q^+(5,q)$. This includes that we will use point-pencils of lines and the lines in fixed planes of $\mathrm{PG}(3,q)$, instead of the planes in $Q^+(5,q)$.

We start with the case $q \equiv 1 \mod 4$.

The first row of B_1 follows by investigating a square tangent line l to Q in $\mathrm{PG}(3,q)$. Since l contains q square points, and no non-square points, l is contained in q point-pencils with vertex a square point, and l is contained in no point-pencils with vertex a non-square point. This explains the first and third element in the first row. For the second and fourth element, q and 0, in the first row, we have that $l \subset R^\perp \iff R \in l^\perp$, with $R \in \mathrm{PG}(3,q)$. From Lemma 10.5.2, we find that $l^\perp$ is a square tangent line, and so that there are q possibilities for R if R is a square point, and no possibilities for R if R is a non-square point. The line l contains one point $P \in Q$ and so it is contained in one point-pencil with vertex in Q and l is contained in one plane $P^\perp$. This gives the last two elements of the first row. The second row of B_1 follows from analogous arguments.

For the third row in B_1, we consider a bisecant l to Q in $\mathrm{PG}(3,q)$. The first and third element of this row follow since l contains $\frac{q-1}{2}$ square points and $\frac{q-1}{2}$ non-square points. Hence, l is contained in $\frac{q-1}{2}$ point-pencils with vertex a square point, and $\frac{q-1}{2}$ point-pencils with vertex a non-square point. For the second and the fourth element of the third row, we use the fact that $l \in R^\perp \iff R \in l^\perp$, and that $l^\perp$ is also a bisecant, see Lemma 10.5.3. Hence, $l^\perp$ contains $\frac{q-1}{2}$ square points and $\frac{q-1}{2}$ non-square points. The last two elements of the row follow since l contains two points $P_1, P_2 \in Q$. Hence, l is contained in the point-pencils through P_1 and P_2, and l is contained in the planes $P_3^\perp$ and $P_4^\perp$, with P_3 and P_4 the two points of Q on $l^\perp$.

For the fourth row in B_1, we consider a line l skew to Q in $\mathrm{PG}(3,q)$. The first and third element of this row follow since l contains $\frac{q+1}{2}$ square points and $\frac{q+1}{2}$ non-square points. Hence, l is contained in $\frac{q+1}{2}$ point-pencils with vertex a square point, and $\frac{q+1}{2}$ point-pencils with vertex a non-square point. For the second and the fourth element, we again use the fact that $l \in R^\perp \iff R \in l^\perp$, and that $l^\perp$ is also skew to Q, see Lemma 10.5.3. Hence, $l^\perp$ contains $\frac{q+1}{2}$ square points and $\frac{q+1}{2}$ non-square points. The last two elements of the row follow since l contains no points in Q.

The last row of B_1 follows since a line l of Q is contained in $q+1$ tangent planes and in $q+1$ point-pencils with vertex a point of l.

The proof for $q \equiv 3 \mod 4$ is analogous. ■

Theorem 10.5.5. *Let q be an odd prime power.*

- *The sets $\mathcal{S}_l \cup \mathcal{S}_g$, $\mathcal{NS}_l \cup \mathcal{NS}_g$ and $\mathcal{O}_l \cup \mathcal{O}_g$ are degree one Cameron-Liebler sets of planes in $Q^+(5,q)$, with parameter $\frac{q(q-1)}{2}$, $\frac{q(q-1)}{2}$ and $q+1$ respectively, for $q \equiv 1 \mod 4$.*
- *The sets $\mathcal{S}_l \cup \mathcal{NS}_g$, $\mathcal{S}_g \cup \mathcal{NS}_l$ and $\mathcal{O}_l \cup \mathcal{O}_g$ are degree one Cameron-Liebler sets of planes in $Q^+(5,q)$, with parameter $\frac{q(q-1)}{2}$, $\frac{q(q-1)}{2}$ and $q+1$ respectively, for $q \equiv 3 \mod 4$.*

Proof. We prove this theorem for $q \equiv 3 \mod 4$. The proof for $q \equiv 1 \mod 4$ is analogous.

From the previous proposition, and from Lemma 1.8.2, we find the following equations. Here, A is the point-plane incidence matrix of $Q^+(5,q)$.

$$\begin{aligned} A^T\chi_{1S} &= q\chi_{\mathcal{S}_l} + q\chi_{\mathcal{NS}_g} + \chi_{\mathcal{O}_l} + \chi_{\mathcal{O}_g} \\ A^T\chi_{1NS} &= q\chi_{\mathcal{S}_g} + q\chi_{\mathcal{NS}_l} + \chi_{\mathcal{O}_l} + \chi_{\mathcal{O}_g} \\ A^T\chi_2 &= \frac{q-1}{2}(\chi_{\mathcal{S}_l} + \chi_{\mathcal{S}_g} + \chi_{\mathcal{NS}_l} + \chi_{\mathcal{NS}_g}) + 2(\chi_{\mathcal{O}_l} + \chi_{\mathcal{O}_g}) \\ A^T\chi_\infty &= (q+1)(\chi_{\mathcal{O}_l} + \chi_{\mathcal{O}_g}). \end{aligned}$$

After some calculations, we find:

$$\begin{aligned} \chi_{\mathcal{S}_l} + \chi_{\mathcal{NS}_g} &= A^T\left(\frac{3q+1}{2q(q+1)}\chi_{1S} + \frac{q-1}{2q(q+1)}\chi_{1NS} - \frac{1}{q+1}\chi_2\right) \\ \chi_{\mathcal{S}_g} + \chi_{NS_l} &= A^T\left(\frac{q-1}{2q(q+1)}\chi_{1S} + \frac{3q+1}{2q(q+1)}\chi_{1NS} - \frac{1}{q+1}\chi_2\right) \\ \chi_{\mathcal{O}_l} + \chi_{\mathcal{O}_g} &= \frac{1}{q+1}A^T\chi_\infty. \end{aligned}$$

The sets $\mathcal{S}_l \cup \mathcal{NS}_g$, $\mathcal{S}_g \cup \mathcal{NS}_l$ and $\mathcal{O}_l \cup \mathcal{O}_g$ are contained in the image of A^T, and so they are degree one Cameron-Liebler sets of planes in $Q^+(5,q)$, for $q \equiv 3 \mod 4$. The parameters of the Cameron-Liebler sets follow immediately from their size, see Lemma 10.2.4.

Analogously, we find that the sets $\mathcal{S}_l \cup \mathcal{S}_g$, $\mathcal{NS}_l \cup \mathcal{NS}_g$ and $\mathcal{O}_l \cup \mathcal{O}_g$ are degree one Cameron-Liebler sets of planes in $Q^+(5,q)$, for $q \equiv 1 \mod 4$. ■

Remark 10.5.6. Note that the Cameron-Liebler sets $\mathcal{O}_l \cup \mathcal{O}_g$ are the union of $q+1$ point-pencils, whose points are the elements of the conic C. Moreover, this set is also the set of point-pencils whose points are the elements of the conic C'. Hence, this example is a well known Cameron-Liebler set. The other determined Cameron-Liebler sets in Theorem 10.5.5 are new examples, in the sense that they are not a union of point-pencils.

Proposition 10.5.7. *The sets* $\mathcal{S}_l \cup \mathcal{S}_g$*, and* $\mathcal{NS}_l \cup \mathcal{NS}_g$*, for* $q \equiv 1 \mod 4$*, and the sets* $\mathcal{S}_l \cup \mathcal{NS}_g$ *and* $\mathcal{S}_g \cup \mathcal{NS}_l$*, for* $q \equiv 3 \mod 4$ *are not the union of point-pencils whose points are pairwise non-collinear.*

Proof. We prove this proposition for the set $\mathcal{L} = \mathcal{S}_l \cup \mathcal{S}_g$, if $q \equiv 1 \mod 4$. The proofs for the other cases are analogous. Suppose from the contrary that $\mathcal{L}$ consists of point-pencils. Since the parameter of $\mathcal{L}$ is $\frac{q(q-1)}{2}$, $\mathcal{L}$ must consist of this many point-pencils. Let P be the base point of one of these point-pencils. By investigating the sum of the first two columns of the matrix B_1 in Proposition 10.5.4, we find that P contains $2q$, 0, $q-1$, $q+1$ or 0 elements of $\mathcal{L}$ for P contained in $\mathcal{X}_{1S}, \mathcal{X}_{1NS}, \mathcal{X}_2, \mathcal{X}_0$, or $\mathcal{X}_\infty$, respectively. Hence, we find in any case that $\mathcal{L}$ cannot contain all planes of $Q^+(5,q)$ through P, which gives the contradiction. ■

Part III

Linear Sets

11 Translation hyperovals and $\mathbb{F}_2-$linear sets of pseudoregulus type

> *I don't believe that life is linear. I think of it as circles - concentric circles that connect.*
>
> —Michelle Williams

In this last part, we discuss a research project on linear sets. dr. Geertrui Van de Voorde and I investigated point sets defined by translation hyperovals in the André/Bruck-Bose representation. The results in this chapter are based on [49].

We show that the affine point sets of translation hyperovals in the André/Bruck-Bose plane representation of $\mathrm{PG}(2, q^k)$ are precisely those that have a scattered $\mathbb{F}_2$-linear set of pseudoregulus type in $\mathrm{PG}(2k-1, q)$ as set of directions. This correspondence is used to generalise the results of Barwick and Jackson who provided a characterisation of translation hyperovals in $\mathrm{PG}(2, q^2)$, see [7].

11.1 Introduction

Recall, from Section 1.6, that a translation hyperoval in $\mathrm{PG}(2, q)$ is a hyperoval H such that there exists a bisecant ℓ of H with the property that the group of elations with axis ℓ acts transitively on the points of H not on ℓ.

In [7], Barwick and Jackson provided a characterisation of translation hyperovals in $\mathrm{PG}(2, q^2)$: they considered a set $\mathcal{C}$ of points in $\mathrm{PG}(4, q)$, q even, with certain combinatorial properties with respect to the planes of $\mathrm{PG}(4, q)$ (see Section 11.3 for details). They proved that the set $\mathcal{C}'$ of directions determined by the points of $\mathcal{C}$ has the property that every line intersects $\mathcal{C}'$ in $0, 1, 3$ or $q-1$ points. They then used this to construct a Desarguesian line spread $\mathcal{S}$ in $\mathrm{PG}(3, q)$, such that in the corresponding André/Bruck-Bose plane $\mathcal{P}(\mathcal{S}) \cong \mathrm{PG}(2, q^2)$, the points corresponding to $\mathcal{C}$ form a translation hyperoval. This extended the work done in [8], where the same authors gave a similar characterisation of André/Bruck-Bose representation of conics for q odd.

We will generalise the combinatorial characterisation provided by Barwick and Jackson for translation hyperovals in $\mathrm{PG}(2, q^k), \forall k \geq 2$. In order to do this, we elaborate on the correspondence between translation hyperovals and linear sets (see e.g. [79, 82]).

11.1.1 Linear sets

Linear sets are a central object in finite geometry and have been studied intensively, mainly due to the connection with other objects such as semifield planes, blocking sets, and more recently, MRD codes (see e.g. [83, 86, 100]).

Let V be an r-dimensional vector space over $\mathbb{F}_{q^n}$, let Ω be the projective space $\mathrm{PG}(V) = \mathrm{PG}(r-1, q^n)$. A set T is said to be an $\mathbb{F}_q$*-linear set* of Ω of rank t if it is defined by the non-zero vectors of an $\mathbb{F}_q$-vector subspace U of V of vector dimension t, i.e.

$$T = L_U = \left\{\langle u\rangle_{\mathbb{F}_{q^n}} | u \in U \setminus \{0\}\right\}.$$

By field reduction, the point set of $\mathrm{PG}(r-1, q^n)$ corresponds to a set $\mathcal{D}$ of $(n-1)$-dimensional subspaces of $\mathrm{PG}(rn-1, q)$, which partitions the point set of $\mathrm{PG}(rn-1, q)$. These subspaces form a Desarguesian $(n-1)$-spread in $\mathrm{PG}(rn-1, q)$. Using coordinates, we see that a point $P = (x_0, x_1, \ldots, x_{r-1})_{q^n} \in \mathrm{PG}(r-1, q^n)$ corresponds to the set $\{(\alpha x_0, \alpha x_1, \ldots, \alpha x_{r-1})_q | \alpha \in \mathbb{F}_{q^n}\}$ in $\mathrm{PG}(rn-1, q)$. Note that we have used r coordinates from $\mathbb{F}_{q^n}$, defined up to $\mathbb{F}_q$-scalar multiple, to define points of $\mathrm{PG}(rn-1, q)$, and the set $\{(\alpha x_0, \alpha x_1, \ldots, \alpha x_{r-1})_q | \alpha \in \mathbb{F}_{q^n}\}$ consists of $\frac{q^n-1}{q-1}$ different points forming an $(n-1)$-dimensional space over $\mathbb{F}_q$. Hence, we find that $\mathcal{D}$ is given by the set of $(n-1)$-spaces

$$\{(\alpha x_0, \alpha x_1, \ldots, \alpha x_{r-1})_q | \alpha \in \mathbb{F}_{q^n}\} \text{ for all } (x_0, x_1, \ldots, x_{r-1}) \in V(r, q^n).$$

Note that these coordinates for points in $\mathrm{PG}(rn-1, q)$ can be transformed into the usual coordinates consisting of rn elements of $\mathbb{F}_q$ by representing the elements of $\mathbb{F}_{q^n}$ as the n coordinates with respect to a fixed basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$.

We also have a more geometric perspective on the notion of a linear set; namely, an $\mathbb{F}_q$-linear set of rank t is a set T of points of $\mathrm{PG}(r-1, q^n)$ for which there exists a subspace π of (projective) dimension $t-1$ in $\mathrm{PG}(rn-1, q)$ such that the points of T correspond to the elements of $\mathcal{D}$ that have a non-empty intersection with π. For more on this approach to linear sets, we refer to [86]. If the subspace π intersects each spread element in at most a point, then π is called *scattered* with respect to $\mathcal{D}$ and the associated linear set is called a *scattered* linear set.

Note that if π is $(n-1)$-dimensional and scattered, then the associated $\mathbb{F}_q$-linear set has rank n and has exactly $\frac{q^n-1}{q-1}$ points, and conversely. We will make use of the following bound on the rank of a scattered linear set.

Result 11.1.1 ([17, Theorem 4.3]). *The rank of a scattered* $\mathbb{F}_q$*-linear set in* $\mathrm{PG}(r-1, q^n)$ *is at most* $rn/2$.

A *maximum scattered linear set* is a scattered $\mathbb{F}_q$-linear set in $\mathrm{PG}(r-1, q^n)$ with rank $rn/2$. In this project we work with maximum scattered linear sets to which a geometric structure, called *pseudoregulus*, can be associated. These linear sets were introduced by G. Marino, O. Polverino and R. Trombetti in [90] and were generalized by M. Lavrauw and G. Van de Voorde in [85]. The name *pseudoregulus* originates from the geometrical construction of Freeman [61]. For more information, we refer to [50, 87].

Definition 11.1.2. Let S be a scattered $\mathbb{F}_q$-linear set of $\mathrm{PG}(2k-1, q^n)$ of rank kn, where $n, k \geq 2$. We say that S is of *pseudoregulus type* if

1. there exist $m = \frac{q^{nk}-1}{q^n-1}$ pairwise disjoint lines of $\mathrm{PG}(2k-1, q^n)$, say $s_1, s_2, \ldots, s_m$, such that
$$|S \cap s_i| = \frac{q^n-1}{q-1} \quad \forall i = 1, \ldots, m,$$
2. there exist exactly two $(k-1)$-dimensional subspaces T_1 and T_2 of $\mathrm{PG}(2k-1, q^n)$ disjoint from S such that $T_j \cap s_i \neq \emptyset$ for each $i = 1, \ldots, m$ and $j = 1, 2$.

The set of lines s_i, $i = 1, \ldots, m$, is called the *pseudoregulus* of $\mathrm{PG}(2k-1, q^n)$ associated with the linear set S and we refer to T_1 and T_2 as *transversal spaces* to this pseudoregulus. Since a maximum scattered linear set spans the whole space, we see that the transversal spaces are disjoint.

For $n = 3$, it is known that every maximum scattered linear set of $\Pi = \mathrm{PG}(2k-1, q^3)$, $k \geq 2$, is of pseudoregulus type, and they are all equivalent under the collineation group of Π, see [84, 85, 90].

More in general, we need the following result of [87]. Applied to $\mathbb{F}_2$-linear sets, this gives us the following result.

Result 11.1.3 ([87, Theorem 3.12]). *Each $\mathbb{F}_2$-linear set of* $\mathrm{PG}(2k-1, q)$*, q even, of pseudoregulus type, is of the form $L_{\rho,f}$ with*

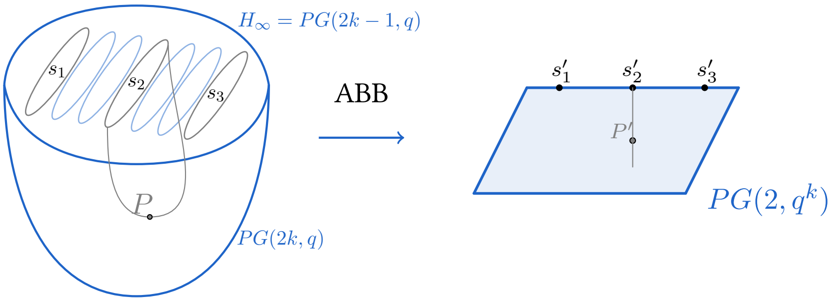
$$L_{\rho,f} = \left\{ (u, \rho f(u))_q \, | u \in U_0 \right\},$$

with $\rho \in \mathbb{F}_q^*$, U_0, U_∞ the k-dimensional vector spaces corresponding to the transversal spaces T_0, T_∞ and with $f : U_0 \to U_\infty$ an invertible semi-linear map with companion automorphism $\sigma \in Aut(\mathbb{F}_q)$, $Fix(\sigma) = \{0, 1\}$.

Note that in the previous result, $\mathrm{PG}(2k-1, q)$ is identified with $\mathrm{PG}(V)$, $V = U_0 \oplus U_\infty$ and a point, corresponding to a vector $v = v_0 + v_\infty \in U_0 \oplus U_\infty$, has coordinates $(v_0, v_\infty)_q$.

11.1.2 The Barlotti-Cofman and André/Bruck-Bose constructions

We start with introducing the André/Bruck-Bose construction (see [1, 24]). Let H_∞ be a hyperplane in $\mathrm{PG}(2k, q)$ and let $\mathcal{S}$ be a $(k-1)$-spread in H_∞. Let $\mathcal{P}$ be the set of affine points, together with the $q^k + 1$ spread elements of $\mathcal{S}$. Let $\mathcal{L}$ be the set of k-spaces in $\mathrm{PG}(2k, q)$ meeting H_∞ in an element of $\mathcal{S}$, together with the hyperplane at infinity H_∞. The incidence structure $(\mathcal{P}, \mathcal{L}, I)$, with I the natural incidence relation, is isomorphic to a projective plane of order q^k, which is called the *André/Bruck-Bose plane*, corresponding with the spread $\mathcal{S}$. The André/Bruck-Bose plane corresponding to a spread $\mathcal{S}$ is Desarguesian if and only if the spread $\mathcal{S}$ is Desarguesian.



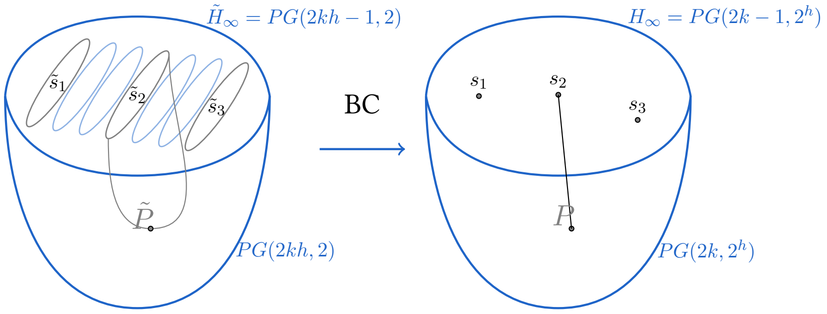
$s_1, s_2, s_3 \in S$,
S is $(k-1)$-spread in H_∞

In this chapter, we will switch between the three different representations of a projective plane $\mathrm{PG}(2, q^k)$, $q = 2^h$. Using the André/Bruck-Bose correspondence, we can, on the one hand, model this plane as a subset of points and k-spaces in $\mathrm{PG}(2k, q)$, determined by a $(k-1)$-spread in a specific hyperplane H_∞ of $\mathrm{PG}(2k, q)$, which we define as the hyperplane at infinity of $\mathrm{PG}(2k, q)$. On the other hand, we can see it as a subset of points and hk-spaces of $\mathrm{PG}(2hk, 2)$ determined by a $(hk-1)$-spread in a specific hyperplane $\tilde{H}_\infty$ of $\mathrm{PG}(2kh, 2)$, which we call the hyperplane at infinity of $\mathrm{PG}(2kh, 2)$. We can switch between the $\mathrm{PG}(2k, q)$-setting and the $\mathrm{PG}(2hk, 2)$-setting by the *Barlotti-Cofman* correspondence, which is a natural generalization of the André/Bruck-Bose correspondence. Note that in this chapter, we use the ˜-symbol for the subspaces in $\mathrm{PG}(2hk, 2)$. This is in contrast with the ˜-symbol in Chapters 4 and 9, used for the projective extension of an affine space.

The Barlotti-Cofman representation of the projective space $\mathrm{PG}(2k, 2^h)$ in $\mathrm{PG}(2hk, 2)$ is defined as follows (see [4]). Let $\mathcal{S}'$ be a Desarguesian $(h-1)$-spread in $\mathrm{PG}(2hk-1, 2)$. Embed $\mathrm{PG}(2hk-1, 2)$ as the hyperplane $\widetilde{H}_\infty$ at infinity in $\mathrm{PG}(2hk, 2)$. Consider the following incidence structure $\mathcal{P}(\mathcal{S}) = (\mathcal{P}, \mathcal{L}, I)$, where incidence is natural:

- The set $\mathcal{P}$ of points consists of the 2^{2hk} affine points P_i in $\mathrm{PG}(2hk, 2)$ (i.e. the points not in $\widetilde{H}_\infty$) together with elements of the $(h-1)$-spread $\mathcal{S}'$ in $\widetilde{H}_\infty$.
- The set $\mathcal{L}$ of lines consists of the following two sets of subspaces in $\mathrm{PG}(2hk, 2)$.
  - The set of h-spaces spanned by an element of $\mathcal{S}'$ and an affine point of $\mathrm{PG}(2hk, 2)$.
  - The set of $(2h-1)$-spaces in $\widetilde{H}_\infty$ spanned by two different elements of $\mathcal{S}'$.

