LOCAL BEHAVIOR OF DISTRIBUTIONS AND APPLICATIONS

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Dedicada a Antonio, Marta, y Raquel

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Abstract

This dissertation studies local and asymptotic properties of distributions (generalized functions) in connection to several problems in harmonic analysis, approximation theory, classical real and complex function theory, tauberian theory, summability of divergent series and integrals, and number theory.

In Chapter 2 we give two new proofs of the Prime Number Theory based on ideas from asymptotic analysis on spaces of distributions.

Several inverse problems in Fourier analysis and summability theory are studied in detail. Chapter 3 provides a complete characterization of point values of tempered distributions and functions in terms of a generalized pointwise Fourier inversion formula. The relation of the Fourier inversion formula with several summability procedures for divergent series and integrals is established. This work also provides formulas for jump singularities, that is, detection of edges from spectral data, which can be used as effective numerical detectors. Chapters 5 and 6 introduce new summability methods for the determination of jump discontinuities. Estimations on orders of summability are given in Chapter 8.

Chapters 4 and 9 give a tauberian theory for distributional point values; this theory recovers important classical tauberians of Hardy and Littlewood, among others, for Dirichlet series.

We make a complete wavelet analysis of asymptotic properties of distributions in Chapter 11. This study connects the boundary asymptotic behavior of the wavelet transform with asymptotics of tempered distributions. It is shown that our tauberian theorems become full characterizations.

Chapter 10 makes a comprehensive study of asymptotic properties of distributions. Open problems in the area are solved in Chapter 10 and new tools are

developed. We obtain a complete structural description of quasiasymptotics in one variable.

We introduce the $\phi-$transform for the local analysis of functions, measures, and distributions. In Chapter 7 the transform is used to study distributionally regulated functions (introduced here). Chapter 12 presents a characterization of measures in terms of the boundary behavior of this transform. We characterize the support of tempered distributions in Chapter 13 by various summability means of the Fourier transform.

Introduction

The theory of Schwartz distributions, and other types of generalized functions, is a very powerful tool in analysis and applied mathematics. There are several approaches to the theory of distributions, but in all of them one quickly learns that distributions do not have point values, as functions do, despite the fact that they are called "generalized functions." Interestingly, many common objects in analysis do not have point values, even though they are referred as "functions": If $f \in L^1(\mathbb{R})$, what is $f(0)$? Recall that the elements of $L^1(\mathbb{R})$ are equivalence classes of functions equal almost everywhere, and thus one may change the values on any set of measure zero, as $\{0\}$ for instance, without changing the element of $L^1(\mathbb{R})$. Nevertheless, point values are a fundamental necessity in most problems of analysis, and this makes analysts look for substitutes, for example the notion of Lebesgue points, which is the actual concept used for point values of L^p-functions.

In a seminal work, Łojasiewicz [128, 129] was the first to give a satisfactory definition of the value of a distribution at a point, which when applied at points where the distribution is locally equal to a continuous function gives the usual value, but can also be applied in more complicated situations (Lebesgue points, Denjoy integrable functions, Peano differentials [34, 128], de la Vallée Poussin derivatives [256], among others). Once a notion of point value is introduced, one can start to ask questions about its relation with other concepts in analysis where pointwise problems play a fundamental role. The concept of Łojasiewicz point value has been shown to be very useful in several areas, such as abelian and tauberian results for integral transforms [139, 149], spectral expansions [47, 236, 237], the summability of cardinal series [239, 240], wavelet analysis [241, 242, 188], or partial differential equations [54, 235]. The idea of Łojasiewicz has been extended to other asymptotic notions which can be used to measure the pointwise behavior, or asymptotic behavior at infinity, of a generalized function.

Asymptotic analysis is an old subject and, as distribution theory, it has found applications in various fields of pure and applied mathematics, physics, and engineering. The requirements of modern mathematics, and mathematical physics [61, 231, 234, 249], have brought the necessity to incorporate ideas from asymptotic analysis to the field of generalized functions, and reciprocally.

During the past five decades, numerous definitions of asymptotic behavior for generalized functions have been elaborated and applied to concrete problems in mathematics and mathematical physics. Some of the main features of these theories and their applications have been collected in various monographs [61, 139, 160, 231].

The core of this dissertation lies in the study of local and asymptotic properties of Schwartz distributions and their interactions with other areas of analysis such as harmonic analysis, asymptotic analysis, classical theory of real and complex functions, summability of divergent series and integrals, tauberian theory, analytic number theory, and applied mathematics. We will use tools from functional analysis and integral transform methods, especially of abelian and tauberian nature, to investigate several problems in the above mentioned areas. Interestingly, a distributional point of view can lead to generalizations of many important theorems in classical analysis which very often can be used to recover the classical result and reveal new information to the problem itself.

In the course of this doctoral investigation, many of my results have appeared published. This dissertation is based on 15 of my articles ([212]–[228]), 12 of which have been already published or accepted for publication. The intention of this document is to explain such contributions in detail. The exposition may differ from that given in the individual articles and I have tried to make it more complete and accessible to non-specialists. I have also added a preliminary chapter (Chapter 1) where the reader can find some background material and references to it, I hope this be useful for the reader.

In the following, I specialize the discussion to the main subjects of interest. Although the study is unified by technique and scope, the topics and applications covered are somehow broad. Therefore, I have decided to divide the rest of this introduction into three categories which better enclose the character of each individual problem and topic. These categories are *inverse problems in Fourier analysis*, *generalized asymptotics*, and *tauberian and abelian theory*. In addition, most of the chapter have their own independent introductions where the reader can find further bibliographical comments.

Inverse Problems in Fourier Analysis

The study of the relationship between the local behavior of a function (or generalized function) and the convergence or summability properties of its Fourier series or Fourier transform is a very rich problem. It has a long tradition and history [62, 63, 89, 105, 131, 184, 236, 256]. Furthermore, it is still a subject of active research [23, 47, 71, 74, 164]. These types of problems have constantly been a new source of ideas for analysts for more than two centuries. They are also of great importance in applied mathematics, since they can be used as the base of many important computational algorithms.

There is an intimate and interesting relation between the value of a periodic distribution and its Fourier series. It was shown in [47] that if a 2π-periodic distribution f has Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ and $x_0 \in \mathbb{R}$, then $f(x_0) = \gamma$ distributionally (i.e., in the sense of Łojasiewicz [128]) if and only if there exists k such that

$$\lim_{x\to\infty} \sum_{-x<n\leq ax} c_n e^{inx_0} = \gamma \qquad (\mathrm{C}, k) \,, \tag{0.0.1}$$

for each $a > 0$, where (C, k) means in the Cesàro sense of order k [61, 85]. Observe that the characterization holds in terms of the slightly asymmetric means of (0.0.1); the summability of symmetric partial sums (i.e., the series in cosines-sines form) is not enough to conclude the existence of the point value.

The characterization (0.0.1) and its extensions is the starting point of our incursions into *Fourier inverse problems*. It is natural to ask whether there is an analog to (0.0.1) for the Fourier transform. The answer to this question is positive. It will be the subject of Chapter 3, where a generalized *pointwise Fourier inversion formula* will be presented. Such a contribution appeared published in [215, 216]. This pointwise Fourier inversion formula has a general character, it is applicable to very general tempered distributions, i.e., elements of the space $\mathcal{S}'(\mathbb{R})$ [180], and is valid at every point where the distribution has a distributional point value. The formula depends on the concept of e.v distributional evaluations in the Cesàro sense, introduced by R. Estrada and myself also in [216]. This result is stated as follows. Here $\hat{f}$ stands for the Fourier transform of f. I refer the reader to Chapter 3 for the notation used in the statement. *For a tempered distribution, we have* $f(x_0) = \gamma$, *distributionally, if and only if there exists a* $k \in \mathbb{N}$ *such that*

$$\frac{1}{2\pi}\text{e.v.}\left\langle \hat{f}\left(t\right), e^{-ix_0 t}\right\rangle = \gamma \qquad (\mathrm{C}, k)\ . \tag{0.0.2}$$

It is remarkable that no such characterizations have been given for classical functions. The notion of e.v. distributional evaluations uses asymmetric differences of the primitives of $e^{ix_0 x}\hat{f}$, just as in the previously mentioned case of Fourier series. Moreover, it includes as immediate corollaries the case of Fourier series and other cases of interest; for instance, if $\hat{f}$ coincides with a locally integrable function, then (0.0.2) reads

$$\lim_{x\to\infty}\frac{1}{2\pi}\int_{-x}^{ax}\hat{f}(t)e^{ix_0 t}\mathrm{d}t = \gamma \qquad (\mathrm{C}, k)\ , \quad \text{for each } a > 0\ . \tag{0.0.3}$$

Therefore, this theory provides a novel unifying approach to pointwise problems in Fourier analysis; indeed, it considers Fourier series and integrals at the same time! It also includes "trigonometric integrals" of distributions. One may also apply this result to non-harmonic series, and characterize distributional point values in terms of Riesz typical means (see Chapter 3 or [85] for the definition of Riesz typical means).

It is worth to say some words about the technique employed to prove (0.0.2) and its relation with some problems we will discuss later. The technique motivated the creation of some new tools in generalized asymptotics by the author and S. Pilipović [227]. When proving (0.0.2), we were led to study the structure of a distributional quasiasymptotic relation of the form (see Section 1.8.1 for quasiasymptotics)

$$g(\lambda x) \sim \frac{\gamma \delta(x)}{\lambda}, \qquad \lambda \to \infty \ , \tag{0.0.4}$$

in the space $\mathcal{D}'(\mathbb{R})$, where δ is the Dirac delta distribution. The difficulty to study the structure of these types of relations was pointed out in [153, 156, 160, 192], and a structural characterization remained as an open question in generalized asymptotics. We basically solved this open question in [216] for the asymptotic relation (0.0.4).

Another interesting related question is that of convergence of Fourier series and integrals in the presence of distributional point values; namely, conditions to ensure convergence, not just the (C) summability, of (0.0.1) and (0.0.3). It is obvious that one needs to impose extra conditions to deduce convergence, that is, so called *tauberian conditions*. It is important to mention that, in particular, any of such tauberian results implies a tauberian theorem for ordinary Cesàro summability of series and integrals. We found in [216] some general conditions over the tails of series (and integrals) to guarantee convergence in this context. In Chapter 3, these results will be discussed in detail.

We will address in Chapter 8 the study of the order of summability in the pointwise Fourier inversion formula (0.0.2). While the results from Chapter 3 provide characterizations of distributional point values, they do not say anything about the order of summability. It is a fundamental discovery of Łojasiewicz that distributional point value is actually an average notion, in the sense that it can be described by taking certain (sufficiently large) number of averages [128]. Therefore, one can assign an order to distributional point values. In [223], we slightly modified Łojasiewicz definition and related it with the order of summability of (0.0.2). We obtained the order of summability of the Fourier inversion formula upon knowledge of the order of the point value, and conversely. Our results can also be connected with the classical Hardy-Littlewood problem of the symmetric (C) summability of a trigonometric series (see [256, Chap.XI] and references therein). We formulated and solved an analog to this problem in our distributional setting using the concept of symmetric (distributional) point values [223]; it can be used to obtain the classical results for trigonometric series, but also can be applied to more complicated cases of interest. Those results will be presented in Chapter 3 without information about the order of summability; the order of summability will be obtained in Chapter 8. These estimates on the order of summability are also important for determination of jumps of functions and distributions [218, 222, 223], as explained below.

One can also apply these ideas to the study of jump singularities of functions (or generalized functions), that is, detection of edges from spectral data. This subject has had recent attention because of its potential applications in numerical algorithms for reconstruction of functions. For example, one has the case of periodic functions. In the spectral data context, one is interested in reconstructing a function f from its Fourier coefficients. While when f is sufficiently smooth the straightforward approximation by the partial sums of the Fourier series provides a highly accurate reconstruction, the situation is radically different for piecewise smooth functions, mainly due to the Gibbs phenomenon. There are several approaches to overcome the difficulties presented in the presence of *edges* (for instance, [77, 134, 211]). However, all these recovery procedures require a priori knowledge of underlying jump discontinuities of the function or its derivatives. Thus, detection of edges is a critical issue in the problem. Detection of edges is also fundamental in a variety of computational algorithms, from spectral accurate schemes for capturing shock discontinuities [133] to image compression [4].

The model results for the determination of jumps by spectral data are those of *Fejér* and *Lukács* [63, 131, 256], and many modern works still follow their ideas. Therefore, it is convenient to state their formulas. Let f be a 2π-periodic function with Fourier series $a_0/2+\sum_{n=1}^{\infty}(a_n \cos nx+\sin b_n x)$. Then, the classical Lukács and Fejér theorems use the conjugate series [256] to calculate the jump of a function, say $[f]_{x=x_0}$, the first one by

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} (a_n \sin nx_0 - b_n \cos nx_0) = -\frac{[f]_{x=x_0}}{\pi} \ , \tag{0.0.5}$$

and the second one by the formula

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} n(a_n \sin nx - b_n \cos nx) = -\frac{[f]_{x=x_0}}{\pi} \ . \tag{0.0.6}$$

One may say that the numerous recent extensions of (0.0.5) and (0.0.6) in the literature ([9, 54, 66, 67, 70, 165, 186, 187, 218, 222, 244, 248, 253], [118]–[121], [140]–[142]) go into three directions: enlargement of the class of functions, extensions of the notion of jump, and the use of different means to determine the jump. Also an important matter, having much relevance in applications, is that of finding higher accurate formulas; here one usually has to sacrifice generality and ask for more from the function (often hypotheses such as piecewise smoothness or so). We will address the first three questions in Chapters 5, 6, 7, and 8.

In [215, 218, 222, 223], we have initiated a distributional comprehensive approach to the problem. Such results will be the subjects of Chapters 5, 6, and 7. This scheme is still in progress, but promises many improvements to the current results, including *numerical* ones. We have left the usual classes of classical functions, and obtained results for very general distributions and tempered distributions. Using the concept of the quasiasymptotic behavior (see Section 1.8.1), we extended the usual notions for jumps to distributional notions for pointwise jumps, namely, the *jump behavior* and the *symmetric jump behavior* (Section 5.2). A complete characterization of the distributional jumps is given in Chapter 5. The distributional jumps include those of classical functions; hence one gains generality considering at the same time all those functions inside the large space of distributions, and most notions for jumps at individual points used in analysis. These concepts being applicable to arbitrary tempered distributions, they also provide a way to treat formulas in terms of Fourier series and integrals in just one approach. The distributional jumps only use very local information from functions and distributions, thus, one can remove global assumptions from the analysis of the problem that have been classically imposed to the functions.

In the case of tempered distributions, we will study the analogs to (0.0.5) and (0.0.6) in Chapter 5 and Chapter 6, respectively. These formulas determine the jumps of distributions in terms of higher order Cesàro averages of the distributional Fourier transforms. In the first case, we use a logarithmic-Cesàro average [218]. We have estimated the order of summability in [223], this result will be presented in Chapter 8. For the distributional generalization of (0.0.6), we introduced in [222] what we named *differentiated means* in the Cesàro and Riesz sense in order to find formulas for jumps. For instance, the differentiated means of order 0 in the Cesàro sense of a series coincide with (0.0.6). When one deals only with distributions in $\mathcal{D}'(\mathbb{R})$ [180], thus one does not have the Fourier transform available, the jump can still be found by using *differentiated Abel-Poisson* means, that is, the jump can be calculated in terms of the asymptotic behavior of harmonic, harmonic conjugates, and analytic representations, as shown in [218, 222]. In [215], we also gave formulas for the jump in terms of the asymptotic behavior of partial derivatives of the $\phi-$transform (see Chapter 7), which in particular provide formulas in terms of the asymptotic boundary behavior of solutions to certain partial differential equations on the upper half-plane. The approach we have taken has also a numerical advantage with respect to others. Making a clear distinction between the jump and symmetric jump behaviors (usually obscured in the literature), one realizes that for the jump behavior one only needs a portion of the spectral data (either the positive or negative part of the spectrum) to recover a jump.

It is important to mention two problems which will not be studied here, *higher accurate formulas* and multidimensional problems. Gelb and Tadmor [66, 67] introduced the so called *concentration factors* in order to accelerate the convergence rate for the unacceptable slow error $O(1/\log N)$ provided by Lukács approximation (0.0.5). Their idea is to consider approximations of the form

$$\sum_{n=1}^{N} \sigma\left(\frac{n}{N}\right)(a_n \sin nx_0 - b_n \cos nx_0)\ . \tag{0.0.7}$$

Imposing conditions over σ, they obtained a considerable better error, $O\left(1/N\right)$, for certain classes of functions. In [187], Sjölin has shown that this method is also effective to approximate jumps of functions by generalized conjugate partial Fourier integrals. Some variants and improvements, in the case of Fourier series, have been recently given in [244], where imposing more regularity restriction they obtained an error of order $O(1/N^2)$. I strongly believe that the distributional method can lead to formulas that provide accuracy as $o\,(1/N^p)$, if one assumes the analyzed function satisfies regularity conditions as being piecewise C^p. For the multidimensional case, F. Móricz [142] has shown analogs to Lukács theorem for functions in two variables in terms of Abel-Poisson means. It is my belief some of the ideas of this dissertation can be pulled up to *multidimensional problems in edge detection.*

The problems we have discussed so far are of local nature. In Chapter 14 we deal with global estimates for integral transforms of a certain class of differentiable functions. These results extend those of R. Berndt's dissertation [13, 14] which were used to study singular integrals with new singularities (suitable generalizations of the Hilbert transform). Berndt's result is only applicable to the sine transform. We shall extend his result in three directions. We first generalize the global estimates to wider classes of oscillatory kernels rather than sine. We then relax Berndt's hypotheses for the sine transform case and obtain estimates for the transform of certain distributions which are regular off the origin but singular at the origin. Finally, we obtain similar results for the Laplace transform.

We now turn our attention to problems in several variables. In recent studies, there have been serious efforts to characterize the support of tempered distributions by summability of the Fourier transform. In the spirit of some results of Kahane and Salem, who studied the case of periodic distributions in [105], González Vieli and Graham have given several characterizations of the support of *certain* tempered distributions in several variables in terms of uniform convergence of the symmetric Cesàro means of the Fourier inversion formula [72, 74, 75, 78, 79]. They proved that for tempered distributions whose Fourier transforms are functions, i.e., $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\hat{f}$ is a function of polynomial growth, if for some $k \in \mathbb{N}$

$$\lim_{r\to\infty} \int_{|\mathbf{u}|\leq r} \hat{f}\,(\mathbf{u})\, e^{i\mathbf{u}\cdot\mathbf{x}} \mathrm{d}\mathbf{u} = 0 \quad (\mathrm{C},k)\ , \tag{0.0.8}$$

uniformly on compacts of an open set $\Omega \subset \mathbb{R}^n$, then $\Omega \subset \mathbb{R}^n \setminus \operatorname{supp} f$, and conversely.

We have characterized the support of *any* tempered distribution in [221]; therefore, we extended the result just described. This result will be the subject of Chapter 13. Our characterization presents three important aspects. First, it only asks for a pointwise verification of (0.0.8) plus L^1-boundedness over compact subsets of Ω, rather than the much stronger hypothesis of uniform convergence. Second, it holds in terms of several summability methods for the pointwise Fourier inversion; that is, Abel-Poisson means, Cesàro means, and the $\psi-$means (introduced in [221]). Finally, our summability means are applicable to the Fourier transform of any tempered distribution; having solved a difficulty shown in previous works, where the Fourier transform had to be assumed to be a function, mainly because of the unavailability of summability methods for arbitrary distributions.

Problems in multidimensional Fourier analysis are, in general, much more difficult than in one dimension. Even the summability procedures for Fourier series are hard to handle [184]. The difficulty often comes from choosing the right arrangement of the lattice points to take the summability. One encounters the same problem for Fourier integrals. Classically, the popular method has been that of *Bochner-Riesz* summability [16, 184]; it corresponds, in the one dimensional case, to the Cesàro means of the symmetric partial sums. Therefore, as the one dimensional case suggests, it is to be expected that Bochner-Riesz summability is not good enough to characterize distributional point values in several variables. While the results of this dissertation provide a complete characterization of point values in one variable, the corresponding multidimensional problem is still an open questions. The solution to such a problem can be a great step forward in the understanding of local properties of functions and distributions of several variables. I invite the interested reader to study the following *open problem: To extend (0.0.2) to several variables, that is, to find a characterization of distributional point values for a tempered distribution (in several variables) in terms of the summability of its Fourier transform.* Intuition suggests the characterization should be in terms of one dimensional Cesàro-Riesz means of averages of the Fourier transform over dilations of certain family of sets; so, the fundamental problem here is to find a suitable family of sets to obtain the desired characterization. Obviously, there many other related open problems in several variables raised by the present doctoral thesis. For instance, the analogs to the results from Chapters 5, 6, 7, and 8 to several variables can be considered as open questions.

Generalized Asymptotics

The term generalized asymptotics refers to asymptotic analysis on spaces of generalized functions. For more than five decades, many approaches have appeared and considerably evolved. A survey of definitions and results up to 1989 can be found in [160]. Perhaps, the most developed approaches to generalized asymptotics are those of Vladimirov, Drozhzhinov and Zavialov [231], and of Estrada and Kanwal [61]. This work makes extensive use of two asymptotic notions for Schwartz distributions: quasiasymptotics [231] and the Cesàro behavior [49, 224]. Actually, the Cesàro behavior is a particular case of the quasiasymptotic behavior, but it is of practical value to make the distinction between them. We will employ a third notion, though with a much more modest use, that of $S-$asymptotics, widely studied by the Novi Sad School [155, 156, 157, 158, 160, 193, 194].

In this work the reader will be introduced into the ideas of generalized asymptotics right from the beginning. In Chapter 2, two quick new distributional proofs of the celebrated *Prime Number Theorem* are provided. The exposition is based on [220], a collaborative work with R. Estrada. The proofs are direct applications of the theory of asymptotic behaviors on spaces of distributions, specifically, of the concepts of quasiasymptotics and $S-$asymptotics. In view of such results, it is then interesting to ask whether techniques from generalized asymptotics could be applied to other problems in number theory. Expected fields where these techniques could be used seem to be in asymptotic estimations for sums of additive functions [102], or in estimates of sums involving prime lattices in $\mathbb{R}^n$; however, these possibilities have been totally unexplored until now (as far as I know). I hope that this first incursion of methods from generalized functions in analytic number theory serves as a motivation for further developments.

The reader will find throughout the first nine chapters of this dissertation how useful the quasiasymptotic behavior is to measure the local behavior of functions and generalized functions. This fact will be enough motivation to devote a full chapter, Chapter 10, to the study of theoretical questions in quasiasymptotic analysis. Chapter 10 makes a major contribution to the one-dimensional quasiasymptotic analysis and can be considered as one the main achievements of my doctoral work.

The introduction of the quasiasymptotic behavior of distributions was one of major steps toward the understanding of asymptotic properties of distributions. The concept is due to Zavialov [249]. The motivation for its introduction came from theoretical questions in quantum field theory, where it was later effectively applied [231, 233, 234]. Roughly speaking, the idea is to study the asymptotic behavior at large or small scale of the dilates of a distribution. So, given a function or a generalized function, one looks for asymptotic representations of the form

$$f(\lambda x) \sim \rho(\lambda) g(x) \ ,$$

where the parameter λ is taken to either ∞ or 0. One can show [61, 160, 231] that the comparison function ρ must be regularly varying in the sense of Karamata [111, 112]. It brings into scene the well developed and powerful theory of *regular* *variation* [15] which has important applications to analytic number theory, the theory of entire functions, differential equations, and probability theory [15, 64, 136].

The study of structural theorems in quasiasymptotic analysis has always had a privileged place in the theory [128, 153, 156, 160, 192, 231]. In general, the word *structural* theorem refers in distribution theory to the description of convergence properties of distributions in terms of ordinary convergence or uniform convergence of continuous functions. Experience has shown that the structure of quasiasymptotics, and other asymptotic notions, plays a very important theoretical role in the application of the notion to other contexts, this makes its study a fundamental problem in the theory. Vladimirov and collaborators gave the first general structural theorems in [231]. Although their results describe the quasiasymptotics for a wide class of tempered distributions, they need to impose restrictions over the support of the distributions. For instance, in the one dimensional case, their results are only applicable to distributions with support bounded at the left. Thereafter, many authors dedicated efforts to extend the structural characterization and remove the support type restrictions [160]. The necessity of a complete solution for this problem has been recognized in several articles [153, 156, 192, 216]. In a series of papers [212, 213, 227], I have solved a question which remained open for long time: a complete structural characterization for quasiasymptotics of Schwartz distributions (in one dimension).

Having solved the structural question for the particular quasiasymptotic behavior (0.0.2), I took over, in collaboration with Pilipović, the general open problem of the characterization of all one dimensional quasiasymptotics at the origin. We succeeded in our goal, the completed solution is presented in [227]. In our solution to this problem, we introduced new tools in the area: *asymptotically and associate* *asymptotically homogeneous functions* (with respect to slowly varying functions). Later, I applied the same technique in [212] to completely describe the case at infinity. In addition, I have also investigated [213], in the one dimensional case, *quasiasymptotic boundedness*, i.e., relations of the form

$$f(\lambda x) = O(\rho(\lambda)) \ ,$$

where ρ is a regularly varying function (in the sense of Karamata). All these results will be the main body of Chapter 10.

Finally, I would like to point out some important open questions in the area. While the complete structure of quasiasymptotic is now known for the one dimensional case, the problem still remains open in the multidimensional case. *Open question: To describe the structure of (multidimensional) quasiasymptotics and quasiasymptotic boundedness.* Some partial results have been already obtained by Zavialov and Drozhzhinov [42, 43]. Their results suggest that spherical representations could be a path to be followed in order to obtain the desired structural description. They have described the structure except for the so called *critical degrees.* The techniques of Chapter 10 are specially effective analyzing critical degrees in the one dimensional case, so one might expect that they give new insights in the multidimensional problem. Finally, it would be interesting to try to apply the same sort of ideas to asymptotic analysis on other spaces of generalized functions such as spaces of ultradistributions, Fourier hyperfunctions, Colombeau generalized functions, and regular convolution quotients [158, 161, 193, 194], where the structural description of asymptotic notions is far from being complete.

Abelian and Tauberian Theory

The name *abelian* (or direct) theorem usually refers to those results which obtain asymptotic information after performing an integral transformation to a (generalized) function. On the other hand, a *tauberian* (or inverse) theorem is the converse to an abelian result, subject to an additional (often surprising!) assumption, the so called *tauberian hypothesis*. In general, tauberian theorems are much deeper and more difficult to show than abelian ones.

Tauberian theory is interesting by itself, but the study of tauberian type results has been historically stimulated by their potential applications in diverse fields of mathematics. It provides striking methods to attack very hard problems! Its applications in *number theory* are vast [15, 102, 115, 246], for instance, one could mention the famous short proof of the Prime Number Theorem by using the classical Wiener-Ikehara tauberian theorem [115]. Numerous applications are found in the area of complex analysis [15]. The great potential of tauberian theory for *probability* was realized more than 40 years ago by W. Feller [64]. Tauberian theory has shown to be of importance in *partial differential equations* for the study of asymptotics of solutions of Cauchy problems [7, 40, 41, 231]. Even mathematical physics has pushed forward developments of the subject; indeed, theoretical questions in *quantum field theory* have motivated the creation of many multidimensional tauberian tools [231, 233, 234].

In the case of functions and measures (Stieltjes integrals) in one variable, tauberian theory is rather advanced. The results of the first half of the last century were gathered by the extensive work of Wiener [246] and the classical book of Hardy [85]. More recent accounts are found in the excellent monographs by Bingham et al [15] (also devoted to regular variation) and Korevaar [115].

The study of abelian and tauberian type results has also attracted the attention of many researchers in the area of generalized functions, and has produced several important generalizations of classical results. Everyone familiar with tauberian theory would absolutely agree to say that Wiener tauberian theory [246] and Karamata theory of regular variation [15, 109, 110] have had a predominant role in classical tauberian theory. It is then important to mention that these two theories admit extensions to the setting of generalized functions; moreover, we emphasize that the tauberian theorems for generalized functions contain as *particular* cases those for classical functions and measures. The Wiener tauberian theorem has been extended for distributions in [39, 149, 157]. Perhaps, the most representative and robust work in an tauberian direction is that of Vladimirov, Drozhzhinov, and Zavialov started in the earlies 70's (see [231] and references therein); their multidimensional tauberian theory for the Laplace transform of distributions is a natural extension of Karamata theory [37, 229, 231]. Tauberian theory for generalized functions also provides a well established machinery for applications to areas such as mathematical physics and partial differential equations [40, 160, 235, 231, 233]. The tauberian and abelian type results for generalized functions have also received an great impulse from the study of various integral transforms [39, 123, 124, 139, 160, 176].

A great part of this dissertation is dedicated to the study of abelian and tauberian type results for functions and generalized functions. Part of it has already been mentioned in detail; for example, many of problems in connection with Fourier inverse problems are of abelian nature. On the other hand, Chapters 2, 4, 9, and 11 deal with some tauberian problems. This study provides abelian and tauberian theorems for distributions in terms of analytic and harmonic representations, the Fourier transform, the Laplace transform, the wavelet transform, and the so called $\phi-$transform (explained below). It is essential that tauberian type results for generalized functions should contain those for functions and measures, or which is the same the theory must provide systematic tools to obtain in a lucid way the classical results. In this work, I have tried to take care of this aspect by developing some tools to link generalized functions to classical tauberian theorems for functions and Stieltjes integrals.

Tauberian theorems in which complex-analytic or boundary properties, usually of global character, of the transform play an important role are called *complex tauberians* [115]. In Chapter 2 a complex tauberian theorem is obtained as a natural consequence of our method for showing the Prime Number Theorem. It is apparently a weaker version of Wiener-Ikehara theorem, though good enough for many applications in number theory. Actually, the same result was obtained by Korevaar in [117] via Newman's contour integration method, and he showed that Wiener-Ikehara theorem may be deduced from it. So, in essence, the complex tauberian theorem from Chapter 2 is as strong as Wiener-Ikehara tauberian theorem. Recently, Korevaar has proposed new distributional versions involving pseudofunction boundary behavior for important complex tauberians [116, 115]. It is my opinion that a combination of the ideas from Chapter 2 and Korevaar's new distributional perspective can lead to improvements in complex tauberian theory and tauberian remainder theory.

Chapters 4 and 9 are dedicated to the study of tauberian theorems for distributional point values. The exposition is based on the results from [217, 226]. A tauberian theorem for distributional point values in terms of the boundary behavior of analytic representations is given in Chapter 4. The tauberian hypothesis is provided by distributional boundedness at a point [26, 254]. This theorem is then used to give a new (and simple) proof of the celebrated Littlewood's theorem [85, 127]. Actually the method is good enough to give the more general version of Ananda Rau for Dirichlet series [5]; this method is a combination of our tauberian theorem for distributional point values and arguments previously applied in [216] to the study of tauberian conditions for convergence of Fourier series (the latter discussed also in Chapter 3). The study is enlarged in Chapter 9, where a more comprehensive approach is taken. A tauberian theory for distributional point values is developed parallel to Tauber's second tauberian theorem. The Cesàro behavior becomes crucial in Chapter 9, where it shows to be the natural framework for applications to classical tauberians for Dirichlet series and Stieltjes integrals.

Chapter 11 makes a complete wavelet analysis of quasiasymptotic properties of distributions via abelian and tauberian theorems. Wavelet analysis is a powerful method for studying local properties of functions [95, 96, 104, 138]. It is also a very convenient tool for the study of local properties of generalized functions. The local asymptotic behavior of distributions in terms of orthogonal wavelets has been studied in [163, 162, 177, 188, 241]. Relations between the wavelet transform and pointwise regularity of certain classes of distributions were explored in [209]. In recent articles [174, 175, 176], Saneva and Bučkovska have studied abelian and tauberian results for the quasiasymptotic behavior of tempered distributions in terms of the wavelet transform. They also pointed out the importance of a more complete tauberian study for this transform. The results of Chapter 11 provide such a complete wavelet analysis. It is remarkable that these results are more than tauberian theorems in various cases; indeed, they are full characterizations of quasiasymptotic properties, at least module polynomials.