This incidence structure $(\mathcal{P}, \mathcal{L}, I)$ is isomorphic to $\mathrm{PG}(2k, 2^h)$, and let H_∞ be the hyperplane containing all points corresponding with the $(h-1)$-spread $\mathcal{S}'$. We use the notation P for the affine point of $\mathrm{PG}(2k, 2^h)$ (i.e. a point not contained in H_∞) which corresponds to the affine point $\widetilde{P} \in \mathrm{PG}(2hk, 2)$. A point, say R in H_∞, corresponds to the element $\mathcal{S}'(R)$ of the $(h-1)$-spread $\mathcal{S}'$ in $\widetilde{H}_\infty$.



$\tilde{s}_1, \tilde{s}_2, \tilde{s}_3 \in \mathcal{X}$,
$\mathcal{S}'$ is a $(h-1)$-spread in $\tilde{H}_\infty$

As already mentioned above, we will work in the following three projective spaces:

- The $2k$-dimensional projective space $\Psi_q = \mathrm{PG}(2k, q), q = 2^h, h > 2$, with the $(2k-1)$-space at infinity called H_∞.
- The projective plane $\Pi_{q^k} = \mathrm{PG}(2, q^k)$, $q = 2^h$, with line at infinity called ℓ_∞. Given a Desarguesian $(k-1)$-spread $\mathcal{S}$ in H_∞ in Ψ_q, the plane Π_{q^k} is obtained by the André-Bruck-Bose construction using $\mathcal{S}$.
- The $2hk$-dimensional projective space $\Lambda_2 = \mathrm{PG}(2hk, 2)$, with the $(2hk-1)$-space $\widetilde{H}_\infty$ at infinity. Note that a Desarguesian $(h-1)$-spread $\mathcal{S}'$ in $\widetilde{H}_\infty$ gives rise to the Barlotti-Cofman representation of Ψ_q. Also vice versa, the Barlotti-Cofman representation of Ψ_q defines a Desarguesian $(h-1)$-spread $\mathcal{S}'$ in $\widetilde{H}_\infty$. Moreover, if $\mathcal{S}$ is the $(k-1)$-spread in H_∞ in Ψ_q such that Π_{q^k} is the corresponding projective plane, the André-Bruck-Bose representation of Π_{q^k} in Λ_2 gives rise to a Desarguesian $(hk-1)$-spread $\widetilde{\mathcal{S}}$ in $\widetilde{H}_\infty$, such that $\mathcal{S}'$ is a subspread of $\widetilde{\mathcal{S}}$.

11.1.3 Main theorem

In this chapter, we prove the following Main Theorem. A consequence of this result is the generalization of the characterization of translation hyperovals in $\mathrm{PG}(2, q^2)$ in [7].

Consider $\Psi_q = \mathrm{PG}(2k, q)$ and the hyperplane H_∞ of $\mathrm{PG}(2k, q)$. Recall that a point of $\mathrm{PG}(2k, q)$ is called affine if it is not contained in H_∞. Likewise, a line is called affine if it is not contained in H_∞. Let P_1, P_2 be affine points, then the point $P_1P_2 \cap H_\infty$ is the *direction* determined by the line P_1P_2. If $\mathcal{Q}$ is a set of affine points, then the *directions determined by* $\mathcal{Q}$ are all points of H_∞ that appear as the direction of a line P_iP_j, for some $P_i, P_j \in \mathcal{Q}$.

Theorem 11.1.4. *Let $\mathcal{Q}$ be a set of q^k affine points in $\mathrm{PG}(2k, q)$, $q = 2^h$, $h \geq 4$, $k \geq 2$, determining a set D of $q^k - 1$ directions in the hyperplane at infinity $H_\infty = \mathrm{PG}(2k-1, q)$. Suppose that every line has $0, 1, 3$ or $q-1$ points in common with the point set D. Then*

(1) D is an $\mathbb{F}_2$-linear set of pseudoregulus type.

(2) There exists a Desarguesian spread $\mathcal{S}$ in H_∞ such that, in the André/Bruck-Bose plane $\mathcal{P}(\mathcal{S}) \cong \mathrm{PG}(2, q^k)$, with H_∞ corresponding to the line l_∞, the points of $\mathcal{Q}$ together with 2 extra points on ℓ_∞, form a translation hyperoval in $\mathrm{PG}(2, q^k)$.

Vice versa, via the André/Bruck-Bose construction, the set of affine points of a translation hyperoval in $\mathrm{PG}(2, q^k)$, $q > 4, k \geq 2$, corresponds to a set $\mathcal{Q}$ of q^k affine points in $\mathrm{PG}(2k, q)$ whose set of determined directions D is an $\mathbb{F}_2$-linear set of pseudoregulus type. Consequently, every line meets D in $0, 1, 3$ or $q-1$ points.

Note that we work with a set of affine points in $\mathrm{PG}(2k, q)$ whose set of directions is a scattered linear set with specific properties. Using this, we can make the link with translation hyperovals in the André/Bruck-Bose-plane $\mathrm{PG}(2, q^k)$. For this, we used the ideas found by V. Jha, N.L. Johnson and M. Lavrauw in [79, 82], in which a scattered $(k-1)$-space π_H, with respect to a $(k-1)$-spread S in the hyperplane at infinity $H_\infty = \mathrm{PG}(2k-1, 2) \subset \mathrm{PG}(2k, 2)$ was used. Since π_H contains $2^k - 1$ points and since $|S| = 2^k + 1$, it follows that there are two spread elements s_1, s_2 disjoint from π_H. Let Π be a k-space in $\mathrm{PG}(2k, 2)$, with $\Pi \cap H_\infty = \pi_H$, then it can be proven that the affine points of Π, together with s_1 and s_2, correspond to the points of a translation hyperoval in the André/Bruck-Bose-plane, using the spread S.

This idea is also used in several other papers. For example, in [5], the authors gave an explicit construction of infinite families of maximal scattered linear sets in $\mathrm{PG}(n-1, q^t), t \geq 4$ even. For $q = 2$, they used a similar technique to find complete caps in $\mathrm{AG}(n, 2^t)$ of size $2^{\frac{nt}{2}}$. We will use a similar idea in this chapter to generalize the results in [7].

11.2 The proof of the main theorem

From now on, we consider a set $\mathcal{Q}$ satisfying the conditions of Theorem 11.1.4:

- $\mathcal{Q}$ is a set of q^k affine points in $\mathrm{PG}(2k, q)$, $q = 2^h$, $h \geq 4$, $k \geq 2$;
- D, the set of directions determined by $\mathcal{Q}$ at the hyperplane at infinity H_∞, has size $q^k - 1$;
- Every line has $0, 1, 3$ or $q-1$ points in common with the point set D.

11.2.1 The $(q-1)$-secants to D are disjoint

Definition 11.2.1. A 0-*point* in H_∞ is a point $P \notin D$ such that P is contained in at least one $(q-1)$-secant to D.

From Proposition 11.2.5, it will follow that a 0-point is contained in precisely one $(q-1)$-secant to D. We first start with two lemmas.

Lemma 11.2.2. *No three points of $\mathcal{Q}$ are collinear.*

Proof. Let l be an affine line in $\mathrm{PG}(2k,q)$ containing $3 \leq t \leq q$ points of $\mathcal{Q}$, and let $P' = l \cap H_\infty$. A point $P_i \in \mathcal{Q} \setminus l$ determines a plane $\alpha_i = \langle P_i, l\rangle$ such that the line $l_i = \alpha_i \cap H_\infty$ is a $(q-1)$-secant: the lines through P_i and a point of $l \cap \mathcal{Q}$ determine $t \geq 3$ directions of D on the line l_i, different from the point $P' \in D$. So l contains more than three points of D, showing that l_i is a $(q-1)$-secant. Furthermore, the plane α_i contains at most q affine points of $\mathcal{Q}$, as every affine line in α through a 0-point of l_i contains at most one element of $\mathcal{Q}$.

This implies that each of the $q^k - t$ points of $\mathcal{Q} \setminus l$ define a plane α, with $\alpha \cap H_\infty$ a $(q-1)$-secant, and so that α contains at most $q-t$ points of $\mathcal{Q} \setminus l$. This shows that the number of such planes α_i through l, and hence the number of $(q-1)$-secants through P', is at least $\frac{q^k-t}{q-t}$. This gives that there are at least $1 + \frac{q^k-t}{q-t}(q-2) > q^k - 1$ points of D, a contradiction since $t \geq 2$. ■

Lemma 11.2.3. *Let γ be a plane in $\mathrm{PG}(2k,q)$ containing 4 points P_1, P_2, P_3 and P_4 of $\mathcal{Q}$, such that $P_1P_2 \cap P_3P_4 \notin \mathcal{Q} \cup D$. Then γ meets H_∞ in a $(q-1)$-secant to D.*

Proof. By Lemma 11.2.2, no three points of P_1, P_2, P_3, P_4 are collinear. Since $P_1P_2 \cap P_3P_4 \notin D$, we see that P_1P_2 and P_3P_4 define two different directions in H_∞. The lines containing two of the four points P_1, P_2, P_3 and P_4 determine at least 4 directions on the line $\gamma \cap H_\infty$. The statement follows since a line contains $0, 1, 3$ or $q-1$ points of D. ■

Corollary 11.2.4. *Let P_0 be a point in $\mathcal{Q}$. Then, all directions in D are determined by the lines P_0P_i with $P_i \in \mathcal{Q} \setminus \{P_0\}$.*

Proof. From Lemma 11.2.2, it follows that two lines P_0P_i and P_0P_j, $P_i \neq P_j$, are different, and so, determine different points at infinity. The corollary follows since $|D| = q^k - 1$, which is equal to the number of points $P_i \in \mathcal{Q}$, different from P_0. ■

Proposition 11.2.5. *Every two $(q-1)$-secants to D are disjoint.*

Proof. Consider a point $P_0 \in \mathcal{Q}$. Then, by Corollary 11.2.4, all directions in D are determined by the lines P_0P_i with $P_i \in \mathcal{Q} \setminus \{P_0\}$. Let P_i' denote the direction of the line P_0P_i, that is, the point $P_0P_i \cap H_\infty$. We see that a line through a point $P_i' \in D$ contains 0 or 2 points of $\mathcal{Q}$.

Let l_α and l_β be two lines, both containing $q-1$ points of D, with $P' = l_\alpha \cap l_\beta$. Let $\alpha = \langle P_0, l_\alpha\rangle$ and $\beta = \langle P_0, l_\beta\rangle$ and let $\{P_{1\alpha}, P_{2\alpha}\}$ and $\{P_{1\beta}, P_{2\beta}\}$ be the 0-points in l_α and l_β. Note that P' may be amongst these points. It follows from the argument above that there are precisely q points in $\alpha \cap \mathcal{Q}$ and that the affine points of $\mathcal{Q}$ in α together with the two points $P_{1\alpha}, P_{2\alpha}$ form a hyperoval H_α. Similarly, we find a hyperoval H_β in β.

We first suppose that $P' \in D$. This implies that there is a point $P \neq P_0$ of $\mathcal{Q}$ on the line P_0P'. Note that P_0 and P are contained in $H_\alpha \cap H_\beta$.

Consider a point $R \in l_\alpha$, different from $P', P_{1\alpha}, P_{2\alpha}$. Then $R \in D$ and through R, there are $\frac{q}{2}$ bisecants to $H_\alpha \neq l_\alpha$. One of these bisecants contains P and another one contains P_0. Since $q > 8$, there exists a bisecant to H_α through R which intersects the line P_0P in a point $R_0 \notin \{P_0, P, P'\}$.

Through R_0, there are $\frac{q}{2}-2$ bisecants r_i to H_β, different from the lines R_0P, $R_0P_{1\beta}$ and $R_0P_{2\beta}$. Let $r_i \cap l_\beta = R_i, i = 1, \dots, \frac{q}{2}-2$. A plane $\langle R, r_i\rangle$ contains two lines, r_i and $m = RR_0$, both containing two points of $\mathcal{Q}$ and $r_i \cap m = R_0 \notin \mathcal{Q}$. Hence, by Lemma 11.2.3, we find that every line RR_i is a $(q-1)$-secant to D.

So the number of $(q-1)$-secants of the form RR_i is $\frac{q}{2}-2$, and the total number of 0-points on these lines is $2(\frac{q}{2}-2) = q-4$. Let Ω be the set of these 0-points. We call a (≤ 3)*-secant* in $\langle l_\alpha, l_\beta\rangle$ a line with at most 3 points of D. A line through P' in $\langle l_\alpha, l_\beta\rangle$ intersects all lines RR_i. The $q-4$ points of Ω lie on the $q-1$ lines through P' different from l_α and l_β. Since every line RR_i contains precisely two 0-points, we find that for $q > 8$ there are at most 3 (≤ 3)-secants through P': if there are at least four (≤ 3)-secants through P' in $\langle l_\alpha, l_\beta\rangle$, then the number of 0-points of Ω on each of these lines is at least $\frac{q}{2}-2-2$, as we supposed that $P' \in D$. This implies that there would be at least $4(\frac{q}{2}-4) > q-4$ 0-points in Ω, which gives a contradiction for $q \geq 16$.

Now we distinguish different cases depending on the number of (≤ 3)-secants through P'. In each of the cases we will show that there exist at least two (≤ 3)-secants l_1, l_2 in $\langle l_\alpha, l_\beta\rangle$, and a point $X \notin D$ not on these lines. This leads to a contradiction since there are at least $q+1-7$ lines through X, both intersecting l_1 and l_2 in a point not in D, and not through $l_1 \cap l_2$. These lines contain at least 3 points not in D so they have to be (≤ 3)-secants. But this implies that there are at least $1 + (q-6)(q-3) = q^2 - 9q + 19$ points in $\langle l_\alpha, l_\beta\rangle$, not contained in D. On the other hand, there are at most three (≤ 3)-secants through P' and the other lines through P' contain two 0-points. This implies that there are at most $3q + 2(q-2) = 5q - 4 < q^2 - 9q + 19$ points in $\langle l_\alpha, l_\beta\rangle$, not contained in D. This gives a contradiction for $q \geq 16$.

It remains to show that in every case there exist at least two (≤ 3)-secants and a point $X \notin D$, not on these lines.

- Suppose first that there are two or three (≤ 3)-secants through P'. These lines are different from l_α, so they do not contain the point $P_{1\alpha}$. Then $X = P_{1\alpha} \notin D$ is a point not on the (≤ 3)-secants.

- Suppose there is a unique (≤ 3)-secant l through P'. Then every other line through P' contains two 0-points. Suppose first that there exists a 0-point P_1 so that $P_{1\alpha}P_1 \cap l \notin D$. Then $l' = P_{1\alpha}P_1$ contains 3 points not in D, so l' is a (≤ 3)-secant. Note that $P_1 \neq P_{2\alpha}$ as otherwise $P_{1\alpha}P_1 \cap l = l_\alpha \cap l = P' \in D$. Hence, $X = P_{2\alpha} \notin D$ is not contained in $l \cup l'$.

  If there is no point P_1 so that $P_{1\alpha}P_1 \cap l \notin D$, then all $2q-4$ 0-points on the $(q-1)$-secants through P', different from l_α, l_β, lie on at most 2 lines $P_{1\alpha}P_1$ and $P_{1\alpha}P_2$, with $P_1, P_2 \in D \cap l \setminus \{P'\}$. But then $P_{1\alpha}P_1$ and $P_{1\alpha}P_2$ are (≤ 3)-secants. Note that these lines are different from l_α, and so, they do not contain $P_{2\alpha}$. Hence, we may take $X = P_{2\alpha}$.

- Suppose all lines through P' are $(q-1)$-secants with Γ the corresponding set of $2q+2$ 0-points. Let $G \in \Gamma$ and consider the $q+1$ lines through G in $\langle l_\alpha, l_\beta\rangle$. The $2q+1$ other points of Γ lie on these lines and since every line contains 2 or at least $q-2$ points not in D, we find that through G there is at least one (≤ 3)-secant l_1. Consider now a point $G' \in \Gamma \setminus l_1$. Through this point there is also a (≤ 3)-secant l_2. The lines $l_1 \cup l_2$ contain at most $2q+1$ points of Γ, so there is at least one 0-point X not contained in these two lines.

This shows that two $(q-1)$-secants cannot meet in a point P' of D. Suppose now that $P' \notin D$. As above, we find for a given point $R \in D \cap l_\alpha$, at least $\frac{q}{2}-2$ $(q-1)$-secants RR_i, different from l_α. But by the previous part, we know that there are no two $(q-1)$-secants through a point $R \in D$. As $\frac{q}{2}-2 \geq 2$, we find a contradiction. ■

We now deduce a corollary that will be useful later.

Corollary 11.2.6. *A $(q-1)$-secant and a 3-secant to D in H_∞ cannot have a 0-point in common.*

Proof. Let l_α be a 3-secant to D, l_β be a $(q-1)$-secant to D, and $P' = l_\alpha \cap l_\beta$ be a 0-point. Pick $P_0 \in \mathcal{Q}$ and let $\alpha = \langle P_0, l_\alpha \rangle$ and $\beta = \langle P_0, l_\beta \rangle$. The points of $Q \cup D$ in α form a Fano plane: let P'_i, $i = 1, 2, 3$, be the three points of D on the line l_α and let P_i, $i = 1, 2, 3$, be the corresponding affine points of $\mathcal{Q}$ so that $P_0P_i \cap l_\alpha = P'_i$. Since there are only three directions P'_1, P'_2, P'_3 of D in α, we find that $\{P_1, P_3, P'_2\}$,$\{P_1, P_2, P'_3\}$ and $\{P_2, P_3, P'_1\}$ are triples of collinear points. Since also $\{P'_1, P'_2, P'_3\}$ and $\{P_0, P_i, P'_i\}$, $i = 1, 2, 3$, are triples of collinear points, we find that the points $\{P_0, P_1, P_2, P_3, P'_1, P'_2, P'_3\}$ define a Fano plane $\mathrm{PG}(2, 2)$. Let R_0 be the point $P'_1P_2 \cap P'P_0$. Note that $R_0 \notin \mathcal{Q}$. As the points of $\mathcal{Q}$ in β form a q-arc, we know that there are at least two lines R_0R_1 and R_0R_2 in β, with $R_1, R_2 \in l_\beta \cap D$, such that both lines contain 2 points of $\mathcal{Q}$. By Lemma 11.2.3 we see that the lines P'_1R_1 and P'_1R_2 are both $(q-1)$-secants through P'_1. This gives a contradiction by Proposition 11.2.5. ■

11.2.2 The set D of directions in H_∞ is a linear set

Recall that we use the notation $\widetilde{P}$ for the affine point in Λ_2, corresponding to the affine point $P \in \Psi_q$. Let $\mathcal{S}'$ be the $(h-1)$-spread in the hyperplane $\widetilde{H}_\infty$ of $\mathrm{PG}(2hk, 2)$ corresponding to the points of the hyperplane H_∞ of Ψ_q. We use the notation $\mathcal{S}'(P')$ for the element of $\mathcal{S}'$ corresponding to the point $P' \in H_\infty$. We will now show that D is an $\mathbb{F}_2$-linear set in H_∞ by showing that its points correspond to spread elements in $\widetilde{H}_\infty$ intersecting some fixed $(hk-1)$-subspace of $\widetilde{H}_\infty$.

Let $\mathscr{Q} = \mathcal{Q} \cup D$, $\widetilde{\mathscr{Q}} = \widetilde{\mathcal{Q}} \cup \widetilde{D}$, with $\widetilde{\mathcal{Q}}$ the union of the points $\widetilde{P}$, with $P \in Q$, and $\widetilde{D}$ the directions in $\widetilde{H}_\infty$ determined by the points of $\widetilde{\mathcal{Q}}$.

Lemma 11.2.7. *Let $P_0, P_1, P_2 \in \mathcal{Q}$ and $P'_i = P_0P_i \cap H_\infty$, $i = 1, 2$. If $P'_1P'_2$ is a 3-secant to D, then the plane in $\mathrm{PG}(2hk, 2)$ spanned by $\widetilde{P}_0$, $\widetilde{P}_1$ and $\widetilde{P}_2$ is contained in $\widetilde{\mathscr{Q}}$.*

Proof. Since $P'_1P'_2$ is not a $(q-1)$-secant, we know that there is a unique point $P'_3 \neq P'_1, P'_2$ in $P'_1P'_2 \cap D$, and a point $P_3 \in \mathcal{Q}$ such that $P'_3 \in P_0P_3$. Let α be the plane spanned by the points P_0, P_1 and P_2. As $\alpha \cap D = \{P'_1, P'_2, P'_3\}$, we find that $\{P_1, P_3, P'_2\}$,$\{P_1, P_2, P'_3\}$ and $\{P_2, P_3, P'_1\}$ are triples of collinear points. As in the proof of Corollary 11.2.6, we find that these points define a Fano plane $\mathrm{PG}(2, 2)$. We claim that the corresponding points $\widetilde{P}_0$, $\widetilde{P}_1$, $\widetilde{P}_2$ and $\widetilde{P}_3$ lie in a plane in $\mathrm{PG}(2hk, 2)$. Suppose these points are not contained in a plane in $\mathrm{PG}(2hk, 2)$, then they span a 3-space β. Since $P'_1 = P_0P_1 \cap P_2P_3$, $\widetilde{P}_0\widetilde{P}_1$ meets $\mathcal{S}'(P'_1)$ in a point, say A_1. Similarly, $\widetilde{P}_2\widetilde{P}_3$ meets $\mathcal{S}'(P'_1)$ in a point, say B_1. Since $\widetilde{P}_0, \widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3$ span a 3-space, $A_1 \neq B_1$. Similarly, the points $A_2 = \widetilde{P}_0\widetilde{P}_2 \cap \mathcal{S}'(P'_2)$ and $B_2 = \widetilde{P}_1\widetilde{P}_3 \cap \mathcal{S}'(P'_2)$ are different and span the line A_2B_2. But now $A_1B_1 \in \mathcal{S}'(\widetilde{P}'_1)$ and $A_2B_2 \in \mathcal{S}'(\widetilde{P}'_2)$ are two lines in the plane $\beta \cap \widetilde{H}_\infty$, so they intersect, a contradiction since the spread elements $\mathcal{S}'(P'_1)$ and $\mathcal{S}'(P'_2)$ are disjoint. ■

Theorem 11.2.8. *The set D is an $\mathbb{F}_2$-linear set.*

Proof. We prove, by induction on $t \in \{2, \ldots, hk\}$, that there exists a t-space β contained in $\widetilde{\mathscr{Q}}$ such that the points in H_∞ corresponding to the spread elements intersecting $\beta \cap \widetilde{H}_\infty$ are not all contained in a single $(q-1)$-secant.

For the induction basis $t = 2$, we use Lemma 11.2.7, and so, we have the following property: if $\widetilde{P}_0$, $\widetilde{P}_1$ and $\widetilde{P}_2$ are three points in $\widetilde{\mathcal{Q}}$ such that the line at infinity of the plane spanned by these points corresponds to a 3-secant in Ψ_q, then we know that all points of $\langle \widetilde{P}_0, \widetilde{P}_1, \widetilde{P}_2 \rangle$ are included in $\widetilde{\mathscr{Q}}$.