I end this discussion with some comments about an integral transform which has been widely use throughout this dissertation. We introduced in [215, 219, 221] the distributional $\phi-$transform in relation with the study of local properties of distributions. This is not a new object; in fact the $\phi-$transform is nothing else than a approximation of the unity used for long time in analysis. However, our perspective is apparently new. Given $f \in \mathcal{D}'(\mathbb{R}^n)$ and ϕ a test function, with $\int_{\mathbb{R}^n} \phi(x)\mathrm{d}x = 1$, its $\phi-$transform F_ϕ is defined as the C^∞-function on $\mathbb{H}^{n+1} := \mathbb{R}^n \times \mathbb{R}_+$

$$F_\phi(\mathbf{x}, t) := \langle f(\mathbf{x} + t\mathbf{y}), \phi(\mathbf{y}) \rangle \ , \qquad \text{for } t > 0,$$

where the evaluation is with respect to $\mathbf{y}$. We have used this transform as a tool for several purposes. In [215], we use it to find formulas for jumps of functions and distributions (Chapter 7). We applied the $\phi-$transform to characterize the support of a tempered distribution by the summability of its Fourier transform (Chapter 13). This transform is also an important tool in the passage from local properties to *global* ones. For example, we made use of it to show that distributionally regulated functions [215] can only have jumps at most in a countable set; this result will be proved in Chapter 7. In the same chapter, we will study many important properties of the $\phi-$transform of distributions in one variable. In Chapter 12, we will study this transform in the multidimensional setting; as an application, a characterization [219] of a positive measure by the behavior of its $\phi-$transform over cones at points of the boundary will be given.

It is worth to mention the potential applications of the $\phi-$transform to study certain classes of *partial differential equations.* The work of Drozhzhinov and Zavialov is important in this direction [40, 41]. They used the $\phi-$transform (called standard average there) for tempered distributions with respect to a rapidly decreasing ϕ with values in Banach spaces to study the Cauchy problem for the heat equation. They also applied their results to problems in mathematical physics. The key point is the flexibility of the $\phi-$transform, for example ϕ can be the Poisson kernel, in such a case one obtains results for harmonic functions [225], or ϕ can be any other kernel associated to a *boundary value problem* or *Cauchy problem* for a PDE [40, 41, 215]. In this approach a problem of vital importance is that *not always* the interesting test functions ϕ are in a standard space of test functions such as $\mathcal{D}$ or $\mathcal{S}$. We have worked [215, 219, 221] in obtaining natural growth conditions over a distributions in order to define its $\phi-$transform with respect to wider classes of test functions; those results can be found in Chapter 7 (in one variable) and Chapter 12 (in several variables) of this dissertation.

Chapter 1
Preliminaries and Notation

In this chapter we collect some notions and tools to be employed in the future. In addition, we comment and fix the general notation to be used in the subsequent chapters. We pay special attention to function and distribution spaces and related concepts. The material to be discussed in Sections 1.2–1.6 can be found in any standard textbook on distribution theory, so readers familiar with distribution theory can skip those sections. In Section 1.8 we introduce some asymptotic notions for Schwartz distributions, they will play a crucial role in our study; further asymptotic concepts will be introduced and developed later.

1.1 Generalities

The set of positive and negative integers are denoted by $\mathbb{Z}_+$ and $\mathbb{Z}_-$, respectively; we will include 0 in the set of natural numbers $\mathbb{N}$. If $x \in \mathbb{R}$, then $[x]$ denotes its integral part. The sets of positive and negative real numbers will be denoted by $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively.

Points in the n-dimensional euclidean space $\mathbb{R}^n$ are denoted by bold fonts. We use the notation $\mathbf{x} \bullet \mathbf{y}$ for the standard euclidean inner product between $\mathbf{x}$ and $\mathbf{y}$. The euclidean norm is simply denoted by $|\mathbf{x}|$. The set $\mathbb{H}^n$ denotes the upper $(n+1)$-dimensional half-space, that is, $\mathbb{H}^n = \mathbb{R}^n \times \mathbb{R}_+$.; whenever the context presents no ambiguities, we just write $\mathbb{H}$ for $\mathbb{H}^n$. Given a complex number $z = x + iy \in \mathbb{C}$, we write $\Re e\, z = x$ and $\Im m\, z = y$. When $n = 1$, we often refer to $\mathbb{H}$ as " the subset $\Im m\, z > 0$ ". The complex conjugate of z is $\bar{z} = x - iy$.

The notation $\bar{A}$ is used for the closure of a set A in a given topological space. Given a continuous complex-valued function g we denote the support of the function by $\operatorname{supp} g$, i.e., the closure of the set where the function does not vanish.

A multi-index is an element $\mathbf{m} \in \mathbb{N}^n$. The length of the multi-index is the sum of its coordinates, that is, $|\mathbf{m}| = m_1 + m_2 + \cdots + m_n$, where $\mathbf{m} = (m_1, m_2, \ldots, m_n)$. Notice that we employ the same notion for the length of multi-indices as for the euclidean norm, but the distinction should be always clear from the context. We use the notations $\mathbf{m}! = m_1! m_2! \ldots m_n!$ and $\mathbf{x}^{\mathbf{m}} = x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$, where $\mathbf{x} = (x_1, x_2, \ldots x_n)$. The differential operators $\mathbf{D}^m = \mathbf{D}_1^{m_1} \mathbf{D}_2^{m_2} \ldots \mathbf{D}_n^{m_n}$, where each $\mathbf{D}_j$ is partial differentiation in the i^{th} variable. For the one variable case, we use the usual calculus notation for derivatives.

Let $\Omega \subseteq \mathbb{R}^n$ be an open subset. The space of complex-valued continuous functions on Ω is denoted by $C(\Omega)$; the space of k-times continuously differentiable functions by $C^k(\Omega)$, that is, $\phi \in C^k(\Omega)$ if $\mathbf{D}^{\mathbf{m}}\phi \in C(\Omega)$, for all $|\mathbf{m}| \leq k$. If $k = \infty$, we also use the notation $\mathcal{E}(\Omega) := C^{\infty}(\Omega)$. The space $C^k(\overline{\Omega})$ is the subspace of $C^k(\Omega)$ consisting of those elements ϕ for which $\mathbf{D}^{\mathbf{m}}\phi$ admits a continuous extension to $\overline{\Omega}$, for all $|\mathbf{m}| \leq k$. The space $C^k_{\mathrm{c}}(\Omega)$ consists of those elements of $C^k(\Omega)$ with compact support; when $k = \infty$, we always denote it by $\mathcal{D}(\Omega) := C^{\infty}_{\mathrm{c}}(\Omega)$. If $\overline{\Omega}$ is compact, then $C_{\mathrm{c}}(\overline{\Omega})$ is the subspace of functions in $C^k(\overline{\Omega})$ which vanish on $\partial\Omega$, the boundary of Ω.

We assume the reader is familiar with measure theory and integration theory, for which we refer to the excellent monographs [76, 173]. Measurability and integrability is always taken with respect to the Lebesgue measure, unless explicitly specified. The (Lebesgue) integral of g over an open set $\Omega \subseteq \mathbb{R}^n$ is given by

$$\int_{\Omega} g(\mathbf{x}) \mathrm{d}\mathbf{x} \, .$$

The classical Lebesgue spaces over Ω are denoted by $L^p(\Omega)$, $1 \leq p \leq \infty$. We say that a (complex-valued, measurable) function g is p-locally integrable in Ω (for $p = 1$ we say it is locally integrable) if $g \in L^p(K)$, for any compact subset K of Ω. We denote the set of p-locally integrable functions by $L^p_{\mathrm{loc}}(\Omega)$. Occasionally, we shall also consider the more general integral in the sense of Denjoy-Perron-Henstock; again, we refer the reader to [76, 173] for definitions and properties.

Techniques from functional analysis will be extensively used in this study. In particular, the theory of locally convex topological spaces and duality. We assume the reader has experience working with Fréchet spaces as well as projective and inductive limits of such spaces. For the fundamental definitions and results we refer to [99, 208] without further comments. The Hahn-Banach [208, p.181] and Banach-Steinhaus [208, p.346] theorems will be very important tools for us.

We shall make use of the Landau order symbols. Let g and h be two complex-valued functions defined in a pointed neighborhood of x_0. We write

$$g(x) = O(h(x)) \ , \quad x \to x_0 \ ,$$

if there exists a positive constant M such that $|g(x)| \leq M\,|h(x)|$, for all x sufficiently close to x_0. We write

$$g(x) = o(h(x)) \ , \quad x \to x_0 \ ,$$

if for any $\varepsilon > 0$ there exists a pointed neighborhood of x_0 such that $|g(x)| \leq \varepsilon\,|h(x)|$, for all values in that pointed neighborhood. We also allow x_0 to be infinity. We say that g is asymptotic to h as $x \to x_0$ if $g(x) = h(x) + o(h(x))$. In this case, we write

$$g(x) \sim h(x) \ , \quad x \to x_0 \ ,$$

When h is non-zero near x_0, it means that

$$\lim_{x \to x_0} \frac{g(x)}{h(x)} = 1 \ .$$

If we write $g(x) \sim Ch(x)$, the constant C might be 0, in that case the asymptotic relation is interpreted as $g(x) = o(h(x))$. Suppose that $\{h_n\}_{n=0}^{\infty}$ is a sequence of functions defined on a pointed neighborhood of x_0 such that $h_{n+1}(x) = o(h_n(x))$, as

$x \to x_0$. We say that g has an asymptotic expansion with respect to $\{h_n\}_{n=0}^{\infty}$ if there exists a sequence of complex numbers $\{c_n\}_{n=0}^{\infty}$ such that $g(x) - \sum_{n=0}^{N} c_{N+1} h_n(x) \sim c_{N+1} h_{N+1}(x)$, for each N. In such a case we write

$$g(x) \sim \sum_{n=0}^{\infty} c_n h_n(x) \ , \quad x \to x_0 \ ,$$

The right hand side of the last relation is called an asymptotic series. Of course, there is no assumption of convergence for asymptotic series.

1.2 Spaces of Test Functions and Distributions

We now present a brief summary of basic definitions and properties of the main spaces of functions and generalized functions (Schwartz distributions) to be employed in the sequel. For further details about the theory of Schwartz distributions and other types of generalized functions we refer to [6, 24, 26, 30, 61, 97, 99, 107, 108, 139, 144, 146, 180, 197, 208, 230, 251, 252].

Let $\Omega \subseteq \mathbb{R}^n$ be an open set.

Radon Measures

Let $K \subset \Omega$ be a compact set with non-empty interior. We equip the space $C_{\mathrm{c}}(K)$ with the topology of uniform convergence. A *Radon measure* on Ω is a continuous (complex) linear functional over the space $C_{\mathrm{c}}(\Omega)$, equipped with the natural inductive limit topology generated by the spaces $C_{\mathrm{c}}(K)$ [208, Chap.21]. Let μ be a Radon measure, by the Riesz representation theorem, we can always associate to it a regular Borel measure which is finite on compacts of Ω; we denote both the measure and the functional by μ, so that the action of μ on $\phi \in C_{\mathrm{c}}(\Omega)$ is given by

$$\langle \mu, \phi \rangle = \int_{\Omega} \phi(\mathbf{x}) \mathrm{d}\mu(\mathbf{x}) \ .$$

Every positive linear functional on $C_{\mathrm{c}}(\Omega)$, i.e., one such that $\langle \mu, \phi \rangle \geq 0$ whenever $\phi \geq 0$, is a Radon measure. Observe that if $f \in L^p_{\mathrm{loc}}(\Omega)$, it can be viewed as the

Radon measure

$$\langle f, \phi \rangle = \int_\Omega \phi(\mathbf{x}) f(\mathbf{x}) \mathrm{d}\mathbf{x} \ .$$

In the one variable case, given an arbitrary Radon measure μ on (a,b), there is a function of local bounded variation s_μ such that $\mu = \mathrm{d}s_\mu$, that is, the action of μ on a function $\phi \in C_{\mathrm{c}}(a,b)$ can be computed as a Stieltjes integral,

$$\langle \mu, \phi \rangle = \int_a^b \phi(x) \mathrm{d}s_\mu(x) \ .$$

The Space of Distributions

Let $K \subset \Omega$ be compact set with non-empty interior. We endow the space $C_{\mathrm{c}}^\infty(K)$ with its canonical Fréchet space topology, i.e., the one of uniform convergence of all partial derivatives [99, 180, 208]. The Schwartz topology of $\mathcal{D}(\Omega)$ is given by the inductive limit topology of the spaces $C_{\mathrm{c}}^\infty(K)$; the space $\mathcal{D}(\Omega)$ has the structure of an LF-space [208]. It is a montel space, and hence it is reflexive [208].

A distributions on Ω is a continuous (complex) linear functional over $\mathcal{D}(\Omega)$, the space of distributions is denoted by $\mathcal{D}'(\Omega)$. Therefore a linear functional over $\mathcal{D}(\Omega)$ is a distribution if its restriction to each $C_{\mathrm{c}}^\infty(K)$ is continuous. Distributions will be denoted by either f or $f(\mathbf{x})$; the variable $\mathbf{x}$ makes no allusion to a point value (unless specified), it only plays the role of a "variable of evaluation" just as the calculus use of variables of integration. The evaluation of $f \in \mathcal{D}'(\Omega)$ at a test function ϕ, that is, an element of $\mathcal{D}(\Omega)$, is denoted by

$$\langle f, \phi \rangle = \langle f(\mathbf{x}), \phi(\mathbf{x}) \rangle \ .$$

Any Radon measure on Ω can be viewed as a distribution. Therefore $L^p_{\mathrm{loc}}(\Omega) \subset \mathcal{D}'(\Omega)$. We call regular distributions to those which arise from locally integrable functions. Furthermore, any positive distribution, i.e., one such that $\langle \mu, \phi \rangle \geq 0$ whenever $0 \leq \phi \in \mathcal{D}(\Omega)$, is a positive Radon measure [180, 208].

We will equip $\mathcal{D}'(\Omega)$ with two main dual topologies [180, 208]: the weak topology of pointwise convergence over elements of $\mathcal{D}(\Omega)$, and the strong topology of uniform convergence over bounded sets of $\mathcal{D}(\Omega)$. The use of the corresponding topology will depend on the context. Since $\mathcal{D}(\Omega)$ is a barrelled space [208], it follows from the Banach-Steinhaus theorem [208, Chap.33] that weak boundedness, strong boundedness, and equicontinuity are equivalent for subsets of distribution spaces. It has the following useful consequence: for sequences $\{f_n\}_{n=0}^{\infty}$ or more generally for filters with a countably basis in $\mathcal{D}'(\Omega)$, weak and strong convergence are equivalent. The last fact follows in view of the Montel property of $\mathcal{D}(\Omega)$ and the Banach-Steinhaus theorem [208, p.348].

Let $f \in \mathcal{D}'(\Omega)$, the restriction of f to an open subset $U \subset \Omega$ makes sense as the transpose of the canonical inclusion $\mathcal{D}(U) \hookrightarrow \mathcal{D}(\Omega)$ [208, Chap.23]. A distribution $g \in \mathcal{D}'(U)$ is called *extendable* to Ω, if there exists $f \in \mathcal{D}'(\Omega)$ whose restriction to U is exactly g. In general, not all distribution defined on U is extendable to Ω.

A distribution $f \in \mathcal{D}'(\Omega)$ is said to vanish on an open subset $U \subset \Omega$ if $\langle f, \phi \rangle$ for all $\phi \in \mathcal{D}(U)$. The support of f, denoted by $\operatorname{supp} f$, is the complement in Ω of the largest open set where it vanishes. Observe that if $f \in L^p_{\text{loc}}(\Omega)$, then the support of f as a distribution is precisely the essential support of f, this justifies the equality $\operatorname{supp} f = \operatorname{ess\,supp} f$.

Let $\mathbf{m}$ be a multi-index and $g \in \mathcal{D}'(\Omega)$. The distribution $\mathbf{D}^{\mathbf{m}} g$ is defined by

$$\langle \mathbf{D}^{\mathbf{m}} g, \phi \rangle = (-1)^{|\mathbf{m}|} \langle g, \mathbf{D}^{\mathbf{m}} \phi \rangle$$

Given any multi-index and $f \in \mathcal{D}'(\Omega)$, there exists a distribution g such that $\mathbf{D}^{\mathbf{m}} g = f$ [180]. We will say that g is an $\mathbf{m}$-primitive of f.

We now focus in structural properties of distributions and distributional convergence. We discuss Schwartz characterization theorems of boundedness and convergence of distributions [180, 230]. Suppose that $\mathfrak{B} \subset \mathcal{D}'(\Omega)$ is a bounded set for the weak (or strong) topology. Then for any given open subset $U \subset \Omega$, with compact closure in Ω, there exists a multi-index $\mathbf{m}_U$ such that any $f \in \mathfrak{B}$ satisfies

$$f = \mathbf{D}^{\mathbf{m}_U} F_f \quad (\text{on } U) \ ,$$

where each $F_f \in C(\overline{U})$ and the family $\{F_f\}_{f\in\mathfrak{B}}$ is uniformly bounded on $\overline{U}$. Conversely, if the last property is satisfied for each such an open subset U, then $\mathfrak{B}$ is bounded in the strong topology of $\mathcal{D}'(\Omega)$. In particular, any distribution is locally equal to the distributional derivative of a continuous function. The description for convergence is similar, suppose that $f_j \to 0$ as $n \to \infty$ in the weak topology, then for each open U with compact closure there exists $\mathbf{m}_U$ and a sequence of continuous functions $F_j \in C(\overline{U})$ such that $f_j = \mathbf{D}^{\mathbf{m}_U} F_j$, on $\overline{U}$, and $F_j \to 0$ uniformly over $\overline{U}$. Naturally, the converse is also true. Obviously, j may be replaced by a continuous parameter $\lambda \in \mathbb{R}$ in the last statement and the result would still be valid.

We now discuss some other operations with distributions. Let $\varphi \in \mathcal{E}(\Omega)$ and $f \in \mathcal{D}'(\Omega)$, the multiplication φf is the distribution given by $\langle \varphi f, \phi\rangle := \langle f, \varphi\phi\rangle$. The multiplication of two distributions is an irregular operation [6], it cannot be defined in general within the framework of the theory of distributions [179], unless additional conditions be imposed [30, 97, 106, 146, 230]. The *change of variables* is defined as follows. Let $f \in \mathcal{D}'(\Omega)$ and $\Psi : U \longrightarrow \Omega$ be a C^∞-diffeomorphism, the distribution $f(\Psi(\mathbf{x})) \in \mathcal{D}'(U)$ is given by

$$\langle f(\Psi(\mathbf{x})), \phi(\mathbf{x})\rangle := \left\langle f(\mathbf{y}), \frac{\phi(\Psi^{-1}(\mathbf{y}))}{|J_\Psi(\Psi^{-1}(\mathbf{y}))|} \right\rangle \ ,$$

where $J_\Psi(\cdot) = \det(\mathrm{d}\Psi(\cdot))$ is the jabobian of the transformation, so that it is consistent with the change of variables for regular distributions. If A is an invertible linear transformation and $\mathbf{x}_0 \in \mathbb{R}^n$, we obtain

$$\langle f(A\mathbf{x} + \mathbf{x}_0)), \phi(\mathbf{x})\rangle = \frac{1}{|\mathrm{det}A|} \left\langle f(\mathbf{y}), \phi(A^{-1}(\mathbf{y} - \mathbf{x}_0))\right\rangle \ .$$

If $f \in \mathcal{D}'(\Omega)$ and $g \in \mathcal{D}'(\Omega_1)$, their tensor product (or direct product) [180, 230] is the distribution $f \otimes g \in \mathcal{D}'(\Omega \times \Omega_1)$ generated by

$$\langle f \otimes g, \phi \otimes \psi \rangle = \langle f, \phi \rangle \langle g, \psi \rangle \ ,$$

where $\phi \in \mathcal{D}(\Omega)$ and $\psi \in \mathcal{D}(\Omega_1)$.

Other Spaces

We need to consider other spaces of distributions.

The space $\mathcal{E}(\Omega) = C^\infty(\Omega)$ is equipped with its usual Fréchet space structure of uniform convergence of all partial derivatives over compact subsets of Ω. Its dual space, $\mathcal{E}'(\Omega)$ coincides then with the distributions of compact support in Ω [180, 208].

The space $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of rapidly decreasing smooth test functions, that is, those functions in $\phi \in \mathcal{E}(\mathbb{R}^n)$ for which there are constants $M_{k,\mathbf{m}}$ such that

$$|\mathbf{x}|^k |\mathbf{D}^{\mathbf{m}} \phi(\mathbf{x})| < M_{k,\mathbf{m}} \ ,$$

for all $\mathbf{x} \in \mathbb{R}^n$, $k \in \mathbb{N}$, and $\mathbf{m} \in \mathbb{N}^n$. It is topologized in the usual way [180, 208, 230]. Its dual, the space of tempered distributions, is denoted by $\mathcal{S}'(\mathbb{R}^n)$. The structure of tempered distributions is simple: $f \in \mathcal{S}'(\mathbb{R}^n)$ if and only if there exist $k \in \mathbb{N}$, $\mathbf{m} \in \mathbb{N}^n$ and a continuous functions F such that $\mathbf{D}^{\mathbf{m}} F = f$ and $F(\mathbf{x}) = O(|\mathbf{x}|^k)$, $|\mathbf{x}| \to \infty$. Clearly, $\mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$.

We now consider spaces of type $\mathcal{K}$ and $\mathcal{K}_\beta$ [61, 82]. We first need the following definition.

Definition 1.1. *Let $\phi \in \mathcal{E}(\mathbb{R}^n)$ and $\beta \in \mathbb{R}$. We say that*

$$\phi(\mathbf{x}) = O(|\mathbf{x}|^\beta) \quad \textit{strongly as} \quad |x| \to \infty \ , \tag{1.2.1}$$

if for each $\mathbf{m} \in \mathbb{N}^n$

$$\mathbf{D}^{\mathbf{m}}\phi(\mathbf{x}) = O(|\mathbf{x}|^{\beta - |\mathbf{m}|}) \quad |\mathbf{x}| \to \infty \ . \tag{1.2.2}$$

The set of test functions ϕ satisfying Definition 1.1 for a particular β forms the space $\mathcal{K}_\beta(\mathbb{R}^n)$. It is topologized in the obvious way [61], having a Fréchet space structure. These spaces and their dual spaces are very important in the theory of asymptotic expansions of distributions [61]. we set $\mathcal{K}(\mathbb{R}^n) = \bigcup \mathcal{K}_\beta(\mathbb{R}^n)$ (the union having a topological meaning), and $\mathcal{K}'(\mathbb{R}^n) = \bigcap \mathcal{K}'_\beta(\mathbb{R}^n)$ (with projective limit topology) is the space of distributional small distributions at infinity [49, 61]. We have the inclusion $\mathcal{K}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$.

We now turn our attention to some other spaces in one variable. Let $a \in \mathbb{R}$, the spaces $\mathcal{D}[a,\infty)$ and $\mathcal{S}[a,\infty)$ consist of restrictions of elements of $\mathcal{D}(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$, respectively, to the interval $[a,\infty)$. They are provided with the inhered canonical topology. Their dual spaces are $\mathcal{D}'[a,\infty)$ and $\mathcal{S}'[a,\infty)$; they coincide [230, 231] with distributions and tempered distributions, respectively, supported in $[a,\infty)$. When $a = 0$, we also use the notations $\mathcal{D}'(\overline{\mathbb{R}}_+) = \mathcal{D}'[0,\infty)$ and $\mathcal{S}'(\overline{\mathbb{R}}_+) = \mathcal{S}'[0,\infty)$.

Remark 1.2. *In general the word* distribution *will be extrictly used for elements (or subspaces) of spaces* $\mathcal{D}'(\Omega)$*. However, in very few occasions (Chapter 11), the author will commit abuse to such a terminology by calling "distributions" to elements of other duals spaces which are not necessarily contained in a distribution space.*

Convolution

The convolution of two distributions is an irregular operation and can only be defined in some special circumstances. We shall make a very modest use of the convolution (mostly in one variable) in the simplest cases. There are many definitions in the literature which may be applied to more complicated situations, for those we refer to [6, 106, 180, 230].

If $f \in \mathcal{A}'(\mathbb{R}^n)$ and $\phi \in \mathcal{A}(\mathbb{R}^n)$, where $\mathcal{A} = \mathcal{D}, \mathcal{E}, \mathcal{S}$, then $f * \phi \in \mathcal{E}(\mathbb{R}^n)$ is given by

$$f * \phi(\mathbf{x}) = \langle f(\mathbf{y}), \phi(\mathbf{x} - \mathbf{y}) \rangle \ .$$

Naturally the above evaluation is with respect to $\mathbf{y}$.

In the one-dimensional case, the convolution can always be defined for two distributions with support bounded at the left (see [230, Section 4]). So, for $a, b \in \mathbb{R}$

$$* : \mathcal{D}'[a, \infty) \times \mathcal{D}'[b, \infty) \longrightarrow \mathcal{D}'[a + b, \infty) \ ,$$

$$* : \mathcal{S}'[a, \infty) \times \mathcal{S}'[b, \infty) \longrightarrow \mathcal{S}'[a + b, \infty) \ ,$$

are separately continuous bilinear maps [230]. In particular, the spaces $\mathcal{D}'[0, \infty)$ and $\mathcal{S}'[0, \infty)$ are convolution algebras.

1.3 Special Distributions

In this section we discuss some particular examples of distributions over the real line. These special distributions are more than examples, since they will often appear throughout all the chapters. Some properties and formulas are stated without proof, we leave to the reader the verification of these well known facts (they may also be found in [61, 68, 97, 230]).

The *Heaviside function* is the regular distribution H given by

$$\langle H(x), \phi(x) \rangle = \int_0^\infty \phi(x) \mathrm{d}x \ . \tag{1.3.1}$$

The *signum* function is $\mathrm{sgn} x = H(x) - H(-x)$.

The *Dirac delta* distribution is the Radon measure defined as

$$\langle \delta(x), \phi(x) \rangle = \phi(0) \ , \tag{1.3.2}$$

observe that $\delta(x) = H'(x) = (1/2)\mathrm{sgn}' x$. The k^{th} derivative of δ, the distribution $\delta^{(k)}$, is then given by $\left\langle \delta^{(k)}(x), \phi(x) \right\rangle = (-1)^k \phi^{(k)}(0)$.

The distribution p.v.$(1/x)$ is defined by the Cauchy principal value integral

$$\begin{aligned}\left\langle \text{p.v.}\left(\frac{1}{x}\right), \phi(x)\right\rangle &= \text{p.v.}\int_{-\infty}^{\infty} \frac{\phi(x)}{x}\,\mathrm{d}x \\ &:= \lim_{\varepsilon\to 0^+}\left(\int_{-\infty}^{-\varepsilon} \frac{\phi(x)}{x}\,\mathrm{d}x + \int_{\varepsilon}^{\infty} \frac{\phi(x)}{x}\,\mathrm{d}x\right) \\ &= \int_0^{\infty} \frac{\phi(x)-\phi(-x)}{x}\,\mathrm{d}x \ ;\end{aligned}$$

it is not a Radon measure. Notice that $(\log|x|)' = \text{p.v.}(1/x)$.

If $\Re e\,\alpha > -1$, the distribution x_+^α is a regular distribution whose action on test functions is given by the integral

$$\left\langle x_+^\alpha, \phi(x)\right\rangle = \int_0^\infty x^\alpha \phi(x)\mathrm{d}x \ ; \tag{1.3.3}$$

when $\Re e\alpha < -1$, $\alpha \notin \mathbb{Z}_-$, then x_+^α is defined as

$$\frac{x_+^\alpha}{\Gamma(\alpha+1)} = \frac{\left(x_+^{\alpha+n}\right)^{(n)}}{\Gamma(\alpha+n+1)} \ , \tag{1.3.4}$$

where Γ is the *Euler Gamma* function and $n = [-\alpha]$. Therefore, x_+^α is well defined for $\alpha \in \mathbb{C}\setminus\mathbb{Z}_-$. The expression (1.3.4) is meaningful for $\alpha = -k \in \mathbb{Z}_-$; indeed

$$\left.\frac{x_+^\alpha}{\Gamma(\alpha+1)}\right|_{\alpha=-k} = \delta^{(k-1)}(x) \ . \tag{1.3.5}$$

Alternatively, we may have defined the distributions x_+^α by the Marcel Riesz analytic continuation procedure [61, 68, 97] of (1.3.3). This analytic continuation produces a family of analytic distributions on $\mathbb{C}\setminus\mathbb{Z}_-$, having simple poles at the negative integers with residues [61, p.65]

$$\mathrm{Res}_{\alpha=-k}\, x_+^\alpha = \frac{(-1)^{k-1}}{(k-1)!}\,\delta^{(k-1)}(x) \ . \tag{1.3.6}$$

The distributions x_-^α are defined as $x_-^\alpha = (-x)_+^\alpha$ so that when $\Re e\,\alpha > -1$,

$$\left\langle x_-^\alpha, \phi(x)\right\rangle = \int_0^\infty x^\alpha \phi(-x)\mathrm{d}x \ , \tag{1.3.7}$$

they also form an analytic family of distributions on $\mathbb{C} \setminus \mathbb{Z}_-$, with residues at the negative integers given by

$$\operatorname{Res}_{\alpha=-k} x_-^\alpha = \frac{\delta^{(k-1)}(x)}{(k-1)!} \ . \tag{1.3.8}$$

Note that the distributions $\Phi_\alpha(x) := x^{\alpha-1}/\Gamma(\alpha)$ form an abelian group under convolution, i.e., $\Phi_\alpha * \Phi_\beta = \Phi_{\alpha+\beta}$.

The distributions $x_+^{\alpha-1}/\Gamma(\alpha)$ can be used to define *fractional derivatives and primitives* for distributions with support bounded at the left [60, 68, 230]. If f has support bounded at the left, its *α-primitive* is defined as

$$f^{(-\alpha)} := f * \frac{x^{\alpha-1}}{\Gamma(\alpha)} \ . \tag{1.3.9}$$

Observe that the α-primitive is nothing else than the fractional derivative [230] of order $-\alpha$. So, $f^{(\alpha)}$ the fractional *α-derivative* of f.

When f is no longer supported on an interval of the form $[b, \infty)$, we cannot in general speak about fractional order primitives. However, if $k \in \mathbb{N}$, we say that F is a k-primitive of f if $F^{(k)} = f$. Primitives of distributions always exist [180, 230]. When f is locally integrable, not necessarily with support bounded on the left, we can still use the k-primitive given by formula (1.3.9) with $\alpha = k$.

Let $k \in \mathbb{Z}_+$. If k is an even positive integer, we define $x^{-k} := (x_-^\alpha + x_+^\alpha)\big|_{\alpha=-k}$; on the other hand, if k is odd, $x^{-k} := (x_+^\alpha - x_-^\alpha)\big|_{\alpha=-k}$. Due to (1.3.6) and (1.3.8), we have cancellation of the poles and these distributions are well defined. Notice that

$$\text{p.v.}\left(\frac{1}{x}\right) = x^{-1} \ ; \tag{1.3.10}$$

we will use both notations for this distribution.

Another useful method for defining distributions out of divergent integrals is that of *Hadamard finite part* [61, p.67]. Assume g is integrable on any compact subset of $(0, a]$, the Hadamard finite part at 0 of an integral $\int_0^a g(x)\mathrm{d}x$ is constructed as follows. Let

$$G(\varepsilon) = \int_{\varepsilon}^{a} g(x)\mathrm{d}x \ . \tag{1.3.11}$$

Suppose that $G(\varepsilon)$ can be split into two parts as

$$G(\varepsilon) = G_1(\varepsilon) + G_2(\varepsilon) \ , \tag{1.3.12}$$

where G_1 is a linear combination of functions of the form $\varepsilon^{-\alpha}(\log\varepsilon)^{\beta}$ and $\varepsilon^{-\gamma}$, $\alpha, \gamma > 0$, and G_2 has a finite limit as $\varepsilon \to 0^+$. We then define the finite part of the integral as

$$\text{F.p.} \int_0^a g(x)\mathrm{d}x = \lim_{\varepsilon\to 0^+} G_2(\varepsilon) \ . \tag{1.3.13}$$

One can show [61, p.68] that

$$\left\langle x_+^{\alpha}, \phi(x)\right\rangle = \text{F.p.} \int_0^{\infty} x^{\alpha}\phi(x)\mathrm{d}x \ . \tag{1.3.14}$$

We will also employ the distributions $\mathrm{Pf}(H(x)/x^k)$, $k \in \mathbb{Z}_+$, here Pf stands for the word pseudo-function. They are defined as

$$\left\langle \mathrm{Pf}\left(\frac{H(x)}{x^k}\right), \phi(x)\right\rangle = \text{F.p.} \int_0^{\infty} \frac{\phi(x)}{x^k}\mathrm{d}x \ . \tag{1.3.15}$$

One defines $\mathrm{Pf}(H(-x)/x^k)$ as

$$\left\langle \mathrm{Pf}\left(\frac{H(-x)}{x^k}\right), \phi(x)\right\rangle = (-1)^k\text{F.p.} \int_0^{\infty} \frac{\phi(-x)}{x^k}\mathrm{d}x \ . \tag{1.3.16}$$

The formulas

$$(H(x)\log x)' = \mathrm{Pf}\left(\frac{1}{x}\right) \ , \tag{1.3.17}$$

and

$$\left(\mathrm{Pf}\left(\frac{H(x)}{x^k}\right)\right)' = -k\mathrm{Pf}\left(\frac{H(x)}{x^{k+1}}\right) + \frac{(-1)^k}{k!}\delta^{(k)}(x) \ , \tag{1.3.18}$$

are readily verified [61, p.68].

1.4 Homogeneous Distributions

A distribution $g \in \mathcal{D}'(\mathbb{R}^n)$ is said to be *homogeneous* of *degree* $\alpha \in \mathbb{C}$ if $g(a\mathbf{x}) = a^\alpha g(\mathbf{x})$, for any $a > 0$. In terms of test functions, it means that

$$\langle g(a\mathbf{x}), \phi(\mathbf{x})\rangle = \frac{1}{a^n}\left\langle g(\mathbf{x}), \phi\left(\frac{\mathbf{x}}{a}\right)\right\rangle = a^\alpha \langle g(\mathbf{x}), \phi(\mathbf{x})\rangle \ , \tag{1.4.1}$$

for each $a > 0$ and $\phi \in \mathcal{D}(\mathbb{R}^n)$. One can find an explicit characterization of homogeneous distribution in [61, p.72] (see also [97, 68]).

In particular, we explicitly know all the homogeneous distributions over the real line. So, if $g \in \mathcal{D}'(\mathbb{R})$ is homogeneous of degree α, then either g has the form

$$g(x) = C_- x_-^\alpha + C_+ x_+^\alpha, \text{ if } \alpha \notin \mathbb{Z}_-, \tag{1.4.2}$$

for some constants C_- and C_+, or

$$g(x) = \gamma\delta^{(k-1)}(x) + \beta x^{-k}, \text{ if } \alpha = -k \in \mathbb{Z}_-, \tag{1.4.3}$$

for some constants γ and β.

Notice that the distributions $\mathrm{Pf}(H(\pm)x)/x^k)$ are not homogeneous. They are rather associate homogeneous [68, 61, 185], that is, their dilates follow the formula:

$$\mathrm{Pf}\left(\frac{H(\pm ax)}{(ax)^k}\right) = \frac{1}{a^k}\mathrm{Pf}\left(\frac{H(\pm x)}{x^k}\right) \mp \frac{(-1)^{k-1}\log a}{a^k(k-1)!}\delta^{(k-1)}(x) \ . \tag{1.4.4}$$

We finally remark that some interesting extensions of homogeneity can be found in [83, 185].

1.5 The Fourier and Laplace Transforms

The Fourier transform is an isomorphism of $\mathcal{S}(\mathbb{R}^n)$ onto itself [68, 180, 197, 252, 230]. It is a very well known tool in analysis, and we assume the reader is familiar with it. We fix the constants so that the Fourier transform of $\phi \in \mathcal{S}(\mathbb{R}^n)$ is given by

$$\mathcal{F}(\phi)(\mathbf{u}) = \hat{\phi}(\mathbf{u}) = \int_{\mathbb{R}^n} e^{-i\mathbf{x}\bullet\mathbf{u}}\phi(\mathbf{x})\mathrm{d}\mathbf{x} \ , \tag{1.5.1}$$

then the Fourier inversion formula becomes

$$\phi(\mathbf{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\mathbf{x}\bullet\mathbf{u}} \hat{\phi}(\mathbf{u}) \mathrm{d}\mathbf{u} \ . \tag{1.5.2}$$

The Fourier transform is defined on $\mathcal{S}'(\mathbb{R}^n)$ by duality, i.e., if f is a tempered distribution, then

$$\left\langle \hat{f}(\mathbf{x}), \phi(\mathbf{x}) \right\rangle := \left\langle f(\mathbf{x}), \hat{\phi}(\mathbf{x}) \right\rangle \ . \tag{1.5.3}$$

We use the notation $\mathcal{F}^{-1}$ for the inverse Fourier transform.

Only in Chapter 2 we will make a different choice of the constants in the Fourier transform which better fit to our purposes.

We will follow the definition of the Laplace transform due to L. Schwartz [11, 180, 230, 231]. It is equivalent to the one given in [251, 252]. We will only consider the Laplace transform of distributions in one-variable. A distribution $f \in \mathcal{D}'(\mathbb{R})$ is said to be Laplace transformable [180] on the strip $a < \Re e\ z < b$ if $e^{-\xi t} f(t)$ is a tempered distribution for $a < \xi < b$; in such a case its Laplace transform is well defined on that strip and can be computed by the evaluation

$$\mathcal{L}\{f; z\} = \left\langle f(t), e^{-zt} \right\rangle \ , \quad a < \Re e\ z < b \ . \tag{1.5.4}$$

In particular if the support of $f \in \mathcal{S}'(\mathbb{R})$ is bounded at the left, then its Laplace transform is well defined on $\Re e\ z > 0$ and is given by (1.5.4). When the support of f is bounded at the right, formula (1.5.4) is applicable but for $\Re e\ z < 0$.

1.6 Analytic and Harmonic Representations

Any distribution $f \in \mathcal{D}'(\mathbb{R})$ may be seen as a hyperfunction [144, 107], that is, $f(x) = F(x + i0) - F(x - i0)$, where F is analytic for $\Im m\ z \neq 0$; moreover, this representation holds distributionally in the sense that

$$f(x) = \lim_{y \to 0^+} \left(F(x + iy) - F(x - iy) \right) \ , \tag{1.6.1}$$

where the last limit is taken in the weak topology of $\mathcal{D}'(\mathbb{R})$ [24]. It means that for each test function $\phi \in \mathcal{D}(\mathbb{R})$

$$\langle f(x), \phi(x) \rangle = \lim_{y \to 0^+} \int_{-\infty}^{\infty} \left(F(x+iy) - F(x-iy) \right) \phi(x) \mathrm{d}x \ . \tag{1.6.2}$$

In such a case, we say that F is an analytic representation of f on $\mathbb{C} \backslash \mathbb{R}$. Note that, initially, we are not assuming that the limits $\lim_{y \to 0^+} F(x \pm iy)$ belong to $\mathcal{D}'(\mathbb{R})$ separately, but that their difference does; however, it is shown in [48, Section 5] that the existence of the distributional jump of F across the real axis implies the existence of $\lim_{y \to 0^+} F(x \pm iy)$, separately, in $\mathcal{D}'(\mathbb{R})$. We write $F(x \pm i0)$ to represent these distributional boundary values.

A necessary and sufficient condition [48, 97] for a function F, analytic on a region $((a,b) \times (-R,R)) \setminus \mathbb{R}$, to have a distributional boundary values on real line is the existence of constants M_K and n_K such that

$$|F(x+iy)| < \frac{M_K}{|y|^{n_K}} \ , \quad 0 < |y| < R \ , \ x \in K \ , \tag{1.6.3}$$

for each compact subset $K \subset (a,b)$.

We recall the well known *edge of the wedge theorem* [24, 11] (in one-dimension). Suppose that F_+ and F_- are analytic in some rectangular regions $(a,b) \pm i(0,R)$, respectively, and that both have distributional boundary values on the real axis. If $F_+(x+i0) = F_-(x-i0)$, in $\mathcal{D}'(a,b)$, then there exists a function F, analytic on $(a,b) \times (-R,R)$, such that $F(z) = F_\pm(z)$, for $\pm \Im m \, z > 0$. So, they are the analytic continuation of each other across the interval (a,b).

There are various standard methods to construct analytic representations for certain distributions. Let us start with distributions from $\mathcal{E}'(\mathbb{R})$. If $f \in \mathcal{E}'(\mathbb{R})$ is a distribution with compact support, then the Cauchy transform is given by

$$F(z) = F\{f; z\} := \frac{1}{2\pi i} \left\langle f(t), \frac{1}{t-z} \right\rangle \ , \quad \Re e \ z \notin \operatorname{supp} f \ . \tag{1.6.4}$$

One can then show [24] that $f(x) = F(x+i0) - F(x-i0)$. For example,

$$F(z) = \frac{(-1)^{k+1}k!}{2\pi i z^{k+1}}$$

is an analytic representation of $\delta^{(k)}$.

When $f \in \mathcal{S}'(\mathbb{R})$. We can use the Fourier transform to produce an analytic representation. Decompose $\hat{f} = \hat{f}_- + \hat{f}_+$, where $\operatorname{supp} \hat{f}_- \subseteq (-\infty, 0]$ and $\operatorname{supp} \hat{f}_+ \subseteq [0, \infty)$, respectively. Then [24],

$$F(z) := \begin{cases} \dfrac{1}{2\pi}\left\langle \hat{f}_+(t), e^{izt} \right\rangle, & \Im m\ z > 0\ , \\ -\dfrac{1}{2\pi}\left\langle \hat{f}_-(t), e^{izt} \right\rangle, & \Im m\ z < 0\ , \end{cases} \tag{1.6.5}$$

is an analytic representation of f. So, $F(z) = \pm\mathcal{L}\left\{\hat{f}_\pm; \mp iz\right\}$, if $\pm\Im m\ z > 0$. We call this analytic representation the *Fourier-Laplace representation.*

Notice that if f has compact support, then $\hat{f}$ is locally integrable; it is actually the restriction to the real axis of an analytic function of exponential type, by Schwartz-Paley-Wiener theorem [97, 208]. If we choose $\hat{f}_\pm$ to be locally integrable functions, then it is not hard to see that (1.6.4) and (1.6.5) give the same analytic function.

Next, we consider representations of distributions by harmonic functions. We say that $U(z)$, harmonic on $\Im m\ z > 0$, is a harmonic representation of $f \in \mathcal{D}'(\mathbb{R})$ if

$$\lim_{y \to 0^+} U(x+iy) = f(x)\ , \quad \text{in } \mathcal{D}'(\mathbb{R})\ , \tag{1.6.6}$$

in the sense that

$$\langle f(x), \phi(x) \rangle = \lim_{y\to 0^+} \int_{-\infty}^{\infty} U(x+iy)\phi(x)\mathrm{d}x\ .$$

We write $f(x) = U(x+i0)$. Any distribution admits a harmonic representation. Indeed, let F be an analytic representation on $\mathbb{C} \setminus \mathbb{R}$, then $U(z) = F(z) - F(\bar{z})$ is harmonic on $\Im m\ z > 0$ and $f(x) = U(x+i0)$.

Suppose that $U(x+i0)=0$ in $\mathcal{D}'(a,b)$; then by applying the reflection principle to the real and imaginary parts of U ([11, Section 4.5], [207, Section 3.4],[32]), we have that U admits a harmonic extension to a (complex) neighborhood of (a,b). We will refer to this result as the distributional *reflection principle* (or just reflection principle).

Recall [32, 207] that V is called a harmonic conjugate to U if they satisfy the Cauchy-Riemann equations,

$$\frac{\partial U}{\partial x}=\frac{\partial V}{\partial y}\ ,\quad \frac{\partial U}{\partial y}=-\frac{\partial V}{\partial x}\ ,$$

then, $U+iV$ is analytic. Observe that, because of the results from [56] and [48, Section 5], one has that if a harmonic function on the upper half-plane admits distributional boundary values, then any harmonic conjugate to it admits distributional boundary values.

1.7 Slowly Varying Functions

Slowly varying functions will be important in several parts of our study. We only comment some basic properties, we will come back to slowly varying functions in due course.

They were introduced by J. Karamata in [111, 112]. The associated theory is usual referred as *Karamata theory* of regular variation. It was later refined by him and others. The standard references to the subject are [15, 183], the first being the most comprehensive one. Both books are a rich source of historical facts about the theory. We also comment the important role that regular variation have had in the modern and classical developments of tauberian theory [15, 109, 110, 115, 160, 231, 232].

We start with regularly varying functions at infinity. We say that a function ρ, measurable, positive and defined on an interval of the form $[A,\infty)$ is *regularly* *varying* at infinity if

$$\lim_{x\to\infty}\frac{\rho(ax)}{\rho(x)} = h(a), \tag{1.7.1}$$

exists and is finite for each $a > 0$. One can then show [15, 183] that $h(a) = a^{\alpha}$, for some α. The number α is called the *index of regular variation.* If $\alpha = 0$, then the function is called *slowly varying function at infinity*; the letter L is commonly used for denoting slowly varying functions, we should follow this convention. Note that ρ is regularly varying if and only if it can be written as $\rho(x) = x^{\alpha}L(x)$, where L is slowly varying. Hence, it is enough to explore the properties of slowly varying functions in order to study those of regularly varying functions. We remark an important result [15, 183], as long as (1.7.1) holds for each $a > 0$ in a set of positive measure, then it holds uniformly on any compact subset of $(0,\infty)$.

One of the most basic (and most important) results in the theory of slowly varying functions is *the representation formula* (see first two pages of Seneta's book [183]). Furthermore, the representation formula completely characterizes all the slowly varying functions; L is slowly varying at the infinity if and only if there exist measurable functions u and w defined on some interval $[B,\infty)$, u being bounded and having a finite limit at infinity and w being continuous on $[B,\infty)$ with $w(x) = o(1)$, such that

$$L(x) = \exp\left(u(x) + \int_B^x \frac{w(t)}{t}\mathrm{d}t\right) , \quad x \in [B,\infty) \ . \tag{1.7.2}$$

This formula is important because it will enable us to obtain some useful estimates for L. For instance, it is clear that if $\sigma > 0$, then

$$L(x) = o(x^{\sigma}) \ , \quad \text{and} \quad \frac{1}{L(x)} = o(x^{\sigma}) \ , \quad x \to \infty \ .$$

The above estimates have a valuable consequence to keep in mind: *regularly varying functions at infinity are tempered distributions for large values of x.*

With the obvious modifications, we define regularly varying and slowly varying functions at the origin. In particular, L is slowly varying at the origin if and only if $L(1/x)$ is slowly varying at infinity, hence a representation formula of type (1.7.2) holds for L with the interval of integration being $[x, B]$. We also remark that slowly varying functions at the origin are regular distributions for small arguments.

1.8 Asymptotic Behavior of Generalized Functions

There are several ways to define the asymptotic behavior for generalized functions. We will consider the three most important asymptotic notions for Schwartz distributions, they will be the natural framework in our future investigations of the local behavior of distributions. We will refer in the future to the asymptotic notions presented in this section as *generalized asymptotics*.

1.8.1 Quasiasymptotics

The quasiasymptotic behavior of distributions was introduced by Zavialov [249] as a result of his investigations in Quantum Field Theory, and further developed by him, Vladimirov and Drozhzhinov [231]. It is fair to mention the contributions of the Novi Sad (Serbian) School to the field [160]. We only consider here Schwartz distributions, but we point out that the quasiasymptotic behavior can also be defined for other classes of generalized functions, the interested reader might want to consult [158, 161, 40, 41].

It is our intension in this section to give a very brief introduction to the subject, paying special attention to some particular cases and properties that will be absolutely necessary requirements for the first chapters of this treatise. We will retake the subject in Chapter 10, where we will make a major contribution toward the understanding of quasiasymptotic properties of distributions in one variable.

In general, we cannot talk about pointwise behavior of distributions, therefore, if we want to study asymptotic properties of distributions, we should usually introduce new parameters in order to give sense to asymptotic relations. The idea of the concept of quasiasymptotic behaviors of distributions is to look for asymptotic representations, at either small scale or large scale, of the dilations of a distribution. Specifically, we look for asymptotic representations of the form

$$f(hx) \sim \rho(h)g(x), \quad \text{as } h \to 0^+ \text{ , or } h \to \infty \text{ ,} \tag{1.8.1}$$

in the distributional sense, that is, holding after evaluation at each test function

$$\langle f(hx), \phi(x) \rangle \sim \rho(h) \langle g(x), \phi(x) \rangle \ . \tag{1.8.2}$$

We now define the concept of quasiasymptotic behavior and quasiasymptotic boundedness of distributions at infinity.

Definition 1.3. *A distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ has quasiasymptotic behavior at infinity in $\mathcal{D}'(\mathbb{R}^n)$ with respect to a real function ρ, which is assumed to be positive and measurable near infinity, if*

$$\lim_{\lambda \to \infty} \left\langle \frac{f(\lambda \mathbf{x})}{\rho(\lambda)}, \phi(\mathbf{x}) \right\rangle \tag{1.8.3}$$

exists (and is finite) for each $\phi \in \mathcal{D}(\mathbb{R}^n)$.

We refer to quasiasymptotic behavior also as *quasiasymptotics.* Observe that, because of Banach-Steinhaus theorem, there must be a distribution $g \in \mathcal{D}'(\mathbb{R}^n)$ such that the above limit (1.8.3) is equal to $\langle g(\mathbf{x}), \phi(\mathbf{x}) \rangle$, for each $\phi \in \mathcal{D}(\mathbb{R}^n)$. One can show that ρ and g cannot be arbitrary. Indeed, if one assumes that g is a non-zero distribution, then relation (1.8.5) forces ρ to be a regularly varying function and g a homogeneous distribution having degree of homogeneity equal to the index of regular variation of ρ [61, 160, 231]; we will not need this fact until Chapter 10, where we reproduce a proof of it.

Definition 1.4. *A distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ is called quasiasymptotically bounded at infinity in $\mathcal{D}'(\mathbb{R}^n)$ with respect to a function real function ρ, which is assumed to be positive and measurable near infinity, if $f(\lambda\mathbf{x})/\rho(\lambda)$ is bounded in the weak topology of $\mathcal{D}'(\mathbb{R}^n)$ for large values of λ, i.e.,*

$$\left\langle \frac{f(\lambda\mathbf{x})}{\rho(\lambda)}, \phi(\mathbf{x}) \right\rangle = O(1) \ , \quad \lambda \to \infty \ . \tag{1.8.4}$$

The quasiasymptotics at finite points are defined in a similar manner.

Definition 1.5. *Let $\mathbf{x}_0 \in \mathbb{R}^n$. A distribution $f \in \mathcal{D}'(\mathbb{R})$ is said to have quasiasymptotic behavior in $\mathcal{D}'(\mathbb{R}^n)$ at the point $x = x_0$ with respect to a function ρ, which is assumed to be measurable and positive near the origin, if there exists $g \in \mathcal{D}'(\mathbb{R}^n)$ such that*

$$\lim_{\varepsilon \to 0^+} \frac{1}{\rho(\varepsilon)} \left\langle f\left(\mathbf{x}_0 + \varepsilon\mathbf{x}\right), \phi(\mathbf{x}) \right\rangle = \langle g(\mathbf{x}), \phi(\mathbf{x}) \rangle \ , \ \ \forall \phi \in \mathcal{D}(\mathbb{R}^n) \ . \tag{1.8.5}$$

Definition 1.6. *Let $\mathbf{x}_0 \in \mathbb{R}^n$. A distribution $f \in \mathcal{D}'(\mathbb{R})$ is said to be quasiasymptotically bounded in $\mathcal{D}'(\mathbb{R}^n)$ at the point $x = x_0$ with respect to a function ρ, which is assumed to be measurable and positive near the origin, if $f(\mathbf{x}_0 + \varepsilon\mathbf{x})/\rho(\varepsilon)$ form a weakly bounded set for ε small enough.*

We now discuss some basic properties of the quasiasymptotics. Let us start with the case at points. Our first trivial observation is that, by shifting to $\mathbf{x}_0$, in most cases is enough to consider $\mathbf{x}_0 = \mathbf{0}$. In addition the quasiasymptotics at a point are local properties; in the sense that if f and h are equal in a neighborhood of $\mathbf{x}_0$ and f has quasiasymptotic behavior (or is quasiasymptotically bounded), then h has the same quasiasymptotic behavior (or quasiasymptotic boundedness) at the point. Hence, to talk about the quasiasymptotic behavior or quasiasymptotic boundedness at $\mathbf{x} = \mathbf{x}_0$, the distribution only needs to be defined in a neighborhood of $\mathbf{x}_0$.

We may also talk about quasiasymptotics or quasiasymptotic boundedness in other spaces of distributions, say $\mathcal{A}'$ the dual of the suitable space of functions $\mathcal{A}$. For quasiasymptotics, it means that $f \in \mathcal{A}'$ and the test functions in (1.8.1), resp. (1.8.2), can be taken from $\mathcal{A}$. In the case of quasiasymptotic boundedness, it means that the corresponding set is weakly bounded in $\mathcal{A}'$. For instance, we will make extensive use of quasiasymptotics in $\mathcal{S}'(\mathbb{R})$. There is an obvious dependence on the space of distributions to be employed, so to denote the quasiasymptotics at infinity, we will indistinctly use the two convenient notations

$$f(\lambda x) \sim \lambda^{\alpha} L(\lambda) g(x) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{A}', \tag{1.8.6}$$

and

$$f(\lambda x) = \lambda^{\alpha} L(\lambda) g(x) + o(\lambda^{\alpha} L(\lambda)) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{A}' . \tag{1.8.7}$$

For quasiasymptotic boundedness, we use the notation

$$f(\lambda x) = O(\lambda^{\alpha} L(\lambda)) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{A}' . \tag{1.8.8}$$

Likewise, an analogous notation will be used at finite points.

In the following, we focus in the one-dimensional case.

One can also consider asymptotic expansions in the sense of quasiasymptotics, that is, expansions of the form

$$f(\lambda x) \sim \sum_{n=0}^{\infty} c_n(\lambda) g_n(x) \quad \text{in } \mathcal{A}' , \tag{1.8.9}$$

in the weak topology of $\mathcal{A}'$, i.e., for each test function $\phi \in \mathcal{A}$

$$\langle f(\lambda x), \phi(x) \rangle \sim \sum_{n=0}^{\infty} c_n(\lambda) \langle g_n(x), \phi(x) \rangle \quad \text{in } \mathcal{A}' . \tag{1.8.10}$$

The asymptotic expansion (1.8.9) is called asymptotic separation of variables or quasiasymptotic expansion [61, 160, 231]. As an example of (1.8.9), we have the Estrada-Kanwal *moment asymptotic expansion* [57, 61]

$$f(\lambda x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n \mu_n}{n! \lambda^{n+1}} \delta^{(n)}(x) \quad \text{as } \lambda \to \infty , \tag{1.8.11}$$

where $\mu_n = \langle f(x), x^n \rangle$. This expansion is valid in the space $\mathcal{K}'(\mathbb{R})$, for any $f \in \mathcal{K}'(\mathbb{R})$ [61].

Since $\mathcal{E}'(\mathbb{R}) \subset \mathcal{K}'(\mathbb{R})$, any distribution of compact support satisfies the moment asymptotic expansion. Actually, for $f \in \mathcal{E}'(\mathbb{R})$, the moment asymptotic expansion (1.8.11) holds in the space $\mathcal{E}'(\mathbb{R})$. Therefore, contrary to the case at points, the quasiasymptotic at ∞ is not a local property.

An advantage of quasiasymptotic relations is that *differentiation is permitted*, since the derivative is a continuous operator on spaces of distributions. From now on, we will make use of this fact without further comments.

We now discuss some basic facts of quasiasymptotics in the case when ρ is a power function and g is a homogeneous distribution. The first result is very well known [61, 160, 231], but we state it and prove it for the convenience of the reader; it relates the ordinary asymptotic behavior of functions and the quasiasymptotic behavior of distributions.

Proposition 1.7. *Let f be a locally integrable function with support on an interval $[b, \infty)$. Suppose that $f(x) = O(x^\alpha)$, $x \to \infty$, where $\alpha > -1$, then $f(\lambda x) = O(\lambda^\alpha)$ as $\lambda \to \infty$ in $\mathcal{S}'(\mathbb{R})$. Furthermore, if $f(x) \sim Cx^\alpha$, $x \to \infty$, then $f(\lambda x) \sim C(\lambda x)_+^\alpha$ as $\lambda \to \infty$ in $\mathcal{S}'(\mathbb{R})$.*

Proof. We can assume that $\operatorname{supp} f \subseteq [1, \infty)$. Otherwise, decompose $f = f_1 + f_2$, where f_2 is supported on $[1, \infty)$ and f_1 has compact support; since f_1 satisfies the moment asymptotic expansion (1.8.11), then $f_1(\lambda x)$ only contributes to $f(\lambda x)$ up to an $O(\lambda^{-1})$ term, thus, we may assume that $f = f_2$. Next, pick M such that $|f(x)| \leq Mx^\alpha$, the same argument we just applied allows to assume that the last inequality holds for all $x > 1$. Take $\phi \in \mathcal{S}(\mathbb{R})$. So we have

$$|\langle f(\lambda x), \phi(x) \rangle| = \lambda^\alpha \int_{\frac{1}{\lambda}}^{\infty} \left| \frac{f(\lambda x)}{(\lambda x)^\alpha} \right| x^\alpha \, |\phi(x)| \, \mathrm{d}x \leq M\lambda^\alpha \int_0^\infty x^\alpha \, |\phi(x)| \, \mathrm{d}x.$$

If now $f(x) \sim Cx^\alpha$, we can apply Lebesgue dominated convergence theorem to conclude that

$$\lim_{\lambda\to\infty} \frac{1}{\lambda^\alpha} \langle f(\lambda x), \phi(x)\rangle = \lim_{\lambda\to\infty} \int_{\frac{1}{\lambda}}^{\infty} \frac{f(\lambda x)}{(\lambda x)^\alpha} x^\alpha \phi(x) \mathrm{d}x = C \int_0^\infty x^\alpha \phi(x) \mathrm{d}x \ .$$

$\square$

A similar result holds for functions with support bounded at the right.

We now present the structural theorem for quasiasymptotics of distributions in $\mathcal{D}'[0,\infty)$. The result was basically obtained in [37].

Proposition 1.8. *A distribution $f \in \mathcal{D}'[0,\infty)$ has quasiasymptotic behavior*

$$f(\lambda x) \sim C \frac{(\lambda x)_+^\alpha}{\Gamma(\alpha+1)} \quad \text{as } \lambda \to \infty \text{ in } \mathcal{D}'(\mathbb{R}) \tag{1.8.12}$$

if and only if $f \in \mathcal{S}'[0,\infty)$ and there exists a non-negative integer $k > -\alpha - 1$ such that $f^{(-k)}$ is an ordinary function and

$$f^{(-k)}(x) \sim C \frac{x_+^{\alpha+k}}{\Gamma(\alpha+k+1)} \ , \quad x \to \infty \ , \tag{1.8.13}$$

in the ordinary sense. Moreover, the quasiasymptotic behavior (1.8.12) holds actually in $\mathcal{S}'[0,\infty)$.

Proof. The converse follows directly from Proposition 1.7. The Banach-Steinhaus theorem, the quasiasymptotic behavior (1.8.12) and the definition of convergence in $\mathcal{D}'[0,\infty)$ imply that there exists n, sufficiently large, such that the evaluation of f at $\phi_n(t) := (1-t)^n(H(t) - H(t-1))$ makes sense and (1.8.12) holds when evaluated at ϕ_n. Here H is the Heaviside functions. Put $k = n+1$, then, as $x \to \infty$,

$$
\begin{aligned}
f^{(-k)}(x) =& \frac{1}{(k-1)!}\left\langle f(t),(x-t)^{k-1}(H(t)-H(t-1))\right\rangle \\
=& \frac{x^{k-1}}{(k-1)!}\left\langle f(t),\phi_n\left(\frac{t}{x}\right)\right\rangle \\
=& \frac{x^{k}}{(k-1)!}\left\langle f(xt),\phi_n\left(t\right)\right\rangle \\
\sim& \frac{Cx^{k+\alpha}}{(k-1)!\Gamma(\alpha+1)}\,\mathrm{F.p.}\int_0^1 t^{\alpha}(1-t)^{k-1}\mathrm{d}t \\
=& \frac{Cx^{k+\alpha}}{\Gamma(\alpha+k+1)}\ .
\end{aligned}
$$

□

Likewise one shows.

Proposition 1.9. *A distribution $f \in \mathcal{D}'[0,\infty)$ satisfies*

$$f(\lambda x) = O(\lambda^{\alpha}) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{D}'(\mathbb{R}) \tag{1.8.14}$$

if and only if $f \in \mathcal{S}'[0,\infty)$ and there exists a non-negative integer $k > -\alpha - 1$ such that $f^{(-k)}$ is an ordinary function and

$$f^{(-k)}(x) = O(x^{\alpha+k})\ , \quad x \to \infty\ , \tag{1.8.15}$$

in the ordinary sense. Furthermore, (1.8.14) holds actually in $\mathcal{S}'[0,\infty)$.

We end our discussion about quasiasymptotics with a bibliographical remark.

Remark 1.10. *In [249, 231] the original definition for the quasiasymptotic behaviors at infinity is given only for $f \in \mathcal{S}'[0,\infty)$; there the function ρ is called an automodel function but we will not follow this terminology. In [150, 151, 152, 153], the definition is extended to the form just presented here. Sometimes, it is also assumed that $g \neq 0$; nevertheless that assumption is not essential for us, and we do allow g to be 0.*

1.8.2 The Cesàro Behavior

Let us define the Cesàro behavior of a distribution at infinity. We follow closely the expositions from [49, 61]. At this point, we shall confine ourselves with the definition for integral Cesàro orders and comparison with respect to power functions; however, we point out that the Cesàro behavior of distributions can also be defined for fractional orders [223, 224, 226] (see also Chapters 8 and 9 below), in addition, regularly varying functions may be included in the theory [224].

It is studied by using the order symbols $O(x^\alpha)$ and $o(x^\alpha)$ in the Cesàro sense.

Definition 1.11. *Let $f \in \mathcal{D}'(\mathbb{R})$, $m \in \mathbb{N}$, and $\alpha \in \mathbb{R} \setminus \mathbb{Z}_-$. We say that $f(x) = O(x^\alpha)$ as $x \to \infty$ in the Cesàro sense of order m (in the (C, m) sense) and write*

$$f(x) = O(x^\alpha) \quad (\mathrm{C}, m), \quad x \to \infty, \tag{1.8.16}$$

if each primitive F of order m, i.e., $F^{(m)} = f$, is an ordinary function for large arguments and satisfies the ordinary order relation

$$F(x) = p(x) + O\left(x^{\alpha+m}\right), \quad x \to \infty, \tag{1.8.17}$$

for some suitable polynomial p of degree at most $m-1$, which in general depends on F. Similarly for the little o symbol. We say that f is asymptotic to Cx^α as $x \to \infty$ in the Cesàro sense of order m and write

$$f(x) \sim Cx_+^\alpha \quad (\mathrm{C}, m), \quad x \to \infty, \tag{1.8.18}$$

if we have $f(x) - Cx_+^\alpha = o(x^\alpha)$ (C, m), $x \to \infty$.

Notice that if $\alpha > -1$, then the polynomial p is irrelevant in (1.8.17). A similar definition applies when $x \to -\infty$. One may also consider the case when $\alpha = -1, -2, -3, \ldots$ [61, Def.6.3.1], but we shall not do so here. Obviously, if f vanishes for large arguments, then $f(x) = o(x^\alpha)$ (C, m), for any m and α. When we do not want to make reference to the order m in (1.8.16) or (1.8.18), we simply write (C), meaning (C, m) for some m.

For $\alpha = 0$, we obtain the notion of Cesàro limits at infinity.

Definition 1.12. *Let $f \in \mathcal{D}'(\mathbb{R})$ and $m \in \mathbb{N}$. We say that f has a limit ℓ at infinity in the Cesàro sense of order m (in the (C, m) sense) and write*

$$\lim_{x \to \infty} f(x) = \ell \quad (\mathrm{C}, m) \ ,$$

if we have that $f(x) = \ell + o(1) \ \ (\mathrm{C}, m)$, $x \to \infty$.

We want discuss the close relation between Cesàro asymptotics and the quasiasymptotic behavior. For further properties, we refer to [61].

The next theorem shows that the Cesàro behavior, in the case $\alpha > -1$, is totally determined by the quasiasymptotic properties of the distribution on intervals being bounded at the left.