Now we suppose that there is a t-space β, with $\beta \subset \widetilde{\mathscr{Q}}$. By the induction hypothesis, we may assume that the points in H_∞, corresponding to the spread elements intersecting $\beta \cap \widetilde{H}_\infty$, are not all contained in a single $(q-1)$-secant.

If $t = hk$, then our proof is finished, so assume that $t < hk$. This implies that there exists a point $\widetilde{G} \in \widetilde{\mathcal{Q}} \setminus \beta$. Let G be the corresponding point in $\mathcal{Q}$ in $\mathrm{PG}(2k, q)$, and let $\gamma = \langle \beta, \widetilde{G} \rangle$. We show that every point $\widetilde{X}$ in $\gamma \setminus \beta$ is a point of $\widetilde{\mathscr{Q}}$. Suppose first that $\widetilde{X}$ is a point at infinity of $\gamma \setminus \beta$, then the line $\widetilde{X}\widetilde{G}$ contains an affine point $\widetilde{Y}$ of β, as β is a hyperplane of γ. But since $\widetilde{G}$ and $\widetilde{Y}$ are points of $\widetilde{\mathcal{Q}}$, we find that $\widetilde{X} \in \widetilde{D} \subset \widetilde{\mathscr{Q}}$.

Suppose now that $\widetilde{X}$ is an affine point in $\gamma \setminus \beta$, and let X be the corresponding point in $\mathrm{PG}(2k, q)$. As the field size in $\mathrm{PG}(2hk, 2)$ is 2, the line $\widetilde{X}\widetilde{G}$ contains 1 extra point $\widetilde{Y}$. This point has to lie in β and in the hyperplane at infinity, so $\widetilde{Y} \in \beta \cap \widetilde{H}_\infty$. Let l_1 be a line through $\widetilde{Y}$ in β corresponding to a 3-secant, which exists since we have seen that not all points corresponding to points of $\beta \cap H_\infty$ are contained in one single $(q-1)$-secant. The plane spanned by $\widetilde{G}$ and l_1 is contained in $\mathscr{Q}$ by Lemma 11.2.7, and hence, since X lies on the line $\widetilde{Y}\widetilde{G}$ which is contained in this plane, $X \in \widetilde{\mathscr{Q}}$. This implies that $\gamma \subseteq \mathscr{Q}$. We can repeat this argument until we find that $\widetilde{\mathscr{Q}}$ is a hk-space in $\mathrm{PG}(2hk, 2)$. ■

Note that D is a scattered linear set since $|D| = q^k - 1 = 2^{hk} - 1 = |\,\mathrm{PG}(hk-1, 2)|$. As D has rank hk, we find that D is maximum scattered.

Remark 11.2.9. In Lemma 11.2.5, we showed that the $(q-1)$-secants to D were disjoint. In Theorem 11.2.8, we have used this to show that D is a maximum scattered $\mathbb{F}_2$-linear set. The fact that $(q-1)$-secants to a maximum scattered $\mathbb{F}_2$-linear set are disjoint, is well-known (see e.g. [87, Proposition 3.2]).

11.2.3 The set D is an $\mathbb{F}_2$-linear set of pseudoregulus type

The proof that D is of pseudoregulus type, is based on some ideas of [85, Lemma 5 and Lemma 7].

Lemma 11.2.10. *There are* $\frac{q^k-1}{q-1}$ *pairwise disjoint* $(q-1)$*-secants to* D *in* $\mathrm{PG}(2k-1, q), q > 4$.

Proof. Let K be the $(hk-1)$-dimensional subspace in $\mathrm{PG}(2hk-1, 2)$ defining the $\mathbb{F}_2$-linear set D and let $\mathcal{S}'$ be the $(h-1)$-spread that corresponds to the point set of $\mathrm{PG}(2k-1, q)$. For every hk-space Y through K in $\mathrm{PG}(2hk-1, 2)$, we find at least one element of $\mathcal{S}'$ that intersects Y in a line since D is maximum scattered. Every line l, through a point of K, such that l lies in an element of $\mathcal{S}'$, defines a hk-space through K, and the number of hk-spaces through K is $2^{hk} - 1$. This implies that there are on average $2^{h-1} - 1 > 2$ lines contained in different spread elements of $\mathcal{S}'$ in a hk-space through K in $\mathrm{PG}(2hk-1, 2)$.

Take a hk-space Y through K with at least two lines contained in spread elements, and let S_1 and S_2 be two elements of $\mathcal{S}'$ that intersect Y in the lines y_1 and y_2 respectively. The $(2h-1)$-space $\langle S_1, S_2 \rangle$ intersects K in at least a plane, as y_1 and y_2 span a 3-space. But this implies that the line l in $\mathrm{PG}(2k-1, q)$, corresponding with $\langle S_1, S_2 \rangle$ contains at least 7 points of D. This implies that l is a $(q-1)$-secant of D, and that $\langle S_1, S_2 \rangle$ intersects K in a $(h-1)$-space α as a $(h-1)$-space contains $2^h - 1 = q - 1$ points. Consider now the h-space $\beta = Y \cap \langle S_1, S_2 \rangle$ through α. Since all of the $2^h + 1$ $(h-1)$-spaces of $\mathcal{S}'$ in $\langle S_1, S_2 \rangle$ intersect β in a point or a line, we find that there are precisely $2^{h-1} - 1$ elements of $\mathcal{S}'$, meeting β, and so Y, in a line. Hence, this proves that a hk-space Y through K, containing at least 2 lines y_1, y_2 in S_1, S_2 respectively, contains at least $2^{h-1} - 1$ lines y_i in different spread elements of $\mathcal{S}'$. Now we prove, by contradiction, that Y cannot contain more lines y_i contained in a spread element. Suppose Y contains another line $y_0 \subset S_0$ with $S_0 \in \mathcal{S}'$, then $y_0 \notin \langle S_1, S_2 \rangle$. Repeating the previous argument for y_1 and y_2 shows that there are two $(2h-1)$-spaces $\langle S_1, S_2\rangle$ and $\langle S_0, S_1\rangle$, both meeting K in a $(h-1)$-space and so, there are two $(q-1)$-secants through $P_1 \in H_\infty$, the point corresponding to the spread element S_1. This gives a contradiction by Proposition 11.2.5.

Since the average number of lines contained in a spread element in a hk-space through K is $2^{h-1} - 1 > 2$, we find that every hk-space through K contains exactly $2^{h-1} - 1$ lines contained in a spread element. In particular, every line $y_i \subset S_i$, with $S_i \in \mathcal{S}'$ and y_i through a point of K, defines a hk-space through K, and so a $(q-1)$-secant. So we find that every point in D is contained in at least one $(q-1)$-secant. As we already proved that two $(q-1)$-secants are disjoint (see Lemma 11.2.5), we find $\frac{q^k-1}{q-1}$ pairwise disjoint $(q-1)$-secants in $\mathrm{PG}(2k-1, q)$. ■

We will first show that the linear set is of pseudoregulus type when $k = 2$. To prove this, we begin with a lemma.

Lemma 11.2.11. *Assume that $k = 2$. Let l be a line in H_∞ through two 0-points, not on the same $(q-1)$-secant, then l contains no points of D.*

Proof. Let l_1 and l_2 be two $(q-1)$-secants in H_∞. Let l be a line through a 0-point of l_1 and through a 0-point of l_2. Recall that l_1 and l_2 are disjoint by Proposition 11.2.5. Every two points $A, B, A \in l_1$, $B \in l_2$, define a third point in D on the line AB. Hence we find, since $|D| = q^2 - 1$, that every point $P \in D \setminus \{l_1, l_2\}$ is uniquely defined as a third point on a line, defined by two points A and B of D in l_1 and l_2 respectively.

Now suppose that l contains a point $X \in D$. Then X lies on a unique line l', intersecting l_1 and l_2 in precisely one point. But then l_1 and l_2 lie in a plane spanned by l and l', a contradiction since l_1 and l_2 are disjoint by Proposition 11.2.5. ■

Proposition 11.2.12. *Assume that $k = 2$. The $(q-1)$-secants to D in $\mathrm{PG}(3, q)$ form a pseudoregulus.*

Proof. By Lemma 11.2.10, it is sufficient to prove that there exist 2 lines in $\mathrm{PG}(3, q)$ that have a point in common with all $(q-1)$-secants to D. Consider three $(q-1)$-secants l_1, l_2 and l_3 and let $P_i, Q_i \in l_i$, $i = 1, 2, 3$, be the corresponding 0-points. Let l_0 be the unique line through P_1 that intersects l_2 and l_3 both in a point, say $R_2 = l_0 \cap l_2$ and $R_3 = l_0 \cap l_3$ respectively. By Proposition 11.2.5 and Corollary 11.2.6, R_2 and R_3 cannot both belong to $\mathcal{Q}$, so suppose R_2 is a 0-point of l_2 (w.l.o.g. $R_2 = P_2$). We see that $l_0 = P_1P_2$ is a line through two 0-points, so R_3 is also a 0-point by Corollary 11.2.11, w.l.o.g. $R_3 = P_3$. By the same argument, we see that Q_1, Q_2 and Q_3 are contained in a line, say l_∞.

Now we want to show that every other $(q-1)$-secant has a 0-point in common with both l_0 and l_∞. Consider a $(q-1)$-secant l_4, different from l_1, l_2, l_3, with 0-points P_4 and Q_4. Consider now again the unique line m through P_4 that intersects l_1 and l_2 in a point. By the previous arguments, m has to contain a 0-point of l_1 and a 0-point of l_2, so $m = l_0$, $m = l_\infty$, $m = P_1Q_2$ or $m = Q_1P_2$. We will show that only the first two possibilities can occur, which then proves that every other 0-point lies on l_0 or l_∞. Suppose to the contrary that $m = P_1Q_2P_4$ (the case $m = Q_1P_2P_4$ is completely analogous). Then the unique line through Q_4, meeting l_1 and l_2, is the line Q_1P_2. Consider now the unique line m' through P_4 meeting l_2 and l_3 in a point. As we supposed that $m \neq l_0$ and $m \neq l_\infty$, we see that P_4 cannot lie on these lines, so m' contains the points P_4, P_2, Q_3 or the points P_4, Q_2, P_3. In the former case, both lines l_0 and l_∞ are contained in the plane spanned by $m' = P_4Q_3P_2$ and $m = P_1Q_2P_4$. This implies that the disjoint lines l_1 and l_2 are contained in this plane, a contradiction. If $m' = P_4P_3Q_2$, then m and m' both contain P_4 and Q_2 but intersect l_0 in different points, a contradiction. We conclude that P_4, and analogously P'_4, is contained in the line l_0 or l_∞. ■

Using the previous proposition, we will prove that for all k, the $\mathbb{F}_2$-linear set D in $\mathrm{PG}(2k-1,q)$ is of pseudoregulus type.

Theorem 11.2.13. *The* $(q-1)$*-secants to* D *in* $\mathrm{PG}(2k-1,q)$ *form a pseudoregulus.*

Proof. By Lemma 11.2.10 it is sufficient to prove that there exist two $(k-1)$-spaces in $\mathrm{PG}(2k-1,q)$ that both have a point in common with all $(q-1)$-secants to D.

Consider a $(q-1)$-secant l_0, and let P_0 and P_0' be the 0-points on l_0. Let l_i be a $(q-1)$-secant, different from l_0. The lines l_0 and l_i span a 3-space γ and since D is a scattered $\mathbb{F}_2$-linear set, $\gamma \cap D$ is also a scattered $\mathbb{F}_2$-linear set. Since γ contains $2(q-1)$ points of D on the lines l_i, l_0 and $(q-1)^2$ points of D defined in a unique way as a third point on the line A_1A_2, with $A_1 \in l_0$, $A_2 \in l_i$, we have that $|D \cap \gamma| = q^2-1$, and hence it is a maximum scattered linear set. By Theorem 11.2.12, we find that $\gamma \cap D$ is of pseudoregulus type. This means that it has transversal lines, say m_i and m_i', where P_0 lies on m_i and P_0' lies on m_i'. This holds for every $(q-1)$-secant l_i. The number of $(q-1)$-secants to D, which are mutually disjoint, is exactly $\frac{q^k-1}{q-1}$, see Lemma 11.2.10, and so, the number of 0-points is exactly $2\frac{q^k-1}{q-1}$. There are $\frac{q^k-1}{q-1} - 1 = \frac{q^k-q}{q-1}$ lines l_i different from l_0, and each such line l_i defines a line m_i full of 0-points. Since this line m_i contains q points different from P_0, we have proven that a 0-point P_0 lies on $\frac{q^{k-1}-1}{q-1}$ lines full of 0-points (call such lines 0-*lines*). Every $(q-1)$-secant l_i also contains a 0-point P_i' on a line m_i', hence every 0-point P_0 is contained in $\frac{q^k-1}{q-1}$ lines containing precisely one other 0-point.

Let A and A' be the set of all points on the lines m_i and m_i' respectively. Then we will show that $A \cup A'$ is the union of two disjoint $(k-1)$-spaces.

Consider a line containing two 0-points P_1, P_2, with l_1 and l_2 the $(q-1)$-secants through P_1, P_2. Then, as seen before, the intersection of the 3-space spanned by l_1 and l_2 with D is a linear set of pseudoregulus type, and hence the line P_1P_2 contains 2 or $q+1$ 0-points. This shows that every line in $\mathrm{PG}(2k-1,q)$ intersects $A \cup A'$ in $0, 1, 2$ or $q+1$ points. This in turn implies that a plane with three 0-lines only contains 0-points. Consider now a point P_3 on a 0-line through P_0, and consider a 0-line $m \neq P_0P_3$ through P_3. If m contains a point $P_4 \neq P_3$ such that P_4P_0 is a 0-line through P_0, then we see that the plane $\langle P_0, m\rangle$ only contains 0-points. In the other case, m contains at least two 0-points on 0-lines through P_0'. In this case, all the points in the plane $\langle P_0', m\rangle$ are 0-points, and hence the line P_3P_0' is a 0-line, a contradiction. So we find that every 0-line through a 0-point of A is contained in A. Since every point of A lies on $\frac{q^{k-1}-1}{q-1}$ 0-lines, and A contains $\frac{q^k-1}{q-1}$ 0-points, we find that every 2 points of A are contained in a 0-line of A. The same argument works for the set A'. This shows that A forms a subspace and likewise A' forms a subspace. Since $|A| = |A'| = \frac{q^k-1}{q-1}$, these subspaces are $(k-1)$-dimensional. ■

11.2.4 There exists a suitable Desarguesian $(k-1)$-spread $\mathcal{S}$ in $\mathrm{PG}(2k-1,q)$

Consider the scattered linear set $D \subset H_\infty$ of pseudoregulus type. Let T_0 and T_∞ be the transversal $(k-1)$-spaces to the pseudoregulus defined by D found in Theorem 11.2.13. Now we want to show that there exists a Desarguesian $(k-1)$-spread $\mathcal{S}$ in $\mathrm{PG}(2k-1,q)$ such that $T_0, T_\infty \in \mathcal{S}$ and such that every other $(k-1)$-space of $\mathcal{S}$ has precisely one point in common with D.

Lemma 11.2.14. *There exists a Desarguesian* $(k-1)$*-spread* $\mathcal{S}$ *in* $\mathrm{PG}(2k-1,q)$*, such that* $T_0, T_\infty \in \mathcal{S}$ *and such that every other element of* $\mathcal{S}$ *has precisely one point in common with* D.

Proof. We prove this lemma using the representation of Result 11.1.3, in which we consider U_0, U_∞ as $\mathbb{F}_{q^k}$. By [87, Theorem 3.7] we find that the linear sets $L_{\rho,f}$ and $L_{\rho',g}$ are equivalent if and only if $\sigma_f = \sigma_g^{\pm 1}$, where σ_f and σ_g are the field automorphisms associated with f and g respectively. Hence, up to equivalence, we may suppose that $\rho = 1$ and $f : \mathbb{F}_{q^k} \to \mathbb{F}_{q^k} : t \to t^{2^i}$, $\gcd(i, hk) = 1$.

It follows that D is equivalent to the set of points P_u with

$$P_u := \left(u, u^{2^i}\right)_q, u \in \mathbb{F}_{q^k}^*.$$

The transversal spaces T_0 and T_∞ are the point sets $T_0 = \{(u,0) | u \in \mathbb{F}_{q^k}^*\}$ and $T_\infty = \{(0,u) | u \in \mathbb{F}_{q^k}^*\}$.

Consider now the set $\mathcal{S}_0$ of $(k-1)$-spaces T_u, $u \in \mathbb{F}_{q^k}^*$, with

$$T_u := \left\{ \left(\alpha u, \alpha u^{2^i}\right)_q \mid \alpha \in \mathbb{F}_{q^k}^* \right\}. \tag{11.1}$$

We will show that the set $\mathcal{S} = \mathcal{S}_0 \cup \{T_0, T_\infty\}$ is a $(k-1)$-spread of $\mathrm{PG}(2k-1, q)$. Suppose that $P = T_{u_1} \cap T_{u_2}$, for some $u_1, u_2 \notin \{0, \infty\}$, then there exist elements $\alpha_1, \alpha_2 \in \mathbb{F}_{q^k}^*, \mu \in \mathbb{F}_q^*$, such that

$$\begin{cases} \alpha_1 u_1 & = \mu \alpha_2 u_2 \\ \alpha_1 u_1^{2^i} & = \mu \alpha_2 u_2^{2^i} \end{cases} \tag{11.2}$$

with $\mu \in \mathbb{F}_q^*$. This implies that $u_1^{2^i - 1} = u_2^{2^i - 1}$ or $\left(\frac{u_1}{u_2}\right)^{2^i} = \frac{u_1}{u_2}$. Hence, $\frac{u_1}{u_2} \in \mathbb{F}_{2^i} \cap \mathbb{F}_{2^{hk}}$ which is $\mathbb{F}_2$ since $\gcd(i, hk) = 1$. Since $u_1, u_2 \in \mathbb{F}_{q^k}^*$, this implies that $u_1 = u_2$, and that $T_{u_1} = T_{u_2}$. In particular, we see that $T_u \neq T_{u'}$ for $u \neq u' \in \mathbb{F}_{q^k}^*$. Since T_0 and T_∞ are distinct from T_u for all $u \in \mathbb{F}_{q^k}^*$, we obtain that $|\mathcal{S}| = q^k + 1$.

We will now show that $T_u \cap T_0 = \emptyset$ for all $u \in \mathbb{F}_{q^k}^*$. If $P = T_u \cap T_0$, $u \notin \{0, \infty\}$ for some $u \in \mathbb{F}_{q^k}^*$, then $P = (u', 0)_q$ with $u' \in \mathbb{F}_{q^k}^*$ and

$$\begin{cases} \alpha u & = \mu u' \\ \alpha u^{2^i} & = 0 \end{cases}$$

for some $\mu \in \mathbb{F}_q^*$ and $\alpha \in \mathbb{F}_{q^k}^*$. The second equality gives a contradiction since $u \neq 0 \neq \alpha$. Hence, $T_u \cap T_0 = \emptyset$. It follows from a similar argument that $T_u \cap T_\infty = \emptyset$. This shows that $\mathcal{S}$ is a spread which is Desarguesian as seen in Subsection 11.1.1. ■

Remark 11.2.15. In [87, Theorem 3.11(i)], a geometric construction of the Desarguesian spread, found in Lemma 11.2.14, using indicator sets, is given.

11.2.5 The point set $\mathcal{Q}$ defines a translation hyperoval in the André/Bruck-Bose plane $\mathcal{P}(S)$

The Desarguesian spread $\mathcal{S}$ found in Lemma 11.2.14 defines the projective plane $\mathcal{P}(\mathcal{S}) = \Pi_{q^k} \cong \mathrm{PG}(2, q^k)$ by the André/Bruck-Bose construction. The transversal $(k-1)$-spaces $T_0, T_\infty \in \mathcal{S}$ to the pseudoregulus associated with D correspond to points P_0, P_∞ contained in the line ℓ_∞ at infinity of $\mathrm{PG}(2, q^k)$.

Theorem 11.2.16. *The set $\mathcal{Q}$, together with T_0 and T_∞, defines a translation hyperoval in $\Pi_{q^k} \cong \mathrm{PG}(2, q^k)$.*

Proof. Let $\mathcal{A}$ be the set of points in Π_{q^k} corresponding to the point set $\mathcal{Q}$ of Ψ_q. Recall that T_0 corresponds to a point P_0 and T_∞ to a point P_∞, contained in the line ℓ_∞ of Π_{q^k}. We first show that every line in $\mathrm{PG}(2, q^k)$ contains at most 2 points of the set $\mathcal{H} = \mathcal{A} \cup P_0 \cup P_\infty$.

- The line ℓ_∞ at infinity only contains the points P_0 and P_∞.
- Consider a line $l \neq \ell_\infty$ through P_0 in $\mathrm{PG}(2, q^k)$. This line corresponds to a k-space through T_0 in $\mathrm{PG}(2k, q)$. As $P_0 \in l \cap \mathcal{H}$, we have to show that this k-space contains at most one affine point of $\mathcal{Q}$. If this space would contain 2 (or more) affine points $X_1, X_2 \in \mathcal{Q}$, then they would define a direction of D at infinity in T_0. But this is impossible as T_0 has no points of D. This argument also works for the lines through P_∞, different from ℓ_∞.
- Consider a line l through a point P_i, $i \notin \{0, \infty\}$, at infinity. This point P_i corresponds to an element $T_i \in \mathcal{S}$ that intersects the pseudoregulus D in a unique point X_i. The line l corresponds to a k-space γ in $\mathrm{PG}(2k, q)$ through T_i. Suppose that γ contains at least 3 points from $\mathcal{Q}$, say X, Y, Z. By Lemma 11.2.2, these points are not collinear, hence they determine at least two different points of D which are contained in T_i, a contradiction by the choice of $\mathcal{S}$, see Lemma 11.2.14. This proves that γ contains at most two points of $\mathcal{Q}$, which implies that the line l contains at most two points of $\mathcal{A}$.

Since $\mathcal{H}$ has size $q^k + 2$, it follows that $\mathcal{H}$ is a hyperoval.

Finally consider the group G of elations in $\mathrm{PG}(2hk, 2)$ with axis the hyperplane at infinity $\widetilde{H}_\infty$. Since the points of $\widetilde{\mathcal{Q}}$ form a subspace, we see that G acts transitively on the points of $\widetilde{\mathcal{Q}}$. Every element of G induces an element of the group G' of elations in $\mathrm{PG}(2, q^k)$ with axis the line P_0P_∞. Hence, G' acts transitively on the points of $\mathcal{A}$ in $\mathrm{PG}(2, q^k)$. This shows that $\mathcal{H}$ is a translation hyperoval. ■

11.2.6 Every translation hyperoval defines a linear set of pseudoregulus type

In this section, we show that the vice versa part of Theorem 11.1.4 holds.