Proposition 1.13. *Let $f \in \mathcal{D}'(\mathbb{R})$, $m \in \mathbb{N}$, and $\alpha > -1$. Let f_+ be any distribution supported on an interval of the form $[a, \infty)$, $a \in \mathbb{R}$, coinciding with f for large arguments, i.e., in some open interval with finite left end point. Then, we have the next equivalences.*

(i) The following two conditions are equivalent,

$$f(x) = O\left(x^{\alpha}\right) \quad (\mathrm{C})\,, \quad x \to \infty \ , \tag{1.8.19}$$

and f_+ belongs to $\mathcal{S}'(\mathbb{R})$ and is quasiasymptotically bounded of degree α, i.e.,

$$f_+(\lambda x) = O(\lambda^{\alpha}) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R}) \ . \tag{1.8.20}$$

(ii) The conditions,

$$f(x) \sim C x_+^{\alpha} \quad (\mathrm{C})\,, \quad x \to \infty \ , \tag{1.8.21}$$

and $f_+ \in \mathcal{S}'(\mathbb{R})$ has the quasiasymptotic behavior

$$f_+(\lambda x) = C\lambda^\alpha x_+^\alpha + o(\lambda^\alpha) \quad \textit{as } \lambda \to \infty \textit{ in } \mathcal{S}'(\mathbb{R}) \ , \tag{1.8.22}$$

are equivalent.

Proof. We can assume that $f = f_+$ and that $f \in \mathcal{D}'[0,\infty)$, and so the equivalence between (1.8.19) and (1.8.22) is precisely the structural theorem for quasiasymptotic boundedness (Proposition 1.9) in this space. On the other hand, the equivalence between (1.8.21) and (1.8.22) is precisely the content of the structural theorem for quasiasymptotic behavior of degree $\alpha > -1$ (Proposition 1.8) in $\mathcal{D}'[0,\infty)$. □

When $\alpha < -1$, we do not exactly obtain a characterization in terms of quasiasymptotics because delta terms could appear in the expansion.

Proposition 1.14. *Let $f \in \mathcal{D}'(\mathbb{R})$, $m \in \mathbb{N}$, and $\alpha < -1$, $\alpha \notin \mathbb{Z}_-$. Let f_+ be any distribution supported on an interval of the form $[a,\infty)$, $a \in \mathbb{R}$, coinciding with f for large arguments. Then, we have the next equivalences.*

(i) The following two conditions are equivalent,

$$f(x) = O\left(x_+^\alpha\right) \quad (\mathrm{C})\,, \quad x \to \infty \ , \tag{1.8.23}$$

and there exist $n > -\alpha$ constants $a_0, \dots, a_{n-1}$, in general depending on f_+, such that f_+ has the asymptotic expansion

$$f_+(\lambda x) = \sum_{j=0}^{n-1} a_j \frac{\delta^{(j)}(x)}{\lambda^{j+1}} + O(\lambda^\alpha) \quad \textit{as } \lambda \to \infty \textit{ in } \mathcal{S}'(\mathbb{R}) \ . \tag{1.8.24}$$

(ii) The conditions,

$$f(x) \sim Cx_+^\alpha \quad (\mathrm{C})\,, \quad x \to \infty \ , \tag{1.8.25}$$

and the existence of constants $n > -\alpha$ constants $a_0, \ldots, a_{n-1}$, in general depending on f_+, such that

$$f_+(\lambda x) = C\lambda^\alpha x_+^\alpha + \sum_{j=0}^{n-1} a_j \frac{\delta^{(j)}(x)}{\lambda^{j+1}} + o(\lambda^\alpha) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R}) \ , \quad (1.8.26)$$

are equivalent.

Proof. We can assume $f = f_+$. We only show (ii), the proof of (i) is similar to this case and is left to the reader. Assume (1.8.25), then there exist G_1, G_2, $m > -\alpha - 1$, and m constants $c_0, \ldots, c_{m-1}$ such that $f = G_1 + G_2$, G_1 has compact support, G_2 is a locally integral functions with support on $[, \infty)$, and

$$G_2(x) = \sum_{j=0}^{m-1} c_j \frac{x^j}{j!} + C\frac{\Gamma(\alpha+1)}{\Gamma(m+\alpha+1)} x^{m+\alpha} + o(x^{m+\alpha}) \ ,$$

$x \to \infty$. Since G_1 has compact support, then $G_1(\lambda x) = O(\lambda^{-1})$, in $\mathcal{S}'(\mathbb{R})$, and so $G^{(m)}(\lambda x) = O(\lambda^{-m-1}) = o(\lambda^\alpha)$; then, since it does not contributes for (1.8.26), we can assume that $G_1 = 0$. On the other hand, by Proposition 1.7, the ordinary asymptotic expansion of G_2 implies

$$G_2(\lambda x) = \sum_{j=0}^{m-1} c_j \frac{(\lambda x)_+^j}{j!} + C\frac{\Gamma(\alpha+1)}{\Gamma(m+\alpha+1)} (\lambda x)_+^{m+\alpha} + o(\lambda^{m+\alpha})$$

in $\mathcal{S}'(\mathbb{R})$. Differentiating m-times the above asymptotic formula, and discarding the irrelevant constants, we obtain (1.8.26) with $a_j = c_{m-1-j}$. The converse follows from the structural theorem, Proposition 1.8, applied to $f_+ - \sum_{j=0}^{n-1} c_j \, \delta^{(j)}$. $\square$

1.8.3 $S-$asymptotics

The final asymptotic notion we shall need is that of S-asymptotics, it stands for *shift-asymptotics*. They were introduced by Pilipović and Stanković in [155] inspired by previous notions from [6, 25, 180]. We only state the definition, since we will not make use of any deep result about $S-$asymptotics, besides basic properties which follow directly from the definition. For a complete account we refer to [160];

for $S-$asymptotics in other spaces of generalized functions the reader may consult [158, 193, 194].

Definition 1.15. *A distribution $f \in \mathcal{D}'(\mathbb{R})$ is said to have S-asymptotic at infinity in $\mathcal{D}'(\mathbb{R}^n)$ with respect to a real function ρ, which is assumed to be positive and measurable near infinity, if there exists $g \in \mathcal{D}'(\mathbb{R})$ such that*

$$\lim_{h\to\infty} \left\langle \frac{f(x+h)}{\rho(h)}, \phi(x) \right\rangle = \langle g(x), \phi(x) \rangle \ , \quad h \to \infty \ , \tag{1.8.27}$$

for each $\phi \in \mathcal{D}(\mathbb{R})$.

We use the notations

$$f(x+h) \sim \rho(h)g(x) \quad \text{as } h \to \infty \text{ in } \mathcal{D}'(\mathbb{R}) \ , \tag{1.8.28}$$

or

$$f(x+h) = \rho(h)g(x) + o(\rho(h)) \quad \text{as } h \to \infty \text{ in } \mathcal{D}'(\mathbb{R}) \ , \tag{1.8.29}$$

for S-asymptotics. Obviously, we can consider S-asymptotics in other spaces of distributions with a clear meaning. As quasiasymptotic relations, S-asymptotic relations still hold if we differentiate them. Observe also, that this asymptotic notion is a local one at infinity.

Chapter 2

A Quick Way to the Prime Number Theorem

2.1 Introduction

This first short chapter serves as a motivation for our further study of local and asymptotic properties of Schwartz distributions. We obtain a non-trivial application of generalized asymptotics. We give two new distributional proofs of the celebrated Prime Number Theorem (in short PNT). Of course, the word distributional refers to Schwartz distributions. So, we show that

$$\pi(x) \sim \frac{x}{\log x} \ , \quad x \to \infty \ , \tag{2.1.1}$$

where

$$\pi(x) = \sum_{p \text{ prime}, \ p<x} 1 \ . \tag{2.1.2}$$

We provide two related proofs. It is remarkable that both proofs are direct and do not use any tauberian argument. Our arguments are based on Chebyshev's elementary estimate [101, p.14]

$$\pi(x) = O\left(x/\log x\right) \ , \quad x \to \infty \ , \tag{2.1.3}$$

and additional properties of the Riemann zeta function on the line $\Re e \ z = 1$.

The author hopes that this first incursion of generalized asymptotics into number theory encourages a future exploration of the range of applicability of techniques from distribution theory to other problems from analytic number theory.

The result of this chapter have already been put into article form [220], but we add to the exposition from [220] a complementary tauberian theorem of Wiener-Ikehara type, this is done in the last section of the chapter.

2.2 Special Functions and Distributions Related to the PNT

In this section we briefly explain some special functions and distributions related to prime numbers.

Throughout this article, the letter p stands only for a prime number. We denote by Λ the von Mangoldt function defined on the natural numbers as

$$\Lambda(n) = \begin{cases} 0\,, & \text{if } n = 1\,, \\ \log p\,, & \text{if } n = p^m \text{ with } p \text{ prime and } m > 0\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{2.2.1}$$

As usually done, we denote by ψ the *Chebyshev function*

$$\psi(x) = \sum_{p^m < x} \log p = \sum_{n<x} \Lambda(n)\ . \tag{2.2.2}$$

It follows easily from Chebyshev's classical estimate (2.1.3) that for some $M > 0$

$$\psi(x) < Mx\ . \tag{2.2.3}$$

It is very well known since the time of Chebyshev that the PNT is equivalent to the statement

$$\psi(x) \sim x\ . \tag{2.2.4}$$

Our approach to the PNT will be to show (2.2.4).

Our proof of the PNT is based on finding the distributional asymptotic behavior of $\psi'(x)$ (the derivative is understood in the distributional sense, of course); observe that

$$\psi'(x) = \sum_{n=1}^{\infty} \Lambda(n)\, \delta(x-n)\ , \tag{2.2.5}$$

where δ is the well known Dirac delta distribution (Section 1.3). For this goal, we shall study the asymptotic properties of the distribution

$$v(x) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} \delta(x - \log n)\ ; \tag{2.2.6}$$

clearly $v \in \mathcal{S}'(\mathbb{R})$.

Consider the Riemann zeta function

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \ , \ \Re e \, z > 1 \ . \tag{2.2.7}$$

Let us first take the Fourier-Laplace transform of v, that is, for $\Im m \, z > 0$

$$\left\langle v(t), e^{izt} \right\rangle = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1-iz}} = -\frac{\zeta'(1-iz)}{\zeta(1-iz)} \ , \tag{2.2.8}$$

a formula that Riemann obtained by logarithmic differentiation of the Euler product for the zeta function $\zeta(z) = \prod_p 1/(1-p^{-z})$.

Taking the boundary values on the real axis, in the distributional sense, we obtain the Fourier transform of v ,

$$\hat{v}(x) = -\frac{\zeta'(1-ix)}{\zeta(1-ix)} \ . \tag{2.2.9}$$

Notice that we are not saying that the right hand side on the last relation is a function but rather that it is a tempered distribution. We shall always interpret (2.2.9) as equality in the space $\mathcal{S}'(\mathbb{R})$, meaning that for each $\phi \in \mathcal{S}(\mathbb{R})$

$$\langle \hat{v}(x), \phi(x) \rangle = -\lim_{y \to 0^+} \int_{-\infty}^{\infty} \phi(x) \frac{\zeta'(1-ix+y)}{\zeta(1-ix+y)} \, \mathrm{d}x \ . \tag{2.2.10}$$

It is implicit in (2.2.9) that the Fourier transform we are using is

$$\hat{\phi}(x) = \int_{-\infty}^{\infty} e^{ixt} \phi(t) \, \mathrm{d}t \ , \quad \text{for } \phi \in \mathcal{S}(\mathbb{R}) \ .$$

We discuss some properties of the distribution $\hat{v}$. From the well known properties of ζ, we conclude that on $\mathbb{R} \setminus \{0\}$ $\hat{v}$ is a locally integrable function. Indeed,

$$\zeta(z) - \frac{1}{z-1} \tag{2.2.11}$$

admits an analytic continuation to a neighborhood of $\Re e \, z = 1$, as one easily proves by applying the Euler-Maclaurin formula [61]; in addition, $\zeta(1+ix)$, $x \neq 0$, is free of zeros [103, 115]. It follows then that

$$\hat{v}(x) - \frac{i}{(x+i0)} \in L^1_{\mathrm{loc}}(\mathbb{R}) \ , \tag{2.2.12}$$

where here we use the notation $1/(x+i0)$ for the distributional boundary value of the analytic function z^{-1}, $\Im m\, z > 0$.

The property (2.2.12) together with Chebyshev's estimate (2.2.3) will be the key ingredients for the proof of the PNT given in Section 2.5.

The proof to be given in Section 2.4 makes use of additional information of the Riemann zeta function on the line $\Re e\, z = 1$; we shall take for granted that $\hat{v}$ has at most polynomial growth as $|x| \to \infty$. In fact, more than this is true: $\hat{v}(x) = O(\log^{\beta}(x))$ as $x \to \infty$, for some $\beta > 0$. The reader can find the proof of this fact in [101, Chap.2] (see also [122]). Summarizing, we have that

$$\hat{v}(x) - \frac{i}{(x+i0)} \in L^1_{\mathrm{loc}}(\mathbb{R}) \text{ and has polynomial growth .} \tag{2.2.13}$$

2.3 Notation from Generalized Asymptotics

The purpose of this section is to clarify the notation to be used in the following two sections. It was basically explained in Section 1.8, but we choose to make some comments. Besides the notation, we do not make use of any deep result from generalized asymptotics.

Let $f \in \mathcal{D}'(\mathbb{R})$, a relation of the form

$$\lim_{h\to\infty} f(x+h) = \beta \ , \quad \text{in } \mathcal{D}'(\mathbb{R}) \ , \tag{2.3.1}$$

means that the limit is taken in the weak topology of $\mathcal{D}'(\mathbb{R})$, that is, for each test function from $\mathcal{D}(\mathbb{R})$ the following limit holds,

$$\lim_{h\to\infty} \langle f(x+h), \phi(x)\rangle = \beta \int_{-\infty}^{\infty} \phi(x)\mathrm{d}x \ . \tag{2.3.2}$$

The meaning of the expression $\lim_{h\to\infty} f(x+h) = \beta$ in $\mathcal{S}'(\mathbb{R})$ is clear. Observe that relation (2.3.1) is an example of the so-called *S-asymptotics*, introduced already in Section 1.8.3.

On the other hand, we will study in connection to the PNT a particular case of the quasiasymptotic behavior (Section 1.8.1), namely, a limit of the form

$$\lim_{\lambda\to\infty} f(\lambda x) = \beta H(x) \ , \quad \text{in } \mathcal{D}'(\mathbb{R}) \ , \tag{2.3.3}$$

where $H(x)$ is the Heaviside function (Section 1.3). Needless to say that (2.3.3) should be always interpreted in the weak topology of $\mathcal{D}'(\mathbb{R})$. We may also talk about (2.3.3) in other spaces of distributions with a clear meaning. For instance, we will consider (2.3.3) in $\mathcal{D}'(0,\infty)$, not in $\mathcal{D}'(\mathbb{R})$, which means that we are, initially, only in the right to evaluate (2.3.3) at test functions with support in $(0,\infty)$.

2.4 First Proof of the PNT

Our first proof is based on (2.2.3) and (2.2.13). We begin with the distribution v given by (2.2.6).

Our first step is to show that

$$\lim_{h\to\infty} v(x+h) = 1, \quad \text{in } \mathcal{S}'(\mathbb{R}) \ . \tag{2.4.1}$$

Recall that $H(x)$ denotes the Heaviside function. Let $\phi \in \mathcal{S}(\mathbb{R})$. Consider $\phi_1 \in \mathcal{S}(\mathbb{R})$ such that $\phi = \hat{\phi_1}$; then as $h \to \infty$

$$\begin{aligned}
\langle v(x+h), \phi(x)\rangle &= \int_{-h}^{\infty} \phi(x)\mathrm{d}x + \langle v(x+h) - H(x+h) \, , \phi(x)\rangle \\
&= \int_{-h}^{\infty} \phi(x)\mathrm{d}x + \left\langle \hat{v}(x) - \frac{i}{(x+i0)} \, , e^{-ihx}\phi_1(x)\right\rangle \\
&= \int_{-h}^{\infty} \phi(x)\mathrm{d}x + \int_{-\infty}^{\infty} e^{-ihx}\phi_1(x)\left(\hat{v}(x) - \frac{i}{(x+i0)}\right)\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \phi(x)\mathrm{d}x + o(1), \quad h\to\infty \ ,
\end{aligned}$$

where the last step follows in view of (2.2.13) and the Riemann-Lebesgue lemma. This shows (2.4.1).

The second step is to show that

$$\lim_{\lambda\to\infty} \psi'(\lambda x) = \lim_{\lambda\to\infty} \sum_{n=1}^{\infty} \Lambda(n)\delta(\lambda x - n) = H(x) \ , \quad \text{in } \mathcal{D}'(0,\infty) \ , \tag{2.4.2}$$

here again $H(x)$ is the Heaviside function and ψ is the Chebyshev function. Indeed (2.4.1) implies that $e^{t+h}v(t+h) \sim e^{t+h}$, as $h \to \infty$, in the weak topology of $\mathcal{D}'(\mathbb{R})$, which readily implies that for each $\phi \in \mathcal{D}(\mathbb{R})$

$$\sum_{n=1}^{\infty} \Lambda(n)\phi(\log n - h) \sim e^h \int_0^{\infty} \phi(\log x)\mathrm{d}x \ , \quad h \to \infty \ .$$

If $\phi_1 \in \mathcal{D}(0,\infty)$, it can be written as $\phi_1(x) = \phi(\log x)$ with $\phi \in \mathcal{D}(\mathbb{R})$, changing $\lambda = e^h$ in the above relation we obtain (2.4.2).

Here comes the final step in our argument, we evaluate (2.4.2) at suitable test functions to deduce that $\psi(x) \sim x$. Let $\sigma > 0$ be an arbitrary number; find ϕ_1 and $\phi_2 \in \mathcal{D}(0,\infty)$ with the following properties: $0 \leq \phi_i \leq 1$, $\operatorname{supp} \phi_1 \subseteq (0,1]$, $\phi_1(x) = 1$ on $[\sigma, 1-\sigma]$, $\operatorname{supp} \phi_2 \subseteq (0, 1+\sigma]$, and finally, $\phi_2(x) = 1$ on $[\sigma, 1]$. Evaluating ϕ_2 in (2.4.2) and using (2.2.3), we obtain that

$$\begin{aligned}
\limsup_{\lambda\to\infty} \frac{1}{\lambda}\sum_{n<\lambda}\Lambda(n) &\leq \limsup_{\lambda\to\infty}\left(\frac{1}{\lambda}\sum_{n<\sigma\lambda}\Lambda(n) + \frac{1}{\lambda}\sum_{n=0}^{\infty}\Lambda(n)\phi_2\left(\frac{n}{\lambda}\right)\right) \\
&\leq M\sigma + \lim_{\lambda\to\infty}\frac{1}{\lambda}\sum_{n=0}^{\infty}\Lambda(n)\phi_2\left(\frac{n}{\lambda}\right) \\
&= M\sigma + \int_0^{\infty}\phi_2(x)\mathrm{d}x \leq 1 + \sigma(M+1) \ .
\end{aligned}$$

Evaluating at ϕ_1, we easily obtain that

$$1 - 2\sigma \leq \liminf_{\lambda\to\infty}\frac{1}{\lambda}\sum_{n<\lambda}\Lambda(n) \ .$$

Since σ was arbitrary, we conclude that $\psi(\lambda) \sim \lambda$ and the PNT follows immediately.

2.5 Second Proof of the PNT

The second proof is based on (2.2.3) and (2.2.12). We present a variant of the proof discussed in Section 2.4. In fact, we show how to avoid the use of the growth properties of $\zeta(z)$ on $\Re e\ z = 1$.

We begin by observing that it is enough to establish (2.4.1). Indeed, once (2.4.1) is obtained, one can proceed identically as in the Section 2.4 and prove the PNT.

Therefore, we shall derive (2.4.1) from (2.2.3) and (2.2.12). In view of (2.2.12) and the argument from the last section involving the Riemann-Lebesgue lemma, we can still deduce that for each test function ϕ with $\operatorname{supp}\hat{\phi}$ compact

$$\lim_{h\to\infty}\langle v(x+h),\phi(x)\rangle = \int_{-\infty}^{\infty}\phi(x)\mathrm{d}x \ . \tag{2.5.1}$$

The set of test functions having this property is dense in $\mathcal{S}(\mathbb{R})$. Then, if one were able to show that $v(x+h)=O(1)$ in $\mathcal{S}'(\mathbb{R})$, that is, that the set of translates of v is a weakly bounded set, then (2.4.1) would follow from the Banach-Steinhaus theorem and the convergence over a dense subset of $\mathcal{S}(\mathbb{R})$. We now show this last property. Let $g(x)=e^{-x}\psi(e^{x})$. Because of (2.2.3), we have that $g(x+h)=O(1)$ in the weak topology of $\mathcal{S}'(\mathbb{R})$. Consequently, we also have that $g'(x+h)=O(1)$ in $\mathcal{S}'(\mathbb{R})$. Hence, $v(x+h)=g'(x+h)+g(x+h)=O(1)$ in $\mathcal{S}'(\mathbb{R})$, as required. The boundedness of $v(x+h)$ together with (2.5.1) imply the PNT.

2.6 A Complex Tauberian Theorem

Our arguments given in the past two sections may be used to show the following complex tauberian theorem. The proof is basically the same as our second proof of the prime number theorem, but we give it for the sake of completeness.

Theorem 2.1. *Let s be a non-decreasing function supported on $[0,\infty)$ satisfying the growth condition $s(x)=O(e^{x})$. Hence, the function*

$$G(z)=\int_{0}^{\infty}e^{-zt}\mathrm{d}s(t) \tag{2.6.1}$$

is analytic on $\Re\mathrm{e}\, z>1$. If there exists a constant β such that

$$G(z)-\frac{\beta}{z-1} \tag{2.6.2}$$

admits a boundary distribution on the line $\Re\mathrm{e}\, z=1$ which belongs to $L^{1}_{\mathrm{loc}}(1+i\mathbb{R})$, then

$$s(x)\sim\beta e^{x}\ ,\quad x\to\infty\ . \tag{2.6.3}$$

Proof. By subtracting $s(0)H(x)$, we may assume that $s(0) = 0$, so the derivative of s is given by the Stieltjes integral $\langle s'(t), \phi(t)\rangle = \int_0^\infty \phi(t)\mathrm{d}s(t)$. Let $M > 0$ such that $s(x) < Me^x$. Define $v(x) = e^{-x}s'(x)$. We have that $e^{-x}s(x)$ is a tempered distribution and its set of translates is, in particular, weakly bounded; since $(e^{-x}s(x))' = -e^{-x}s(x) + v(x)$, we conclude that $v \in \mathcal{S}'(\mathbb{R})$ and $v(x+h) = O(1)$ in $\mathcal{S}'(\mathbb{R})$. The Fourier-Laplace transform of v on $\Im m\, z$ is given by

$$\left\langle v(t), e^{izt}\right\rangle = \int_0^\infty e^{(iz-1)t}\mathrm{d}s(t) = G(1-iz) \ ,$$

Hence, $\hat{v}(x) - i\beta/(x+i0)$ is locally integrable, therefore $e^{-ihx}(\hat{v}(x) - i\beta/(x+i0)) = o(1)$ as $h \to \infty$ in $\mathcal{D}'(\mathbb{R})$. Taking Fourier inverse transform, we conclude that $v(x+h) = \beta + o(1)$ as $h \to \infty$ in $\mathcal{F}(\mathcal{D}'(\mathbb{R}))$, the Fourier transform image of $\mathcal{D}'(\mathbb{R})$. Using the density of $\mathcal{F}(\mathcal{D}(\mathbb{R}))$ and the boundedness of $v(x+h)$, we conclude that $v(x+h) = \beta + o(1)$ actually in $\mathcal{S}'(\mathbb{R})$. Multiplying by e^{x+h}, we obtain $s'(x+h) \sim e^{x+h}$ in $\mathcal{D}'(\mathbb{R})$. Let $g(x) = s(\log x)$, then $\lim_{\lambda\to\infty} g'(\lambda x) = \beta H(x)$ in $\mathcal{D}'(0,\infty)$; indeed,

$$\begin{aligned}
\langle g'(\lambda x), \phi(x)\rangle &= -\frac{1}{\lambda^2}\int_0^\infty s(\log x)\phi'\left(\frac{x}{\lambda}\right)\mathrm{d}x \\
&= -\frac{1}{\lambda}\int_{-\infty}^\infty s(t+\log\lambda)e^t\phi'(e^t)\mathrm{d}t \\
&= \frac{1}{\lambda}\left\langle s'(t+\log\lambda), \phi(e^t)\right\rangle \\
&= \int_{-\infty}^\infty e^t\phi(e^t)\mathrm{d}t + o(1) \\
&= \int_0^\infty \phi(x)\mathrm{d}x + o(1) \ , \quad \lambda \to \infty \ .
\end{aligned}$$

We now choose σ, ϕ_1, and ϕ_2 as in Section 2.4. Evaluating ϕ_2 at the quasiasymptotic limit of g', we obtain that

$$\begin{aligned}
&\limsup_{\lambda\to\infty}\frac{g(\lambda)}{\lambda} = \limsup_{\lambda\to\infty}\frac{1}{\lambda}\int_0^\lambda \mathrm{d}g(t) \le \limsup_{\lambda\to\infty}\left(\frac{g(\sigma\lambda)}{\lambda} + \frac{1}{\lambda}\int_0^\infty \phi_2\left(\frac{t}{\lambda}\right)\mathrm{d}g(t)\right) \\
&\le M\sigma + \lim_{\lambda\to\infty}\langle g'(\lambda x), \phi(x)\rangle = M\sigma + \beta\int_0^\infty \phi_2(x)\mathrm{d}x \le \beta + \sigma(M+\beta) \ .
\end{aligned}$$

Evaluating at ϕ_1, we easily obtain that

$$\beta - 2\sigma\beta \leq \liminf_{\lambda\to\infty} \frac{g(\lambda)}{\lambda} .$$

Since σ was arbitrary, we conclude (2.6.3). $\square$

Theorem 2.1 implies the following result for Dirichlet series. It was obtained by Korevaar [117] via purely complex variable methods; here we use purely distributional methods! We remark that this result was used in [117] to conclude the classical Wiener-Ikehara theorem.

Theorem 2.2. *Let $\sum_{n=1}^{\infty} c_n$ be a series with terms terms bounded from below, i.e., there exists $K > 0$ such that $c_n > -K$ for all n. Suppose that the partial sums satisfy $\sum_{n=1}^{N} c_n = O(N)$. Let*

$$G(z) = \sum_{n=1}^{\infty} \frac{c_n}{n^z} , \tag{2.6.4}$$

it is analytic on $\Re e\, z > 1$. If there exists a constant β such that the distributional boundary value of

$$G(z) - \frac{\beta}{z-1} \tag{2.6.5}$$

on the line $\Re e\, z = 1$ belongs to $L^1_{\mathrm{loc}}(1 + i\mathbb{R})$, then

$$\sum_{n=1}^{N} c_n \sim \beta N , \quad N \to \infty . \tag{2.6.6}$$

Proof. Set $s(x) = \sum_{n\leq e^x}(c_n + K)$. Then $s(x) = O(e^x)$, and

$$\int_0^{\infty} e^{-zt}\mathrm{d}s(t) = K\zeta(z) + \sum_{n=1}^{\infty}\frac{c_n}{n^z} ;$$

thus, s satisfies the hypothesis of Theorem 2.1, and so

$$s(x) \sim (\beta + K)e^x,$$

from where (2.6.6) follows. $\square$

Naturally, Theorem 2.2, applied to $\sum_{n=0}^{\infty} \Lambda(n)$ directly, implies the PNT; furthermore, the proof, as has been given here, is essentially the same as our distributional method for the proof of the PNT itself.

Chapter 3

Summability of the Fourier Transform and Distributional Point Values

3.1 Introduction

The study of the relationship between the local behavior of a periodic function and the convergence or summability of its Fourier series is an old and interesting problem. It has a long tradition [62, 256, 93, 92]. Since the convergence fails in many interesting cases, the study is usually carried out by means of summability methods. In the famous monograph [256], it was said by A. Zygmund that the problem of summability of Fourier series of classical functions at individual points could be considered as a closed chapter in Mathematics. However, since the introduction of the so called Generalized Functions, new problems were opened.

Interestingly, one can extend many results from the classical theory of Fourier series of functions to Fourier series of distributions. For example, one of the most basic results in the classical theory is that of L. Fejér which asserts that the Fourier series of a continuous functions, although not necessarily convergent, is $(\mathrm{C},1)$ summable; furthermore, if $f\in L^1[0,2\pi]$ then its (symmetric) Fourier series is $(\mathrm{C},1)$ summable at every Lebesgue point [62, 93, 256]. This admits an extension. The first extension to periodic distributions was given by G. Walter [236, 237].

A distributional point of view of Fourier series is sometimes more convenient because it provides new interpretations of summability of trigonometric series that the classical point of view hides in somehow. For instance, it is possible to completely characterize the value of periodic distributions at a point in terms of summability of the Fourier series. For periodic distributions, that is, elements f of $\mathcal{D}'(\mathbb{R})$, it was shown in [47] that if f has Fourier series $\sum_{n=-\infty}^{\infty}c_n e^{inx}$ and $x_0\in\mathbb{R}$, then

$f(x_0)=\gamma$, distributionally, if and only if there exists k such that

$$\lim_{x\to\infty}\sum_{-x<n\leq ax} c_n e^{inx_0}=\gamma\ (\mathrm{C},k)\,, \tag{3.1.1}$$

for each $a>0$. It should be stressed that the characterization holds in terms of the slightly asymmetric means of (3.1.1), but it is not true for symmetric sums, i.e., if we just take $a=1$, leading to consider, as has been classically done, the cosines-sines series. The characterization also fails if we consider the means $\sum_{0\leq n\leq x} c_n e^{inx_0}$ and $\sum_{-x\leq n<0} c_n e^{inx_0}$, separately. It is remarkable that such a type of characterization has not been given for classical functions but for generalized functions.

It is also to be observed that the characterization holds for the *distributional point value.* The notion of the value of a function at a point is somewhat complicated. Indeed, while it is clear what $f\,(x_0)$ is if $f\in C\,(\mathbb{R})$, the same question becomes hard to answer if $f\in L^p\,(\mathbb{R})$ since the elements of this space are not functions but equivalence classes of functions equal almost everywhere. If f is a distribution, the problem seems hopeless since distributions are not defined pointwise, but are the elements of certain dual spaces, that is, global objects. It is therefore very interesting that there is a notion of point value for distributions, introduced by Łojasiewicz in [128], that not only reduces to the usual one for distributions locally equal to continuous functions, but that has many interesting and useful properties. The concept of distributional point value has shown to be of importance in several areas, such as abelian and tauberian results for integral transforms [139, 149, 231, 243], the study of local properties of distributions [72, 74, 75, 78, 79, 215, 217, 223], spectral expansions [61, 216, 223, 236, 237], the boundary behavior of solutions of partial differential equations [54, 238], the summability of cardinal series [239, 240], or pointwise convergence of wavelet expansions [241, 242].

In the case of Fourier integrals of classical functions the situation is similar to that of Fourier series, summability methods must be employed as well. One has also a Cesàro summability version for the Fourier inversion integral formula in a theorem due to Plancherel [166, 206]. Other methods of summability are also studied in classical books [17, 19]. Actually the approach given in [17, 19] is very close to distributional point values. Indeed, what they do is to consider pointwise inversion formulas of the type

$$\lim_{x\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(t)e^{ix_0t}\phi\left(\frac{t}{x}\right)\mathrm{d}t=\phi(0)f(x_0)\ , \tag{3.1.2}$$

which is what one usually does in distribution theory when dealing with distributional point values.

The scope of this chapter is to investigate extensions of (3.1.1) to general tempered distributions and their Fourier transforms. We will take a comprehensive approach, it includes at the same time Fourier series and integral, and more generally, the Fourier transform of arbitrary tempered distributions. Therefore we first show that the distributional point values of a tempered distribution are characterized by their Fourier transforms in a way similar to those of periodic distributions are characterized by their Fourier series as in (3.1.1), that is, we show that they are determined by a generalized Fourier inversion formula. In particular, it will follow from our analysis that if $f\in\mathcal{S}'(\mathbb{R})$ and $x_0\in\mathbb{R}$, and $\hat{f}$ is locally integrable, then $f(x_0)=\gamma$ distributionally if and only if there exists k such that

$$\frac{1}{2\pi}\lim_{x\to\infty}\int_{-x}^{ax}\hat{f}(t)e^{ix_0t}\mathrm{d}t=\gamma\quad(\mathrm{C},k)\,, \tag{3.1.3}$$

for each $a>0$.