Proposition 11.2.17. *Via the André/Bruck-Bose construction, the set of affine points of a translation hyperoval in $\mathrm{PG}(2, q^k)$, $q = 2^h$, where $h, k \geq 2$ corresponds to a set $\mathcal{Q}$ of q^k affine points in $\mathrm{PG}(2k, q)$ whose set of determined directions D is an $\mathbb{F}_2$-linear set of pseudoregulus type.*

Proof. Consider a translation hyperoval H of $\mathrm{PG}(2, q^k)$. Without loss of generality we may suppose that $H = \{(1, t, t^{2^i})_{q^k} | t \in \mathbb{F}_{q^k}\} \cup \{(0, 1, 0)_{q^k}, (0, 0, 1)_{q^k}\}$ with $\gcd(i, hk) = 1$. Let $l_\infty = \langle (0, 1, 0)_{q^k}, (0, 0, 1)_{q^k} \rangle$ be the line at infinity. The set of affine points of H corresponds to the set of points $H' = \{(1, t, t^{2^i})_q \in \mathbb{F}_q \oplus \mathbb{F}_{q^k} \oplus \mathbb{F}_{q^k} | t \in \mathbb{F}_{q^k}\}$ in $\mathrm{PG}(2k, q)$ (for more information about the use of these coordinates for H and H', see [105]). The determined directions in the hyperplane at infinity $H_\infty : X_0 = 0$ have coordinates $(0, t_1 - t_2, t_1^{2^i} - t_2^{2^i})_q$ where $t_1, t_2 \in \mathbb{F}_{q^k}$. So the set $D = \{(0, u, u^{2^i})_q | u \in \mathbb{F}_{q^k}\}$ is precisely the set of directions determined by the points of H. By Result 11.1.3, we find that this set of directions D is an $\mathbb{F}_2$-linear set of pseudoregulus type in the hyperplane H_∞. ■

We will now show that every line in $\mathrm{PG}(2k-1, q)$ intersects the points of the linear set D in $0, 1, 3$ or $q-1$ points.

Proposition 11.2.18. *Let D be the set of points of an $\mathbb{F}_2$-linear set of pseudoregulus type in $\mathrm{PG}(2k-1,q)$, $q=2^h$, $h>2$, $k\geq 2$. Then every line of $\mathrm{PG}(2k-1,q)$ meets D in $0,1,3$ or $q-1$ points.*

Proof. We use the representation of Result 11.1.3 for the points of D. Let $R_1=(u_1,f(u_1))_q$ and $R_2=(u_2,f(u_2))_q$, $u_1,u_2\in U_0$, be two points of D not on the same line of the pseudoregulus, so the vectors $\langle u_1\rangle$ and $\langle u_2\rangle$ in $V(k,q)$ are not an $\mathbb{F}_q$-multiple (in short $\langle u_1\rangle_q\neq\langle u_2\rangle_q$). Recall that f is an invertible semi-linear map with automorphism $\sigma\in Aut(\mathbb{F}_q)$, $Fix(\sigma)=\{0,1\}$. A third point $R_3=(u_3,f(u_3))_q\in D$ is contained in R_1R_2 if and only if there are $\mu,\lambda\in\mathbb{F}_q$ such that

$$\begin{aligned}
&\begin{cases} u_1+\lambda u_2 &= \mu u_3\\ f(u_1)+\lambda f(u_2) &= \mu f(u_3)\end{cases}\\
\Leftrightarrow &\begin{cases} f(u_1)+\lambda^\sigma f(u_2) &= \mu^\sigma f(u_3)\\ f(u_1)+\lambda f(u_2) &= \mu f(u_3)\end{cases}\\
\Leftrightarrow &\begin{cases} f(u_1)+\lambda^\sigma f(u_2) &= \mu^\sigma f(u_3)\\ (\lambda^\sigma-\lambda)f(u_2) &= (\mu^\sigma-\mu)f(u_3)\end{cases}\\
\Leftrightarrow &\begin{cases} f(u_1+\lambda u_2) &= f(\mu u_3)\\ f((\lambda-\lambda^{\sigma^{-1}})u_2) &= f((\mu-\mu^{\sigma^{-1}})u_3)\end{cases}\\
\Leftrightarrow &\begin{cases} u_1+\lambda u_2 &= \mu u_3\\ (\lambda^\sigma-\lambda)^{\sigma^{-1}}u_2 &= (\mu-\mu^{\sigma^{-1}})u_3\end{cases}
\end{aligned}$$

As R_2 and R_3 lie on different $(q-1)$-secants to D, we have that $R_2\neq R_3$ and so, $\langle u_2\rangle_q\neq\langle u_3\rangle_q$. It follows that $\lambda^\sigma-\lambda=\mu-\mu^{\sigma^{-1}}=0$, so $\lambda,\mu\in Fix(\sigma)=\{0,1\}$. We find that there is only one solution of this system, such that $R_1\neq R_3$ (i.e. $\langle u_1\rangle_q\neq\langle u_3\rangle_q$), namely when $\lambda=\mu=1$. Hence, given two points R_1,R_2 in D, there is a unique point $R_3\in D\cap R_1R_2$, different from R_1 and R_2. ■

11.3 The generalisation of a characterisation of Barwick and Jackson

Using Theorem 11.1.4, we are now able to generalise the following result of Barwick-Jackson which concerns translation hyperovals in $\mathrm{PG}(2,q^2)$ ([7]).

Result 11.3.1 ([7, Theorem 1.2]). *Consider $\mathrm{PG}(4,q)$, q even, $q>2$, with the hyperplane at infinity denoted by Σ_∞. Let $\mathcal{C}$ be a set of q^2 affine points, called $\mathcal{C}$-points and consider a set of planes called $\mathcal{C}$-planes which satisfies the following:*

(A1) Each $\mathcal{C}$-plane meets $\mathcal{C}$ in a q-arc.

(A2) Any two distinct $\mathcal{C}$-points lie in a unique $\mathcal{C}$-plane.

(A3) The affine points that are not in $\mathcal{C}$ lie on exactly one $\mathcal{C}$-plane.

(A4) Every plane which meets $\mathcal{C}$ in at least 3 points either meets $\mathcal{C}$ in 4 points or is a $\mathcal{C}$-plane.

Then there exists a Desarguesian spread $\mathcal{S}$ in Σ_∞ such that in the Bruck-Bose plane $\mathcal{P}(\mathcal{S})\cong\mathrm{PG}(2,q^2)$, the $\mathcal{C}$-points, together with 2 extra points on ℓ_∞, form a translation hyperoval in $\mathrm{PG}(2,q^2)$.

Remark 11.3.2. At two different points, the proofs of [7] are inherently linked to the fact that they are dealing with hyperovals in $\mathrm{PG}(2, q^2)$. In [7, Lemma 4.1] the authors show the existence of a design which is isomorphic to an affine plane, of which they later need to use the parallel classes. In [7, Theorem 4.11], they use the Klein correspondence to represent lines in $\mathrm{PG}(3, q)$ in $\mathrm{PG}(5, q)$. Both techniques cannot be extended in a straightforward way to q^k, $k > 2$.

The following Proposition shows that a set of $\mathcal{C}$-planes as defined by Barwick and Jackson in [7] (using $\mathrm{PG}(2k, q)$ instead of $\mathrm{PG}(4, q)$) satisfies the conditions of Theorem 11.1.4.

Proposition 11.3.3. *Consider* $\mathrm{PG}(2k, q)$*,* q *even,* $q > 2$*, with the hyperplane at infinity denoted by* H_∞*. Let* $\mathcal{C}$ *be a set of* q^k *affine points, called* $\mathcal{C}$*-points and consider a set of planes called* $\mathcal{C}$*-planes which satisfies the following:*

(A1) Each $\mathcal{C}$*-plane meets* $\mathcal{C}$ *in a* q*-arc.*

(A2) Any two distinct $\mathcal{C}$*-points lie in a unique* $\mathcal{C}$*-plane.*

(A3) The affine points that are not in $\mathcal{C}$ *lie on exactly one* $\mathcal{C}$*-plane.*

(A4) Every plane which meets $\mathcal{C}$ *in at least* 3 *points either meets* $\mathcal{C}$ *in* 4 *points or is a* $\mathcal{C}$*-plane.*

Then $\mathcal{C}$ *determines a set of* $q^k - 1$ *directions* D *in* H_∞ *such that every line of* H_∞ *meets* D *in* $0, 1, 3$ *or* $q - 1$ *points.*

Proof. Note that all $\mathcal{C}$-points are affine. Since every two $\mathcal{C}$-points lie on a $\mathcal{C}$-plane which meets $\mathcal{C}$ in a q-arc, we have that no three $\mathcal{C}$-points are collinear.

Let P_0 be a $\mathcal{C}$-point and let D_0 be the set of points of the form $P_0P_i \cap H_\infty$, where $P_i \neq P_0$ is a point of $\mathcal{C}$. We first show that every line meets D_0 in $0, 1, 3$ or $q - 1$ points. Let M be a line of H_∞ containing 2 points of D_0, say $R'_1 = P_0R_1 \cap H_\infty$, $R'_2 = P_0R_2 \cap H_\infty$, where $R_1, R_2 \in \mathcal{C}$. Then $\langle M, P_0\rangle$ contains at least 3 points of $\mathcal{C}$, and hence, by (A4), either it is a $\mathcal{C}$-plane or it contains exactly 4 points of $\mathcal{C}$. If $\langle M, P_0\rangle$ is a $\mathcal{C}$-plane, it contains q points of $\mathcal{C}$ forming a q-arc, and hence, M contains $q - 1$ points of D_0. Now suppose that $\langle M, P_0\rangle$ contains exactly 4 $\mathcal{C}$-points, then M contains 3 points of D_0.

Now let $P_1 \neq P_0$ be a point of $\mathcal{C}$ and let D_1 be the set of points of the form $P_1P_i \cap H_\infty$, where $P_i \neq P_1$ is a point of $\mathcal{C}$. We claim that $D_0 = D_1$. Let $P'_1 = P_0P_1 \cap H_\infty$. We see that $P'_1 \in D_0 \cap D_1$. Consider a point $P'_2 \neq P'_1$ in D_0, then $P_0P_2 \cap H_\infty = P'_2$ for some $P_2 \in \mathcal{C}$. Consider the plane $\pi = \langle P_0, P_1, P_2\rangle$.

Suppose first that π is not a $\mathcal{C}$-plane, then, by (A4), π contains exactly one extra point, say P_3 of $\mathcal{C}$. The lines P_0P_1 and P_2P_3 lie in π and hence, meet in a point Q. By $(A2)$, there is a $\mathcal{C}$-plane μ through P_0P_1, and likewise, there is a $\mathcal{C}$-plane μ' through P_2P_3. Since π is not a $\mathcal{C}$-plane, μ and μ' are two distinct $\mathcal{C}$-planes through Q. By (A3) this implies that Q is a point of H_∞. Likewise, $P_0P_2 \cap P_1P_3$ and $P_0P_3 \cap P_1P_2$ are points of H_∞. It follows that $D_0 \cap \pi = D_1 \cap \pi$. This argument shows that for all points $R \neq P'_1 \in D_0$ such that $\langle P_0, P_1, R\rangle$ is not a $\mathcal{C}$-plane, we have that $R \in D_1$. Now P_0P_1 lies on a unique $\mathcal{C}$-plane, say ν. Let $\nu \cap H_\infty = L$, then we have shown that $\langle P_0, P_1, R\rangle$ is not a $\mathcal{C}$-plane as long as $R \in H_\infty$ is not on L. We conclude that $D_0 \setminus L = D_1 \setminus L$.

Now assume that $D_0 \neq D_1$ and let X be a point in D_1 which is not contained in D_0. Then $X \in L$ and P_1X contains a point $Y \neq P_1 \in \mathcal{C}$. Consider a point $P'_4 \in D_1$, not on L, then $P_1P'_4$ contains a point $P_4 \neq P_1$ of $\mathcal{C}$. Since $P'_4 \in D_1 \setminus L$, $P'_4 \in D_0$ so the line P'_4P_0 contains a point $P_5 \neq P_1$ of $\mathcal{C}$.

The plane $\langle P_1, P'_4, X\rangle$ is not a $\mathcal{C}$-plane since otherwise, the points P_1 and Y of $\mathcal{C}$ would lie in two different $\mathcal{C}$-planes. This implies that $\langle P_1, P_4, X\rangle$, which contains the $\mathcal{C}$-points P_1, P_4, Y, contains exactly one extra point of $\mathcal{C}$, say P_6. Denote $P_1P_6 \cap H_\infty$ by P_6'. We see that there are exactly 3 points of D_1 on the line $P_4'X$, namely P_4', X and P_6'.

Now P_6' is a point of D_1, not on L, so $P_6' \in D_0$. Hence, there is a point $S \neq P_0 \in \mathcal{C}$ on the line P_0P_6'.

If $\langle P_4', P_6', P_0\rangle$ is not a $\mathcal{C}$-plane, then, since it contains P_0, P_5, S of $\mathcal{C}$, it contains precisely 3 points of D_0 at infinity. These are the points P_4', P_6' and one other point, say T, which needs to be different from X by our assumption that $X \notin D_0$. That implies that T is not on L, and hence, $T \in D_1$. This is a contradiction since we have seen that the only points of D_1 on $P_4'X$ are P_4', X and P_6'. Now if $\langle P_4', P_6, P_0\rangle$ is a $\mathcal{C}$-plane, we find $q-1$ points of D_0 on $P_4'X$, all of them are not on L. Hence, we find $q-1$ points of D_1 on $P_4'X$, not on L. This is again a contradiction since $P_4'X$ has only the points P_4' and P_6' of D_1 not on L.

This proves our claim that $D_0 = D_1$. Since P_1 was chosen arbitrarily, different from P_0, and $D_0 = D_1$, we find that the set D of directions determined by $\mathcal{C}$ is precisely the set D_0. The statement now follows from the fact that a line meets D_0 in $0, 1, 3$ or $q-1$ points. ■

Proposition 11.3.3 shows that the set $\mathcal{C}$ satisfies the criteria of Theorem 11.1.4. Hence, we find the following generalisation of Result 11.3.1.

Theorem 11.3.4. *Consider* $\mathrm{PG}(2k, q)$, *q even, $q > 2$, with the hyperplane at infinity denoted by H_∞. Let $\mathcal{C}$ be a set of q^k affine points, called $\mathcal{C}$-points, and consider a set of planes, called $\mathcal{C}$-planes, which satisfies the following:*

- *(A1) Each $\mathcal{C}$-plane meets $\mathcal{C}$ in a q-arc.*
- *(A2) Any two distinct $\mathcal{C}$-points lie in a unique $\mathcal{C}$-plane.*
- *(A3) The affine points that are not in $\mathcal{C}$ lie on exactly one $\mathcal{C}$-plane.*
- *(A4) Every plane which meets $\mathcal{C}$ in at least 3 points either meets $\mathcal{C}$ in 4 points or is a $\mathcal{C}$-plane.*

Then there exists a Desarguesian spread $\mathcal{S}$ in H_∞ such that in the Bruck-Bose plane $\mathcal{P}(\mathcal{S}) \cong \mathrm{PG}(2, q^k)$, the $\mathcal{C}$-points, together with 2 extra points on ℓ_∞, form a translation hyperoval in $\mathrm{PG}(2, q^k)$.

Part IV

Appendix

A English summary

> *I guess ice cream is one of those things that are beyond imagination.*
>
> —Lucy Maud Montgomery

In this chapter, we give a short summary on the most important concepts and results in this thesis. For more details, and for the proofs of the results, we refer to the chapters above.

This thesis consist of three large parts. The first part handles several intersection problems in projective and affine geometries. In the second part, we discuss Cameron-Liebler sets in affine, projective and polar spaces. The last part concerns translation hyperovals in $\mathrm{PG}(4, q)$, q even, for which we use linear sets.

A.1 Introduction

Before we start with the first main part, we give a short introduction. In Chapter 1.1 several incidence geometries are defined. The most commonly used incidence geometry in this thesis is the *projective space* $\mathrm{PG}(n, q)$ of dimension n over the field $\mathbb{F}_q$ with q elements, q a prime power. This is the geometry of subspaces of an $(n + 1)$-dimensional vector space over the same field. The projective dimension of a subspace in $\mathrm{PG}(n, q)$ is the vector dimension of the corresponding vector space, minus one. In this thesis, we only work with projective dimensions. Subspaces of dimension k are also called k-spaces. The number of points in an n-dimensional projective space is $\theta_n = \frac{q^{n+1}-1}{q-1}$, while the number of k-spaces in an n-dimensional projective space is given by the *Gaussian binomial coefficient* $\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q$.

An *affine space* $\mathrm{AG}(n, q)$ is the incidence geometry obtained from a projective space $\mathrm{PG}(n, q)$, by removing an $(n-1)-$dimensional space, or hyperplane H, together with all its incident subspaces. This hyperplane is also called the *hyperplane at infinity*.

The finite classical polar spaces are incidence geometries embedded in a projective space $\mathrm{PG}(n, q)$. They consist of the totally isotropic subspaces of a vector space $V(n + 1; q)$, with respect to a quadratic, symplectic or Hermitian form, and are equipped with the natural incidence relation.

A.2 Intersection problems

The first main part of this thesis handles intersection problems. In this part, we discuss the classification of several (large) families of subspaces in projective and affine spaces, that meet pre-established conditions.

A.2.1 Sets of k-spaces pairwise intersecting in at least a $(k-2)$-space

In this first research project, large families of k-spaces, pairwise intersecting in at least a $(k-2)$-space in $\mathrm{PG}(n,q)$, are studied. The largest set is a $(k-2)$*-pencil*. This is the set of k-spaces containing a fixed $(k-2)$-space. This was proven for general t-spaces by P. Frankl and R.M. Wilson.

Theorem A.2.1. ***[60, Theorem 1]*** *Let t and k be integers, with $0 \leq t \leq k$. Let $\mathcal{S}$ be a set of k-spaces in $\mathrm{PG}(n,q)$, pairwise intersecting in at least a t-space.*

- *(i) If $n \geq 2k+1$, then $|\mathcal{S}| \leq \begin{bmatrix} n-t \\ k-t \end{bmatrix}$. Equality holds if and only if $\mathcal{S}$ is the set of all the k-spaces, containing a fixed t-space of $\mathrm{PG}(n,q)$, or $n = 2k+1$ and $\mathcal{S}$ is the set of all the k-spaces in a fixed $(2k-t)$-space.*
- *(ii) If $2k-t \leq n \leq 2k$, then $|\mathcal{S}| \leq \begin{bmatrix} 2k-t+1 \\ k-t \end{bmatrix}$. Equality holds if and only if $\mathcal{S}$ is the set of all the k-spaces in a fixed $(2k-t)$-space.*

In this thesis, the case $t = k-2$ is studied. We classify the ten largest maximal examples of sets of k-spaces pairwise intersecting in at least a $(k-2)$-space. For figures of the examples below, we refer to Chapter 3.

Example A.2.2. *Examples of maximal sets $\mathcal{S}$ of k-spaces in $\mathrm{PG}(n,q)$ pairwise intersecting in at least a $(k-2)$-space.*

- (i) $(k-2)$-pencil: *the set $\mathcal{S}$ is the set of all k-spaces that contain a fixed $(k-2)$-space. Then $|\mathcal{S}| = \begin{bmatrix} n-k+2 \\ 2 \end{bmatrix}$.*
- (ii) Star: *there exists a k-space ζ such that $\mathcal{S}$ contains all k-spaces that have at least a $(k-1)$-space in common with ζ. Then $|\mathcal{S}| = q\theta_k\theta_{n-k-1} + 1$.*
- (iii) Generalized Hilton-Milner example: *there exists a $(k+1)$-space ν and a $(k-2)$-space $\pi \subset \nu$ such that $\mathcal{S}$ consists of all k-spaces in ν, together with all k-spaces of $\mathrm{PG}(n,q)$, not in ν, through π that intersect ν in a $(k-1)$-space. Then $|\mathcal{S}| = \theta_{k+1} + q^2(q^2+q+1)\theta_{n-k-2}$.*
- (iv) *There exists a $(k+2)$-space ρ, a k-space $\alpha \subset \rho$ and a $(k-2)$-space $\pi \subset \alpha$ so that $\mathcal{S}$ contains all k-spaces in ρ that meet α in a $(k-1)$-space not through π, all k-spaces in ρ through π, and all k-spaces in $\mathrm{PG}(n,q)$, not in ρ, that contain a $(k-1)$-space of α through π. Then $|\mathcal{S}| = (q+1)\theta_{n-k} + q^3(q+1)\theta_{k-2} + q^4 - q$.*
- (v) *There is a $(k+2)$-space ρ, and a $(k-1)$-space $\alpha \subset \rho$ such that $\mathcal{S}$ contains all k-spaces in ρ that meet α in at least a $(k-2)$-space, and all k-spaces in $\mathrm{PG}(n,q)$, not in ρ, through α. Note that all k-spaces in $\mathrm{PG}(n,q)$ through α are contained in $\mathcal{S}$. Then $|\mathcal{S}| = \theta_{n-k} + q^2(q^2+q+1)\theta_{k-1}$.*
- (vi) *There are two $(k+2)$-spaces ρ_1, ρ_2 intersecting in a $(k+1)$-space $\alpha = \rho_1 \cap \rho_2$. There are two $(k-1)$-spaces $\pi_A, \pi_B \subset \alpha$ with $\pi_A \cap \pi_B$ the $(k-2)$-space λ, there is a point $P_{AB} \in \alpha \backslash \langle \pi_A, \pi_B \rangle$, and let $\lambda_A, \lambda_B \subset \lambda$ be two different $(k-3)$-spaces. Then $\mathcal{S}$ contains*
  - *all k-spaces in α,*
  - *all k-spaces of $\mathrm{PG}(n,q)$ through $\langle P_{AB}, \lambda \rangle$, not in ρ_1 and not in ρ_2.*
  - *all k-spaces in ρ_1, not in α, through P_{AB} and a $(k-2)$-space in π_A through λ_A,*
  - *all k-spaces in ρ_1, not in α, through P_{AB} and a $(k-2)$-space in π_B through λ_B,*
  - *all k-spaces in ρ_2, not in α, through P_{AB} and a $(k-2)$-space in π_A through λ_B,*
  - *all k-spaces in ρ_2, not in α, through P_{AB} and a $(k-2)$-space in π_B through λ_A.*

Then $|\mathcal{S}| = \theta_{n-k} + q^2\theta_{k-1} + 4q^3$.

(vii) *There is a* $(k-3)$*-space* γ *contained in all* k*-spaces of* $\mathcal{S}$*. In the quotient space* $\mathrm{PG}(n,q)/\gamma$*, the set of planes corresponding to the elements of* $\mathcal{S}$ *is the set of planes of example* $VIII$ *in [33]: Let* Ψ *be an* $(n-k+2)$*-space, disjoint from* γ*, in* $\mathrm{PG}(n,q)$*. Consider two solids* σ_1 *and* σ_2 *in* Ψ*, intersecting in a line* l*. Take the points* P_1 *and* P_2 *on* l*. Then* $\mathcal{S}$ *is the set containing all* k*-spaces through* $\langle\gamma, l\rangle$*, all* k*-spaces through* $\langle\gamma, P_1\rangle$ *that contain a line in* σ_1 *and a line in* σ_2 *skew to* γ*, and all* k*-spaces through* $\langle\gamma, P_2\rangle$ *in* $\langle\gamma, \sigma_1\rangle$ *or in* $\langle\gamma, \sigma_2\rangle$*. Then* $|\mathcal{S}| = \theta_{n-k} + q^4 + 2q^3 + 3q^2$.