It is worth to mention that these ideas are related to the classical problem of (C) summability of Fourier series (see [256, Chap.XI] and references therein). The first to formulate the problem were Hardy and Littlewood [89, 90]. It basically aims to characterize trigonometric series such that their sines and cosines series,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx_0 + b_n \sin nx_0) \ ,$$

are Cesàro summable at a given point x_0 and whose coefficients are of slow growth (hence they are tempered distributions!). If we do not care about the order of (C) summability, then distributional point values provide an easy and quick solution to this problem [61, Thm.6.14.5]. A classical approach to this problem is presented in [256, Chap.XI], where the problem of (C) summability of the symmetric partial sums is investigated with generalized symmetric derivatives in the sense of *de la Vallée Poussin* (notion which can be interpreted as distributional symmetric point values as shown in Section 3.10).

We will also study conditions which allow us to conclude that the asymmetric means in (3.1.1) *converge* to γ. In case of series of the power series type such results are the so-called tauberian theorems. We show that in case the sequence $\{c_n\}_{n=-\infty}^{\infty}$ belongs to the space l^p for some $p \in [1, \infty)$ and the tails satisfy the estimate $\sum_{|n| \geq N}^{\infty} |c_n|^p = O\left(N^{1-p}\right)$, as $N \to \infty$, then the asymmetric partial sums converge to $f(x_0)$ at any point x_0 where the distributional point value exists. We also give several other conditions that guarantee the convergence in (3.1.1). We then proceed to obtain results on the convergence of the asymmetric partial integral when $\hat{f}$ belongs to $L^p(\mathbb{R})$ and in other cases.

The author would like to mention that the main results of the chapter are already published by the author and R. Estrada in [216]; however, the exposition presented here is more complete and contains some complementary results which naturally arise from the context of our investigations of distributional point values and summability of the Fourier transform.

The plan of the chapter is as follows. In Section 3.2 we review the Łojasiewicz notion of distributional point values and some of its properties. Section 3.3 is of preliminary character, we discuss several summability procedures for divergent series and integrals; we then discuss how to extend the summability method to distributional evaluations. The main results of the chapter are found in Sections 3.4 and 3.5, where we prove the characterization of the point values of tempered distributions in terms of asymmetric evaluations of their Fourier transforms. The crucial argument to obtain such a result is the structural characterization of the quasiasymptotic behavior $g(\lambda x) \sim \gamma\delta(\lambda x)$. We also show that the corresponding results for the symmetric evaluations or for the separate evaluations over the positive and negative parts of the spectrum do not hold. The results for the convergence of asymmetric partial sums of Fourier series are given in Section 3.6. Next we show in Section 3.7 that our results have direct applications to the convergence of asymmetric partial sums of lacunary Fourier series; in particular we show how we can construct continuous functions whose derivatives do not have distributional point values at any point. In Section 3.8 we extend the results of Section 3.6 to the convergence of asymmetric partial integrals in the Fourier inversion formula. Abel summability of the Fourier inversion formula is investigated in Section 3.9. Finally, we formulate and solve the Hardy-Littlewood (C)-summability problem for tempered distributions in Section 3.11; this is done in terms of distributional symmetric point values, which will be introduced in Section 3.10.

3.2 Distributional Point Values

The notion of the of the value of distribution at point was introduced by S. Łojasiewicz in [128]. He defined the value of a distribution $f \in \mathcal{D}'(\mathbb{R})$ at the point $x = x_0$ as the limit

$$\gamma = \lim_{\varepsilon \to 0} f(x_0 + \varepsilon x)\,, \tag{3.2.1}$$

if the limit exists in the weak topology of $\mathcal{D}'(\mathbb{R})$, that is, if

$$\lim_{\varepsilon \to 0} \langle f(x_0 + \varepsilon x), \phi(x) \rangle = \lim_{\varepsilon \to 0} \left\langle f(x), \frac{1}{\varepsilon}\phi\left(\frac{x - x_0}{\varepsilon}\right)\right\rangle = \gamma \int_{-\infty}^{\infty} \phi(x)\,\mathrm{d}x\,, \tag{3.2.2}$$

for each $\phi \in \mathcal{D}(\mathbb{R})$. In such a case, one declares $f(x_0) := \gamma$.

Observe that distributional point values in the sense of Łojasiewicz forms part of a more general notion of behavior of a distribution at a point, the notion of *quasiasymptotics*, as defined in Section 1.8.1. In particular, this is a local concept. So, in the notation of quasiasymptotics, the limit (3.2.1) may be written as

$$f(x_0 + \varepsilon x) = \gamma + o(1) \quad \text{as } \varepsilon \to 0 \text{ in } \mathcal{D}'(\mathbb{R}) \ . \tag{3.2.3}$$

We will refer to Łojasiewicz point values as *distributional point values*, and will use the following *notation* for the existence of the distributional point value at $x = x_0$ with value γ,

$$f(x_0) = \gamma, \quad \text{distributionally.} \tag{3.2.4}$$

Łojasiewicz gave himself a *structural characterization* of distributional point values. It was shown by him [128] that the existence of the distributional point value $f(x_0) = \gamma$, distributionally, is equivalent to the existence of $n \in \mathbb{N}$, and a primitive of order n of f, that is, $F^{(n)} = f$, which is continuous in a neighborhood of x_0 and satisfies

$$\lim_{x \to x_0} \frac{n! F(x)}{(x - x_0)^n} = \gamma \ . \tag{3.2.5}$$

Therefore, the existence of a distributional point value is actually an average notion. Such a structural characterization allows us to relate distributional point values with the the classical concept of Peano differentials ([34],[256, Chap.XI]. Indeed, if F_1 is another n-primitive of f, different form F, then there exist n constants $a_0, a_1, \ldots, a_{n-1}$ such that

$$F_1(x) = a_0 + a_1(x - x_0) + \cdots + a_{n-1}(x - x_0)^{n-1} + \frac{\gamma}{n!}(x - x_0)^n + o(|x - x_0|^n), \quad x \to x_0 \ .$$

Hence, any n-primitive of F_1 admits a Peano n-differential, and its Peano n-derivative is actually equal to γ.

A priori, relation (3.2.2) is only assumed to hold for $\phi \in \mathcal{D}(\mathbb{R})$. Suppose now that $f \in \mathcal{S}'(\mathbb{R})$ and $f(x_0) = \gamma$, distributionally; initially, (3.2.2) does not have to be true for $\phi \in \mathcal{S}(\mathbb{R})$. However, it is shown in [54, Corollary 1] and [153] that if (3.2.2) holds for $\phi \in \mathcal{D}(\mathbb{R})$, it will remain true for $\phi \in \mathcal{S}(\mathbb{R})$. Actually, this fact holds for any quasiasymptotic behavior, as will be seen in Chapter 10.

Let us provide some examples.

Example 3.1. *(Lebesgue points) Recall the classical definition of Lebesgue points. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. We say that f has a Lebesgue point at $x = x_0$ if*

$$\lim_{h\to 0} \frac{1}{h} \int_{x_0}^{x_0+h} |f(x) - \gamma_{x_0}| \,\mathrm{d}x = 0 \ ,$$

for some constant γ_{x_0}. Then, one can show [256] that $f(x_0) = \gamma_{x_0}$ a.e.; we refer to the set of points where Lebesgue points exist as the Lebesgue set of f. Observe that at a Lebesgue point, we have that $f(x_0) = \gamma_{x_0}$, distributionally. Hence, distributional point values include the Lebesgue points, which is actually the notion of point value used by analysts for L^p-functions.

Example 3.2. *The functions $x^\alpha e^{i/x}$, where $\alpha \in \mathbb{R}$, have regularizations $f_\alpha \in \mathcal{D}'(\mathbb{R})$ that have distributional point values at $x = 0$, and, in fact, $f_\alpha(0) = 0$, distributionally. This fact was established by Lojasiewicz in [128]. Observe that if $\alpha < 0$ then f_α is unbounded near $x = 0$ in the ordinary sense.*

Example 3.3. *In general the behavior of distributional point values with respect to non-linear operations could be very complicated. If $f(x) = \sin x^{-1}$, then $f(0) = 0$ distributionally, but $f^2(x) = (1 - \cos 2x^{-1})/2$, and thus $f^2(0) = 1/2$ distributionally. If $g(x)$ is the usual regularization of $x^{-1} \sin x^{-1}$ then $g(0) = 0$ distributionally, but $g^2(0)$ does not exist. It is not known if such behavior occurs at a small set of points only. It would be very interesting to study the relationship of distributional point values and the non-linear theories of generalized functions.*

3.3 Cesàro and Abel Summability

It is the intension of this section to introduce two methods of summability for distributional evaluations. The are defined via the Cesàro behavior (Section 1.8.2). We are only interested in the one-dimensional case; for the multidimensional case we refer to Chapter 13 (see also [221]).

We start by presenting a very brief introduction to summability of divergent series and integrals. It will serve as a motivation to the study of more general notions applicable to Schwartz distribution. There is a very rich and extensive literature on this traditional subject; for instance, the reader is referred to [28, 85, 91]. See also [93, 206, 256] for connections with Fourier series and integrals.

We will then discuss the Abel and Cesàro methods for distributional evaluations. For the second part, we follow closely [61].

3.3.1 Cesàro, Riesz, and Abel Summability of Series and Integrals

We shall discuss the summability methods by Abel, Cesàro and Riesz means for series and integrals.

Let us start with *Cesàro* summability. In general we say that a numerical series $\sum_{n=0}^{\infty} c_n$, possibly divergent, is summable to a complex number γ in the average, or Cesàro sense of order 1, if the averages of its partial sums converge to γ, that is,

$$\lim_{n\to\infty} \frac{s_0 + s_1 + s_2 + \cdots + s_n}{n+1} = \gamma \ , \tag{3.3.1}$$

where $s_n = \sum_{j=0}^{n} c_j$, in such a case one writes

$$\sum_{n=0}^{\infty} c_n = \gamma \quad (\mathrm{C}, 1) \ . \tag{3.3.2}$$

It is elementary to check that if the series is convergent, then it is summable by the $(\mathrm{C}, 1)$ method, but the converse is naturally false. For example, one may take

$\sum_{n=0}^{\infty}(-1)^n$, which is evidently divergent; but its average converges to 1/2, hence $\sum_{n=0}^{\infty}(-1)^n = 1/2 \quad (\mathrm{C},1)$.

The Cesàro method of summability is important in the analysis of several series expansions of functions and generalized functions; in particular for Fourier series. In fact, it is a famous result of Fejér that the Fourier series of a continuous function, although not necessarily convergent, is $(\mathrm{C},1)$ summable to the value of the function at any point [62, 93, 256]. Furthermore, Kolmogorov proved [256, Chap.VIII] that there are functions in the class $L^1[0,2\pi]$ whose Fourier series diverge everywhere; therefore, even in the case of classical functions, it is imperative the use of summability methods for the pointwise analysis of trigonometric series. In Section 3.5, we will generalize Fejér's classical result to include periodic distributions, for that we will use higher order Cesàro means.

We can extend the $(\mathrm{C},1)$ to higher order average means. There are several approaches, and all of them are equivalent. Perhaps the simplest, but analytically inadequate, is that of Hölder means. We can define recursively the sequences, $s_n^k := (\sum_{j=0}^{n} s_j^{k-1})/(n+1)$, with $s_n^0 := s_n = \sum_{j=0}^{n} c_j$. Then, we call s_n^k the *Hölder means* of order k of the series, and say that $\sum_{n=0}^{\infty} c_n = \gamma \quad (\mathrm{H},k)$, if $s_n^k \to \gamma$ as $n \to \infty$. As we remarked before, Hölder means present serious difficulties associated with their analytical manipulation [85], we shall therefore avoid their use in the future.

Another approach to the extension of (3.3.1) is via higher order *Cesàro means.* Given a series $\sum_{n=0}^{\infty} c_n$ we define its Cesàro means of order β, $\beta > -1$, by

$$C_n^{\beta} = \frac{\Gamma(\beta+1)}{n^{\beta}} \sum_{j=0}^{n} \binom{\beta+j}{\beta} c_{n-j} \ , \tag{3.3.3}$$

then we say that the series is Cesàro summable of order β to γ, and write $\sum_{n=0}^{\infty} c_n = \gamma \quad (\mathrm{C},\beta)$, if $C_n^{\beta} \to \gamma$ as $n \to \infty$. An interesting example is $\sum_{n=0}^{\infty}(-1)^n n^{\alpha}$, $\alpha > -1$, which is (C,β) summable whenever $\beta > \alpha$, oscillates finitely when $\beta = \alpha$, and oscillates infinitely for $\beta < \alpha$; we refer to [85] for a proof of this fact.

We shall also discuss the method of Marcel *Riesz* by typical means [28, 85, 91, 172]. Actually, the Riesz method will be the most important for us in the subsequent sections. Let $\{\lambda_n\}_{n=0}^{\infty}$ be an increasing sequence of non-negative numbers such that $\lambda_n \to \infty$ as $n \to \infty$. We say that a series is summable by the Riesz means, with respect to $\{\lambda_n\}$, of order $\beta \geq 0$ if

$$\lim_{x\to\infty} \sum_{0\leq\lambda_n<x} c_n \left(1-\frac{\lambda_n}{x}\right)^{\beta} = \gamma \; ; \tag{3.3.4}$$

and then we write

$$\sum_{n=0}^{\infty} c_n = \gamma \quad (\mathrm{R}, \{\lambda_n\}, \beta) \; . \tag{3.3.5}$$

These three methods of summability can be compared. If $\beta = m \in \mathbb{N}$, then the (H, m) and the (C, m) methods are equivalent [85]. While if $\beta \geq 0$ and $\lambda_n = n$, the (C,β) and the $(\mathrm{R}, \{n\}, \beta)$ methods sum the same series to the same value, and so they are also equivalent [100, 85, 94, 172]. Here the use of a continuous variable in (3.3.5) is absolutely necessary for the equivalence [85]. The Riesz method has an advantage over the other two methods, it is easily generalizable to integrals, even to distributions as we shall see later in Section 3.3.2. Therefore, we advise the reader that whenever we talk about Cesàro summability, even if we write (C,β), the means should be thought as Riesz means.

Let now f be a locally integrable function supported in $[0,\infty)$. Let $\beta > 0$. We write

$$\lim_{x\to\infty} f(x) = \gamma \quad (\mathrm{C},\beta) \; , \tag{3.3.6}$$

if

$$\lim_{x\to\infty} \beta \int_0^x f(t)\left(1-\frac{t}{x}\right)^{\beta-1} \mathrm{d}t = \gamma \; . \tag{3.3.7}$$

Note that (3.3.7) basically says that $f^{-\beta}(x)$, the β-primitive of f, is asymptotic to $\gamma x^{\beta}/\Gamma(\beta+1)$ as $x \to \infty$. So (3.3.6) coincides with Definition 1.12, which is applicable to distributions. Suppose that f is a function of local bounded variation, then its distributional derivative is a Radon measure, a continuous linear functional over the space of continuous functions with compact support, say $f' = \mu$. Hence integration by parts in (3.3.7) shows that it is equivalent to

$$\lim_{x\to\infty} \int_0^x \left(1 - \frac{t}{x}\right)^{\beta} \mathrm{d}\mu(t) = \gamma \ . \tag{3.3.8}$$

The latter can be taken as the definition of the relation

$$\int_0^{\infty} \mathrm{d}\mu(x) = \gamma \quad (\mathrm{C}, \beta) \ . \tag{3.3.9}$$

Observe that (3.3.5) holds if and only (3.3.9) holds for the Radon measure $\mu = \sum_{n=0}^{\infty} c_n \delta(\,\cdot\, - \lambda_n)$.

We end this discussion by considering *Abel* summability of series [85]. For a series $\sum_{n=0}^{\infty} c_n$, we consider its Abel means, that is, the power series $\sum_{n=0}^{\infty} c_n r^n$. We say that the series is Abel summable to γ, if $\sum_{n=0}^{\infty} c_n r^n$ is convergent for $|r| < 1$ and the power series approaches to the limit γ at the boundary point $r = 1$, i.e.,

$$\lim_{r\to 1_-} \sum_{n=0}^{\infty} c_n r^n = \gamma \ , \tag{3.3.10}$$

we write

$$\sum_{n=0}^{\infty} c_n = \gamma \quad (\mathrm{A}) \ . \tag{3.3.11}$$

It will be more convenient for us to write $r = e^{-y}$, so that the power series becomes a Dirichlet series. So, we have a natural extension for measures supported in $[0, \infty)$ in terms of the Laplace transform. We say that $\int_0^{\infty} \mathrm{d}\mu(x)$ is Abel summable to γ and write

$$\int_0^{\infty} \mathrm{d}\mu(x) = \gamma \quad (\mathrm{A}) \ , \tag{3.3.12}$$

if for any $y > 0$ the integral $\int_0^\infty e^{-yt} \mathrm{d}\mu(t)$ exists as an improper integral, and

$$\lim_{y \to 0^+} \int_0^\infty e^{-yt} \mathrm{d}\mu(t) = \gamma. \tag{3.3.13}$$

When the Radon measure is given by $\sum_{n=0}^\infty c_n \delta(x - \lambda_n)$, we write

$$\sum_{n=0}^\infty c_n = \gamma \quad (\mathrm{A}, \{\lambda_n\}) \ , \tag{3.3.14}$$

if (3.3.13) holds, that is, if the Dirichlet series $\sum_{n=0}^\infty c_n e^{-y\lambda_n}$ is convergent for $y > 0$ and it tends to γ as $y \to 0^+$.

We finally comment some inclusion between the Cesàro and Abel method of summation, if (3.3.9) holds then (3.3.12) is satisfied, this fact is actually recovered below (Corollary 3.10). In the case of power series this fact is the well known *Abel's theorem* [85]. Naturally, the converse is not true. The reader may wish to verify that the series whose coefficients are given by those of the power series $e^{\frac{1}{1-r}} = \sum_{n=0}^\infty c_n r^n$ is an explicit example of a series which is (A) summable but not (C, β) summable [85], no matter what value of β be taken. Furthermore, in [52], it is constructed a series which is Abel summable with coefficients $c_n = O(n^m)$, but it is not (C, β) summable for any β. The study of additional hypotheses to ensure the converse of Abel's theorem motivated the beginning of the tauberian theory. For instance, Littlewood tauberian condition $c_n = O(1/n)$ together with Abel summability imply the convergence of the series [127, 85]. We will obtain a simple and quick proof of Littlewood's theorem in Section 4.4 of Chapter 4, as a direct consequence of our distributional methods. In Section 3.6, we discuss some tauberian conditions for Cesàro summability.

3.3.2 Summability of Distributional Evaluations

We now study two methods of summability for distributional evaluations, the two-sided Cesàro method, and Abel summability. Two more methods will be introduced in Sections 3.5 and 3.11 (Definitions 3.18 and 3.59).

We start with summability in the Cesàro sense. First we assume that our distributions have support bounded at the left. Recall that H denotes the Heaviside function (Section 1.3), i.e., the characteristic function of $(0,\infty)$.

Definition 3.4. *Let $f \in \mathcal{D}'(\mathbb{R})$ have support bounded at the left. Let $\phi \in \mathcal{E}(\mathbb{R})$ and $m \in \mathbb{N}$. We say the evaluation $\langle f(x), \phi(x)\rangle$ has a value γ in the Cesàro sense of order m, and write*

$$\langle f(x), \phi(x)\rangle = \gamma \quad (\mathrm{C}, m) \tag{3.3.15}$$

if $F = (\phi f)^{(-1)} = (\phi f) * H$, the first order primitive of ϕf with support bounded at the left, satisfies $\lim_{x\to\infty} F(x) = \gamma \quad (\mathrm{C}, m)$.

Example 3.5. *Let μ be a Radon measure with support on $[0,\infty)$. Then $\int_0^\infty \mathrm{d}\mu(x) = \gamma \ \ (\mathrm{C}, m)$ if and only if $\langle \mu(x), 1\rangle = \gamma \ \ (\mathrm{C}, m)$. In particular*

$$\sum_{n=0}^{\infty} c_n = \gamma \quad (\mathrm{R}, \{\lambda_n\}, m)$$

if and only if

$$\left\langle \sum_{n=0}^{\infty} c_n \delta(x-\lambda_n), 1 \right\rangle = \gamma \quad (\mathrm{C}, m)\ .$$

If f has support bounded at the right then we say that $\langle f(x), \phi(x)\rangle \ \ (\mathrm{C})$ exists if and only if $\langle f(-x), \phi(-x)\rangle = \gamma \ \ (\mathrm{C})$ exists and we define $\langle f(x), \phi(x)\rangle = \gamma \ \ (\mathrm{C})$.

The distributional evaluations with respect to compactly supported distributions can always be computed in the (C) sense, actually with order $m = 0$.

Lemma 3.6. *Let $f \in \mathcal{E}'(\mathbb{R})$ and $\phi \in \mathcal{E}(\mathbb{R})$. Then $\langle f(x), \phi(x)\rangle \ \ (\mathrm{C}, 0)$ always exists.*

Proof. We can assume that $\phi \equiv 1$. Consider $f^{(-1)}$, it is obviously constant for large arguments, we must show that satisfies $f^{(-1)} = \langle f(x), 1\rangle$ (a constant distribution) on certain interval (a,∞). Decompose $f^{(-1)}(x) = g(x) + cH(x-a)$, where g has compact support and c and a are constants. Then $\langle f(x), 1\rangle = \langle g'(x), 1\rangle + \langle cH'(x-a), 1\rangle = 0 + c\,\langle \delta(x-a), 1\rangle = c$, from where the result follows. □

We now define two-sided Cesàro evaluations

Definition 3.7. *Let $f \in \mathcal{D}'(\mathbb{R})$, $\phi \in \mathcal{E}(\mathbb{R})$, and $m \in \mathbb{N}$. We say the evaluation $\langle f(x), \phi(x)\rangle$ exists in the Cesàro sense of order m if there is a decomposition $f = f_- + f_+$, $\operatorname{supp} f_- \subseteq (-\infty, 0]$ and $\operatorname{supp} f_+ \subseteq [0, \infty)$, such that both evaluations $\langle f_\pm(x), \phi(x)\rangle = \gamma_\pm \quad (\mathrm{C}, m)$ exist. In this case we write*

$$\langle f(x), \phi(x)\rangle = \gamma \quad (\mathrm{C}, m) \ , \tag{3.3.16}$$

where $\gamma = \gamma_- + \gamma_+$.

We must check the consistence of Definition 3.7. Let $f = f_1 + f_2 = g_1 + g_2$ be two decompositions such that f_2 and g_2 have supports bounded at the left, respectively, f_1 and g_1 have supports bounded at the right. Then $h = g_1 - f_1 = f_2 - g_2$ has compact support. If both $\langle f_j(x), \phi(x)\rangle = \gamma_j \quad (\mathrm{C}, m)$ exist, then, by Lemma 3.6, both $\langle g_j(x), \phi(x)\rangle = \beta_j \quad (\mathrm{C}, m)$ exist, and we have the two equalities $\beta_1 = \gamma_1 + \beta$ and $\beta_2 = \gamma_2 - \beta$, where $\beta = \langle f_j(x), \phi(x)\rangle$. Hence the number $\gamma = \gamma_1 + \gamma_2 = \beta_1 + \beta_2$ is independent on the choice of the decomposition.

Let us now define Abel summability for distributional evaluations.

Definition 3.8. *Let $f \in \mathcal{D}'(\mathbb{R})$ and $\phi \in \mathcal{E}(\mathbb{R})$. We say the evaluation $\langle f(x), \phi(x)\rangle$ exists in the Abel sense if there is a decomposition $f = f_- + f_+$, $\operatorname{supp} f_- \subseteq (-\infty, 0]$ and $\operatorname{supp} f_+ \subseteq [0, \infty)$, such that both $e^{\mp yx}\phi(x) f_\pm \in \mathcal{S}'(\mathbb{R})$, for each $y > 0$, and*

$$\lim_{y \to 0^+} \left(\langle \phi(x) f_-(x), e^{yx}\rangle + \left\langle \phi(x) f_+(x), e^{-yx}\right\rangle\right) = \gamma \ , \tag{3.3.17}$$

in this case we write $\langle f(x), \phi(x)\rangle = \gamma \quad (\mathrm{A})$.

The notion of distributional evaluations in the Cesàro sense admits a characterization in terms of the quasiasymptotic behavior.

Proposition 3.9. *Let $f \in \mathcal{D}'(\mathbb{R})$ and $\phi \in \mathcal{E}(\mathbb{R})$. Then $\langle f(x), \phi(x)\rangle = \gamma \quad (\mathrm{C})$ if and only if there exist a decomposition $f = f_- + f_+$, where $\operatorname{supp} f_- \subseteq (-\infty, 0]$*

and $\operatorname{supp} f_+ \subseteq [0,\infty)$*, and a constant* β *such that the following quasiasymptotic behaviors hold*

$$\phi(\lambda x)f_+(\lambda x) = \left(\frac{\gamma}{2}+\beta\right)\frac{\delta(x)}{\lambda} + o\left(\frac{1}{\lambda}\right) \quad \textit{as } \lambda\to\infty \textit{ in } \mathcal{S}'(\mathbb{R}) \tag{3.3.18}$$

and

$$\phi(\lambda x)f_-(\lambda x) = \left(\frac{\gamma}{2}-\beta\right)\frac{\delta(x)}{\lambda} + o\left(\frac{1}{\lambda}\right) \quad \textit{as } \lambda\to\infty \textit{ in } \mathcal{S}'(\mathbb{R})\ . \tag{3.3.19}$$

In particular, we obtain that $\phi f \in \mathcal{S}'(\mathbb{R})$ *and it has the quasiasymptotic behavior,*

$$\phi(\lambda x)f(\lambda x) = \gamma\frac{\delta(x)}{\lambda} + o\left(\frac{1}{\lambda}\right) \quad \textit{as } \lambda\to\infty \textit{ in } \mathcal{S}'(\mathbb{R})\ . \tag{3.3.20}$$

Proof. We may assume that $\phi\equiv 1$. Put $f_-^{(-1)}$ equal to the primitive of $f_-(-x)$ with support on $[0,\infty)$. Because of the assumptions on the supports, note that (3.3.18) and (3.3.19) are equivalent to $\lim_{\lambda\to\infty} f_\pm^{(-1)}(\lambda x) = ((\gamma/2)\pm\beta)H(x)$ in $\mathcal{S}'(\mathbb{R})$. By Proposition 1.13, the latter are equivalent to $\lim_{\lambda\to\infty} f_\pm^{(-1)}(x) = (\gamma/2)\pm\beta\ \ (\mathrm{C})$, which are equivalent to $\langle f_\pm(x),1\rangle = (\gamma/2)\pm\beta\ \ (\mathrm{C})$. And so we obtain the equivalence with $\langle f(x),\phi(x)\rangle = \gamma\ \ (\mathrm{C})$. □

We can use Proposition 3.9 to obtain Abel's theorem in the context of distributional evaluations. The converse is false [52].

Corollary 3.10. *Let* $f\in\mathcal{D}'(\mathbb{R})$ *and* $\phi\in\mathcal{E}(\mathbb{R})$*. Suppose that* $\langle f(x),\phi(x)\rangle=\gamma\ \ (\mathrm{C})$*, then* $\langle f(x),\phi(x)\rangle=\gamma\ \ (\mathrm{A})$*.*

Proof. Using Proposition 3.9, we obtain that, as $\lambda\to\infty$,

$$\begin{aligned}\left\langle \phi(x)f_-(x), e^{\frac{x}{\lambda}}\right\rangle + \left\langle \phi(x)f_+(x), e^{-\frac{x}{\lambda}}\right\rangle &= \lambda\left\langle \phi\left(\lambda x\right) f_-\left(\lambda x\right), e^{x}\right\rangle \\ &\quad + \lambda\left\langle \phi\left(\lambda x\right) f_+\left(\lambda x\right), e^{-x}\right\rangle \\ &= \left(\frac{\gamma}{2}-\beta\right)\left\langle\delta(x), e^{x}\right\rangle \\ &\quad + \left(\frac{\gamma}{2}+\beta\right)\left\langle\delta(x), e^{-x}\right\rangle + o(1)\ .\end{aligned}$$

□

3.4 Distributional Point Values and Asymptotic Behavior of the Fourier Transform

It is our intension to characterize distributional point values by summability of the Fourier transform, to this end, we shall study in the present section the close connection between the value of a distribution at a point and the quasiasymptotic properties of the Fourier transform. The desired characterization will be given in the next section, Section 3.5, by means of a pointwise Fourier inversion formula.

Let $f \in \mathcal{S}'(\mathbb{R})$ have distributional point γ at x_0. Then, we have the following quasiasymptotic behavior.

$$f(x_0 + \varepsilon x) = \gamma + o(1) \quad \text{as } \varepsilon \to 0^+ \ \mathcal{D}'(\mathbb{R}) \ . \tag{3.4.1}$$

As pointed out in Section 3.2, this quasiasymptotic behavior actually holds in $\mathcal{S}'(\mathbb{R})$. Therefore, we can take Fourier transform in (3.4.1) and obtain the equivalent quasiasymptotic expression

$$e^{i\lambda x_0 x}\hat{f}(\lambda x) = 2\pi\gamma\frac{\delta(x)}{\lambda} + o\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R}) \ . \tag{3.4.2}$$

Let us state this simple, but useful, observation as a lemma

Lemma 3.11. *Let $f \in \mathcal{S}'(\mathbb{R})$. Then, $f(x_0) = \gamma$, distributionally, if and only if the Fourier transform satisfies the quasiasymptotic behavior (3.4.2).*

Therefore, on the Fourier side, distributional point values look like (3.4.2). Since our ultimate goal is to characterize distributional point values by certain type of summability of the Fourier transform, it is clear that our summability method should provide a characterization of the quasiasymptotic behavior

$$g(\lambda x) = \gamma\frac{\delta(x)}{\lambda} + o\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R}) \ . \tag{3.4.3}$$

We now study the structure of (3.4.3). Before to go on, the author would like to make some comments. In [231], many structural theorems are provided for quasiasymptotics. Actually, we already faced one of such results in Proposition 1.8, applicable to one-sided quasiasymptotics. However, the results of ([231]) do not cover the case which we are interested in. Moreover, the general structural description of quasiasymptotics remained as an open question for long time. In this section, we basically solve this question for (3.4.3), the solution was obtained by the author and R. Estrada in [216]. Moreover, the method to be given, was actually extended by the author in [212, 213, 227] in order to give a complete answer to the structural problem for quasiasymptotic properties of distributions; we will discuss this in detail in Chapter 10.

3.4.1 Asymptotically Homogeneous Functions

The concept of asymptotically homogeneous functions of degree zero will be needed for the next theorems.

We say that a measurable function c, defined in an interval of the form $[A, \infty) \subset (0, \infty)$, is asymptotically homogeneous of degree 0 if for each $a > 0$, we have

$$c(ax) = c(x) + o(1) \quad \text{as } x \to \infty \ . \tag{3.4.4}$$

No uniformity on a is assumed. Such functions were used in [47] to characterize distributional point values of Fourier series, and by the author and collaborators to study the structure of quasiasymptotics [212, 213, 216, 227]. These functions are also known as de Haan functions [15]. This class has been very well studied for several authors; however, the author was not aware of this fact and learned recently about the existence of such results. In the meantime, he rediscovered by himself some already known results. Some of which are presented in this section. We will extend this class of functions in Chapter 10.

Suppose c satisfies (3.4.4) for each $a > 0$, we may assume that c is real valued, otherwise we consider its real and imaginary parts separately. Then $L(x) = e^{c(x)}$

is positive and measurable on $[A, \infty)$ and for each $a > 0$

$$\lim_{x \to \infty} \frac{L(ax)}{L(x)} = 1 \ . \tag{3.4.5}$$

Therefore, L is a slowly varying function. It is very well known that (3.4.5) must hold uniformly for a in compact subsets of $(0, \infty)$ [183], so should (3.4.4). Actually, if one only assumes that (3.4.4) holds in a set of positive measure, then it holds for every $a > 0$; we will use this property implicitly sometimes in the future. From the very well known representation formula for slowly varying functions (Section 1.7), we obtain two estimates for the growth of c, first,

$$c(x) = o\left(\log x\right) \ , \quad \text{as } x \to \infty \ ; \tag{3.4.6}$$

secondly, there are two constants A_0 and A_1 such that

$$|c(ax) - c(x)| \leq A_0 \left|\log a\right| + A_1, \tag{3.4.7}$$

for $x \geq B$ and $ax \geq B$. The last inequality implies the following lemma.

Lemma 3.12. *Let c be an asymptotically homogeneous function of degree 0 defined on $(0, \infty)$. Let g be a function such that $g(t)(1 + |\log t|)$ is in $L^1(0, \infty)$. Suppose that at least one of the following two condition is satisfied:*

i) c is bounded in each finite subinterval of $(0, \infty)$

ii) $c \in L^1_{\text{loc}}([0, \infty))$ and g is bounded near the origin

then we have that

$$\int_0^\infty c(xt) g(t) \mathrm{d}t = c(x) \int_0^\infty g(t) \mathrm{d}t + o(1) \ , \quad \text{as } x \to \infty \ .$$

Proof. Choose B as in (3.4.7), we keep $x > B$. Consider

$$\int_0^\infty \left(c(xt) - c(x)\right) g(t) \mathrm{d}t = J_1(x) + J_2(x) - J_3(x) \ ,$$

where $J_1(x) = \int_{B/x}^{\infty}(c(xt)-c(x))\,g(t)\mathrm{d}t$, $J_2(x) = \int_0^{B/x} c(xt)g(t)\mathrm{d}t$, and $J_3(x) = c(x)\int_0^{B/x} g(t)\mathrm{d}t$. Because of (3.4.7) and the assumption over g, we can apply Lebesgue Dominated Convergence Theorem to conclude that $J_1(x) = o(1)$ as $x \to \infty$. That $J_2(x) = o(1)$ as $x \to \infty$ follows easily from the assumptions. Finally, by using (3.4.6), we obtain that

$$|J_3(x)| \leq \frac{|c(x)|}{\log x + 1 - \ln B}\int_0^{\infty}(1+|\log t|)\,|g(t)|\,\mathrm{d}t = o(1)\ , \quad x\to\infty\ .$$

$\square$

In particular, we obtain

Corollary 3.13. *Let $c \in L^1_{\mathrm{loc}}(\mathbb{R})$. Suppose that c is asymptotically homogeneous of degree 0. Then*

$$c(\lambda x)H(x) = c(\lambda)H(x) + o(1) \quad \text{as } \lambda\to\infty \text{ in } \mathcal{S}'(\mathbb{R})\ .$$

Let us show that, when we are only interested the behavior for large arguments, then c can be assumed to be C^∞.

Lemma 3.14. *Let c be an asymptotically homogeneous functions of degree zero. Then there exists $c_1 \in C^\infty[0,\infty)$ such that $c(x) = c_1(x) + o(1)$, $x \to \infty$. In particular, c_1 is also asymptotically homogeneous functions of degree zero.*

Proof. Suppose that c is defined and locally bounded on $[A,\infty)$, redefine c as 0 on $[0,B)$. Take $\phi \in \mathcal{D}((0,\infty))$ with integral equal to 1. Set $c_1(x) = \int_{B/x}^{\infty} c(xt)\phi(t)\mathrm{d}t$, then, by Corollary 3.13, c_1 satisfies the requirements. $\square$

Another tool that we will use is the behavior at infinity of a continuous function defined in an interval of the form $[A,\infty)$, with $A > 0$, satisfying

$$\tau(ax) = a^\alpha\tau(x) + o(1)\ ,\ x\to\infty,$$

for each $a > 0$. They are called asymptotically homogeneous of degree α. One can show that if τ satisfies the last condition with $\alpha < 0$, then $\tau(x) = o(1)$, $x \to \infty$ (for the proof see [61, Lemma 6.15.1]). We will show a more general result in Chapter 10 (Proposition 10.16).

3.4.2 Structure of $g(\lambda x) \sim \gamma\delta(\lambda x)$

In the next lemma, we study the asymptotic properties of the primitives of distributions in $\mathcal{D}'(\mathbb{R})$ satisfying $f(\lambda x) = o\,(1/\lambda)$.

Lemma 3.15. *Let $f_0 \in \mathcal{D}'(\mathbb{R})$. For each $n \in \mathbb{N}$, pick an n-primitive of f, f_n in $\mathcal{D}'(\mathbb{R})$. Suppose*

$$f_0(\lambda x) = o\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{D}'(\mathbb{R}). \tag{3.4.8}$$

Then there exists an asymptotically homogeneous function of degree 0, c such that

$$f_n(\lambda x) = \frac{\lambda^{n-1} x^{n-1} c(\lambda)}{(n-1)!} + o\left(\lambda^{n-1}\right) \quad \text{as } \ \lambda \to \infty \tag{3.4.9}$$

in $\mathcal{D}'(\mathbb{R})$ for each $n \geq 1$. There exists n_0 such that the convergence in (3.4.9) is uniform on $[-1, 1]$ for $n \geq n_0$. Conversely, if (3.4.9) holds for some $n \geq 1$, then (3.4.8) holds in $\mathcal{D}'(\mathbb{R})$.

Proof. Suppose $f_0(\lambda x) = o\,(1/\lambda)$. Then there exists a smooth function $c(\lambda)$ such that $f_1(\lambda x) = c(\lambda) + o(1)$ as $\lambda \to \infty$ in $\mathcal{S}'(\mathbb{R})$. Replacing λx by $\lambda x a$ and grouping in two different ways, we obtain $c(a\lambda) = c(\lambda) + o(1)$, as $\lambda \to \infty$, for each $a > 0$. Thus c is asymptotically homogeneous of degree 0. Hence (3.4.9) holds for $n = 1$. Suppose now, it holds for some $n \geq 1$. Then integrating again we obtain $f_{n+1}(x) = \lambda^n x^n c(\lambda)/n! + \rho(\lambda) + o(\lambda^n)$, $\lambda \to \infty$, for some function ρ. Evaluating at λa, this yields $\rho(\lambda a) = \rho(\lambda) + o(\lambda^n)$ and thus the function $\tau(\lambda) = \lambda^{-n}\rho(\lambda)$ satisfies $\tau(a\lambda) = a^{-n}\tau(\lambda) + o(1)$ as $\lambda \to \infty$; it follows that $\tau(\lambda) = o(1)$ as $\lambda \to \infty$, thus $\rho(\lambda) = o(\lambda^n)$, and hence (3.4.9) is obtained for $n + 1$. That the convergence in (3.4.9) holds uniformly on $[-1, 1]$ sets if n is large enough follows from the definition of the convergence of distributions. The converse is obtained by differentiating (3.4.9) n-times with respect to x. □

We now aboard the general case. Let us state and proof the structural theorem for the quasiasymptotic behavior (3.4.3).

Theorem 3.16. *Let $g \in \mathcal{D}'(\mathbb{R})$, then,*

$$g(\lambda x) = \gamma \frac{\delta(x)}{\lambda} + o\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{D}'(\mathbb{R}) \ , \tag{3.4.10}$$

if and only if there exist $m \in \mathbb{N}$ and an $(m+1)$- primitive G_{m+1} of g, i.e., $G^{(m+1)} = g$, which is locally integrable for large positive and negative arguments, and an asymptotically homogeneous function c of degree zero, such that

$$G_{m+1}(x) = \frac{\gamma \operatorname{sgn} x}{2m!} x^m + c(|x|)\frac{x^m}{m!} + o(|x|^m), \quad |x| \to \infty \ , \tag{3.4.11}$$

in the ordinary sense. Furthermore, (3.4.11) is equivalent to the limits

$$\lim_{x \to \infty} (G(ax) - G(-x) = \gamma \quad (\mathrm{C}, m) \ . \tag{3.4.12}$$

for each $a > 0$. We also have that g is a tempered distribution and (3.4.10) holds in $\mathcal{S}'(\mathbb{R})$.

Proof. That (3.4.10) implies (3.4.11) follows from Lemma 3.15 applied to $f_0 = g - \gamma\delta$ by taking $m+1 = n_0$, $G_{m+1}(x) = (\gamma/(2m!)x^m \operatorname{sgn} x + f_{m+1}$, $x = \pm 1$ and replacing λ by x. The converse follows from Corollary 3.13, Lemma 3.14, and $(m+1)$-differentiations. The same results used for the equivalence show that g is tempered and that the quasiasymptotic holds in $\mathcal{S}'(\mathbb{R})$. Let us now show the equivalence between (3.4.11) and (3.4.12).

Set $F_m(a,x) = a^{-m}G_{m+1}(ax) - (-1)^m G_{m+1}(-x)$, observe it is an m-primitive of $G(ax) - G(x)$. Assume (3.4.11), then

$$\begin{aligned} F_m(a,x) &= a^{-m}G_{m+1}(ax) - (-1)^m G_{m+1}(-x) \\ &= \frac{\gamma \operatorname{sgn}(x) x^m}{2m!} + \frac{\gamma \operatorname{sgn}(x) x^m}{2m!} + o\left(x^m\right) \\ &= \frac{\gamma x^m}{m!} + o\left(x^m\right) \,,\ x \to \infty \,, \end{aligned}$$

uniformly for a in compact sets. From where we obtain (3.4.12).

Conversely, suppose that (3.4.12) holds. So for each a

$$\lim_{x \to \infty} m! \frac{F_m(a,x)}{x^m} = \gamma \,.$$

Define $c(x) = m! x^{-m} G_{m+1}(x) - \gamma$, for $x > 0$. A direct calculation shows that c is asymptotically homogeneous of degree zero and that (3.4.12) holds. □

3.5 Characterization of Distributional Point Values in $\mathcal{S}'(\mathbb{R})$

In this section, we characterize the distributional point values of arbitrary tempered distributions by proving the Fourier inversion formula in a generalized sense. This is a pointwise inversion formula for the Fourier transform which holds at any point where the tempered distribution has a point value in the distributional sense.

We want to find a suitable summability method for the Fourier transform which characterizes distributional point values. Because of Lemma 3.6, the problem reduces to characterize the quasiasymptotic $g(\lambda x) \sim \gamma \delta(\lambda x)$. A naive first attempt to this problem might lead us to consider directly Cesàro summability. However, Proposition 3.9 tell us that it is not going to work: Cesàro summability is too strong to give a characterization. Let us be more precise on this matter. Observe that if $\langle g(x), 1 \rangle = \gamma \quad (\mathrm{C})$, then Proposition 3.9 implies that $g(\lambda x) \sim \gamma \delta(\lambda x)$. However the converse is not true.

Example 3.17. *Consider the regular distributions $g(x) = (1/(x\log|x|))H(|x|-3)$. Note that for any $m \geq 0$, $x \to \infty$*

$$\int_3^x \frac{1}{t\log t}\left(1-\frac{t}{x}\right)^m \mathrm{d}t = -\frac{1}{3\log 3} + \frac{m}{x}\int_3^x \log(\log t)\left(1-\frac{t}{x}\right)^{m-1}\mathrm{d}t$$
$$\sim \log(\log x) \ .$$

Then, the evaluation $\langle g(x), 1\rangle$ does not exist in the Cesàro sense. However, $g(\lambda x) = o(\lambda^{-1})$ as $\lambda \to \infty$ in $\mathcal{S}'(\mathbb{R})$. In fact, if $\phi \in \mathcal{S}(\mathbb{R})$, then

$$\langle g(\lambda x), \phi(x)\rangle = \frac{1}{\lambda}\int_{\frac{3}{\lambda}}^{\infty} \frac{\phi(t)-\phi(-t)}{t\log(\lambda t)}\mathrm{d}t = o\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty \ .$$

Therefore, the Cesàro summability is not adequate for the characterization of distributional point values. If we now think carefully, Theorem 3.16 implicitly suggests the method of summability: it is implicit in (3.4.12). Hence, we have found the right summability method!

Definition 3.18. *Let $g \in \mathcal{D}'(\mathbb{R})$, $\phi \in \mathcal{E}(\mathbb{R})$ and $m \in \mathbb{N}$. We say that the special value of $\langle g(x), \phi(x)\rangle$ exists in the Cesàro sense of order m (*e.v. *Cesàro sense), and write*

$$\text{e.v.}\ \langle g(x), \phi(x)\rangle = \gamma \quad (\mathrm{C}, m) \ , \tag{3.5.1}$$

if for some primitive G of ϕg, i.e., $G' = \phi g$, and each $a > 0$ we have

$$\lim_{x\to\infty}(G(ax) - G(-x)) = \gamma \quad (\mathrm{C}, m) \ . \tag{3.5.2}$$

As a corollary of Theorem 3.16, we obtain.

Corollary 3.19. *Let $g \in \mathcal{D}'(\mathbb{R})$, $\phi \in \mathcal{E}(\mathbb{R})$. Then*

$$\text{e.v.}\ \langle g(x), \phi(x)\rangle = \gamma \quad (\mathrm{C}) \tag{3.5.3}$$

if and only if

$$\phi(\lambda x)g(\lambda x) = \gamma\frac{\delta(x)}{\lambda} + o\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R}) \ . \tag{3.5.4}$$

In addition, we have that $\phi g \in \mathcal{S}'(\mathbb{R})$.

As expected, the Cesàro method is strictly stronger than e.v Cesàro summability (see also Example 3.17).

Proposition 3.20. *Let* $g \in \mathcal{D}'(\mathbb{R})$, $\phi \in \mathcal{E}(\mathbb{R})$. *Any evaluation summable* (C, m) *is also summable in* e.v.(C, m) *sense, that is, the evaluation* $\langle g(x), \phi(x)\rangle = \gamma \quad (\mathrm{C}, m)$, *implies* e.v. $\langle g(x), \phi(x)\rangle = \gamma \quad (\mathrm{C}, m)$.

Proof. Let G be a first order primitive of ϕg. Decompose it as $G = G_- + G_+$, with $\operatorname{supp} G_- \subseteq (-\infty, 0]$ and $\operatorname{supp} G_- \subseteq [0, \infty)$. Then, by Proposition 3.9,

$$\lim_{x\to\infty} \pm G_\pm(\pm x) = \frac{\gamma}{2} \pm \beta \quad (\mathrm{C}, m) \ ,$$

for some β. Thus

$$\begin{aligned}
\lim_{x\to\infty} (G(ax) - G(-x)) &= \lim_{x\to\infty} (G_+(ax) - G_-(-x)) \\
&= (\frac{\gamma}{2} + \beta) + (\frac{\gamma}{2} - \beta) \\
&= \gamma \quad (\mathrm{C}, m) \ .
\end{aligned}$$

□

In summary, we succeeded characterizing distributional point values in terms of the summability of the Fourier inversion formula.

Theorem 3.21. *Let* $f \in \mathcal{S}'(\mathbb{R})$. *We have* $f(x_0) = \gamma$, *distributionally, if and only if there exists an* $m \in \mathbb{N}$ *such that*

$$\frac{1}{2\pi} \text{ e.v.} \left\langle \hat{f}(x), e^{ix_0x} \right\rangle = \gamma \quad (\mathrm{C}, m) \ . \tag{3.5.5}$$

Proof. Combine Lemma 3.11 with Corollary 3.19. □

In order to obtain further results, let us introduce some notation. Let $g = \mu$ be a Radon measure. It convenient in this case to write

$$\text{e.v.} \int_{-\infty}^{\infty} \phi(x) \mathrm{d}\mu(x) = \gamma \quad (\mathrm{C}, m) \tag{3.5.6}$$

for (3.5.1). When $m=0$, we suppress $(\mathrm{C},0)$ from the notation, and just write

$$\text{e.v.} \int_{-\infty}^{\infty} \phi(x)\mathrm{d}\mu(x) = \gamma \ .$$

In particular, if $\mu = \sum_{n=-\infty}^{\infty} c_n \delta(\,\cdot\, - n)$ and $\phi \equiv 1$, we use the notation

$$\text{e.v.} \sum_{n=-\infty}^{\infty} c_n = \gamma \quad (\mathrm{C}, m) \ , \tag{3.5.7}$$

omitting again $(\mathrm{C},0)$ when $m=0$.

Observe that if we use the family of summability kernels

$$\phi_a^m(x) = (1+x)^m (H(-x) - H(-1-x)) + \left(1 - \frac{x}{a}\right)^m (H(x) - H(x-a)) \ , \tag{3.5.8}$$

where H is the Heaviside function, then (3.5.6) holds if and only if

$$\lim_{x\to\infty} \int_{-\infty}^{\infty} \phi_a^m\left(\frac{t}{x}\right) \phi(t)\mathrm{d}\mu(t) = \gamma \ , \quad \text{for each } a > 0 \ . \tag{3.5.9}$$

For series we obtain that (3.5.7) holds if and only if

$$\lim_{x\to\infty} \sum_{n=-\infty}^{\infty} \phi_a^m\left(\frac{n}{x}\right) c_n = \gamma \ , \quad \text{for each } a > 0 \ . \tag{3.5.10}$$

Let us now discuss some immediate consequences of Theorem 3.21.

Corollary 3.22. *Let $f \in \mathcal{S}'(\mathbb{R})$ be such that $\hat{f} = \mu$ is a Radon measure. Then, we have $f(x_0) = \gamma$, distributionally, if and only if there exists an $m \in \mathbb{N}$ such that*

$$\frac{1}{2\pi}\text{e.v.} \int_{-\infty}^{\infty} e^{ix_0 x}\mathrm{d}\mu(x) = \gamma \quad (\mathrm{C}, m) \ , \tag{3.5.11}$$

or which amounts to the same,

$$\lim_{x\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix_0 t} \phi_a^m\left(\frac{t}{x}\right) \mathrm{d}\mu(t) = \gamma \ , \quad \textit{for each } a > 0 \ . \tag{3.5.12}$$

The next corollary is a result of R. Estrada [46], the characterization of Fourier series having a distributional point value.

Corollary 3.23. *Let $f \in \mathcal{S}'(\mathbb{R})$ be a 2π-periodic distribution having Fourier series*

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \ . \tag{3.5.13}$$

Then, we have $f(x_0) = \gamma$, distributionally, if and only if there exists an $m \in \mathbb{N}$ such that

$$\text{e.v.} \sum_{n=-\infty}^{\infty} c_n e^{inx_0} = \gamma \quad (\mathrm{C}, m) \ , \tag{3.5.14}$$

or which amounts to the same,

$$\lim_{x\to\infty} \sum_{n=-\infty}^{\infty} \phi_a^m \left(\frac{n}{x}\right) c_n e^{inx_0} = \gamma \ , \quad \textit{for each } a > 0 \ . \tag{3.5.15}$$

Proof. We have that $\hat{f}(x) = 2\pi \sum_{n=-\infty}^{\infty} c_n \delta(x-n)$, the rest follows from Corollary 3.22. $\square$

Let us state Corollary 3.22 when $\hat{f} \in L^1_{\mathrm{loc}}(\mathbb{R})$. A particular case is obtained if $f \in L^p(\mathbb{R})$ with $1 \leq p \leq 2$, since $\hat{f} \in L^q(\mathbb{R})$ with $q = p/(p-1)$ [206, Thm.74].

Corollary 3.24. *Let $f \in \mathcal{S}'(\mathbb{R})$ be such that $\hat{f} \in L^1_{\mathrm{loc}}(\mathbb{R})$. Then, we have $f(x_0) = \gamma$, distributionally, if and only if there exists an $m \in \mathbb{N}$ such that*

$$\frac{1}{2\pi} \text{e.v.} \int_{-\infty}^{\infty} e^{ix_0 x} \hat{f}(x) \mathrm{d}x = \gamma \quad (\mathrm{C}, m) \ , \tag{3.5.16}$$

or which amounts to the same,

$$\lim_{x\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_a^m \left(\frac{t}{x}\right) e^{ix_0 t} \hat{f}(t) \mathrm{d}t = \gamma \ , \quad \textit{for each } a > 0 \ . \tag{3.5.17}$$

It is important to observe that the characterization of distributional point values is given in terms of slightly asymmetric means and that the corresponding result for symmetric means does not hold. The result for separate integration over both the positive and negative parts of the spectrum does not hold either (we already discussed the latter in Example 3.17). Let us provide two further examples.

Example 3.25. *If $f \in \mathcal{S}'(\mathbb{R})$ and $f(x_0) = \gamma$, distributionally, then by taking $a = 1$ we obtain that the symmetric means converge to γ, in the Cesàro sense, so that, in case $\hat{f}(t)e^{ix_0 t}$ is locally integrable,*

$$\lim_{x\to\infty} \frac{1}{2\pi} \int_{-x}^{x} \hat{f}(t)e^{ix_0 t}\mathrm{d}t = \gamma \quad (\mathrm{C}, m)\,, \tag{3.5.18}$$

for some m. However, (3.5.18) does not imply, in general, the existence of the distributional value $f(x_0)$. A simple example is provided by $f(x) = \delta'(x)$ at $x = 0$, since $\hat{f}(t) = -it$, so that (3.5.18) exists and equals 0, but $f(0)$ does not exist.

Example 3.26. *If $f \in \mathcal{S}'(\mathbb{R})$, $\hat{f} \in L^1_{\mathrm{loc}}(\mathbb{R})$, and the two Cesàro limits*

$$\lim_{x\to+\infty} \frac{1}{2\pi} \int_{0}^{x} \hat{f}(t)e^{ix_0 t}\mathrm{d}t = \gamma_+ \quad (\mathrm{C}, m)\,, \tag{3.5.19}$$

$$\lim_{x\to+\infty} \frac{1}{2\pi} \int_{-x}^{0} \hat{f}(t)e^{ix_0 t}\mathrm{d}t = \gamma_- \quad (\mathrm{C}, m)\,, \tag{3.5.20}$$

exist then the distributional value $f(x_0)$ exists and equals $\gamma = \gamma_+ + \gamma_-$. However, the existence of the distributional point value $f(x_0)$ does not imply the existence of both Cesàro limits. For instance, if

$$f(x) = \int_0^\infty \frac{\sin xt\,\mathrm{d}t}{t\ln(t^2 + a^2)}\,, \tag{3.5.21}$$

for some $a > 1$, then f is continuous and $f(0) = 0$, but we have that $\hat{f}(t) = -\pi i t^{-1}(\ln(t^2 + a^2))^{-1}$, and in that case both limits (3.5.19) and (3.5.20) give infinite results, i.e., $|\gamma_+| = |\gamma_-| = \infty$.

There is one case in which the distributional point values can be characterized by Cesàro summability of the Fourier inversion formula without using the asymmetric means, that is, when the distribution has support on a half-ray. This result is an earlier inversion formula for tempered distributions, essentially obtained in [243] (with a different language from ours).

Theorem 3.27. *Let $f \in \mathcal{S}'(\mathbb{R})$ have support bounded at the left . We have $f(x_0) = \gamma$, distributionally, if and only if there exists an $m \in \mathbb{N}$ such that*

$$\frac{1}{2\pi}\left\langle \hat{f}\,(x)\,, e^{ix_0x}\right\rangle = \gamma \qquad (\mathrm{C}, m)\ . \tag{3.5.22}$$

Proof. The converse follows from Proposition 3.20. Let now F be the primitive of $(1/2\pi)e^{ix_0x}\hat{f}$ with support bounded at the left. Then, by Theorem 3.21, we have that

$$\lim_{x\to\infty} F(x) = (\lim_{x\to\infty} F(x) - F(-x)) = \gamma \quad (\mathrm{C}, m)\ .$$

□

Corollary 3.28. *Let $f \in \mathcal{S}'(\mathbb{R})$ be such that $\hat{f} = \mu$ is a Radon measure supported on $[0,\infty)$. Then, we have $f(x_0) = \gamma$, distributionally, if and only if*

$$\frac{1}{2\pi}\int_0^\infty e^{ix_0x}\mathrm{d}\mu(x) = \gamma \quad (\mathrm{C})\ . \tag{3.5.23}$$

We also obtain a corresponding result for Riesz summability.

Corollary 3.29. *Let $f(x) = \sum_{n=0}^\infty c_n e^{i\lambda_n x}$ in $\mathcal{S}'(\mathbb{R})$, where $\lambda_n \nearrow \infty$. Then, we have $f(x_0) = \gamma$, distributionally, if and only if*

$$\sum_{n=0}^\infty c_n e^{i\lambda_n x_0} = \gamma \quad (\mathrm{R}, \{\lambda_n\})\ . \tag{3.5.24}$$

These ideas can be applied to study some types of multiple series. It is not our scope to investigate problems in several variables in this chapter, but the next theorem shows that some problems in summability of multiple series can be solved using this theory. The next result is an example of that.

Theorem 3.30. *Let $f \in \mathcal{S}'(\mathbb{R})$ and ρ be a real-valued function defined on $\mathbb{R}^d$ which only takes non-negative values at points $\mathbf{j} \in \mathbb{N}^d$. Suppose that*

$$f(x) = \sum_{\mathbf{j}\in\mathbb{N}^d} c_{\mathbf{j}} e^{i\rho(\mathbf{j})x} \ \textit{in}\ \mathcal{S}'(\mathbb{R}).$$

Enumerate the image $\rho(\mathbb{N}^d)$ by an increasing sequence $\{\lambda_n\}_{n=0}^{\infty}$. Then, $f(x_0) = \gamma$, distributionally, if and only if there exists an $m \in \mathbb{N}$ such that

$$\sum_{n=0}^{\infty} \left(\sum_{\rho(\mathbf{j})=\lambda_n} c_{\mathbf{j}} e^{i\rho(\mathbf{j})x_0} \right) = \gamma \quad (\mathrm{R}, \{\lambda_n\}, m) \ , \tag{3.5.25}$$

or equivalently,

$$\lim_{\lambda \to \infty} \sum_{\rho(\mathbf{j}) \leq \lambda} c_{\mathbf{j}} e^{i\rho(\mathbf{j})x_0} \left(1 - \frac{\rho(\mathbf{j})}{\lambda}\right)^m = \gamma \ . \tag{3.5.26}$$

Proof. It follows immediately from Corollary 3.29, since

$$f(x) = \sum_{n=0}^{\infty} \left(\sum_{\rho(\mathbf{j})=\lambda_n} c_{\mathbf{j}} e^{i\rho(\mathbf{j})x_0} \right) e^{i\lambda_n x}$$

□

If in particular we take $\rho(\mathbf{y}) = |\mathbf{y}|^2$ (here $\mathbf{y} \in \mathbb{R}^d$ and $|\cdot|$ is the standard euclidean norm) in Theorem 3.30, we obtain that $f(x_0) = \gamma$, distributionally, if and only if the multiple series is Bochner-Riesz summable by spherical means [28].

3.6 Convergence of Fourier Series

We now analyze sufficient conditions under which the existence of the distributional point value implies the convergence of the Fourier series at the point. Note that, in particular, any result of this type gives a tauberian condition for Cesàro summability of series. The next theorem is our first result in this direction. We denote by l^p, $1 \leq p < \infty$, the set of those sequences $\{c_n\}_{n=-\infty}^{\infty}$ such that $\sum_{n=-\infty}^{\infty} |c_n|^p < \infty$.

Theorem 3.31. *Let $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ in $\mathcal{S}'(\mathbb{R})$. Suppose that $\{c_n\} \in l^p$, $1 \leq p < \infty$ and*

$$r_{N,p} = \sum_{|n| \geq N}^{\infty} |c_n|^p = O\left(\frac{1}{N^{p-1}}\right) \ , \ N \to \infty \, . \tag{3.6.1}$$

Then, $f(x_0) = \gamma$, distributionally, implies

$$\text{e.v.} \sum_{n=-\infty}^{\infty} c_n e^{inx_0} = \gamma\,, \tag{3.6.2}$$

or which amounts to the same

$$\lim_{x\to\infty} \sum_{-x\le n\le ax} c_n e^{inx_0} = \gamma\,, \tag{3.6.3}$$

for each $a > 0$.

Proof. If $p = 1$, it is trivial. Let us assume $1 < p < \infty$, and let us find q so that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. If $f(x_0) = \gamma$, we have

$$\lim_{\varepsilon\to 0^+} \sum_{n=-\infty}^{\infty} c_n e^{ix_0 n} \psi(\varepsilon n) = \gamma\psi(0)\,,$$

for each $\psi \in \mathcal{S}(\mathbb{R})$. Choose $\phi \in \mathcal{D}(\mathbb{R})$ such that $0 \le \phi \le 1$, and $\phi(x) = 1$ for $x \in [-1, a]$. Hence

$$\sum_{n=-\infty}^{\infty} c_n e^{inx_0}\phi(n\varepsilon) = \sum_{-\frac{1}{\varepsilon}\le n\le\frac{a}{\varepsilon}} c_n e^{ix_0 n} + \sum_{\frac{a}{\varepsilon}<n} c_n e^{ix_0 n}\phi(n\varepsilon) + \sum_{\frac{1}{\varepsilon}<n} c_{-n} e^{-ix_0 n}\phi(-n\varepsilon) + o(1)\,,$$

as $\varepsilon \to 0^+$. Therefore,

$$\limsup_{\varepsilon\to 0+} \left| \sum_{-\frac{1}{\varepsilon}\le n\le\frac{a}{\varepsilon}} c_n e^{inx_0} - \gamma \right| \le \limsup_{\varepsilon\to 0^+} \left(\sum_{\frac{a}{\varepsilon}<n} |c_n|\,|\phi(\varepsilon n)| + \sum_{\frac{1}{\varepsilon}<n} |c_{-n}|\,|\phi(-\varepsilon n)| \right) .$$

But,

$$\sum_{\frac{a}{\varepsilon}<n} |c_n|\,|\phi(\varepsilon n)| \le \left\{ \sum_{\frac{a}{\varepsilon}<n} |c_n|^p \right\}^{\frac{1}{p}} \left\{ \sum_{\frac{a}{\varepsilon}<n} |\phi(\varepsilon n)|^q \right\}^{\frac{1}{q}} ,$$

By (3.6.1), we can find $M > 0$ such that

$$\sum_{\frac{a}{\varepsilon}<n} |c_n|\,|\phi(\varepsilon n)| \le M a^{-\frac{1}{q}} \left\{ \varepsilon \sum_{\frac{a}{\varepsilon}<n} |\phi(\varepsilon n)|^q \right\}^{\frac{1}{q}} .$$

Then,

$$\limsup \lim_{\varepsilon\to 0+} \sum_{\frac{a}{\varepsilon}<n} |c_n|\,|\phi(\varepsilon n)| \le M a^{-\frac{1}{q}} \left\{ \int_a^{\infty} |\phi(x)|^q \, dx \right\}^{\frac{1}{q}} .$$

Similarly, $\exists M' > 0$ such that

$$\limsup \lim_{\varepsilon\to 0+} \sum_{\frac{1}{\varepsilon}<n} |c_{-n}|\,|\phi(-\varepsilon n)| \le M' a^{-\frac{1}{q}} \left\{ \int_{-\infty}^{-1} |\phi(x)|^q \, dx \right\}^{\frac{1}{q}} .$$

Now, we are free to choose ϕ such that the right sides of the last two inequalities are both less than $\sigma/2$. Therefore,

$$\limsup \lim_{\varepsilon\to 0+} \left| \sum_{-\frac{1}{\varepsilon}\le n\le \frac{a}{\varepsilon}} c_n e^{inx_0} - \gamma \right| < \sigma.$$

Since this can be done for each $\sigma > 0$, we conclude that

$$\lim_{x\to\infty} \sum_{-x\le n\le ax} c_n e^{inx_0} = \gamma\,,$$

as required. □

As an example of the use of Theorem 3.31, let us obtain a classical tauberian result of Hardy for Cesàro summability of series [85, p.121].

Corollary 3.32. *Suppose that* $\sum_{n=0}^{\infty} c_n = \gamma \quad (\mathrm{C})$. *The tauberian condition* $nc_n = O(1)$ *implies the convergence of the series to* γ.

Proof. We associate to the sequence a Fourier series, $f(x) = \sum_{n=0}^{\infty} c_n e^{inx}$. The (C) summability to γ implies $f(0) = \gamma$, distributionally. Now, Hardy's tauberian hypothesis obviously implies (3.6.1) for actually any $p > 1$, so Theorem 3.31 gives the convergence. □

We can generalize Theorem 3.31 to other norms.