$(viii)$ *There is a* $(k-3)$*-space* γ *contained in all* k*-spaces of* $\mathcal{S}$*. In the quotient space* $\mathrm{PG}(n,q)/\gamma$*, the set of planes corresponding to the elements of* $\mathcal{S}$ *is the set of planes of example* IX *in [33]: Let* Ψ *be an* $(n-k+2)$*-space, disjoint from* γ*, in* $\mathrm{PG}(n,q)$*, and let* l *be a line and* σ *a solid skew to* l*, both in* Ψ*. Denote* $\langle l, \sigma\rangle$ *by* ρ*. Let* P_1 *and* P_2 *be two points on* l *and let* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *be a regulus and its opposite regulus in* σ*. Then* $\mathcal{S}$ *is the set containing all* k*-spaces through* $\langle\gamma, l\rangle$*, all* k*-spaces through* $\langle\gamma, P_1\rangle$ *in the* $(k+1)$*-space generated by* γ, l *and a fixed line of* $\mathcal{R}_1$*, and all* k*-spaces through* $\langle\gamma, P_2\rangle$ *in the* $(k+1)$*-space generated by* γ, l *and a fixed line of* $\mathcal{R}_2$*. Then* $|\mathcal{S}| = \theta_{n-k} + 2q^3 + 2q^2$.

(ix) *There is a* $(k-3)$*-space* γ *contained in all* k*-spaces of* $\mathcal{S}$*. In the quotient space* $\mathrm{PG}(n,q)/\gamma$*, the set of planes corresponding to the elements of* $\mathcal{S}$ *is the set of planes of example* VII *in [33]: Let* Ψ *be an* $(n-k+2)$*-space, disjoint from* γ *in* $\mathrm{PG}(n,q)$ *and let* ρ *be a* 5*-space in* Ψ*. Consider a line* l *and a* 3*-space* σ *disjoint from* l*. Choose three points* P_1*,* P_2*,* P_3 *on* l *and choose four non-coplanar points* Q_1, Q_2, Q_3, Q_4 *in* σ*. Denote* $l_1 = Q_1Q_2, \bar{l}_1 = Q_3Q_4, l_2 = Q_1Q_3, \bar{l}_2 = Q_2Q_4$*,* $l_3 = Q_1Q_4$*, and* $\bar{l}_3 = Q_2Q_3$*. Then* $\mathcal{S}$ *is the set containing all* k*-spaces through* $\langle\gamma, l\rangle$ *and all* k*-spaces through* $\langle\gamma, P_i\rangle$ *in* $\langle\gamma, l, l_i\rangle$ *or in* $\langle\gamma, l, \bar{l}_i\rangle$*,* $i = 1, 2, 3$*. Then* $|\mathcal{S}| = \theta_{n-k} + 6q^2$.

(x) $\mathcal{S}$ *is the set of all* k*-spaces contained in a fixed* $(k+2)$*-space* ρ*. Then* $|\mathcal{S}| = \begin{bmatrix} k+3 \\ 2 \end{bmatrix}$.

Main Theorem A.2.3. *Let* $\mathcal{S}$ *be a maximal set of* k*-spaces pairwise intersecting in at least a* $(k-2)$*-space in* $\mathrm{PG}(n,q)$*,* $n \geq 2k$*,* $k \geq 3$ *. Let*

$$f(k,q) = \begin{cases} 3q^4 + 6q^3 + 5q^2 + q + 1 & \text{if } k = 3, q \geq 2 \text{ or } k = 4, q = 2 \\ \theta_{k+1} + q^4 + 2q^3 + 3q^2 & \text{else.} \end{cases}$$

If $|\mathcal{S}| > f(k,q)$*, then* $\mathcal{S}$ *is one of the families described in Example A.2.2. Note that for* $n > 2k+1$*, the examples* $(i) - (ix)$ *are stated in decreasing order of the sizes.*

A.2.2 Hilton-Milner problems in $\mathrm{PG}(n,q)$ and $\mathrm{AG}(n,q)$

As already mentioned above, we know that the largest set of k-spaces, pairwise intersecting in a t-space in $\mathrm{PG}(n,q)$, $n \geq 2k+1$ is a t-pencil. This example is often called the trivial example. Guo and Xu proved that the largest set of k-spaces pairwise intersecting in a t-space in $\mathrm{AG}(n,q)$, $n \geq 2k+t+2$ is t-pencil as well, see [69]. In Chapter 4 the two largest non-trivial examples of k-spaces pairwise intersecting in at least a t-space, in both $\mathrm{PG}(n,q)$ and $\mathrm{AG}(n,q)$ are classified for $n \geq 2k+t+3$ and $q \geq 3$. For this, we suppose that $k \geq t+1$.

We start with examples of t-intersecting sets in the projective setting.

Example A.2.4. *Suppose* $k \geq t+1$ *and let* γ *be a* $(t+2)$*-space in* $\mathrm{PG}(n,q)$*,* $n \geq 2k-t+1$*. Let* $\mathcal{S}$ *be the set of all* k*-spaces in* $\mathrm{PG}(n,q)$*, meeting* γ *in at least a* $(t+1)$*-space.*

Example A.2.5. *Let δ be a t-space, $t \leq k-1$, in $\mathrm{PG}(n,q)$, $n \geq 2k-t+1$, and let ξ be a $(k+1)$-space in $\mathrm{PG}(n,q)$ with $\delta \subset \xi$. Let S_1 be the set of all k-spaces in ξ. Let S_2 be the set of all k-spaces through δ and meeting ξ in at least a $(t+1)$-space. Let $\mathcal{S}$ be the union of the sets S_1 and S_2.*

Note that these examples correspond to Examples A.2.2(ii) and (iii) respectively for $t = k-2$. These are the largest non-trivial examples of t-intersecting sets of k-spaces in $\mathrm{PG}(n,q)$.

Theorem A.2.6. *Let $\mathcal{S}_p$ be a maximal set of k-spaces, pairwise intersecting in at least a t-space in $\mathrm{PG}(n,q)$, $k \geq t+2$, $t \geq 1$, with $q \geq 3$, and $n \geq 2k+t+3$. If $\mathcal{S}_p$ is not a t-pencil, then*

$$|\mathcal{S}_p| \leq \begin{cases} \theta_{k+1} - \theta_{k-t} + \begin{bmatrix} n-t \\ k-t \end{bmatrix} - q^{(k-t+1)(k-t)} \begin{bmatrix} n-k-1 \\ k-t \end{bmatrix} & \text{if } k > 2t+2 \\ \theta_{t+2} \cdot \left(\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} - \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \right) + \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} & \text{if } k \leq 2t+2. \end{cases} .$$

Equality occurs if and only if $\mathcal{S}_p$ is Example A.2.4 for $k \leq 2t+2$ or Example A.2.5 for $k \geq 2t+3$.

Now we give two examples of large t-intersecting sets of k-spaces in $\mathrm{AG}(n,q)$. For an affine space α we denote the projective extension of α by $\tilde{\alpha}$, and let $H_\infty = \mathrm{PG}(n,q) \setminus \mathrm{AG}(n,q)$ be the hyperplane at infinity.

Example A.2.7. *Suppose $k \geq t+1$. Let γ be an affine $(t+2)$-space in $AG(n,q)$, and let $\mathcal{R}$ be a set of θ_{t+1} affine $(t+1)$-spaces in γ such that for every two distinct elements $\sigma_1, \sigma_2 \in \mathcal{R}$, $\tilde{\sigma}_1 \cap H_\infty \neq \tilde{\sigma}_2 \cap H_\infty$. Note that every two different elements of R meet in an affine t-space. Let $\mathcal{S}$ be the set of all k-spaces in $AG(n,q)$, containing γ or meeting γ in an element of $\mathcal{R}$.*

Example A.2.8. *Let δ be a t-space, $k \geq t+1$, in $\mathrm{AG}(n,q)$, and let ξ be a $(k+1)$-space in $\mathrm{AG}(n,q)$ with $\delta \subset \xi$. Let S_1 be a maximal set of affine k-spaces in ξ, such that for any two elements π_1, π_2 of S_1, $\tilde{\pi}_1 \cap H_\infty \neq \tilde{\pi}_2 \cap H_\infty$, and such that for every $\pi_1 \in S_1$: $\tilde{\delta} \cap H_\infty \not\subseteq \tilde{\pi}_1$. Let S_2 be the set of all k-spaces through δ and meeting ξ in at least a $(t+1)$-space. Let $\mathcal{S}$ be the union of the sets S_1 and S_2.*

We find that the largest non-trivial t-intersecting sets in $\mathrm{AG}(n,q)$ arise from one of these two examples; which one depends on whether $k \geq 2t+2$ or not.

Theorem A.2.9. *Let $\mathcal{S}_a$ be a maximal set of k-spaces, pairwise intersecting in at least a t-space in $\mathrm{AG}(n,q)$, $k \geq t+2$, $t \geq 1$, with $q \geq 3$, and $n \geq 2k+t+3$. If $\mathcal{S}_a$ is not a t-pencil, then*

$$|\mathcal{S}_a| \leq \begin{cases} \theta_{k} - \theta_{k-t} + \begin{bmatrix} n-t \\ k-t \end{bmatrix} - q^{(k-t+1)(k-t)} \begin{bmatrix} n-k-1 \\ k-t \end{bmatrix} & \text{if } k > 2t+1 \\ \theta_{t+1} \cdot \left(\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} - \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \right) + \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} & \text{if } k \leq 2t+1. \end{cases}$$

Equality occurs if and only if $\mathcal{S}_a$ is Example A.2.7 for $k \leq 2t+1$ or Example A.2.8 for $k \geq 2t+2$.

A.2.3 The Sunflower bound

In the previous sections, we investigate subspaces pairwise intersecting in *at least* a subspace of a certain dimension. In Chapter 5 we investigate sets of k-spaces in $\mathrm{PG}(n,q)$ pairwise intersecting in *precisely* a point. More generally, a $(k+1, t+1)$-SCID is a set of k-spaces, pairwise intersecting in exactly a t-space. An example of such a SCID is the set S of k-spaces, such that for each $\pi, \tau \in S$ it holds that $\pi \cap \tau = \gamma$, for a t-space γ. This example is a *sunflower with vertex* γ. The *Sunflower bound* states that if the number of elements in a $(k+1, t+1)$-SCID S surpasses the Sunflower bound, then S must be a sunflower.

Theorem A.2.10. *[56, Theorem 1] A $(k+1, t+1)$-SCID S in $\mathrm{PG}(n,q)$, is a sunflower if*

$$|S| > \left(\frac{q^{k+1} - q^{t+1}}{q-1} \right)^2 + \left(\frac{q^{k+1} - q^{t+1}}{q-1} \right) + 1.$$

In Chapter 5 we improve this bound for $k \geq 3$ and $q \geq 7$. For $k = 1$ and $k = 2$, a complete classification is known: every $(k+1, k)$-SCID is a sunflower or consists of all k-spaces in a fixed $(k+1)$-space. For the classification of $(3, 1)$-SCIDs, we refer to [9].

Theorem A.2.11. *A $(k+1, 1)$-SCID in $\mathrm{PG}(n, q)$, $k \geq 3, q \geq 7$, with more than $F_q \theta_k^2$ elements is a sunflower. Here we use*

$$F_q = \frac{1}{2}\left(\frac{B_q}{c_q^2} - \frac{1}{q} - \sqrt{\left(\frac{1}{q} - \frac{B_q}{c_q^2}\right)^2 - 4B_q\left(\frac{1}{c_q^2} - 1\right)}\right)$$

with

$$\begin{aligned} B_q &= (1-c_q)^2\left(1 - c_q - \frac{1}{q^3}\right)^2\left(1 - c_q - \frac{c_q}{q}\right)\left(1 - c_q - \frac{1+c_q}{q}\right)q, \\ c_q &= 1 - \frac{1}{\sqrt[6]{q}} - \frac{1}{2\sqrt[3]{q}}. \end{aligned}$$

In particular, we have that a $(k+1, 1)$-SCID in $\mathrm{PG}(n, q)$, with more than $\left(\frac{2}{\sqrt[6]{q}} + \frac{4}{\sqrt[3]{q}} - \frac{5}{\sqrt{q}}\right)\theta_k^2$ elements is a sunflower.

A.2.4 The chromatic number of some q-Kneser graphs

A *flag* in $\mathrm{PG}(n, q)$ is a set F of non-trivial subspaces of $\mathrm{PG}(n, q)$ (that is, different from $\emptyset$ and $\mathrm{PG}(n, q)$) such that for all $\alpha, \beta \in F$ one has $\alpha \subset \beta$ or $\beta \subset \alpha$. The subset $\{\dim(\alpha) + 1 \mid \alpha \in F\}$, where we use the projective dimension, is called the *type* of F and it is a subset of $\{1, 2, \dots, n\}$. Two flags F and G are in *general position* if $\alpha \cap \beta = \emptyset$ or $\langle \alpha, \beta \rangle = \mathrm{PG}(n, q)$ for all $\alpha \in F$ and $\beta \in G$.

For $\Omega \subseteq \{1, 2, \dots, n\}$ the q-Kneser graph $qK_{n+1;\Omega}$ is the graph whose vertices are all flags of type Ω of $\mathrm{PG}(n, q)$ with two vertices adjacent when the corresponding flags are in general position. We are interested in the chromatic number of these graphs.

For any point $P \in \mathrm{PG}(n, q)$, we define the set $\mathcal{F}_\Omega(P)$ as the set of all flags F of type $\Omega \subseteq \{2, 3, \dots, n\}$ for which $F \cup \{P\}$ is a flag. We call $\mathcal{F}_\Omega(P)$ the *point-pencil* (of flags of type Ω) with base point P.

We determine the chromatic number of the graphs $qK_{5;\Omega}$ for $\Omega = \{2, 4\}$ and $q \neq 2$, and for $qK_{2d+1;\{d,d+1\}}$, with $d \geq 2$ and q very large.

We used the independence number as well as structural information on large cocliques of $qK_{5;\{2,4\}}$ (see [14]), and of $qK_{2d+1,\{d,d+1\}}$ (see [11] for $d = 2$ and [94] for $d = 3$). For $d \geq 4$, no structural information on large cocliques is known yet, and so, in this case, we need an extra assumption, see Conjecture A.2.15. We could prove the following results.

Theorem A.2.12. *For $q \geq 3$ the chromatic number of the Kneser graph $qK_{5;\{2,4\}}$ is θ_3. Moreover, each color class of a minimum coloring is contained in a unique point-pencil and the base points of the obtained points-pencil are the points in a fixed solid.*

Theorem A.2.13. *For $q > 160 \cdot 36^5$, the chromatic number of the Kneser graph $qK_{5;\{2,3\}}$ is $q^3 + q^2 + 1$. Up to duality, for each color class C of a minimum coloring there is a unique point-pencil F such that $F \cup C$ is independent, and the base points of these point-pencils are $q^3 + q^2 + 1$ distinct points of a solid.*

Theorem A.2.14. *For $q > 3 \cdot 7^{15} \cdot 2^{56}$, the chromatic number of the Kneser graph $qK_{7;\{3,4\}}$ is $q^4 + q^3 + q^2 + 1$. Up to duality, for each color class C of a minimum coloring there is a unique point-pencil F such that $F \cup C$ is independent, and the base points of these point-pencils are $q^4 + q^3 + q^2 + 1$ distinct points of a solid.*

Conjecture A.2.15. *For every integer $d \geq 4$ there is an integer $\rho(d)$ such that every maximal coclique of the Kneser graph $qK_{2d+1,\{d,d+1\}}$ contains a point-pencil, the dual of a point-pencil, or has at most $\rho(d) \cdot q^{d^2+d-2}$ elements.*

Theorem A.2.16. *If Conjecture A.2.15 is true for some integer $d \geq 4$, then*

$$\chi(qK_{2d+1,\{d,d+1\}}) = \theta_{d+1} - q,$$

for sufficiently large q, depending on d and $\rho(d)$. Moreover, if $\mathfrak{F}$ is a family of this many maximal cocliques that cover the vertex set, then — up to duality — there exists a $(d+1)$-dimensional subspace U in $\mathrm{PG}(2d, q)$ and an injective map μ from $\mathfrak{F}$ to set of points of U such that the point-pencil $\mathcal{F}(\mu(C))$ is contained in C for all $C \in \mathfrak{F}$.

A.3 Cameron-Liebler sets

In the second part of the thesis, Cameron-Liebler sets in different contexts are investigated. The central thread in this part can be summarized into two questions: What are the equivalent definitions for these sets, and for which parameters x do there exists Cameron-Liebler sets? We investigate both questions in projective, affine and polar spaces.

A.3.1 Cameron-Liebler sets of k-spaces in $\mathrm{PG}(n, q)$

We investigate Cameron-Liebler sets of k-spaces in $\mathrm{PG}(n, q)$. For this, we list several equivalent definitions for these Cameron-Liebler sets, by generalizing the known results about Cameron-Liebler line sets in $\mathrm{PG}(n, q)$, see [51], and Cameron-Liebler sets of k-spaces in $\mathrm{PG}(2k+1, q)$, see [104].

Let A be the incidence matrix of the points and the k-spaces of $\mathrm{PG}(n, q)$: the rows of A are indexed by the points and the columns by the k-spaces. Let V_i, $0 \leq i \leq k$, be the eigenspaces of the related Grassmann scheme, using the classical ordering (see Subsection 10.1.1).

Theorem A.3.1. *Let $\mathcal{L}$ be a non-empty set of k-spaces in $\mathrm{PG}(n, q), n \geq 2k+1$, with characteristic vector χ, and x so that $|\mathcal{L}| = x \begin{bmatrix} n \\ k \end{bmatrix}$. Then the following properties are equivalent.*

1. *$\chi \in \mathrm{im}(A^T)$.*
2. *$\chi \in \ker(A)^\perp$.*
3. *For every k-space π, the number of elements of $\mathcal{L}$ disjoint from π is $(x - \chi(\pi)) \left[\begin{smallmatrix} n-k-1 \\ k \end{smallmatrix}\right] q^{k^2+k}$.*
4. *The vector $\chi - x\frac{q^{k+1}-1}{q^{n+1}-1}\boldsymbol{j}$ is a vector in V_1.*
5. *$\chi \in V_0 \perp V_1$.*

6. *For a given $i \in \{1, \dots, k+1\}$ and any k-space π, the number of elements of $\mathcal{L}$, meeting π in a $(k-i)$-space is given by:*

$$\begin{cases} \left((x-1)\frac{q^{k+1}-1}{q^{k-i+1}-1} + q^{i}\frac{q^{n-k}-1}{q^{i}-1}\right) q^{i(i-1)} \begin{bmatrix} n-k-1 \\ i-1 \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} & \text{if } \pi \in \mathcal{L} \\ x \begin{bmatrix} n-k-1 \\ i-1 \end{bmatrix} \begin{bmatrix} k+1 \\ i \end{bmatrix} q^{i(i-1)} & \text{if } \pi \notin \mathcal{L} \end{cases}.$$

7. *for every pair of conjugate switching k-sets $\mathcal{R}$ and $\mathcal{R}'$, we have that $|\mathcal{L} \cap \mathcal{R}| = |\mathcal{L} \cap \mathcal{R}'|$.*

If $\mathrm{PG}(n,q)$ admits a k-spread, then the following properties are equivalent to the previous ones.

8. *$|\mathcal{L} \cap \mathcal{S}| = x$ for every k-spread $\mathcal{S}$ in $\mathrm{PG}(n,q)$.*
9. *$|\mathcal{L} \cap \mathcal{S}| = x$ for every Desarguesian k-spread $\mathcal{S}$ in $\mathrm{PG}(n,q)$.*

Definition A.3.2. A set $\mathcal{L}$ of k-spaces in $\mathrm{PG}(n,q)$ that fulfills one of the statements in Theorem A.3.1 (and consequently all of them) is called a *Cameron-Liebler set of k-spaces* in $\mathrm{PG}(n,q)$ with parameter $x = |\mathcal{L}| \begin{bmatrix} n \\ k \end{bmatrix}^{-1}$.

Using the information we get from the equivalent definitions, together with some more properties that we derived, we found classification results for Cameron-Liebler sets of k-spaces in $\mathrm{PG}(n,q)$. First note that a Cameron-Liebler set of k-spaces with parameter 0 is the empty set.
In the following lemma we start with the classification for the parameters $x \in]0,2[$.

Lemma A.3.3. *There are no Cameron-Liebler sets of k-spaces in $\mathrm{PG}(n,q)$ with parameter $x \in]0,1[$, and if $n \geq 3k+2$, then there are no Cameron-Liebler sets of k-spaces with parameter $x \in]1,2[$. Let $\mathcal{L}$ be a Cameron-Liebler set of k-spaces with parameter $x = 1$ in $\mathrm{PG}(n,q)$, $n \geq 2k+1$. Then $\mathcal{L}$ is a point-pencil or $n = 2k+1$ and $\mathcal{L}$ is the set of all k-spaces in a hyperplane of $\mathrm{PG}(2k+1,q)$.*

We end with the main classification result of this project.

Theorem A.3.4. *There are no Cameron-Liebler sets of k-spaces in $\mathrm{PG}(n,q)$, $n \geq 3k+2$ and $q \geq 3$, with parameter $2 \leq x \leq \frac{1}{\sqrt[8]{2}} q^{\frac{n}{2}-\frac{k^2}{4}-\frac{3k}{4}-\frac{3}{2}} (q-1)^{\frac{k^2}{4}-\frac{k}{4}+\frac{1}{2}} \sqrt{q^2+q+1}$.*

A.3.2 Cameron-Liebler sets of k-spaces in $\mathrm{AG}(n,q)$

In Section 4.4.3, we give an overview of the most important (equivalent) definition and classification results for Cameron-Liebler sets in affine spaces, proven in [46] and [44]. Similar to the definition of Cameron-Liebler sets of k-spaces in $\mathrm{PG}(n,q)$, we have the following definition in the affine context.

Definition A.3.5. A set $\mathcal{L}$ of k-spaces in $\mathrm{AG}(n,q)$ is a Cameron-Liebler set of k-spaces of parameter x in $\mathrm{AG}(n,q)$ if every k-spread in $\mathrm{AG}(n,q)$ has x elements in common with $\mathcal{L}$.

In contrast to k-spreads in $\mathrm{PG}(n,q)$, we note that there exist k-spreads in $\mathrm{AG}(n,q)$, for every $n \geq k$, which implies that the definition above is well defined.

Due to the immediate link between $\mathrm{PG}(n,q)$ and $\mathrm{AG}(n,q)$, it is possible to classify Cameron-Liebler sets in $\mathrm{AG}(n,q)$, by using the ideas for the same research project in projective spaces.

Theorem A.3.6. *There are no Cameron-Liebler sets of k-spaces in $\mathrm{AG}(n,q)$, $n \geq 3k+2$ and $q \geq 3$, with parameter $2 \leq x \leq \frac{1}{\sqrt[8]{2}} q^{\frac{n}{2}-\frac{k^2}{4}-\frac{3k}{4}-\frac{3}{2}} (q-1)^{\frac{k^2}{4}-\frac{k}{4}+\frac{1}{2}} \sqrt{q^2+q+1}$.*

A.3.3 Degree one Cameron-Liebler sets in finite classical polar spaces

We study the sets of generators defined by the following definition, where A is the incidence matrix of points and generators.

Definition A.3.7. A *degree one Cameron-Liebler set* of generators in a finite classical polar space $\mathcal{P}$ is a set of generators in $\mathcal{P}$, with characteristic vector χ, such that $\chi \in \operatorname{im}(A^T)$. The parameter x of a Cameron-Liebler set $\mathcal{L}$ in the polar space $\mathcal{P}$ of rank d and parameter e is equal to $\frac{|\mathcal{L}|}{\prod_{i=0}^{d-2}(q^{e+i}+1)}$.

This definition coincides with the definition of Boolean degree one functions for generators in polar spaces, given in [59] by Y. Filmus and F. Ihringer. Their definition corresponds to the fact that the corresponding characteristic vector lies in $V_0 \perp V_1$, which are eigenspaces of the related association scheme (see Subsection 10.1.1). In [36], M. De Boeck, M. Rodgers, L. Storme and A. Švob introduced Cameron-Liebler sets of generators in the finite classical polar spaces. In this article, Cameron-Liebler sets of generators in the polar spaces are defined by the *disjointness-definition* and the authors give several equivalent definitions for these Cameron-Liebler sets. Note that this definition is the polar-space-equivalent for the disjointness-definition in the projective context, see Theorem A.3.1.3.

Definition A.3.8 ([36]). Let $\mathcal{P}$ be a finite classical polar space with parameter e and rank d. A set $\mathcal{L}$ of generators in $\mathcal{P}$ is a Cameron-Liebler set of generators in $\mathcal{P}$, with parameter x, if and only if for every generator π in $\mathcal{P}$, the number of elements of $\mathcal{L}$, disjoint from π equals $(x-\chi(\pi))q^{\binom{d-1}{2}+e(d-1)}$.

Using association scheme notation we can interpret the previous definition as follows. The characteristic vector of a Cameron-Liebler set is contained in $V_0 \perp W$, with W the eigenspace of the disjointness matrix A_d corresponding to a specific eigenvalue. It can be seen that W always contains V_1, but it does not necessarily coincide with V_1. Hence, every degree one Cameron-Liebler set is a Cameron-Liebler set, and for some polar spaces Cameron-Liebler sets and degree one Cameron-Liebler sets will coincide, but for others this will not be the case.

Note that we defined degree one Cameron-Liebler sets in an algebraic way. In general, Cameron-Liebler sets in different contexts can often be defined by using both algebraic and combinatorial definitions. For these degree one Cameron-Liebler sets, we also found that this is possible, and we could give an equivalent combinatorial definition.

Theorem A.3.9. *Let $\mathcal{P}$ be a finite classical polar space, of rank d with parameter e, let $\mathcal{L}$ be a set of generators of $\mathcal{P}$ and i be an integer with $1 \leq i \leq d$. If $\mathcal{L}$ is a degree one Cameron-Liebler set of generators in $\mathcal{P}$, with parameter x, then the number of elements of $\mathcal{L}$ meeting a generator π in a $(d-i-1)$-space equals*

$$\begin{cases} \left((x-1)\begin{bmatrix} d-1 \\ i-1 \end{bmatrix} + q^{i+e-1}\begin{bmatrix} d-1 \\ i \end{bmatrix}\right) q^{\binom{i-1}{2}+(i-1)e} & \text{If } \pi \in \mathcal{L} \\ x\begin{bmatrix} d-1 \\ i-1 \end{bmatrix} q^{\binom{i-1}{2}+(i-1)e} & \text{If } \pi \notin \mathcal{L}. \end{cases} \tag{A.1}$$

Moreover, if this property holds for a polar space $\mathcal{P}$ and an integer i such that

- *i is odd for $\mathcal{P} = Q^+(2d-1,q)$, or*
- *$i \neq d$ for $\mathcal{P} = Q(2d,q)$ or $\mathcal{P} = W(2d-1,q)$ both with d odd, or*
- *i is arbitrary otherwise,*

then $\mathcal{L}$ is a degree one Cameron-Liebler set with parameter x.

Apart from these definitions, we also investigated for which values of the parameter x there exists degree one Cameron-Liebler sets. For degree one Cameron-Liebler sets in $W(5,q)$ and $Q(6,q)$ we found the following classification result.

Theorem A.3.10. *A degree one Cameron-Liebler set $\mathcal{L}$ of generators in $W(5,q)$ or $Q(6,q)$ with parameter $2 \leq x \leq \sqrt[3]{2q^2} - \frac{\sqrt[3]{4q}}{3} + \frac{1}{6}$ is the union of α embedded hyperbolic quadrics $Q^+(5,q)$, that pairwise have no plane in common, and $x - 2\alpha$ point-pencils whose vertices are pairwise non-collinear and not contained in the α hyperbolic quadrics $Q^+(5,q)$. For the polar space $Q(6,q)$ or $W(5,q)$ with q even, $\alpha \in \{0, ..., \lfloor \frac{x}{2} \rfloor\}$, for the polar space $W(5,q)$ with q odd, $\alpha = 0$.*

A.3.4 New example of a degree one Cameron-Liebler set of generators in $Q^+(5,q)$

We give an example of a new, non-trivial Cameron-Liebler set of generators in $Q^+(5,q)$, q odd. To explain the construction of the example, we use the Klein correspondence between the lines of $\mathrm{PG}(3,q)$ and the points of $Q^+(5,q)$.

Consider the hyperbolic quadric $Q = Q^+(3,q)$ in $\mathrm{PG}(3,q)$, defined by the equation $x_0x_1 + x_2x_3 = 0$. The lines of Q correspond to the set of points of two conics $C \cup C'$ in $Q^+(5,q)$, such that for the planes $\alpha = \langle C \rangle$ and $\alpha' = \langle C' \rangle$, it holds that α' is the image of α under the polarity of $Q^+(5,q)$.

Every point $P \in \mathrm{PG}(3,q)$ gives rise to a Latin plane π_l^P and a Greek plane π_g^P in $Q^+(5,q)$: the points of π_l^P corresponds to all lines through P in $\mathrm{PG}(3,q)$, and the points of π_g^P corresponds to all lines in the plane $P^\perp$. Here, $\perp$ is the polarity related to the quadric Q in $\mathrm{PG}(3,q)$.

Definition A.3.11. A point $P(x_0, x_1, x_2, x_3) \in \mathrm{PG}(3,q)$ is a *square point* if $x_0x_1 + x_2x_3$ is a square different from 0 in $\mathbb{F}_q$. A point $P(x_0, x_1, x_2, x_3) \in \mathrm{PG}(3,q)$ is a *non-square point* if $x_0x_1 + x_2x_3$ is a non-square in $\mathbb{F}_q$.

Now we can partition the set of planes in $Q^+(5,q)$ into the following sets.

- $\mathcal{S}_l = \{\pi_l^P | P \text{ is a square point}\}$
- $\mathcal{NS}_l = \{\pi_l^P | P \text{ is a non-square point}\}$
- $\mathcal{O}_l = \{\pi_l^P | P \in Q\}$
- $\mathcal{S}_g = \{\pi_g^P | P \text{ is a square point}\}$
- $\mathcal{NS}_g = \{\pi_g^P | P \text{ is a non-square point}\}$
- $\mathcal{O}_g = \{\pi_g^P | P \in Q\}$

For a tangent line ℓ to Q, there are two possibilities; ℓ contains q square points, or ℓ contains q non-square points, see [72, Table 15.5(c)]. In the first case ℓ is a *square tangent line.* In the later case, ℓ is a *non-square tangent line.*

We partition the set of points in $Q^+(5,q)$ into the following sets.

- The set $\mathcal{X}_{1S}$ of points in $Q^+(5,q)$ corresponding to the square tangent lines to Q.
- The set $\mathcal{X}_{1NS}$ of points in $Q^+(5,q)$ corresponding to the non-square tangent lines to Q.
- The set $\mathcal{X}_2$ of points in $Q^+(5,q)$ corresponding to the 2-secants to Q.
- The set $\mathcal{X}_0$ of points in $Q^+(5,q)$ corresponding to the lines disjoint from Q.
- The set $\mathcal{X}_\infty = C \cup C'$ of points in $Q^+(5,q)$ corresponding to the lines of Q.

We could prove that the partitions $\{\mathcal{X}_{1S}, \mathcal{X}_{1NS}, \mathcal{X}_2, \mathcal{X}_0, \mathcal{X}_\infty\}$ and $\{\mathcal{S}_l, \mathcal{S}_g, \mathcal{NS}_l, \mathcal{NS}_g, \mathcal{O}_l, \mathcal{O}_g\}$ form a point-tactical decomposition. By grouping the right partition classes together, we found new Cameron-Liebler sets in $Q^+(5,q)$.

Theorem A.3.12. *Let q be an odd prime power.*

- *The sets $\mathcal{S}_l \cup \mathcal{S}_g$ and $\mathcal{NS}_l \cup \mathcal{NS}_g$ are degree one Cameron-Liebler sets of planes in $Q^+(5,q)$, with parameter $\frac{q(q-1)}{2}$, $\frac{q(q-1)}{2}$ and $q+1$ respectively, for $q \equiv 1 \mod 4$.*
- *The sets $\mathcal{S}_l \cup \mathcal{NS}_g$ and $\mathcal{S}_g \cup \mathcal{NS}_l$ are degree one Cameron-Liebler sets of planes in $Q^+(5,q)$, with parameter $\frac{q(q-1)}{2}$, $\frac{q(q-1)}{2}$ and $q+1$ respectively, for $q \equiv 3 \mod 4$.*

A.4 Linear sets

In the last part of this thesis, we discuss a research project about translation hyperovals and $\mathbb{F}_2$-linear sets. We give a link between the affine points of a translation hyperoval in $\mathrm{PG}(2,q^k)$ and the set of points of a scattered $\mathbb{F}_2$-linear set of pseudoregulus type in $\mathrm{PG}(2k-1,q)$, seen as a set of directions. For this, we used the Barlotti-Cofman construction, which is a generalization of the André/Bruck-Bose construction.

The original aim of this research project was to generalize the following result of Barwick and Jackson.

Result A.4.1 ([7, Theorem 1.2]). *Consider $\mathrm{PG}(4,q)$, q even, $q > 2$, with the hyperplane at infinity denoted by Σ_∞. Let $\mathcal{C}$ be a set of q^2 affine points, called $\mathcal{C}$-points and consider a set of planes called $\mathcal{C}$-planes which satisfies the following properties.*

(A1) Each $\mathcal{C}$-plane meets $\mathcal{C}$ in a q-arc.

(A2) Any two distinct $\mathcal{C}$-points lie in a unique $\mathcal{C}$-plane.

(A3) The affine points that are not in $\mathcal{C}$ lie on exactly one $\mathcal{C}$-plane.

(A4) Every plane which meets $\mathcal{C}$ in at least 3 points either meets $\mathcal{C}$ in 4 points or is a $\mathcal{C}$-plane.

Then there exists a Desarguesian spread $\mathcal{S}$ in Σ_∞ such that in the André/Bruck-Bose plane $\mathcal{P}(\mathcal{S}) \cong \mathrm{PG}(2,q^2)$, the $\mathcal{C}$-points, together with 2 extra points on ℓ_∞ form a translation hyperoval in $\mathrm{PG}(2,q^2)$.

In the search for a generalisation, we examined a collection C of q^k affine points in $\mathrm{PG}(2k,q)$, q even, $q > 2$, with similar combinatorial properties. The techniques used by Barwick and Jackson in the proof of the above result were not generalizable. Hence, we had to look for new techniques, including the use of linear sets, more specifically, those of pseudoregulus type. We were able to prove the following result.

Theorem A.4.2. *Let $\mathcal{Q}$ be a set of q^k affine points in $\mathrm{PG}(2k,q)$, $q = 2^h$, $h \geq 4$, $k \geq 2$, determining a set D of $q^k - 1$ directions in the hyperplane at infinity $H_\infty = \mathrm{PG}(2k-1,q)$. Suppose that every line has $0, 1, 3$ or $q-1$ points in common with the point set D. Then*

(1) D is an $\mathbb{F}_2$-linear set of pseudoregulus type.

(2) There exists a Desarguesian spread $\mathcal{S}$ in H_∞ such that, in the André/Bruck-Bose plane $\mathcal{P}(\mathcal{S}) \cong \mathrm{PG}(2,q^k)$, with H_∞ corresponding to the line l_∞, the points of $\mathcal{Q}$ together with 2 extra points on ℓ_∞, form a translation hyperoval in $\mathrm{PG}(2,q^k)$.

Vice versa, via the André/Bruck-Bose construction, the set of affine points of a translation hyperoval in $\mathrm{PG}(2,q^k)$, $q > 4, k \geq 2$, corresponds to a set $\mathcal{Q}$ of q^k affine points in $\mathrm{PG}(2k,q)$ whose set of determined directions D is an $\mathbb{F}_2$-linear set of pseudoregulus type. Consequently, every line meets D in $0, 1, 3$ or $q-1$ points.

An immediate corollary of this theorem is the generalization of Result A.4.1.

Theorem A.4.3. *Consider* $\mathrm{PG}(2k, q)$, *q even, $q > 2$, with the hyperplane at infinity denoted by Σ_∞. Let $\mathcal{C}$ be a set of q^k affine points, called $\mathcal{C}$-points and consider a set of planes called $\mathcal{C}$-planes which satisfies the following properties.*

(A1) Each $\mathcal{C}$-plane meets $\mathcal{C}$ in a q-arc.

(A2) Any two distinct $\mathcal{C}$-points lie in a unique $\mathcal{C}$-plane.

(A3) The affine points that are not in $\mathcal{C}$ lie on exactly one $\mathcal{C}$-plane.

(A4) Every plane which meets $\mathcal{C}$ in at least 3 points either meets $\mathcal{C}$ in 4 points or is a $\mathcal{C}$-plane.

Then there exists a Desarguesian spread $\mathcal{S}$ in Σ_∞ such that in the André/Bruck-Bose plane $\mathcal{P}(\mathcal{S}) \cong \mathrm{PG}(2, q^k)$, the $\mathcal{C}$-points, together with 2 extra points on ℓ_∞ form a translation hyperoval in $\mathrm{PG}(2, q^k)$.

B Nederlandstalige samenvatting

> *Wiskunde is als zuurstof, als het er is, merk je het niet.*
> *Als het er niet zou zijn, merk je dat je niet zonder kunt.*
>
> —Lex Schrijver

In deze Nederlandstalige samenvatting geven we een kort overzicht van de belangrijkste begrippen en resulaten uit deze thesis. Voor meer details en de bewijzen van de resultaten, verwijzen we naar bovenstaande Engelstalige hoofdstukken.

Deze thesis bestaat uit drie delen. In het eerste deel bespreken we verschillende intersectieproblemen in projectieve en affiene ruimten. In het tweede deel worden Cameron-Lieblerverzamelingen in affiene, projectieve en polaire ruimten besproken. Het laatste deel van deze thesis gaat over translatiehyperovalen in $\mathrm{PG}(4, q)$, q even, waarbij we gebruik maken van lineaire verzamelingen.

B.1 Inleiding

Voordat we starten met het eerste grote deel, geven we een korte inleiding. In Hoofdstuk 1.1 worden incidentiemeetkundes gedefinieerd. De meest gebruikte incidentiemeetkunde in deze thesis is de *projectieve ruimte* $\mathrm{PG}(n, q)$ van dimensie n over het veld $\mathbb{F}_q$ met q elementen, q een priemmacht. Dit is de meetkunde van de deelruimten van een $(n+1)$-dimensionale vectorruimte over hetzelfde veld. De projectieve dimensie van een deelruimte in $\mathrm{PG}(n, q)$ is de vectoriële dimensie van de overeenkomstige vectorruimte min één. In deze thesis werken we steeds met projectieve dimensies en deelruimten van dimensie k worden ook k-ruimten genoemd. Het aantal punten in een n-ruimte is gelijk aan $\theta_n = \frac{q^{n+1}-1}{q-1}$ en het aantal k-ruimten in een n-ruimte wordt gegeven door de *Gaussische binomiaalcoëfficiënt* $\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q$.

Een *affiene ruimte* $\mathrm{AG}(n, q)$ is de incidentiemeetkunde die men verkrijgt door in een projectieve ruimte $\mathrm{PG}(n, q)$ een $(n-1)-$ruimte, of hypervlak H, samen met alle incidente deelruimten te verwijderen. Dit hypervlak wordt ook het *hypervlak op oneindig* genoemd.

De *eindige klassieke polaire ruimten* zijn incidentiemeetkundes, ingebed in een projectieve ruimte $\mathrm{PG}(n, q)$. Ze bestaan uit de totaal isotrope deelruimten van een vectorruimte $V(n+1; q)$, met betrekking tot een kwadratische, symplectische of Hermitische vorm, en zijn voorzien van de natuurlijke incidentierelatie.

B.2 Intersectie problemen

Het eerste deel van deze thesis gaat over intersectie problemen. In dit gedeelte bespreken we de classificatie van verschillende (grote) verzamelingen van deelruimten in projectieve en affiene ruimten, die voldoen aan voorop opgestelde voorwaarden betreffende hun paarsgewijze doorsnede.

B.2.1 Verzamelingen van k-ruimten die paarsgewijs snijden in een $(k-2)$-ruimte

In dit eerste onderzoeksproject werden grote verzamelingen van k-ruimten, die paarsgewijs snijden in minstens een $(k-2)$-ruimte in $\mathrm{PG}(n,q)$ bestudeerd. Het grootste voorbeeld hiervan is een $(k-2)$-*bundel*, of de verzameling van k-ruimten die een vaste $(k-2)$-ruimte bevatten. Dit werd bewezen, voor algemene t-ruimten door P. Frankl en R.M. Wilson.

Stelling B.2.1 ([60, Theorem 1]). *Zij k en t gehele getallen, met $0 \leq t \leq k$, en zij $\mathcal{S}$ een verzameling van k-ruimten in $\mathrm{PG}(n,q)$, paarsgewijs snijdend in minstens een t-ruimte.*

(i) Als $n \geq 2k+1$, dan geldt er dat $|\mathcal{S}| \leq \begin{bmatrix} n-t \\ k-t \end{bmatrix}$. Gelijkheid geldt enkel en alleen in het geval dat $\mathcal{S}$ de verzameling is van alle k-ruimten die een vaste t-ruimte bevatten, of $n = 2k+1$, en $\mathcal{S}$ is de verzameling van alle k-ruimten in een vaste $(2k-t)$-ruimte.

(ii) Als $2k-t \leq n \leq 2k$, dan geldt er dat $|\mathcal{S}| \leq \begin{bmatrix} 2k-t+1 \\ k-t \end{bmatrix}$. Gelijkheid geldt enkel en alleen in het geval dat $\mathcal{S}$ de verzameling is van alle k-ruimten in een vaste $(2k-t)$-ruimte.

In deze thesis wordt het geval $t = k-2$ behandeld. Hierin worden de tien grootste maximale voorbeelden, van k-ruimten paarsgewijs snijdend in minstens een $(k-2)$-ruimte besproken. Voor figuren van onderstaande voorbeelden verwijzen we naar Hoofdstuk 3.

Voorbeeld B.2.2. *Voorbeelden van maximale verzamelingen $\mathcal{S}$ van k-ruimten in $\mathrm{PG}(n,q)$ paarsgewijs snijdend in een $(k-2)$-ruimte.*

(i) $(k-2)$-bundel: *de verzameling $\mathcal{S}$ van alle k-ruimten die een vaste $(k-2)$-ruimte bevatten. Dan is $|\mathcal{S}| = \begin{bmatrix} n-k+2 \\ 2 \end{bmatrix}$.*

(ii) Ster: *er bestaat een k-ruimte ζ zodat $\mathcal{S}$ alle k-ruimten bevat die minstens een $(k-1)$-ruimte gemeen hebben met ζ. Dan is $|\mathcal{S}| = q\theta_k\theta_{n-k-1}+1$.*

(iii) Veralgemeend Hilton-Milner voorbeeld: *er bestaat een $(k+1)$-ruimte ν en een $(k-2)$-ruimte $\pi \subset \nu$ zodat $\mathcal{S}$ bestaat uit alle k-ruimten in ν, samen met alle k-ruimten door π die ν snijden in minstens een $(k-1)$-ruimte. Dan is $|\mathcal{S}| = \theta_{k+1} + q^2(q^2+q+1)\theta_{n-k-2}$.*

(iv) *Er bestaat een $(k+2)$-ruimte ρ, een k-ruimte $\alpha \subset \rho$ en een $(k-2)$-ruimte $\pi \subset \alpha$, zodat $\mathcal{S}$ alle k-ruimten in ρ bevat die α snijden in een $(k-1)$-ruimten niet door π, alle k-ruimten in ρ door π, en alle k-ruimten in $\mathrm{PG}(n,q)$, niet in ρ, die een $(k-1)$-ruimte van α door π bevatten. Dan is $|\mathcal{S}| = (q+1)\theta_{n-k} + q^3(q+1)\theta_{k-2} + q^4 - q$.*

(v) *Er bestaat een $(k+2)$-ruimte ρ, en een $(k-1)$-ruimte $\alpha \subset \rho$ zodat $\mathcal{S}$ alle k-ruimten van ρ bevat die α snijden in minstens een $(k-2)$-ruimte, en alle k-ruimten in $\mathrm{PG}(n,q)$, door α en niet in ρ. Merk op dat alle k-ruimten in $\mathrm{PG}(n,q)$ door α bevat zijn in $\mathcal{S}$. Dan is $|\mathcal{S}| = \theta_{n-k} + q^2(q^2+q+1)\theta_{k-1}$.*

(vi) *Er bestaan twee $(k+2)$-ruimten ρ_1, ρ_2, snijdend in een $(k+1)$-ruimte $\alpha = \rho_1 \cap \rho_2$. Daarnaast zijn er twee $(k-1)$-ruimten $\pi_A, \pi_B \subset \alpha$ met $\pi_A \cap \pi_B$ gelijk aan de $(k-2)$-ruimte λ, en een punt $P_{AB} \in \alpha \setminus \langle \pi_A, \pi_B \rangle$. Stel $\lambda_A, \lambda_B \subset \lambda$ gelijk aan twee verschillende $(k-3)$-ruimten. Dan bevat $\mathcal{S}$ de volgende elementen*

- *alle k-ruimten in α,*
- *alle k-ruimten van $\mathrm{PG}(n,q)$ door $\langle P_{AB}, \lambda \rangle$, maar niet bevat in ρ_1 of ρ_2.*
- *alle k-ruimten in ρ_1, niet in α, door het punt P_{AB} en een $(k-2)$-ruimte in π_A door λ_A,*
- *alle k-ruimten in ρ_1, niet in α, door het punt P_{AB} en een $(k-2)$-ruimte in π_B door λ_B,*

- *alle k-ruimten in ρ_2, niet in α, door het punt P_{AB} en een $(k-2)$-ruimte in π_A door λ_B,*
- *alle k-ruimten in ρ_2, niet in α, door het punt P_{AB} een een $(k-2)$-ruimte in π_B door λ_A.*

Dan is $|\mathcal{S}| = \theta_{n-k} + q^2\theta_{k-1} + 4q^3$.