Theorem 3.33. *Let* $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ *in* $\mathcal{S}'(\mathbb{R})$. *Suppose that*

$$\sum_{|n|\ge N} |c_n|^p \, |n|^{-pr} = O\left(\frac{1}{N^{rp+p-1}}\right) \tag{3.6.4}$$

for some r *and* p *with* $1 < p < \infty$. *If* $f(x_0) = \gamma$, *distributionally, then*

$$\text{e.v.} \sum_{n=-\infty}^{\infty} c_n e^{inx_0} = \gamma\,, \tag{3.6.5}$$

or which amounts to the same

$$\lim_{x\to\infty} \sum_{-x\le n\le ax} c_n e^{inx_0} = \gamma\,, \tag{3.6.6}$$

for each $a>0$.

Proof. Use the inequality

$$\sum_{\frac{a}{\varepsilon}<n} |c_n|\,|\phi(\varepsilon n)| \le \left\{\sum_{\frac{a}{\varepsilon}<n} |c_n|^p\, n^{-rp}\right\}^{\frac{1}{p}} \left\{\sum_{\frac{a}{\varepsilon}<n} n^{rq}\,|\phi(\varepsilon n)|^q\right\}^{\frac{1}{q}}$$

and follow a similar argument as the one in the proof of Theorem 3.31. □

If we take $r=(1/p)-1$ in the last theorem, we obtain the following Hardy and Littlewood tauberian condition for (C) summability [88, p.140-141].

Corollary 3.34. *If*

$$\sum_{n=0}^{\infty} c_n = \gamma \quad (\mathrm{C}, k)\,, \tag{3.6.7}$$

for some $k\in\mathbb{N}$, then the tauberian condition ($p\ge 1$)

$$\sum_{n=0}^{\infty} n^{p-1}\,|c_n|^p < \infty \tag{3.6.8}$$

implies that $\sum_{n=0}^{\infty} c_n$ is convergent to γ.

Theorem 3.33 has an interesting generalization if we replace n^r in (3.6.4) by a regularly varying function of index r (Section 1.7).

Theorem 3.35. *Let $f(x)=\sum_{n=0}^{\infty} c_n e^{inx}$ in $\mathcal{S}'(\mathbb{R})$. Let ρ be a regularly varying function of index r. Suppose that*

$$\sum_{n=N}^{\infty} \frac{|c_n|^p}{(\rho(n))^p} = O\left(\frac{1}{N^{p(r+\sigma)+p-1}}\right)\,, \quad N\to\infty\,, \tag{3.6.9}$$

for some p, $1<p<\infty$, and $\sigma>0$. If $f(x_0)=\gamma$, distributionally, then for any fixed $\varepsilon>0$,

$$\sum_{n=0}^{N} c_n e^{ix_0 n} = \gamma + o\left(\frac{1}{N^{\sigma-\varepsilon}}\right)\,, \quad N\to\infty\,. \tag{3.6.10}$$

Proof. Pick $\phi \in \mathcal{D}(\mathbb{R})$ such that $0 \le \phi \le 1$, $\phi(x) = 1$ for $x \in [0,1]$. Then, we have

$$\sum_{\lambda<n} \left|\phi\left(\frac{n}{\lambda}\right)\right| |c_n| \le \left\{\sum_{\lambda<n} \frac{|c_n|^p}{(\rho(n))^p}\right\}^{\frac{1}{p}} \left\{\sum_{\lambda<n} \left|\phi\left(\frac{n}{\lambda}\right)\right|^q (\rho(n))^q\right\}^{\frac{1}{q}}$$

where q is so that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then, we can find $M_1 > 0$ such that

$$\sum_{\lambda<n} \left|\phi\left(\frac{n}{\lambda}\right)\right| |c_n| \le \frac{M_1}{\lambda^{r+1/q+\sigma}} \left\{\sum_{\lambda<n} \left|\phi\left(\frac{n}{\lambda}\right)\right|^q n^{rq} \left(\frac{\rho(n)}{n^r}\right)^q\right\}^{\frac{1}{q}}. \tag{3.6.11}$$

Set

$$L(x) = \frac{\rho(x)}{x^r},$$

then L is a slowly varying function; hence (Section 1.7), there exists a positive number B such that for all $x \ge B$ we have

$$L(x) = \exp\left\{u(x) + \int_B^x \frac{w(t)}{t}\,\mathrm{d}t\right\}, \tag{3.6.12}$$

where u is a bounded measurable function on $[B,\infty)$ such that $u(x) \to C$ ($|C| < \infty$), and w is a continuous function on $[B,\infty)$ such that $w(x) \to 0$, $x \to \infty$. Let $M_2 > 0$ such that that $|u(x)| \le M_2$, $\forall\, x \ge B$. In addition, given $\varepsilon > 0$ we can find $A > \max\{B,1\}$ such that $|w(x)| < \varepsilon$, $\forall\, x > A$. Therefore, by (3.6.12), we have that for $x \ge A$,

$$L(x) \le \exp\left\{M_2 + \int_B^A \frac{w(t)}{t}\,\mathrm{d}t\right\} x^{\varepsilon}. \tag{3.6.13}$$

Combining (3.6.11) and (3.6.13), we obtain that for $\lambda > A$

$$\sum_{\lambda<n} |c_n| \left|\phi\left(\frac{n}{\lambda}\right)\right| < \frac{M_3}{\lambda^{\sigma-\varepsilon}} \left\{\frac{1}{\lambda}\sum_{\lambda<n} \left|\phi\left(\frac{n}{\lambda}\right)\right|^q \left(\frac{n}{\lambda}\right)^{(r+\varepsilon)q}\right\}^{\frac{1}{q}},$$

where $M_3 = M_1 \exp\left(M_2 + \int_B^A (w(t)/t)\,\mathrm{d}t\right)$. So,

$$\limsup_{\lambda\to\infty} \lambda^{\sigma-\varepsilon} \sum_{n>\lambda} |c_n| \left|\phi\left(\frac{n}{\lambda}\right)\right| < M_3 \left(\int_1^\infty |\phi(x)|^q\, x^{(r+\varepsilon)q}\mathrm{d}x\right)^{\frac{1}{q}}.$$

Now, since the right side of the last inequality holds for every $\phi \in \mathcal{D}(\mathbb{R})$ with $0 \leq \phi \leq 1$ and $\phi(x) = 1$ for $x \in [0,1]$, we conclude that

$$\limsup_{\lambda \to \infty} \lambda^{\sigma-\varepsilon} \left| \sum_{0 \leq n \leq \lambda} c_n e^{ix_0 n} - \gamma \right| = 0\,,$$

and our result follows. $\square$

Observe, in particular, that the last theorem can be applied to functions such as $\rho(x) = x^r \left|\ln x\right|^\alpha$, which are regularly varying functions of index r.

Next, we would like to make some comments about the results we just discussed. If $\{c_n\} \in l^p$ for $1 \leq p \leq 2$, then $f(x) = \sum c_n e^{inx}$ belongs to $L^q[0, 2\pi]$, but the converse is not true. Similarly, if $f \in L^p[0, 2\pi]$, $1 \leq p \leq 2$, then $\{c_n\}$, belongs to l^q, but the converse is not necessarily true. Hence, the results for $\{c_n\} \in l^p$ with $1 \leq p \leq 2$ are about functions. However, for $p > 2$, these results are about *distributions,* in general. For example, as follows from [256, Chapter V], if $\{c_n\} \in l^p \setminus l^2$ for some $p > 2$ then for almost all choices of signs $\rho_n = \pm$ the distribution $\sum_{n=0}^\infty \rho_n c_n e^{inx}$ is not locally integrable; or if $\{c_n\} \in l^p \setminus l^2$ is lacunary then $\sum_{n=0}^\infty c_n e^{inx}$ is never a regular distribution.

We conclude this section discussing a type of tauberian result in summability of Fourier series of distributions, where the conclusion is not the convergence of the series but the (C,m) summability for a specific m. As it has been mentioned before, any result of this type gives a result in the theory of Cesàro summability of series. Let us suppose that f is a periodic distribution of period 2π, and $f(x) = \sum_{n=-\infty}^\infty c_n e^{inx}$. We want to find sufficient conditions under which the existence of $f(x_0)$, distributionally, implies that

$$\lim_{x \to \infty} \sum_{-x \leq n \leq ax} c_n e^{inx_0} = f(x_0) \quad (\mathrm{C}, m)\,, \tag{3.6.14}$$

for an specific positive integer m. A partial answer to this question is given in Theorem 3.36.

Theorem 3.36. *Let $f \in \mathcal{S}'(\mathbb{R})$ such that $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$. Suppose that $f(x_0) = \gamma$, distributionally. If for a fixed a*

$$\sum_{-x \le n \le ax} c_n e^{inx_0} = O(1) \quad (\mathrm{C}, m)\,, \tag{3.6.15}$$

then

$$\lim_{x\to\infty} \sum_{-x \le n \le ax} c_n e^{inx_0} = \gamma \quad (\mathrm{C}, m+1)\,. \tag{3.6.16}$$

Proof. For $x > 0$, set

$$g_a(x) = \sum_{-x \le n \le ax} c_n e^{inx_0}\ ,$$

and put $g_a(x) = 0$ for $x \le 0$. Condition (3.6.15) means that there is an m-primitive G of g_a, such that $\operatorname{supp} G \subseteq [0, \infty)$ and

$$G(x) = O\left(x^m\right)\ , \quad x \to \infty\ .$$

In addition, since $f(x_0) = \gamma$, we have that

$$G(\lambda x) = \frac{\gamma \lambda^m x_+^m}{m!} + o\left(\lambda^m\right) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{D}'(\mathbb{R}),$$

i.e., for each $\phi \in \mathcal{D}(\mathbb{R})$

$$\int_0^\infty G(\lambda x)\phi(x)\,\mathrm{d}x = \frac{\gamma\lambda^m}{m!}\int_0^\infty x^m \phi(x)\,\mathrm{d}x + o\left(\lambda^m\right)\ .$$

Pick $\phi \in \mathcal{D}(\mathbb{R})$ such that $\phi(x) = 1$ for $x \in [-1, 1]$ and $\operatorname{supp}\phi \subseteq [-1, 2]$. Evaluating G at ϕ, we obtain

$$\begin{aligned}&\frac{1}{\lambda}\int_0^\lambda G(x)\,\mathrm{d}x + \frac{1}{\lambda}\int_\lambda^{2\lambda} G(x)\phi\left(\frac{x}{\lambda}\right)\,\mathrm{d}x \\ &\qquad = \frac{\gamma\lambda^m}{(m+1)!} + \frac{\gamma\lambda^m}{m!}\int_1^2 x^m\phi(x)\,\mathrm{d}x + o\left(\lambda^m\right),\ \lambda \to \infty,\end{aligned}$$

which implies

$$\begin{aligned}&\left|\frac{(m+1)!}{\lambda^{m+1}}\int_0^\lambda G(x)\,\mathrm{d}x - \gamma\right| \\ &\le o(1) + \gamma(m+1)\int_1^2 x^m\phi(x)\,\mathrm{d}x + \frac{(m+1)!}{\lambda^{m+1}}\int_\lambda^{2\lambda} |G(x)|\,\phi\left(\frac{x}{\lambda}\right)\,\mathrm{d}x \\ &= o(1) + \{\gamma(m+1) + (m+1)!O(1)\}\int_1^2 x^m\phi\left(x\right)\,\mathrm{d}x,\ \lambda \to \infty\,,\end{aligned}$$

since we can choose ϕ such that $\int_1^2 x^m \phi(x)\, \mathrm{d}x$ is as small as we want, we conclude that

$$\lim_{\lambda \to \infty} \frac{(m+1)!}{\lambda^{m+1}} \int_0^{\lambda} G(x)\, \mathrm{d}x = \gamma\,,$$

and the result follows. □

We obtain the following interesting corollary of Theorem 3.36, known as convexity theorem [85, p.127].

Corollary 3.37. *Let $\{c_n\}_{n=0}^{\infty}$ be a sequence of complex numbers. Suppose that*

$$\sum_{n=0}^{\infty} c_n = \gamma \ (\mathrm{C}, m)\,, \tag{3.6.17}$$

for some $m \in \mathbb{N}$. If the m-Cesàro mean is bounded then

$$\sum_{n=0}^{\infty} c_n = \gamma \ (\mathrm{C}, m+1)\,. \tag{3.6.18}$$

3.7 Series with Gaps

In this section we apply the ideas of the last section to series with gaps. In particular, we shall find examples of continuous functions whose distributional derivatives do not have distributional point values at any point.

Theorem 3.38. *Let $f(x) = \sum_{n=0}^{\infty} c_n e^{inx}$, in $\mathcal{S}'(\mathbb{R})$. In addition, suppose that $\{c_n\}_{n=0}^{\infty}$ is lacunary, in the sense of Hadamard, i.e., $c_n = 0$ except for a sequence $n_k \in \mathbb{N}$ with $n_{k+1} > \alpha n_k$ for some $\alpha > 1$. Then $f(x_0) = \gamma$, distributionally, if and only if*

$$\sum_{n=0}^{\infty} c_n e^{inx_0} = \gamma\,. \tag{3.7.1}$$

In particular, $c_{n_k} = o(1)$, $k \to \infty$.

Proof. Let $\phi \in \mathcal{D}(\mathbb{R})$ such that $0 \le \phi \le 1$, $\phi(x) = 1$ for $x \in [0,1]$ and $\operatorname{supp}\phi \subseteq [-1, \alpha]$. Set $b_n = c_n e^{inx_0}$, for each $n \in \mathbb{N}$. We have that

$$M(\lambda) = \sum_{n_k \le \lambda} b_{n_k} + \sum_{\lambda < n_k < \alpha\lambda} b_{n_k} \phi\left(\frac{n_k}{\lambda}\right) - \gamma = o(1)\,,\ \lambda \to \infty\,.$$

Note that given $\lambda > 0$ there exists at most one k_λ such that $\lambda < k_\lambda < \alpha\lambda$. Therefore if $\lambda = n_m$, we obtain

$$M(n_m) = o(1)\,,\ m \to \infty\,,$$

which is the same as

$$\sum_{k=0}^{m} b_{n_k} - \gamma = o(1)\,,\ m \to \infty\,.$$

This completes the proof. $\square$

Moreover, with a little modification of the last argument, we obtain the following result.

Theorem 3.39. *Let $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$, in $\mathcal{S}'(\mathbb{R})$. Suppose that $\{c_n\}_{n\in\mathbb{Z}}$ is lacunary in both directions; then, $f(x_0) = \gamma$, distributionally, if and only if*

$$\lim_{x\to\infty} \sum_{-x\le n\le ax} c_n e^{inx_0} = \gamma\,,$$

for each $a > 0$.

We obtain several interesting corollaries from the last two theorems. The second part of the following corollary is a result of Kolmogorov [256].

Corollary 3.40. *If $f \in L^1[0, 2\pi]$ and $\{c_n\}_{n\in\mathbb{Z}}$ lacunary, then the Fourier series of f converges to $f(x_0)$ at every point where $f(x_0)$ exists distributionally in the sense of Łojasiewicz. In particular, it converges almost everywhere.*

Proof. Indeed, the first part follows directly from Theorem 3.39, while the second statement is true because f has distributional point values at every point of the Lebesgue set of f. $\square$

Corollary 3.41. *Let $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$, in $\mathcal{S}'(\mathbb{R})$. If $\{c_n\}_{n\in\mathbb{Z}}$ is lacunary, but $c_n \neq o(1)$, then the distributional value $f(x_0)$ does not exist at any point x_0.*

The next corollary allows us to find examples of continuous functions whose distributional derivatives do not have point values anywhere.

Corollary 3.42. *Let $\{c_n\}_{n\in\mathbb{Z}}$ be a lacunary sequence such that $c_n \neq o(1)$ but $c_n = O(1)$, $|n| \to \infty$. Then*

$$g(x) = \sum_{n=-\infty}^{\infty} \frac{c_n}{n} e^{inx}, \tag{3.7.2}$$

is continuous but $g'(x)$ does not have distributional point values at any point; in particular, g is nowhere differentiable.

That g' does not have distributionally point values at any point is stronger than the fact that g is nowhere differentiable. For example, consider $g(x) = x \sin x^{-1}$; $g'(0)$ does not exist in the usual sense, but g' has the value 0 at $x = 0$, distributionally [128]. As we mentioned in Section 3.2, the existence of a point value in implies the existence of a continuous n-primitive having an n-differential at the point in the Denjoy sense [34, 128]; however, even if a distribution has distributional point values everywhere (and hence a function can be associated pointwise to it) the distribution does not correspond to a classical function (at least as far as is known). It is interesting to see how this problems of global existence of distributional point values is related with alternative integrals like the Denjoy-Perron-Henstock integral [76]. For instance, it is clear that if g is a function such that $g'(x_0)$ exists (in the classical sense) for every $x_0 \in \mathbb{R}$, then g' has distributional point values at every point, however, the classical convention of declaring that a distribution is a function if it corresponds to the distribution induced by a locally Lebesgue integrable functions leads to the conclusion that g' (as a distribution) is not a classical function, even though, g' is a function in a wide sense of the word! Therefore the concept should be reinterpreted. On the other hand, if we use the convention that a distribution is a function if it corresponds to the distribution induced by a Denjoy-Perron-Henstock integral, then g' can be interpreted as the distribution induced by this process of integration. Under the same circumstances, other interesting questions can be asked, for example if it is known that g is differentiable for almost every point, is there any reasonable way to associate the distribution g' to the function that assigns $x_0 \longrightarrow g'(x_0)$ which is an almost everywhere defined function? Well, the answer to this question is unknown, since these conditions are not sufficient to deduce the Denjoy-Perron-Henstock integrability of the function g'. In [215] (see also Chapter 7 below), we considered a global problem on the study of distributions having distributional point values almost everywhere; furthermore, the techniques employed there give some evidence of relation with Colombeau theory of generalized functions. It seems that these kind of global problems in general are extremely difficult and not too much is known about global properties of distributional point values. With this short discussion, the author's intention is to indicate some global problems in theory of distributional point values. We now continue with our discussion of series with gaps.

A good illustration of Corollary 3.41 is obtained when we consider the two Weierstrass functions

$$f_\alpha(x) = \sum_{n=0}^{\infty} b^{-n\alpha} \cos(b^n x) \ , \tag{3.7.3}$$

and

$$g_\alpha(x) = \sum_{n=0}^{\infty} b^{-n\alpha} \sin(b^n x) \ , \tag{3.7.4}$$

where $b > 1$ is an integer and α is a positive number less or equal to 1. Observe that f_α and g_α are continuous. Weierstrass showed that for α small enough they are nowhere differentiable. The extension to $0 < \alpha \leq 1$ was first proved by Hardy. Using Corollary 3.41, we obtain a stronger result for it, namely, f'_α and g'_α do not have distributional point values at any point.

Sometimes, even if a distribution with a lacunary Fourier series does not have a point value at a point, it is possible to obtain its local distributional behavior. For example, we will find the behavior of

$$h_\alpha(x) = \sum_{n=0}^{\infty} e^{i\alpha^n x}, \text{ in } \mathcal{S}'(\mathbb{R}), \tag{3.7.5}$$

at $x = 0$, where $\alpha > 1$.

Theorem 3.43. *If h_α is given by (3.7.5), then*

$$h_\alpha(\varepsilon x) = -\frac{\log \varepsilon}{\log \alpha} + O(1) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{S}'(\mathbb{R}). \tag{3.7.6}$$

Proof. It will be enough if we show that

$$\sum_{n=0}^{\infty} \phi(\varepsilon \alpha^n) = -\frac{\log \varepsilon}{\log \alpha} \phi(0) + O(1)\,, \quad \varepsilon \to 0^+, \tag{3.7.7}$$

for any fixed $\phi \in \mathcal{S}'(\mathbb{R})$; this is because if we replace ϕ by $\hat{\phi}$ in the last relation, we obtain the conclusion of Theorem 3.43. Fix α and set

$$F(x) = \sum_{\alpha^n \leq x} 1 = [\log x / \log \alpha] = \log x / \log \alpha + O(1)\,,$$

where $[\cdot]$ stands for the integral part. It follows that

$$F(\lambda x) = \frac{\log \lambda x}{\log \alpha} H(\lambda x - 1) + O(1) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R})\ ,$$

where H is the Heaviside function. Differentiating the last relation, we obtain

$$\lambda \sum_{n=0}^{\infty} \delta\left(\lambda x - \alpha^n\right) = \frac{1}{x \log \alpha} H(\lambda x - 1) + O(1) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R})\ . \tag{3.7.8}$$

Now, we take $\phi \in \mathcal{S}'(\mathbb{R})$ into (3.7.8),

$$\sum_{n=0}^{\infty} \phi\left(\frac{\alpha^n}{\lambda}\right) = \frac{1}{\log \alpha} \int_{1/\lambda}^{\infty} \frac{\phi(x)}{x}\, \mathrm{d}x + O(1)\,, \quad \lambda \to \infty.$$

Replacing $1/\lambda$ by ε,

$$\sum_{n=0}^{\infty} \phi(\varepsilon \alpha^n) = \frac{1}{\log \alpha} \int_{1}^{\infty} \frac{\phi(x)}{x}\, \mathrm{d}x + \frac{1}{\log \alpha} \int_{\varepsilon}^{1} \frac{\phi(x) - \phi(0)}{x}\, \mathrm{d}x$$

$$+\frac{1}{\log\alpha}\int_{\varepsilon}^{1}\frac{\phi(0)}{x}\,\mathrm{d}x+O(1)=-\frac{\log\varepsilon}{\log\alpha}\phi(0)+O(1)\,,\quad \varepsilon\to 0^{+}.$$

Finally, if we replace ϕ by $\hat{\phi}$, we obtain

$$\langle h_{\alpha}(\varepsilon x),\phi(x)\rangle=-\frac{\log\varepsilon}{\log\alpha}\int_{-\infty}^{\infty}\phi(x)\,\mathrm{d}x+O(1)\,,\quad \varepsilon\to 0^{+}$$

for any $\phi\in\mathcal{S}'(\mathbb{R})$. □

Theorem 3.43 allows to find the radial behavior at $z=1$ of the analytic function on the unit disk given by

$$G_{\alpha}(z)=\sum_{n=0}^{\infty}z^{\alpha^{n}}, \tag{3.7.9}$$

when $r\to 1^{-}$.

Corollary 3.44. *If G_{α} is defined by (3.7.9), then*

$$G_{\alpha}(r)=\frac{|\log|\log r||}{\log\alpha}+O(1)\,,\quad r\to 1^{-}, \tag{3.7.10}$$

where r is taken real.

Proof. From Theorem 3.43,

$$\sum_{n=0}^{\infty}\delta\left(x-\varepsilon\alpha^{n}\right)=-\frac{\log\varepsilon}{\log\alpha}\delta(x)+O(1)\,,\quad \varepsilon\to 0^{+}.$$

Define $\phi(x)=e^{-x}$ for $x\geq 0$ and extend it to $\mathbb{R}$ in any smooth way so that $\phi\in\mathcal{S}'(\mathbb{R})$. Then,

$$\sum_{n=0}^{\infty}e^{-\varepsilon\alpha^{n}}=-\frac{\log\varepsilon}{\log\alpha}+O(1)\,,\quad \varepsilon\to 0^{+}.$$

Changing $e^{-\varepsilon}$ by r, we obtain

$$g_{\alpha}(r)=\frac{|\log|\log r||}{\ln\alpha}+O(1)\,,\quad r\to 1^{-},$$

as required. □

3.8 Convergence of Fourier Integrals

We now extend the results of Sections 3.6 and 3.7 to Fourier integrals.

Theorem 3.45. *Let $f \in \mathcal{S}'(\mathbb{R})$. Assume that $\hat{f} \in L^p(\mathbb{R})$, $1 \le p < \infty$, and*

$$r_{p,x} = \int_{|t|\ge x}^{\infty} \left|\hat{f}(t)\right|^p \mathrm{d}t = O\left(\frac{1}{x^{p-1}}\right), \quad x \to \infty\,. \tag{3.8.1}$$

Then, $f(x_0) = \gamma$, distributionally, if and only if

$$\frac{1}{2\pi}\text{e.v.}\int_{-\infty}^{\infty} \hat{f}(t) e^{ix_0 t}\mathrm{d}t = \gamma\,. \tag{3.8.2}$$

Proof. We only consider the case $1 < p < \infty$, since $p = 1$ is trivial. Assume that $f(x_0) = \gamma$, distributionally. Fix $a > 0$. Taking Fourier transform in

$$f\left(x_0 + \frac{x}{\lambda}\right) = \gamma + o(1) \quad \text{as } \lambda \to \infty\,, \tag{3.8.3}$$

we obtain

$$e^{ix_0\lambda x}\hat{f}(\lambda x) = \frac{2\pi\gamma\delta(x)}{\lambda} + o\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R}). \tag{3.8.4}$$

Set $g(x) = e^{ix_0 x}\hat{f}(x)$. Take $\phi \in \mathcal{D}(\mathbb{R})$, such that $\phi(x) = 1$ for $x \in [-1, a]$ and $0 \le \phi \le 1$. Take q such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Thus, we have

$$\int_{-\lambda}^{\lambda a} g(t)\,\mathrm{d}t - 2\pi\gamma = -\int_{-\infty}^{-\lambda} g(t)\phi\left(\frac{t}{\lambda}\right)\,\mathrm{d}t - \int_{a\lambda}^{\infty} g(t)\phi\left(\frac{t}{\lambda}\right)\,\mathrm{d}t + o(1)\,,$$

as $\lambda \to \infty$. We show that

$$\lim_{\lambda\to\infty}\int_{a\lambda}^{\infty} g(t)\phi\left(\frac{t}{\lambda}\right)\,\mathrm{d}t = 0\,, \tag{3.8.5}$$

and

$$\lim_{\lambda\to\infty}\int_{-\infty}^{-\lambda} g(t)\phi\left(\frac{t}{\lambda}\right)\,\mathrm{d}t = 0\,. \tag{3.8.6}$$

We have

$$\begin{aligned}\left|\int_{a\lambda}^{\infty} g(t)\phi\left(\frac{t}{\lambda}\right)\,\mathrm{d}t\right| &\le O\left(\lambda^{-\frac{1}{q}}\right)\left\{\lambda\int_{a}^{\infty}|\phi(t)|^q\,\mathrm{d}t\right\}^{\frac{1}{q}}\\ &= O(1)\left\{\int_{a}^{\infty}|\phi(t)|^q\,\mathrm{d}t\right\}^{\frac{1}{q}}.\end{aligned}$$

Since $\left\{\int_a^\infty |\phi(t)|^q \, \mathrm{d}t\right\}^{\frac{1}{q}}$ can be made arbitrarily small, we conclude (3.8.5). Similarly, (3.8.6) follows. $\square$

Likewise, one can show.

Theorem 3.46. *Let $f \in \mathcal{S}'(\mathbb{R})$. Suppose that $\hat{f}$ is locally integrable and*

$$\int_{|t|\geq x} \left|\hat{f}(t)\right|^p |t|^{-rp} \, \mathrm{d}t = O\left(\frac{1}{x^{pr+p-1}}\right), \quad x \to \infty\,.$$

for some $1 < p < \infty$ and $r \in \mathbb{R}$. Then, $f(x_0) = \gamma$, distributionally, if and only if

$$\frac{1}{2\pi}\mathrm{e.v.} \int_{-\infty}^{\infty} \hat{f}(t) e^{ix_0 t} \mathrm{d}t = \gamma\,.$$

Theorem 3.47. *Let $f \in \mathcal{S}'(\mathbb{R})$ so that $\hat{f}$ is locally integrable with essential support in $[0, \infty)$. Let ρ be a regularly varying function of index r. Suppose that*

$$\int_x^\infty \frac{\left|\hat{f}(t)\right|^p}{(\rho(t))^p} \, \mathrm{d}t = O\left(\frac{1}{x^{p(r+\sigma)+p-1}}\right)\,, \quad x \to \infty\,,$$

for some p, $1 < p < \infty$, r and $\sigma > 0$. If $f(x_0) = \gamma$, distributionally, and ε is any positive number, then

$$\int_0^x \hat{f}(t) e^{ix_0 t} \mathrm{d}t = 2\pi\gamma + o\left(\frac{1}{x^{\sigma-\varepsilon}}\right)\,, \quad x \to \infty\,.$$

If we take $r = (1/p) - 1$ in Theorem 3.46, we obtain the next corollary.

Corollary 3.48. *If $f \in \mathcal{S}'(\mathbb{R})$, $\hat{f}$ is locally integrable, and*

$$\int_{-\infty}^{\infty} |t|^{p-1} \left|\hat{f}(t)\right|^p \mathrm{d}t = O(1)\,, \tag{3.8.7}$$

then, $f(x_0) = \gamma$, distributionally, if and only if

$$\frac{1}{2\pi}\mathrm{e.v.} \int_{-\infty}^{\infty} \hat{f}(t) e^{ix_0 t} \mathrm{d}t = \gamma\,.$$

We conclude this section with a very simple result for integral with gaps, this result generalizes Theorem 3.38.

Theorem 3.49. *Let $f \in \mathcal{S}'(\mathbb{R})$. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that*

$$\frac{\lambda_{n+1} - \eta}{\lambda_n} \geq \alpha > 1\,, \forall n \geq n_0\,, \tag{3.8.8}$$

for some α, n_0 and $\eta > 0$. Suppose that $\hat{f} = \mu$, where μ is a Radon measure supported in a set of the form

$$[a, b] \cup \bigcup_{n=N}^{\infty} [\lambda_n - \eta, \lambda_n]$$

where $[a, b]$ is a compact interval in $(0, \infty)$. If $f(x_0) = \gamma$, distributionally, then

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_0^{\lambda_n} e^{ix_0 t} \mathrm{d}\mu(t) = \gamma\,.$$

Proof. The proof is very easy; take $\phi \in \mathcal{D}(\mathbb{R})$ such that $0 \leq \phi \leq 1$, $\phi(x) = 1$ for $x \in [0, 1]$ and $\operatorname{supp} \phi \subseteq [-1, \alpha]$. Thus, we have

$$\frac{1}{2\pi} \int_0^{\lambda_n} e^{ix_0 t} \mathrm{d}\mu(t) + \frac{1}{2\pi} \int_{\lambda_n}^{\alpha\lambda_n} e^{ix_0 t} \phi\left(\frac{t}{\lambda_n}\right) \mathrm{d}\mu(t) = \gamma + o(1)\,,$$

as $n \to \infty$, but $\hat{f}(t) = 0$ on $[\lambda_n, \alpha\lambda_n]$. $\square$

3.9 Abel Summability

We now analyze Abel summability of the Fourier inversion formula in the presence of distributional point values. Some of the results of this section were previously obtained in [54] by studying the Poisson kernel. Our approach will be via the Fourier transform.

Let us first state an interesting theorem, which we may be regarded as a decomposition theorem for the quasiasymptotic behavior (3.4.3).

Theorem 3.50. *Let $g \in \mathcal{S}'(\mathbb{R})$. Then*

$$g(\lambda x) = \gamma \frac{\delta(x)}{\lambda} + o\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R}) \tag{3.9.1}$$

if and only if there exist a decomposition $g = g_- + g_+$, where $\operatorname{supp} g_- \subseteq (-\infty, 0]$ and $\operatorname{supp} g_+ \subseteq [0, \infty)$, and an asymptotically homogeneous function c of degree *zero such that the following asymptotic relations hold*

$$g_{+}(\lambda x)=\left(\frac{\gamma}{2}+c(\lambda)\right)\frac{\delta(x)}{\lambda}+o\left(\frac{1}{\lambda}\right)\quad \text{as } \lambda\to\infty \text{ in } \mathcal{S}'(\mathbb{R}) \tag{3.9.2}$$

and

$$g_{-}(\lambda x)=\left(\frac{\gamma}{2}-c(\lambda)\right)\frac{\delta(x)}{\lambda}+o\left(\frac{1}{\lambda}\right)\quad \text{as } \lambda\to\infty \text{ in } \mathcal{S}'(\mathbb{R})\ . \tag{3.9.3}$$

Proof. Theorem 3.16 implies the existence of an $(m+1)$-primitive of g, say G, such that

$$G(x)=\frac{\gamma\operatorname{sgn} x}{2m!}x^{m}+c(|x|)\frac{x^{m}}{m!}+o(|x|^{m}),\quad |x|\to\infty\ . \tag{3.9.4}$$

Set $G_{\pm}(x)=G(x)H(\pm x)$, where H is the Heaviside function. We have that (Corollary 3.13),

$$c(\lambda x)H(x)=c(\lambda)H(x)+o(1)\quad \text{as } \lambda\to\infty \text{ in } \mathcal{S}'(\mathbb{R})\ ,$$

which implies

$$G_{\pm}(\lambda x)=(\pm 1)^{m+1}\frac{\gamma}{2m!}(\lambda x)_{\pm}^{m}+(\pm 1)^{m}c(\lambda)\frac{(\lambda x)_{\pm}^{m}}{m!}+o(\lambda^{m})\quad \text{as } \lambda\to\infty \text{ in } \mathcal{S}'(\mathbb{R})\ .$$

If we set $g_{\pm}=G_{\pm}^{(m+1)}$, differentiating $(m+1)$-times the last two asymptotic expressions we obtain (3.9.2) and (3.9.3). Conversely, setting $h_{\pm}(x)=g_{\pm}(x)\mp(c(x)H(x))'$, an application of the structural theorem for the quasiasymptotic behavior of degree -1 with one-sided support to each $h_{\pm}$ implies that there exists m such that (3.9.4) is satisfied, and hence (3.9.1) follows. □

Due to Corollary 3.19, Theorem 3.50 may also be stated in the following equivalent form.