(vii) *Er bestaat een $(k-3)$-ruimte γ bevat in alle k-ruimten van $\mathcal{S}$. In de quotiëntruimte $\mathrm{PG}(n,q)/\gamma$, is de verzameling van vlakken, komende van de elementen van $\mathcal{S}$, de verzameling van de vlakken van voorbeeld $VIII$ in [33]: beschouw een $(n-k+2)$-ruimte Ψ, scheef aan γ, in $\mathrm{PG}(n,q)$. Beschouw twee drie-ruimten σ_1 en σ_2 in Ψ, sijdend in een rechte l. Neem twee punten P_1 en P_2 op l. Dan is $\mathcal{S}$ de verzameling van alle k-ruimten door $\langle\gamma, l\rangle$, alle k-ruimten door $\langle\gamma, P_1\rangle$ die een rechte in σ_1 en een rechte in σ_2 scheef aan γ bevatten, en alle k-ruimten door $\langle\gamma, P_2\rangle$ in $\langle\gamma, \sigma_1\rangle$ of in $\langle\gamma, \sigma_2\rangle$. Dan is $|\mathcal{S}| = \theta_{n-k} + q^4 + 2q^3 + 3q^2$.*

$(viii)$ *Er bestaat een $(k-3)$-ruimte γ bevat in alle k-ruimten van $\mathcal{S}$. In de quotiëntruimte $\mathrm{PG}(n,q)/\gamma$, is de verzameling van vlakken, komende van de elementen van $\mathcal{S}$, de verzameling van de vlakken van voorbeeld IX in [33]: Beschouw een $(n-k+2)$-ruimte Ψ, scheef aan γ, in $\mathrm{PG}(n,q)$, en beschouw een rechte l en een drie-ruimte σ scheef aan l, en beide bevat in Ψ. Stel $\rho = \langle l, \sigma\rangle$. Beschouw twee punten P_1 en P_2 op l, en beschouw een regulus $\mathcal{R}_1$ en zijn tegenovergestelde regulus $\mathcal{R}_2$ in σ. Dan is $\mathcal{S}$ de verzameling van alle k-ruimten door $\langle\gamma, l\rangle$, alle k-ruimten door $\langle\gamma, P_1\rangle$ in de $(k+1)$-ruimte opgespannen door γ, l en een vaste rechte van $\mathcal{R}_1$, en alle k-ruimten door $\langle\gamma, P_2\rangle$ in de $(k+1)$-ruimte opgespannen door γ, l en een vaste rechte van $\mathcal{R}_2$. Dan is $|\mathcal{S}| = \theta_{n-k} + 2q^3 + 2q^2$.*

(ix) *Er bestaat een $(k-3)$-ruimte γ bevat in alle k-ruimten van $\mathcal{S}$. In de quotiëntruimte $\mathrm{PG}(n,q)/\gamma$, is de verzameling van vlakken, komende van de elementen van $\mathcal{S}$, de verzameling van de vlakken van voorbeeld VII in [33]: Zij Ψ een $(n-k+2)$-ruimte, disjunct aan γ in $\mathrm{PG}(n,q)$ en zij ρ een 5-ruimte in Ψ. Beschouw een rechte l en een 3-ruimte σ, disjunct aan l. Kies drie punten P_1, P_2, P_3 op l en kies vier niet-coplanaire punten Q_1, Q_2, Q_3, Q_4 in σ. Stel $l_1 = Q_1Q_2$, $\bar{l}_1 = Q_3Q_4$, $l_2 = Q_1Q_3$, $\bar{l}_2 = Q_2Q_4$, $l_3 = Q_1Q_4$, en $\bar{l}_3 = Q_2Q_3$. Dan is $\mathcal{S}$ de verzameling van alle k-ruimten door $\langle\gamma, l\rangle$ en alle k-ruimten door $\langle\gamma, P_i\rangle$ in $\langle\gamma, l, l_i\rangle$ of in $\langle\gamma, l, \bar{l}_i\rangle$, $i = 1, 2, 3$. Dan is $|\mathcal{S}| = \theta_{n-k} + 6q^2$.*

(x) *$\mathcal{S}$ is de verzameling van alle k-ruimten in een vaste $(k+2)$-ruimte ρ. Dan is $|\mathcal{S}| = \begin{bmatrix} k+3 \\ 2 \end{bmatrix}$.*

Hoofdstelling B.2.3. *Zij $\mathcal{S}$ een maximale verzameling van k-ruimten, paarsgewijs snijdend in minstens een $(k-2)$-ruimte in $\mathrm{PG}(n,q)$, $n \geq 2k$, $k \geq 3$. Zij*

$$f(k,q) = \begin{cases} 3q^4 + 6q^3 + 5q^2 + q + 1 & \text{als } k = 3, q \geq 2 \text{ of } k = 4, q = 2, \\ \theta_{k+1} + q^4 + 2q^3 + 3q^2 & \text{anders.} \end{cases}$$

Als $|\mathcal{S}| > f(k,q)$, dan is $\mathcal{S}$ één van de verzamelingen beschreven in Voorbeeld B.2.2. Merk op dat voor $n > 2k+1$, de voorbeelden $(i) - (ix)$ vermeld staan in dalende volgorde van grootte.

B.2.2 Hilton-Milner problemen in $\mathrm{PG}(n,q)$ en $\mathrm{AG}(n,q)$

Zoals hierboven reeds vermeld, is het geweten dat het grootste voorbeeld van k-ruimten, paarsgewijs snijden in een t-ruimte in $\mathrm{PG}(n,q)$, $n \geq 2k+1$ een t-bundel is. Dit voorbeeld wordt soms ook het triviale voorbeeld genoemd. Guo en Xu bewezen dat het grootste voorbeeld voor k-ruimten paarsgewijs snijdend in een t-ruimte in $\mathrm{AG}(n,q)$, $n \geq 2k+t+2$ ook een t-bundel is, zie [69]. In hoofstuk 4 worden de twee grootste niet-triviale voorbeelden van k-ruimten, paarsgewijs snijdend in een t-ruimte, in zowel $\mathrm{PG}(n,q)$ als $\mathrm{AG}(n,q)$ geclassificeerd voor $n > 2k+t+2$ en $q \geq 3$. Hierbij veronderstellen we dat $k > t$.

We starten met t-snijdende verzamelingen in een projectieve setting.

Voorbeeld B.2.4. *Zij Γ een $(t+2)$-ruimte in $\mathrm{PG}(n,q)$, $n \geq 2k-t+1$. Stel $\mathcal{S}$ gelijk aan de verzameling van alle k-ruimten in $\mathrm{PG}(n,q)$, die Γ snijden in minstens een $(t+1)$-ruimte.*

Voorbeeld B.2.5. *Zij δ een t-ruimte in $\mathrm{PG}(n,q)$, $n \geq 2k-t+1$, en zij ξ een $(k+1)$-ruimte in $\mathrm{PG}(n,q)$ met $\delta \subset \xi$. Zij S_1 de verzameling van alle k-ruimten in ξ. Zij S_2 de verzameling van alle k-ruimten door δ die ξ snijden in minstens een $(t+1)$-ruimte. De verzameling $\mathcal{S}$ is de unie van de verzamelingen S_1 en S_2.*

Merk op dat bovenstaande voorbeelden, voor $t = k-2$ overeenkomen met Voorbeeld B.2.2(ii) en (iii) respectievelijk. Deze voorbeelden zijn de grootste niet-triviale voorbeelden van t-snijdende veramelingen van k-ruimten in $\mathrm{PG}(n,q)$.

Stelling B.2.6. *Zij $\mathcal{S}_p$ een maximale verzameling van k-ruimten, paarsgewijs snijdend in minstens een t-ruimte in $\mathrm{PG}(n,q)$, $k \geq t+2$, $t \geq 1$, met $q \geq 3$, en $n \geq 2k+t+3$. Als $\mathcal{S}_p$ verschillend is van een t-bundel, dan is*

$$|\mathcal{S}_p| \leq \begin{cases} \theta_{k+1} - \theta_{k-t} + \begin{bmatrix} n-t \\ k-t \end{bmatrix} - q^{(k-t+1)(k-t)} \begin{bmatrix} n-k-1 \\ k-t \end{bmatrix} & \text{als } k > 2t+2 \\ \theta_{t+2} \cdot \left(\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} - \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \right) + \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} & \text{als } k \leq 2t+2. \end{cases}.$$

Gelijkheid geldt als en slechts als $\mathcal{S}_p$ gelijk is aan Voorbeeld B.2.4 voor $k \leq 2t+2$ of Voorbeeld B.2.5 voor $k \geq 2t+3$.

Nu geven we twee voorbeelden van grote t-snijdende verzamelingen van k-ruimten in $\mathrm{AG}(n,q)$. Voor een affiene ruimte α noteren we de projectieve uitbreiding van α als $\tilde{\alpha}$, en stel vervolgens $H_\infty = \mathrm{PG}(n,q) \setminus \mathrm{AG}(n,q)$ gelijk aan het hypervlak op oneindig.

Voorbeeld B.2.7. *Zij Γ een affiene $(t+2)$-ruimte in $\mathrm{AG}(n,q)$, en zij $\mathcal{R}$ een verzameling van θ_{t+1} affiene $(t+1)$-ruimten in Γ zodat voor elke twee verschillende elementen $\sigma_1, \sigma_2 \in \mathcal{R}$, $\tilde{\sigma}_1 \cap H_\infty \neq \tilde{\sigma}_2 \cap H_\infty$. Merk op dat elke twee verschillende elementen van $\mathcal{R}$ snijden in een affiene t-ruimte. Dan is $\mathcal{S}$ de verzameling van alle k-ruimten in $\mathrm{AG}(n,q)$, die Γ bevatten of Γ snijden in een element van $\mathcal{R}$.*

Voorbeeld B.2.8. *Zij δ een t-ruimte in $\mathrm{AG}(n,q)$, en zij ξ een $(k+1)$-ruimte in $AG(n,q)$ met $\delta \subset \xi$. Stel S_1 een maximale verzameling van affiene k-ruimten in ξ, zodat voor elke twee elementen π_1, π_2 van S_1, $\tilde{\pi}_1 \cap H_\infty \neq \tilde{\pi}_2 \cap H_\infty$, en zodat voor elke $\pi_1 \in S_1$: $\tilde{\delta} \cap H_\infty \nsubseteq \tilde{\pi}_1$. Stel S_2 de verzameling van alle k-ruimten door δ die ξ snijden in minstens een affiene $(t+1)$-ruimte. Dan is $\mathcal{S}$ de unie van de twee verzamelingen S_1 en S_2.*

We vinden dat de grootste niet triviale voorbeelden van t-snijdende verzamelingen in $\mathrm{AG}(n,q)$ komen van bovenstaande voorbeelden.

Stelling B.2.9. *Zij $\mathcal{S}_a$ een maximale verzameling van k-ruimten, paarsgewijs snijdend in minstens een t-ruimte in $\mathrm{AG}(n,q)$, $k \geq t+2$, $t \geq 1$, met $q \geq 3$, en $n \geq 2k+t+3$. Als $\mathcal{S}_a$ verschillend is van een t-bundel, dan is*

$$|\mathcal{S}_a| \leq \begin{cases} \theta_{k} - \theta_{k-t} + \begin{bmatrix} n-t \\ k-t \end{bmatrix} - q^{(k-t+1)(k-t)} \begin{bmatrix} n-k-1 \\ k-t \end{bmatrix} & \text{als } k > 2t+1 \\ \theta_{t+1} \cdot \left(\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} - \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \right) + \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} & \text{als } k \leq 2t+1. \end{cases}$$

Gelijkheid geldt als en slechts als $\mathcal{S}_a$ gelijk is aan Voorbeeld B.2.7 voor $k \leq 2t+1$ of Voorbeeld B.2.8 voor $k \geq 2t+2$.

B.2.3 De Zonnebloemgrens

In de vorige hoofdstukken bestudeerden we deelruimten paarsgewijs snijdend in *minstens* een deelruimte van een zeker dimensie. In Hoofdstuk 5 worden verzamelingen S van k-ruimten in $\mathrm{PG}(n,q)$ onderzocht, met de eigenschap dat de elementen van S paarsgewijs snijden in *precies* een punt. Meer algemeen is een $(k+1,t+1)$-SCID een verzameling van k-ruimten, paargeswijs snijdend in precies een t-ruimte in $\mathrm{PG}(n,q)$. Een voorbeeld van zo een SCID is de verzameling S van k-ruimten, zodat voor elke $\pi,\tau\in S$ er geldt dat $\pi\cap\tau=\gamma$ voor een vaste t-ruimte γ. Dit voorbeeld is een *zonnebloem met centrum* γ. De *Zonnebloemgrens* stelt dat, als het aantal elementen van $(k+1,t+1)$-SCID S, deze grens overschrijdt, dan moet S een zonnebloem zijn.

Stelling B.2.10 ([56, Theorem 1]). *Een $(k+1,t+1)$-SCID S in $\mathrm{PG}(n,q)$, is een zonnebloem als*

$$|S|>\left(\frac{q^{k+1}-q^{t+1}}{q-1}\right)^2+\left(\frac{q^{k+1}-q^{t+1}}{q-1}\right)+1.$$

In Hoofdstuk 5 wordt bewezen dat deze grens, voor $t=0$, kan verbeterd worden voor $k\geq 3$ en $q\geq 7$. Voor $k=1$ en $k=2$, is er een complete classificatie gekend: Elke $(k+1,k)$-SCID is een zonnebloem of bestaat uit alle k-ruimten in een vaste $(k+1)$-ruimte. Voor de classificatie van $(3,1)$-SCID's, verwijzen we naar [9].

Stelling B.2.11. *Een verzameling an k-ruimten in $\mathrm{PG}(n,q)$, $k\geq 3, q\geq 7$, die paarsgewijs snijden in precies een punt, met meer dan $F_q\theta_k^2$ elementen is een zonnebloem. Hierbij gebruiken we*

$$F_q=\frac{1}{2}\left(\frac{B_q}{c_q^2}-\frac{1}{q}-\sqrt{\left(\frac{1}{q}-\frac{B_q}{c_q^2}\right)^2-4B_q\left(\frac{1}{c_q^2}-1\right)}\right)$$

met

$$B_q=(1-c_q)^2\left(1-c_q-\frac{1}{q^3}\right)^2\left(1-c_q-\frac{c_q}{q}\right)\left(1-c_q-\frac{1+c_q}{q}\right)q,$$
$$c_q=1-\frac{1}{\sqrt[6]{q}}-\frac{1}{2\sqrt[3]{q}}.$$

In het bijzonder vinden we dat een dergelijke verzameling met meer dan $\left(\frac{2}{\sqrt[6]{q}}+\frac{4}{\sqrt[3]{q}}-\frac{5}{\sqrt{q}}\right)\theta_k^2$ elementen een zonnebloem is.

B.2.4 Het chromatisch getal van enkele q-Kneser grafen

Een *vlag* in $\mathrm{PG}(n,q)$ is een verzameling F van niet-triviale deelruimten van $\mathrm{PG}(n,q)$ (dus, deelruimten verschillend van $\emptyset$ en $\mathrm{PG}(n,q)$) zodat voor alle $\alpha,\beta\in F$ er geldt dat $\alpha\subset\beta$ of $\beta\subset\alpha$. De deelverzameling $\{\dim(\alpha)+1\mid\alpha\in F\}$, waarbij we gebruik maken van de projectieve dimensie, wordt het *type* van F genoemd, en is bevat in $\{1,2,\ldots,n\}$. Twee vlaggen F en G zijn in *algemene positie* als $\alpha\cap\beta=\emptyset$ of $\langle\alpha,\beta\rangle=\mathrm{PG}(n,q)$ voor alle $\alpha\in F$ en $\beta\in G$.

Voor $\Omega\subseteq\{1,2,\ldots,n\}$ is de q-Knesergraaf $qK_{n+1;\Omega}$ de graaf waarin de toppen overeenkomen met de vlaggen van type Ω in $\mathrm{PG}(n,q)$, en waarin twee toppen zijn adjacent, als de overeenkomstige vlaggen in algemene positie zijn. Wij zijn geïnteresseerd in het chromatisch getal van deze grafen.

Voor een punt $P \in \mathrm{PG}(n,q)$, definiëren we de verzameling $\mathcal{F}_\Omega(P)$ als de verzameling van alle vlaggen F van type $\Omega \subseteq \{2,3,\ldots,n\}$ waarvoor $F \cup \{P\}$ ook een vlag is. We noemen deze verzameling $\mathcal{F}_\Omega(P)$ de *punt-bundel* (van vlaggen van type Ω) met basispunt P.

We bepaalden het chromatisch getal van de grafen $qK_{5;\Omega}$ voor $\Omega = \{2,4\}$ en $q \neq 2$, en voor $qK_{2d+1;\{d,d+1\}}$, met $d \geq 2$ en q heel groot.

We gebruikten het cokliekgetal, samen met structurele informatie over grote coklieken van $qK_{5;\{2,4\}}$ en $qK_{2d+1,\{d,d+1\}}$, $q \geq 2$. Deze structurele informatie is te vinden in de Hilton-Milner type resultaten in [14] voor $qK_{5;\{2,4\}}$, in [11] voor $qK_{2d+1,\{d,d+1\}}$, met $d = 2$ en in [94] voor $qK_{2d+1,\{d,d+1\}}$, met $d = 3$. Voor $d \geq 4$ is er geen structurele informatie gekend over grote coklieken in $qK_{2d+1,\{d,d+1\}}$. Daarom nemen we, in dit geval, een extra assumptie aan, zie Vermoeden B.2.15. We vonden de volgende resultaten.

Stelling B.2.12. *Voor $q \geq 3$ is het chromatisch getal van de Knesergraaf $qK_{5;\{2,4\}}$ gelijk aan θ_3. Daarnaast is elke kleurklasse van een minimale kleuring bevat in een punt-bundel. De basispunten van deze punt-bundels zijn de punten van een drie-ruimte.*

Stelling B.2.13. *Voor $q > 160 \cdot 36^5$, is het chromatisch getal van de Knesergraaf $qK_{5;\{2,3\}}$ gelijk aan $\theta_3 - q$. Op dualiteit na, is er voor elke kleurklasse van een minimale kleuring een unieke punt-bundel F, zodat $F \cup C$ een cokliek is. De basispunten van deze punt-bundels zijn $\theta_3 - q$ verschillende punten van een drie-ruimte.*

Stelling B.2.14. *Voor $q > 3 \cdot 7^{15} \cdot 2^{56}$, is het chromatisch getal van de Knesergraaf $qK_{7;\{3,4\}}$ gelijk aan $\theta_4 - q$. Op dualiteit na, is er voor elke kleurklasse van een minimale kleuring een unieke punt-bundel F, zodat $F \cup C$ een cokliek is. De basispunten van deze punt-bundels zijn $\theta_4 - q$ verschillende punten van een vier-ruimte.*

Vermoeden B.2.15. *Voor elk natuurlijk getal $d \geq 4$ bestaat er een $\rho(d) \in \mathbb{N}$, zodat elke maximale cokliek van de Knesergraaf $qK_{2d+1,\{d,d+1\}}$ een punt-bundel, het duale van een punt-bundel, of hoogstns $\rho(d) \cdot q^{d^2+d-2}$ elementen bevat.*

Stelling B.2.16. *Als Vermoeden B.2.15 waar is voor een zeker natuurlijk getal $d \geq 4$, dan is*

$$\chi(qK_{2d+1,\{d,d+1\}}) = \theta_{d+1} - q,$$

voor q voldoende groot, afhankelijk van d en $\rho(d)$. Bijkomend, als $\mathfrak{F}$ een familie is van dit aantal maximale coklieken die de volledige toppenverzameling bedekt, dan bestaat er – op dualiteit na – een $(d+1)$-ruimte U in $\mathrm{PG}(2d,q)$ en een injectieve afbeelding μ van $\mathfrak{F}$ naar een verzameling van punten van U, zodat de punt-bundel $\mathcal{F}(\mu(C))$ bevat is in C voor alle $C \in \mathfrak{F}$.

B.3 Cameron-Lieblerverzamelingen

In het tweede deel van deze thesis worden Cameron-Lieblerveramelingen, in verschillende contexten onderzocht. De rode draad in dit deel kan samengevat worden met twee centrale vragen; wat zijn de equivalent definities voor deze verzamelingen, en voor welke parameters x bestaan er Cameron-Lieblerverzamelingen? We onderzoeken beide vragen in projectieve, affiene en polaire ruimten.

B.3.1 Cameron-Liebler k-ruimten in $\mathrm{PG}(n, q)$

We onderzoeken Cameron-Lieblerverzamelingen van k-ruimten in $\mathrm{PG}(n, q)$. Hiervoor lijsten we verschillende equivalente definities op voor deze verzamelingen, door de gekende resultaten voor Cameron-Liebler rechte verzamelingen in $\mathrm{PG}(n, q)$, zie [51], en Cameron-Lieblerverzamelingen van k-ruimten $\mathrm{PG}(2k + 1, q)$, zie [104], te veralgemenen.

Zij A de incidentiematrix van de punten en k-ruimten van $\mathrm{PG}(n, q)$: de rijen van A zijn gelabeld door de punten, en de kolommen door de k-ruimten. Zij V_i, $0 \leq i \leq k$, de eigenruimten van het bijhorende Grassmannschema, in de klassieke ordening, zie Hoofdstuk 10.1.1.

Stelling B.3.1. *Zij $\mathcal{L}$ een niet-ledige verzameling van k-ruimten in $\mathrm{PG}(n, q), n \geq 2k + 1$, met karakteristieke vector χ, en x zodat $|\mathcal{L}| = x \begin{bmatrix} n \\ k \end{bmatrix}$. Dan zijn de volgende eigenschappen equivalent.*

1. $\chi \in \mathrm{im}(A^T)$.
2. $\chi \in \ker(A)^{\perp}$.
3. *Voor elke k-ruimte π is het aantal elementen van $\mathcal{L}$ scheef aan π gelijk aan* $(x - \chi(\pi)) \begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k}$.
4. *De vector* $\chi - x\frac{q^{k+1}-1}{q^{n+1}-1}\boldsymbol{j}$ *is een vector in* V_1.
5. $\chi \in V_0 \perp V_1$.
6. *Voor een gegeven $i \in \{1, \ldots, k + 1\}$ en een k-ruimte π, is het aantal elementen van $\mathcal{L}$, die π snijden in een $(k - i)$-ruimte, gegeven door:*

$$\begin{cases} \left((x-1)\frac{q^{k+1}-1}{q^{k-i+1}-1} + q^i\frac{q^{n-k}-1}{q^i-1}\right) q^{i(i-1)} \begin{bmatrix} n-k-1 \\ i-1 \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} & \textit{als } \pi \in \mathcal{L} \\ x \begin{bmatrix} n-k-1 \\ i-1 \end{bmatrix} \begin{bmatrix} k+1 \\ i \end{bmatrix} q^{i(i-1)} & \textit{als } \pi \notin \mathcal{L} \end{cases}.$$

7. *Voor elk paar van toegevoegde omwisselende k-verzamelingen $\mathcal{R}$ en $\mathcal{R}'$, geldt er dat* $|\mathcal{L} \cap \mathcal{R}| = |\mathcal{L} \cap \mathcal{R}'|$.

Als er k-spreads bestaan in $\mathrm{PG}(n, q)$, dan zijn de volgende eigenschappen equivalent aan de vorige.

8. $|\mathcal{L} \cap \mathcal{S}| = x$ *voor elke k-spread $\mathcal{S}$ in* $\mathrm{PG}(n, q)$.
9. $|\mathcal{L} \cap \mathcal{S}| = x$ *voor elke Desarguesiaanse k-spread $\mathcal{S}$ in* $\mathrm{PG}(n, q)$.

Definitie B.3.2. Een verzameling $\mathcal{L}$ van k-ruimten in $\mathrm{PG}(n, q)$ die voldoet aan één van de eigenschappen in Stelling A.3.1 (en dus aan ze allemaal) wordt een *Cameron-Lieblerverzameling van k-ruimten* in $\mathrm{PG}(n, q)$ genoemd, met parameter $x = |\mathcal{L}| \begin{bmatrix} n \\ k \end{bmatrix}^{-1}$.

Gebruik makend van de informatie uit de equivalente definities, samen met enkele extra eigenschappen, vonden we verschillende classificatieresultaten voor Cameron-Lieblerverzamelingen van k-ruimten in $\mathrm{PG}(n, q)$. Merk op dat een Cameron-Lieblerverzameling van k-ruimten met parameter 0 gelijk is aan de ledige verzameling.

In het volgende lemma geven we de classificatie van de parameters $x \in]0, 2[$.

Lemma B.3.3. *Er bestaat geen Cameron-Lieblerverzameling van k-ruimten in $\mathrm{PG}(n,q)$ met parameter $x \in]0,1[$, en voor $n \geq 3k+2$, bestaan er ook geen Cameron-Lieblerverzamelingen van k-ruimten met parameter $x \in]1,2[$. Zij $\mathcal{L}$ een Cameron-Lieblerverzameling van k-ruimten met parameter $x = 1$ in $\mathrm{PG}(n,q)$, $n \geq 2k+1$. Dan is $\mathcal{L}$ een punt-bundel, of $n = 2k+1$ en $\mathcal{L}$ is de verzameling van alle k-ruimten in een hypervlak van $\mathrm{PG}(2k+1,q)$.*

We eindigen met het belangrijkste classificatieresultaat uit dit project.

Stelling B.3.4. *Er bestaan geen Cameron-Lieblerverzamelingen van k-ruimten in $\mathrm{PG}(n,q)$, $n \geq 3k+2$ en $q \geq 3$, met parameter $2 \leq x \leq \frac{1}{\sqrt[8]{2}} q^{\frac{n}{2}-\frac{k^2}{4}-\frac{3k}{4}-\frac{3}{2}}(q-1)^{\frac{k^2}{4}-\frac{k}{4}+\frac{1}{2}}\sqrt{q^2+q+1}$.*

B.3.2 Cameron-Liebler k-ruimten in $\mathrm{AG}(n,q)$

In Hoofdstuk 4.4.3, geven we een overzicht van de belangrijkste (equivalente) definities en classificatieresultaten voor Cameron-Lieblerverzamelingen in affiene ruimten. De resultaten in dit hoofdstuk werden bewezen in [46] en [44]. Vergelijkbaar met de definitie van Cameron-Lieblerverzamelingen van k-ruimten in $\mathrm{PG}(n,q)$, kunnen we Cameron-Lieblerverzamelingen in $\mathrm{AG}(n,q)$ als volgt definiëren.

Definitie B.3.5. Een verzameling $\mathcal{L}$ van k-ruimten in $\mathrm{AG}(n,q)$ is een *Cameron-Lieblerverzameling van k-ruimten in* $\mathrm{AG}(n,q)$ *met parameter* x als en slechts als elke k-spread in $\mathrm{AG}(n,q)$ x elementen gemeen heeft met $\mathcal{L}$.

In tegenstelling tot k-spreads in $\mathrm{PG}(n,q)$ zien we dat er k-spreads bestaan in $\mathrm{AG}(n,q)$, voor elke $n \geq k$, wat impliceert dat de bovenstaande definitie goed gedefinieerd is.

Door het onmiddellijke verband tussen $\mathrm{PG}(n,q)$ en $\mathrm{AG}(n,q)$ is het mogelijk om Cameron-Lieblerverzamelingen in $\mathrm{AG}(n,q)$ te classificeren, door gebruik te maken van de ideeën voor hetzelfde onderzoeksproject in projectieve ruimten.

Stelling B.3.6. *Er bestaan geen Cameron-Lieblerverzamelingen van k-ruimten in $\mathrm{AG}(n,q)$, $n \geq 3k+2$ en $q \geq 3$, met parameter $2 \leq x \leq \frac{1}{\sqrt[8]{2}} q^{\frac{n}{2}-\frac{k^2}{4}-\frac{3k}{4}-\frac{3}{2}}(q-1)^{\frac{k^2}{4}-\frac{k}{4}+\frac{1}{2}}\sqrt{q^2+q+1}$.*

B.3.3 Cameron-Lieblerverzamelingen van graad één in eindige klassieke polaire ruimten

In dit hoofdstuk bestuderen we Cameron-Lieblerverzamelingen van graad één, van generatoren in eindige klassieke polaire ruimten. De matrix A is de incidentiematrix van punten en generatoren.

Definitie B.3.7. Een *Cameron-Lieblerverzameling van graad één van generatoren in een eindige klassieke polaire ruimte $\mathcal{P}$* is een verzameling van generatoren in $\mathcal{P}$, met karakteristieke vector χ zodat $\chi \in \mathrm{im}(A^T)$.

Deze definitie kan gelinkt worden aan de definitie van een Boolean *degree one* functie voor generatoren in polaire ruimten, zie [59]. De definitie in dit artikel komt overeen met het feit dat de karakteristieke vector van de verzameling gelegen is in $V_0 \perp V_1$. Dit zijn de eigenruimten van het bijhorende associatie schema (zie Sectie 1.9). In [36], M. De Boeck, M. Rodgers, L. Storme en A. Švob introduceerden Cameron-Lieblerverzamelingen van generatoren in eindige klassieke polaire ruimten. In dit artikel, worden Cameron-Lieblerverzamelingen van generatoren in een polaire ruimte gedefinieerd door de *disjunctheidsdefinitie*. Daarbij geven de auteurs verschillende equivalente definities voor deze verzamelingen. Merk op dat deze definitie de polaire-ruimte-versie is voor de disjunctheidsdefinitie in de projectieve context, zie Stelling B.3.1.3.

Definitie B.3.8 ([36]). Zij $\mathcal{P}$ een eindige klassieke polaire ruimte met parameter e en rang d. Een verzameling $\mathcal{L}$ van generatoren in $\mathcal{P}$ is een Cameron-Lieblerverzameling van generatoren in $\mathcal{P}$, met parameter x, als en slechts als voor elke generator π in $\mathcal{P}$, het aantal elementen van $\mathcal{L}$, disjunct aan π is gelijk aan $(x-\chi(\pi))q^{\binom{d-1}{2}+e(d-1)}$.

We kunnen deze definitie, gebruik makend van de notatie van associatie schema's, als volgt interpreteren. De karakteristieke vector van een Cameron-Lieblerverzameling is bevat in $V_0 \perp W$, met W de eigenruimte van de disjunctie matrix A_d, horende bij een specifieke eigenwaarde. Men kan inzien dat V_1 steeds bevat is in W, maar het is er niet steeds aan gelijk. Hieruit volgt dat elke Cameron-Lieblerverzameling van graad één ook een Cameron-Lieblerverzameling is.

Elke Cameron-Lieblerverzameling van graad één is dus een Cameron-Lieblerverzameling, en voor sommige polaire ruimten vallen Cameron-Lieblerverzamelingen en Cameron-Lieblerverzamelingen van graad één samen, maar voor andere zal dit niet het geval zijn.

Merk op dat we Cameron-Lieblerverzamelingen van graad één op een algebraïsche manier gedefinieerd hebben. Over het algemeen kunnen Cameron-Lieblerverzamelingen, in verschillende contexten, gedefinieerd worden door zowel algebraïsche als combinatorische definities te gebruiken. Voor deze Cameron-Lieblerverzamelingen van graad één vonden we ook dat dit mogelijk is, en vonden we een equivalente combinatorische definitie.

Stelling B.3.9. *Zij $\mathcal{P}$ een eindige klassieke polaire ruimte, van rang d met parameter e, zij $\mathcal{L}$ een verzameling van generatoren van $\mathcal{P}$ en i een natuurlijk getal met $1 \leq i \leq d$. Als $\mathcal{L}$ een Cameron-Lieblerverzameling van graad één, van generatoren in $\mathcal{P}$ is, met parameter x, dan is het aantal elementen van $\mathcal{L}$ dat een generator π snijdt in een $(d-i-1)$-ruimte gelijk aan*

$$\begin{cases} \left((x-1)\begin{bmatrix} d-1 \\ i-1 \end{bmatrix} + q^{i+e-1}\begin{bmatrix} d-1 \\ i \end{bmatrix}\right) q^{\binom{i-1}{2}+(i-1)e} & \textit{als } \pi \in \mathcal{L} \\ x\begin{bmatrix} d-1 \\ i-1 \end{bmatrix} q^{\binom{i-1}{2}+(i-1)e} & \textit{als } \pi \notin \mathcal{L}. \end{cases}$$

Bovendien, als deze eigenschap geldt voor een polaire ruimte $\mathcal{P}$ en een geheel getal i zo dat

- *i is oneven voor $\mathcal{P} = Q^+(2d-1,q)$,*
- *$i \neq d$ voor $\mathcal{P} = Q(2d,q)$ of $\mathcal{P} = W(2d-1,q)$, beide met d oneven of*
- *i is willekeurig in de andere gevallen,*

dan is $\mathcal{L}$ een Cameron-Lieblerverzameling van graad één met parameter x.

Verder onderzochten we ook voor welke waarden van de parameter x er een Cameron-Lieblerverzameling van graad één bestaat. Voor Cameron-Lieblerverzamelingen van graad één in $W(5,q)$ en $Q(6,q)$ vonden we het volgende classificatieresultaat.

Stelling B.3.10. *Een Cameron-Lieblerverzameling $\mathcal{L}$ van graad één van generatoren in $W(5,q)$ of $Q(6,q)$ met parameter $2 \leq x \leq \sqrt[3]{2q^2} - \frac{\sqrt[3]{4q}}{3} + \frac{1}{6}$ is de unie van α ingebedde hyperbolische kwadrieken $Q^+(5,q)$, die paarsgewijs geen enkel vlak gemeen hebben, en $x-2\alpha$ punt-bundels waarvan de basispunten paarsgewijs niet-collineair zijn en niet bevat in de α hyperbolische kwadrieken $Q^+(5,q)$. Voor de polaire ruimte $Q(6,q)$ of $W(5,q)$ met q even, $\alpha \in \{0,\dots,\lfloor \frac{x}{2} \rfloor\}$, voor de polaire ruimte $W(5,q)$ met q oneven, $\alpha = 0$.*

B.3.4 Nieuw voorbeeld van een Cameron-Lieblerverzameling van graad één van generatoren in $Q^+(5,q)$

We geven een voorbeeld van een nieuwe, niet-triviale Cameron-Lieblerverzameling van generatoren in $Q^+(5,q)$, q oneven. Om de constructie van het voorbeeld uit te leggen, maken we gebruik van de Klein-correspondentie tussen de rechten van $Q^+(3,q)$ en de punten van $Q^+(5,q)$.

Beschouw de hyperbolische kwadriek $Q = Q^+(3,q)$ in $\mathrm{PG}(3,q)$, gedefinieerd door de vergelijking $x_0x_1 + x_2x_3 = 0$. De rechten van Q corresponderen met de puntenverzameling van twee kegels $C \cup C'$ in $Q^+(5,q)$, zo dat voor de vlakken $\alpha = \langle C \rangle$ en $\alpha' = \langle C' \rangle$ geldt dat α' het beeld is van α onder de polariteit van $Q^+(5,q)$.

Elk punt $P \in \mathrm{PG}(3,q)$ geeft aanleiding tot een Latijns vlak π_l^P en een Grieks vlak π_g^P in $Q^+(5,q)$: de punten van π_l^P corresponderen met alle rechten door P in $\mathrm{PG}(3,q)$, en de punten van π_g^P corresponderen met alle rechten in het vlak $P^\perp$. Hierbij is $\perp$ de polariteit gerelateerd aan de kwadriek Q in $\mathrm{PG}(3,q)$.

Definitie B.3.11. Een punt $P(x_0,x_1,x_2,x_3) \in \mathrm{PG}(3,q)$ is een *kwadraatpunt* als $x_0x_1 + x_2x_3$ een kwadraat verschillend van 0 is in $\mathbb{F}_q$. Een punt $P(x_0,x_1,x_2,x_3) \in \mathrm{PG}(3,q)$ is een *niet-kwadraatpunt* als $x_0x_1 + x_2x_3$ een niet-kwadraat is in $\mathbb{F}_q$.

Nu kunnen we de verzameling vlakken in $Q^+(5,q)$ verdelen in de volgende verzamelingen.

- $\mathcal{S}_l = \{\pi_l^P | P \text{ is een kwadraatpunt}\}$
- $\mathcal{NS}_l = \{\pi_l^P | P \text{ is een niet-kwadraatpunt}\}$
- $\mathcal{O}_l = \{\pi_l^P | P \in Q\}$
- $\mathcal{S}_g = \{\pi_g^P | P \text{ is een kwadraatpunt}\}$
- $\mathcal{NS}_g = \{\pi_g^P | P \text{ is een niet-kwadraatpunt}\}$
- $\mathcal{O}_g = \{\pi_g^P | P \in Q\}$

Voor een raaklijn ℓ aan Q zijn er twee mogelijkheden; ℓ bevat q kwadraatpunten, of ℓ bevat q niet-kwadraatpunten, zie [72, Tabel 15.5(c)]. In het eerste geval is ℓ een kwadraatraaklijn. In het tweede geval is ℓ een niet-kwadraatraaklijn.

We verdelen de punten in $Q^+(5,q)$ op in de volgende verzamelingen.

- De verzameling $\mathcal{X}_{1S}$ van punten in $Q^+(5,q)$ die overeenkomen met de kwadraatraaklijnen aan Q.
- De verzameling $\mathcal{X}_{1NS}$ van punten in $Q^+(5,q)$ die overeenkomen met de niet-kwadraatraaklijnen aan Q.
- De verzameling $\mathcal{X}_2$ van punten in $Q^+(5,q)$ die overeenkomen met de twee-secanten aan Q.
- De verzameling $\mathcal{X}_0$ van punten in $Q^+(5,q)$ die overeenkomen met de rechten disjunct aan Q.
- De verzameling $\mathcal{X}_\infty = C \cup C'$ van punten in $Q^+(5,q)$ die overeenkomen met de rechten in Q.

We konden aantonen dat de partities $\{\mathcal{X}_{1S}, \mathcal{X}_{1NS}, \mathcal{X}_2, \mathcal{X}_0, \mathcal{X}_\infty\}$ en $\{\mathcal{S}_l, \mathcal{S}_g, \mathcal{NS}_l, \mathcal{NS}_g, \mathcal{O}_l, \mathcal{O}_g\}$ een punt-tactische decompositie vormen. Door de juiste partitieklassen te groeperen, vinden we nieuwe Cameron-Lieblerverzamelingen in $Q^+(5,q)$.

Stelling B.3.12. *Zij q een oneven priemmacht.*

- *De verzamelingen $\mathcal{S}_l \cup \mathcal{S}_g$ en $\mathcal{NS}_l \cup \mathcal{NS}_g$ zijn Cameron-Lieblerverzamelingen van graad één van vlakken in $Q^+(5,q)$, met parameter $\frac{q(q-1)}{2}$, $\frac{q(q-1)}{2}$ en $q+1$ respectievelijk, voor $q \equiv 1 \mod 4$.*
- *De verzamelingen $\mathcal{S}_l \cup \mathcal{NS}_g$ en $\mathcal{S}_g \cup \mathcal{NS}_l$ zijn Cameron-Lieblerverzamelingen van graad één van vlakken in $Q^+(5,q)$, met parameter $\frac{q(q-1)}{2}$, $\frac{q(q-1)}{2}$ en $q+1$ respectievelijk, voor $q \equiv 3 \mod 4$.*

B.4 Lineaire verzamelingen

In het laatste deel van deze thesis bespreken we een onderzoeksproject over translatiehyperovalen en $\mathbb{F}_2$-lineaire verzamelingen. We geven een verband tussen de affiene punten van een translatiehyperovaal in $\mathrm{PG}(2,q^k)$ en de puntenverzameling van een geschatterde $\mathbb{F}_2$-lineaire verzameling van het pseudoregulustype in $\mathrm{PG}(2k-1,q)$, gezien al een verzameling van richtingen. Hiervoor gebruikten we de Barlotti-Cofman constructie, die een veralgemening is van de André/Bruck-Boseconstructie.

Het oorspronkelijke doel van dit onderzoeksproject was om het volgende resultaat van Barwick en Jackson te veralgemenen.

Resultaat B.4.1 ([7, Theorem 1.2]). *Beschouw $\mathrm{PG}(4,q)$, q even, $q > 2$, met het hypervlak op oneindig, aangeduid door Σ_∞. Zij C een verzameling van q^2 affiene punten, genaamd $\mathcal{C}$-punten en beschouw een verzameling vlakken, genaamd $\mathcal{C}$-vlakken, die voldoet aan de volgende eigenschappen.*

(A1) Elk $\mathcal{C}$-vlak snijdt $\mathcal{C}$ in een q-boog.

(A2) Elke twee verschillende $\mathcal{C}$-punten liggen in een uniek $\mathcal{C}$-vlak.

(A3) De affiene punten, niet in $\mathcal{C}$, liggen op precies één $\mathcal{C}$-vlak.

(A4) Elk vlak dat minstens 3 punten van $\mathcal{C}$ bevat, bevat precies 4 punten van $\mathcal{C}$ of is een $\mathcal{C}$-vlak.

Dan bestaat er een Desarguesiaanse spread $\mathcal{S}$ in Σ_∞ zodat dat in het André/Bruck-Bose vlak $\mathcal{P}(\mathcal{S}) \cong \mathrm{PG}(2,q^2)$ de $\mathcal{C}$-punten samen met 2 extra punten op ℓ_∞ een translatiehyperovaal vormen in $\mathrm{PG}(2,q^2)$.

Bij de zoektocht naar een veralgemening onderzochten we een verzameling C van q^k affiene punten in $\mathrm{PG}(2k,q)$, q even, $q > 2$, met gelijkaardige combinatorische eigenschappen. De technieken die Barwick en Jackson gebruikten in het bewijs van bovenstaand resultaat waren niet veralgemeenbaar. Daardoor zijn we op zoek gegaan naar andere technieken, waaronder het gebruik van lineaire verzamelingen, in het bijzonder deze van pseudoregulustype. Tijdens dit onderzoek konden we het volgende belangrijke resultaat bewijzen.

Stelling B.4.2. *Zij $\mathcal{Q}$ een verzameling van q^k affiene punten in $\mathrm{PG}(2k,q)$, $q = 2^h$, $h \geq 4$, $k \geq 2$, die een verzameling D van $q^k - 1$ richtingen in het hypervlak op oneindig $H_\infty = PG(2k-1,q)$ bepaalt. Stel dat elke rechte 0, 1, 3 of $q-1$ punten gemeen heeft met de puntenverzameling D. Dan geldt het volgende.*

(1) D is een $\mathbb{F}_2$-lineaire verzameling van het pseudoregulustype.

(2) Er bestaat een Desarguesiaanse spread $\mathcal{S}$ in H_∞ zodanig dat in het André/Bruck-Bose vlak $\mathcal{P}(\mathcal{S}) \cong \mathrm{PG}(2,q^k)$, met H_∞ corresponderend met de rechte l_∞, de punten van $\mathcal{Q}$ samen met 2 extra punten op ℓ_∞ een translatiehyperovaal vormen in $\mathrm{PG}(2,q^k)$.

Omgekeerd komt, via de André/Bruck-Boseconstructie, de verzameling affiene punten van een translatiehyperovaal in $\mathrm{PG}(2,q^k)$, $q>4, k\geq 2$, *overeen met een verzameling* $\mathcal{Q}$ *van* q^k *affiene punten in* $\mathrm{PG}(2k,q)$ *waarvan de verzameling bepaalde richtingen* D *een* $\mathbb{F}_2$*-lineaire verzameling is van het pseudoregulustype. Bijgevolg bevat elke rechte* $0,1,3$ *of* $q-1$ *punten van* D.

Een onmiddelijk gevolg van deze stelling is de veralgemening van Resultaat B.4.1.

Stelling B.4.3. *Beschouw* $\mathrm{PG}(2k,q)$, q *even,* $q>2$, *met het hypervlak op oneindig, aangeduid door* Σ_∞. *Zij* $\mathcal{C}$ *een verzameling van* q^k *affiene punten, genaamd* $\mathcal{C}$*-punten en beschouw een verzameling vlakken, genaamd* $\mathcal{C}$*-vlakken, die voldoet aan de volgende eigenschappen.*

(A1) *Elk* $\mathcal{C}$*-vlak snijdt* $\mathcal{C}$ *in een* q*-boog.*

(A2) *Elke twee verschillende* $\mathcal{C}$*-punten liggen in een uniek* $\mathcal{C}$*-vlak.*

(A3) *De affiene punten, niet in* $\mathcal{C}$, *zijn bevat in precies één* $\mathcal{C}$*-vlak.*

(A4) *Elke vlak dat minstens* 3 *punten bevat van* $\mathcal{C}$, *bevat precies* 4 *punten van* $\mathcal{C}$ *of is een* $\mathcal{C}$*-vlak.*

Dan bestaat er een Desarguesiaanse spread $\mathcal{S}$ *in* Σ_∞ *zodanig dat in het André/Bruck-Bose vlak* $\mathcal{P}(\mathcal{S})\cong\mathrm{PG}(2,q^k)$ *de* $\mathcal{C}$*-punten samen met* 2 *extra punten op* ℓ_∞ *een translatiehyperovaal vormen in* $\mathrm{PG}(2,q^k)$.

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> “ *On ne voit bien qu’avec le coeur. L’essentiel est invisible pour les yeux.* ”
>
> —Antoine de Saint-Exupéry, *Le Petit Prince*

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Index