Theorem 3.51. *Let $g\in\mathcal{D}'(\mathbb{R})$ and $\phi\in\mathcal{E}(\mathbb{R})$. Then* e.v. $\langle f(x),\phi(x)\rangle=\gamma$ (C) *if and only if there exist a decomposition $g=g_{-}+g_{+}$, where* $\operatorname{supp} g_{-}\subseteq(-\infty,0]$ *and*

$\operatorname{supp} g_+ \subseteq [0,\infty)$, *and an asymptotically homogeneous function* c *of degree zero such that the following asymptotic relations hold*

$$\phi(\lambda x)g_+(\lambda x) = \left(\frac{\gamma}{2} + c(\lambda)\right)\frac{\delta(x)}{\lambda} + o\left(\frac{1}{\lambda}\right) \quad \textit{as } \lambda \to \infty \textit{ in } \mathcal{S}'(\mathbb{R}) \tag{3.9.5}$$

and

$$\phi(\lambda x)g_-(\lambda x) = \left(\frac{\gamma}{2} - c(\lambda)\right)\frac{\delta(x)}{\lambda} + o\left(\frac{1}{\lambda}\right) \quad \textit{as } \lambda \to \infty \textit{ in } \mathcal{S}'(\mathbb{R})\ . \tag{3.9.6}$$

We can now obtain the next abelian result.

Proposition 3.52. *Let* $g \in \mathcal{D}'(\mathbb{R})$ *and* $\phi \in \mathcal{E}(\mathbb{R})$. *Suppose that* e.v. $\langle g(x), \phi(x)\rangle = \gamma$ (C). *Then,* $\langle g(x), \phi(x)\rangle = \gamma$ (A). *Moreover, Let* $g = g_- + g_+$ *be a decomposition satisfying the support requirements of Theorem 3.51, then*

$$\lim_{z\to 0}\left(\left\langle g_-(t), \phi(t)e^{i\bar{z}}\right\rangle + \left\langle g_+(t), \phi(t)e^{iz}\right\rangle\right) = \gamma\ , \tag{3.9.7}$$

in any sector $\Im m\, z \geq M\,|\Re e\, z|$, *with* $M > 0$.

Proof. We may assume that $\phi \equiv 1$. We use (3.9.2) and (3.9.3). Write $z = (1/\lambda)(\tau + i)$, so $|\tau| \leq (1/M)$, hence, as $\lambda \to \infty$,

$$\begin{aligned}\left\langle g_-(t), e^{i\bar{z}}\right\rangle + \left\langle g_+(t), e^{iz}\right\rangle &= \lambda\left(\left\langle g_-(\lambda t), e^{(i\tau+1)t}\right\rangle + \left\langle g_+(t), e^{(i\tau-1)t}\right\rangle\right)\\ &= \left(\frac{\gamma}{2} - c(\lambda)\right) + \left(\frac{\gamma}{2} + c(\lambda)\right) + o(1)\\ &= \gamma + o(1)\ ,\end{aligned}$$

with uniform convergence since $\left\{e^{(i\tau-1)t}H(t)\right\}_{M|t|\leq 1}$ is compact in $\mathcal{S}[0,\infty)$. $\square$

So, we obtain the Fourier inversion formula in the Abel sense.

Corollary 3.53. *Let* $g \in \mathcal{S}'(\mathbb{R})$. *Suppose* $f(x_0) = \gamma$, *distributionally. Then the Fourier inversion formula holds in the Abel sense, i.e.,*

$$\frac{1}{2\pi}\left\langle \hat{f}(x), e^{ix_0x}\right\rangle = \gamma \quad \text{(A)}\ . \tag{3.9.8}$$

Moreover, let $\hat{f} = \hat{f}_- + \hat{f}_+$*, with* $\operatorname{supp} \hat{f}_- \subseteq (-\infty, 0]$ *and* $\operatorname{supp} \hat{f}_+ \subseteq [0, \infty)$*, then*

$$\lim_{z \to x_0} \frac{1}{2\pi} \left(\left\langle \hat{f}_-(t), e^{i\bar{z}} \right\rangle + \left\langle \hat{f}_+(t), e^{iz} \right\rangle \right) = \gamma \ , \tag{3.9.9}$$

in any sector $\Im m\, z \geq M \left| \Re e\, z - x_0 \right|$*, with* $M > 0$*.*

In the case of Fourier series, we obtain a result from [237].

Corollary 3.54. *Let* $f \in \mathcal{S}'(\mathbb{R})$ *be a* 2π*-periodic distribution having Fourier series*

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \ . \tag{3.9.10}$$

Suppose $f(x_0) = \gamma$*, distributionally. Then*

$$\lim_{z \to x_0} \left(c_0 + \sum_{n=1}^{\infty} \left(c_{-n} e^{-in\bar{z}} + c_n e^{inz} \right) \right) = \gamma \ , \tag{3.9.11}$$

in any sector $\Im m\, z \geq M \left| \Re e\, z - x_0 \right|$*, with* $M > 0$*. In particular, if* $a_n = c_{-n} + c_n$ *and* $b_n = i(c_n - c_{-n})$*, we obtain that*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx_0 + b_n \sin nx_0 \right) = \gamma \quad (\mathrm{A}) \ . \tag{3.9.12}$$

Proof. Relation (3.9.11) follows directly form Corollary 3.53. If we set $z = x_0 + iy$ in (3.9.11), we obtain

$$\lim_{y \to 0^+} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx_0 + b_n \sin nx_0 \right) e^{-ny} \right) = \gamma$$

which gives (3.9.12) $\square$

Let now $f \in \mathcal{D}'(\mathbb{R})$ have $f(x_0) = \gamma$, distributionally. We cannot longer talk about Abel summability of the Fourier inversion formula, since the Fourier transform is not available in $\mathcal{D}'(\mathbb{R})$. Nevertheless, there is a substitute of Abel summability, if we interpreted it as the boundary limit at $x = x_0$ of a harmonic representation (Section 1.6).

Theorem 3.55. *Let $f \in \mathcal{D}'(\mathbb{R})$. Suppose that U is a harmonic representation of f in the upper half-plane $\Im m\, z > 0$. If $f(x_0) = \gamma$, distributionally, then*

$$\lim_{z \to x_0} U(z) = \gamma, \tag{3.9.13}$$

in any sector $\Im m\, z \geq M\,|\Re e\, z - x_0|$, with $M > 0$.

Proof. We first see that we may assume $f \in \mathcal{S}'(\mathbb{R})$. Indeed we can decompose $f = f_1 + f_2$ where f_2 is zero in a neighborhood of x_0 and $f_1 \in \mathcal{S}'(\mathbb{R})$. Let U_1 and U_2 be harmonic representations of f_1 and f_2, respectively; then U_2 represents the zero distribution in a neighborhood of x_0. Then by applying the reflection principle to the real and imaginary parts of U_2 ([11, Section 4.5], [206, Section 3.4]), we have that U admits a harmonic extension to a (complex) neighborhood of x_0, and so it is real analytic, therefore, $U(z) - U_1(z) = U_2(z) = O\left(|z - x_0|\right)$ as $z \to x_0$. Additionally, $f_1(x_0) = \gamma$, distributionally, thus, we can assume that $f = f_1$. The same argument with the reflection principle shows that (3.9.13) is independent of the choice of U. Therefore, we can assume that U is the Fourier-Laplace representation [24] of f, that is, let $\hat{f} = \hat{f}_+ + \hat{f}_-$ be a decomposition such that $\operatorname{supp} \hat{f}_- \subseteq (-\infty, 0]$ and $\operatorname{supp} \hat{f}_+ \subseteq [0, \infty)$, we can assume that

$$U(z) = \frac{1}{2\pi}\left(\left\langle \hat{f}_-(t), e^{i\bar{z}} \right\rangle + \left\langle \hat{f}_+(t), e^{iz} \right\rangle\right), \quad \Im m\, z > 0.$$

But in this case, Corollary 3.53 yields (3.9.13) $\square$

Naturally, the converse of Theorem 3.55 is not true.

3.10 Symmetric Point Values

This section is devoted to the study of *symmetric point values* of distributions. They are studied by means of the symmetric part of a distribution about at given point $x = x_0$, that is, the distribution

$$\chi^f_{x_0}(x) := \frac{f(x + x_0) + f(x_0 - x)}{2} \tag{3.10.1}$$

Notice that $\chi^f_{x_0}$ is an even distribution.

Definition 3.56. *Let $f \in \mathcal{D}'(\mathbb{R})$ and $x_0 \in \mathbb{R}$. We say that f has a (distributional) symmetric point value γ at $x = x_0$ if its symmetric part about x_0 has a distributional point value at $x = 0$, that is, $\chi^f_{x_0}(0) = \gamma$, distributionally. In this case we write $f_{\mathrm{sym}}(x_0) = \gamma$, distributionally.*

Of course, the existence of the symmetric value at x_0 is equivalent to the quasi-asymptotic behavior

$$\chi^f_{x_0}(\varepsilon x) = \frac{f(x_0 + \varepsilon x) + f(x_0 - \varepsilon x)}{2} = \gamma + o(1) \quad \text{as } \epsilon \to 0^+ \text{ in } \mathcal{D}'(\mathbb{R}) \ , \qquad (3.10.2)$$

in other words,

$$\lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \left\langle f(x), \phi\left(\frac{x - x_0}{\varepsilon}\right) + \phi\left(\frac{x_0 - x}{\varepsilon}\right)\right\rangle = \gamma \int_{-\infty}^{\infty} \phi(x) \mathrm{d}x \ , \qquad (3.10.3)$$

for each $\phi \in \mathcal{D}(\mathbb{R})$.

Observe that if $\chi^f_{x_0} \in \mathcal{S}'(\mathbb{R})$, then (3.10.2) actually holds in the space $\mathcal{S}'(\mathbb{R})$.

If $f(x_0) = \gamma$, distributionally, then, obviously, $f_{\mathrm{sym}}(x_0) = \gamma$, distributionally. However, the existence of a symmetric point value is weaker than the existence of a distributional point value. For example $\delta'_{\mathrm{sym}}(0) = 0$, distributionally, but the usual distributional point value of δ' does not exist at $x = 0$.

We may use Łojasiewicz characterization of distributional point values (3.2.5) to characterize symmetric point values.

Theorem 3.57. *Let $f \in \mathcal{D}'(\mathbb{R})$ and $x_0 \in \mathbb{R}$. We have that $f_{\mathrm{sym}}(x_0) = \gamma$, distributionally, if and only if there exists $n \in \mathbb{N}$ and an n-primitive F of f such that $F(x_0 + x) + (-1)^n F(x_0 - x)$ is locally integrable in a neighborhood of the origin and*

$$F(x_0 + h) + (-1)^n F(x_0 - h) = 2\gamma \frac{h^n}{n!} + o(h^n), \quad h \to 0. \qquad (3.10.4)$$

Example 3.58. *(Symmetric Lebesgue points) Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. We say that f has a symmetric Lebesgue point value at $x = x_0$ if*

$$\lim_{h\to 0^+} \frac{1}{h}\int_0^h |f(x+x_0)+f(x_0-x)-2\gamma_{x_0}|\,\mathrm{d}x = 0\ ,$$

for some constant γ_{x_0}. Observe that at a Lebesgue point, we have that $f_{\mathrm{sym}}(x_0) = \gamma_{x_0}$, distributionally. Hence, distributional symmetric point values include symmetric Lebesgue points, which is usually the notion of symmetric point value used by analysts for L^p-functions.

Using Theorem 3.57, we can also describe distributional symmetric point values in terms of *de la Vallée Poussin derivatives* ([210],[256, Chapter XI]). Given a distribution f define its *jump distribution* at $x = x_0$ by

$$\psi^f_{x_0}(x) = f(x_0+x) - f(x_0-x)\ . \tag{3.10.5}$$

So that, $(1/2)\psi^f_{x_0}$ is the antisymmetric part of f about $x = x_0$. Then in the case that n is even in Theorem 3.57, we obtain that $\chi^F_{x_0}(h) = \gamma h^n/n! + o(h^n)$; but on the other hand when n is odd $\psi^F_{x_0}(h) = 2\gamma h^n/n! + o(h^n)$. Let now F_1 be an arbitrary n-primitive of f, then we obtain that F_1 is de la Vallée Poussin n-differentiable at $x = x_0$, that is, either

$$\chi^{F_1}_{x_0}(h) = a_0 + a_2h^2 + \cdots + \gamma h^n/n! + o(h^n), \quad \text{as } h \to 0\ ,$$

for some constants $a_0, a_2 \ldots$, when n is even, or

$$\frac{1}{2}\psi^{F_1}_{x_0}(h) = b_1h + b_3h^3 + \cdots + \gamma h^n/n! + o(h^n), \quad \text{as } h \to 0\ ,$$

for some constants $b_1, b_3 \ldots$, when n is odd.

3.11 Solution to the Hardy-Littlewood (C) Summability Problem for Distributions

As an application of Theorem 3.21, we now formulate and solve the so called Hardy-Littlewood (C) summability problem in the context of tempered distributions.

This classical problem aims to characterize trigonometric series, in cosines-sines form, which are (C) summable to some value at a point $x = x_0$, that is,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx_0 + b_n \sin nx_0) = \gamma \quad (\mathrm{C}, m) \ ,$$

for some γ and $m \in \mathbb{N}$. One also imposes the restrictions $a_n = O(n^k)$ and $b_n = O(n^k)$, for some k; thus, the trigonometric series represents a tempered distribution! The problem for trigonometric series was first formulated by Hardy and Littlewood in [89]; a complete treatment with historical remarks is found in [256, Chap.XI]; see also [61, pp.357–361] for a quick distributional solution.

In order to formulate the problem for tempered distributions, we need the following summability notion for distributional evaluations.

Definition 3.59. *Let $g \in \mathcal{D}'(\mathbb{R})$, $\phi \in \mathcal{E}(\mathbb{R})$, and $m \in \mathbb{N}$. We say that the principal value evaluation* p.v. $\langle g(x), \phi(x)\rangle$ *exists and is equal to γ in the Cesàro sense of order m, and write*

$$\text{p.v.}\ \langle g(x), \phi(x)\rangle = \gamma \quad (\mathrm{C}, m) \ , \tag{3.11.1}$$

if some first order primitive G of ϕg, i.e., $G' = \phi g$, satisfies

$$\lim_{x\to\infty}(G(x) - G(-x)) = \gamma \quad (\mathrm{C}, m) \ . \tag{3.11.2}$$

Note that e.v. $\langle g(x), \phi(x)\rangle = \gamma \ \ (\mathrm{C}, m)$ implies p.v. $\langle g(x), \phi(x)\rangle = \gamma \ \ (\mathrm{C}, m)$, as the reader can easily verify. On the other hand the converse is not true; take for example p.v. $\langle x, 1\rangle = 0 \ \ (\mathrm{C}, 0)$, but clearly the evaluation e.v. $\langle x, 1\rangle \ \ (\mathrm{C})$ does not exist.

When $g = \mu$ is a Radon measure, we write

$$\text{p.v.} \int_{-\infty}^{\infty} \phi(x)\mathrm{d}\mu(x) = \gamma \quad (\mathrm{C}, m) \ , \tag{3.11.3}$$

for (3.11.1). Observe that (3.11.3) explicitly means that

$$\lim_{x\to\infty} \int_{-x}^{x} \phi(t)\left(1 - \frac{t}{|x|}\right)^m \mathrm{d}\mu(t) = \gamma \ . \tag{3.11.4}$$

If $\mu = \sum_{n=-\infty}^{\infty} c_n \delta(\,\cdot\, - n)$ and $\phi \equiv 1$, then we write (3.11.3) as

$$\text{p.v.} \sum_{n=-\infty}^{\infty} c_n = \gamma \quad (\mathrm{C}, m)\ , \tag{3.11.5}$$

which is equivalent to have

$$c_0 + \sum_{n=1}^{\infty} (c_n + c_{-n}) = \gamma \quad (\mathrm{C}, m)\ . \tag{3.11.6}$$

Example 3.60. *Consider the trigonometric series* $\sum_{n=-\infty}^{\infty} c_n e^{inx_0}$ *then*

$$\text{p.v.} \sum_{n=-\infty}^{\infty} c_n e^{inx_0} = \gamma \quad (\mathrm{C}, m)$$

if and only if

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx_0 + b_n \sin nx_0) = \gamma \quad (\mathrm{C}, m)\ ,$$

with $a_n = c_n + c_{-n}$ *and* $b_n = i(c_n - c_{-n})$.

We can now formulate our problem: *we want to characterize tempered distributions* f *such that*

$$\frac{1}{2\pi} \text{p.v.} \left\langle \hat{f}(x), e^{ix_0 x} \right\rangle = \gamma \quad (\mathrm{C})\ . \tag{3.11.7}$$

We study some properties of the principal value evaluations in the (C) sense. They admit a quasiasymptotic characterization, but unlike e.v. Cesàro evaluations, the existence of p.v. $\langle g(x), \phi(x)\rangle = \gamma$ (C) does not imply that $\phi g \in \mathcal{S}'(\mathbb{R})$. We first need the following lemmas.

Lemma 3.61. *Let* $g \in \mathcal{D}'(\mathbb{R})$ *be an even distribution. There exists* $h \in \mathcal{D}'(\mathbb{R})$ *such that* $\operatorname{supp} h \subseteq [0, \infty)$ *and* $g(x) = h(x) + h(-x)$.

Proof. Decompose $g = g_- + g_+$, where $\operatorname{supp} g_- \subseteq (-\infty, 0]$ and $\operatorname{supp} g_+ \subseteq [0, \infty)$. The parity of g implies that $g_+(x) - g_-(-x)$ is concentrated at the origin, and so there exist constants such that

$$g_-(x) = g_+(-x) + \sum_{j=0}^{n} a_j \delta^{(j)}(x)\ , \tag{3.11.8}$$

Since, $g(x) - g_+(-x) - g_+(x) = \sum_{j=0}^{n} a_j \delta^{(j)}(x)$ is even, it follows that $a_j = 0$ whenever j is odd. So, $n = 2k$, and hence $h = g_+ + (1/2)\sum_{j=0}^{k} a_{2j}\delta^{(2j)}$ satisfies the requirements. $\square$

Lemma 3.62. *Let $g \in \mathcal{S}'(\mathbb{R})$ be an even distribution. Then*

$$g(\lambda x) = \gamma \frac{\delta(x)}{\lambda} + o\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R}) \tag{3.11.9}$$

if and only if any $h \in \mathcal{S}'(\mathbb{R})$ such that $\operatorname{supp} h \subseteq [0,\infty)$, *and $g(x) = h(x) + h(-x)$, satisfies*

$$h(\lambda x) = \frac{\gamma\delta(x)}{2\lambda} + o\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R})\ . \tag{3.11.10}$$

Proof. The converse is clear. On the other hand take h as in Lemma 3.61. Proposition 3.16 implies the existence of m such that

$$h^{(-2m)}(x) = \frac{\gamma x^{2m-1}}{2(2m-1)!} + c(x)\frac{x^{2m-1}}{(2m-1)!} + o(x^{2m-1})$$

and

$$h^{(-2m)}(x) = \frac{\gamma x^{2m-1}}{2(2m-1)!} - c(|x|)\frac{x^{2m-1}}{(2m-1)!} + o(x^{2m-1})\ ,$$

$x \to \infty$, but comparison between the last two expressions gives that $c(x) = o(1)$, and hence

$$h^{(-2m)}(x) = \frac{\gamma x^{2m-1}}{2(2m-1)!}, \quad x \to \infty\ ,$$

which implies (3.11.10). $\square$

Proposition 3.63. *Let $g \in \mathcal{D}'(\mathbb{R})$ and $\phi \in \mathcal{E}(\mathbb{R})$. Then,*

$$\text{p.v.}\ \langle g(x), \phi(x)\rangle = \gamma \quad (\mathrm{C}) \tag{3.11.11}$$

if and only if

$$\phi(-\lambda x)g(-\lambda x) + \phi(\lambda x)g(\lambda x) = 2\gamma\frac{\delta(x)}{\lambda} + o\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R})\ . \tag{3.11.12}$$

if and only if for any decomposition $g = g_- + g_+$*, where* $\operatorname{supp} g_- \subseteq (-\infty, 0]$ *and* $\operatorname{supp} g_+ \subseteq [0, \infty)$*,*

$$\phi(-\lambda x)g_-(-\lambda x)+\phi(\lambda x)g_+(\lambda x) = \gamma\frac{\delta(x)}{\lambda}+o\left(\frac{1}{\lambda}\right) \quad \textit{as } \lambda \to \infty \textit{ in } \mathcal{S}'(\mathbb{R}) \; . \quad (3.11.13)$$

In particular, we obtain that $\phi(-x)g(-x) + \phi(x)g(x) \in \mathcal{S}'(\mathbb{R})$*.*

Proof. Assume that $\phi \equiv 1$. We have that $g(-x)+g(x)$ is an even distribution, then, by Lemma 3.61, we can find h with $\operatorname{supp} h \subseteq [0, \infty)$ such that $g(-x)+g(x) = h(x)+h(-x)$. It is easy to see that (3.11.11) is equivalent to $\lim_{x\to\infty} h^{(-1)}(x) = \gamma \quad (\mathrm{C})$ which holds if and only if (3.11.10), and by Lemma 3.62, it is equivalent to (3.11.12). The equivalence with (3.11.13) follows by taking $h(x) = g_-(-x) + g_+(x)$. □

The right notion to characterize (3.11.7) is that of distributional symmetric point values from Section 3.10. We have already set the ground to solve our problem. The following theorem is the solution to the Hardy-Littlewood (C)-problem for tempered distributions.

Theorem 3.64. *Let* $f \in \mathcal{S}'(\mathbb{R})$*. Then*

$$\frac{1}{2\pi}\mathrm{p.v.} \left\langle \hat{f}(x), e^{ix_0x} \right\rangle = \gamma \quad (\mathrm{C}) \qquad (3.11.14)$$

if and only if $f_{\mathrm{sym}}(x_0) = \gamma$*, distributionally.*

Proof. By definition $f_{\mathrm{sym}}(x_0) = \gamma$, distributionally, if and only if,

$$f(x_0 - \varepsilon x) + f(x_0 + \varepsilon x) = \gamma + o(1) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{S}'(\mathbb{R}) \; ,$$

which, by taking Fourier transform, is equivalent to

$$e^{-i\lambda x_0 x}\hat{f}(-\lambda x) + e^{i\lambda x_0 x}\hat{f}(\lambda x) = 2\pi\gamma\frac{\delta(x)}{\lambda} + o\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R}) \; ,$$

and, by Proposition 3.63, the latter is equivalent to (3.11.14). □

We immediately obtain, by Theorem 3.64 and Example 3.60, the following result of Hardy and Littlewood. Naturally, the language in the original statement differs from ours, at that time distribution theory and quasiasymptotics did not even exist!

Corollary 3.65. *Let $f \in \mathcal{S}'(\mathbb{R})$ be a 2π periodic distribution having Fourier series, in cosines-sines form,*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \ . \tag{3.11.15}$$

Then,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx_0 + b_n \sin nx_0) = \gamma \quad \text{(C)} \tag{3.11.16}$$

if and only if $f_{\mathrm{sym}}(x_0) = \gamma$, distributionally.

We end this section by showing three abelian results.

Theorem 3.66. *Let $g \in \mathcal{D}'(\mathbb{R})$ and $\phi \in \mathcal{E}(\mathbb{R})$. If*

$$\text{p.v.} \left\langle g(x), \phi(x) \right\rangle = \gamma \quad \text{(C)} \ , \tag{3.11.17}$$

then,

$$\left\langle g(x), \phi(x) \right\rangle = \gamma \quad \text{(A)} \ . \tag{3.11.18}$$

Proof. Take g_- and g_+ as in Proposition 3.63, then, by (3.10.2), as $\lambda \to \infty$,

$$\begin{aligned} \left(\left\langle \phi(x) g_-(x), e^{\frac{x}{\lambda}} \right\rangle + \left\langle \phi(x) g_+(x), e^{-\frac{x}{\lambda}} \right\rangle \right) &= \lambda \left\langle \phi(-\lambda x) g_-(-\lambda x) + \phi(\lambda x) g_+(\lambda x), e^{-x} \right\rangle \\ &= \gamma \left\langle \delta(x), e^{-x} \right\rangle + o(1) \\ &= \gamma + o(1) \ . \end{aligned}$$

□

For symmetric point values, we get a radial version of Theorem 3.55.

Theorem 3.67. *Let $f \in \mathcal{D}'(\mathbb{R})$. Let U be a harmonic representation of f on the upper half-plane $\Im m\, z > 0$. If $f_{\text{sym}}(x_0)$, then*

$$\lim_{y \to 0^+} U(x_0 + iy) = \gamma. \tag{3.11.19}$$

Proof. As in the proof of Theorem 3.55, we may assume that f is a tempered distribution. If $\hat{f} = \hat{f}_+ + \hat{f}_-$ is a decomposition such that $\operatorname{supp} \hat{f}_- \subseteq (-\infty, 0]$ and $\operatorname{supp} \hat{f}_+ \subseteq [0, \infty)$, we can assume that

$$U(z) = \frac{1}{2\pi} \left(\left\langle \hat{f}_-(t), e^{i\bar{z}} \right\rangle + \left\langle \hat{f}_+(t), e^{iz} \right\rangle \right), \quad \Im m\, z > 0.$$

But in this case, Theorem 3.66 yields (3.11.19). □

The next corollary extends a result of Walter from [237].

Corollary 3.68. *Let $f \in \mathcal{S}'(\mathbb{R})$ be a 2π periodic distribution having Fourier series, in cosines-sines form,*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \ . \tag{3.11.20}$$

If $f_{\text{sym}}(x_0) = \gamma$, distributionally, then,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx_0 + b_n \sin nx_0) = \gamma \quad \text{(A)}. \tag{3.11.21}$$

Proof. Notice that

$$U(z) = \frac{a_0}{2} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n - ib_n)\, e^{izn} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n + ib_n)\, e^{-i\bar{z}n}$$

is a harmonic representation of f, so by Theorem 3.67,

$$\lim_{y \to 0^+} U(x_0 + iy) = \lim_{y \to \infty} \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx_0 + b_n \sin nx_0) e^{-yn} = \gamma \ .$$

Of course, we could have also used Corollary 3.65 to conclude (3.11.21), since (C) summability implies (A) summability. □

Chapter 4

Tauberian Theorems for Distributional Point Values

4.1 Introduction

The study of abelian and tauberian results for integral transforms of distributions has attracted the attention of several authors, and has produced several important generalizations of classical results [139, 149, 157, 159, 231]. These type of results have historically stimulated important developments in the theory. Also, the study of distributions as boundary values of analytic functions has shown to be quite important in the understanding of generalized functions [11, 24, 230, 231].

The aim of this chapter is to present a tauberian theorem for distributional point values. The following abelian result is well known, it was originally due to Constantinescu [31]:

Suppose that $f \in \mathcal{D}'(\mathbb{R})$ *is the boundary value of a function* F, *analytic in the upper half-plane, that is* $f(x) = F(x + i0)$; *if* $f(x_0) = \gamma$ *distributionally, then* $F(x_0 + iy) \to \gamma$ *as* $y \to 0^+$.

Notice that the above result is a particular case of Theorem 3.55, which we already remarked that can be viewed as Abel summability for non-tempered distributions. On the other hand [52], as pointed out in Section 3.9, the converse result is false.

In Theorem 4.7 we give a tauberian condition under which the converse of the abelian result holds, namely, we prove that the distribution has to be distributionally bounded at the point. The notions of distributional point values and distributional boundedness are reviewed in Section 4.2, we will use the approach introduced by J. Campos Ferreira [26] . We also show that when the distribution f is a bounded function near the point, then the distributional point value is of order 1. Furthermore, we give a general tauberian result of this kind for analytic functions that have distributional limits on a contour.

In Section 4.4, we apply our tauberian theorem to obtain a simple proof of a celebrated tauberian theorem of Hardy and Littlewood [5, 87, 88, 127].

Our results from Section 4.3 are used to give a tauberian theorem for the existence of distributional point values in terms of the Fourier transform, this is done in Section 4.5. It is remarkable that such result is more than a tauberian one, it is a characterization of distributional point values of tempered distributions whose Fourier transform is supported on $[0,\infty)$; the tauberian characterization, Theorem 4.5.4, is in terms of Abel summability of the Fourier inversion formula plus a Littlewood-type $O(1/\lambda)$ tauberian condition.

We study in Section 4.6 other related tauberian results related to boundary values of analytic functions and distributional point values.

The author would like to remark that some of the results of the chapter have been already published in [217].

4.2 Distributional Boundedness at a Point

Let us define distributional boundedness at a point. It was introduced by Z. Zieleźny in [254].

Definition 4.1. *Let $f \in \mathcal{D}'(\mathbb{R})$. We say that f is distributionally bounded at $x = x_0$ if it is quasiasymptotically bounded with respect to the constant function, that is,*

$$f(x_0+\varepsilon x) = O(1) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{D}'(\mathbb{R}) \ . \tag{4.2.1}$$

Observe the Definition 4.1 is meaningful if f is just defined in a neighborhood of $x = x_0$, since the quasiasymptotics are local properties. Because of the results of [54], if f is tempered, then (4.2.1) holds actually in the space $\mathcal{S}'(\mathbb{R})$; this fact actually holds for general quasiasymptotic boundedness (see Chapter 10).

We shall introduce the equivalent approach of J. Campos Ferreira to distributional point values and distributional boundedness [26]. It is in somehow connected with the structure of these two quasiasymptotic concepts. Let us introduce the operator μ_a which is defined on complex valued locally integrable functions defined in $\mathbb{R}$ as

$$\mu_a \{f(t); x\} = \frac{1}{x-a} \int_a^x f(t)\, \mathrm{d}t\,, \quad x \neq a\,, \tag{4.2.2}$$

while the operator ∂_a is the inverse of μ_a,

$$\partial_a (g) = ((x-a)\, g(x))'\,, \tag{4.2.3}$$

and it is defined on distributions. Suppose first that $f_0 = f$ is *real.* Then if it is bounded near $x = a$, we can define

$$\overline{f_0}(a) = \limsup_{x \to a} f(x)\,, \qquad \underline{f_0}(a) = \liminf_{x \to a} f(x)\,. \tag{4.2.4}$$

Then $f_1 = \mu_a(f)$ will be likewise bounded near $x = a$ and actually

$$\underline{f_0}(a) \le \underline{f_1}(a) \le \overline{f_1}(a) \le \overline{f_0}(a) \tag{4.2.5}$$

and, in particular, if $f(a) = f_0(a)$ exists, then $f_1(a)$ also exists and $f_1(a) = f_0(a)$. The next lemma is not difficult to show, we leave the verification as an exercise to the reader (see also [26]).

Lemma 4.2. *A distribution $f \in \mathcal{D}'(\mathbb{R})$ is distributionally bounded at $x = x_0$ if and only if there exist $n \in \mathbb{N}$ and $f_n \in \mathcal{D}'(\mathbb{R})$, bounded in a pointed neighborhood $(x_0 - \varepsilon, x_0) \cup (x_0, x_0 + \varepsilon)$ of x_0, such that $f = \partial_{x_0}^n f_n$.*

If f_0 is distributionally bounded at $x = x_0$, then there exists a *unique* distributionally bounded distribution near $x = x_0$, f_1, with $f_0 = \partial_{x_0} f_1$. Therefore, ∂_{x_0} and μ_{x_0} are isomorphisms of the space of distributionally bounded distributions near $x = x_0$. Given f_0 we can form a sequence of distributionally bounded distributions $\{f_n\}_{n=-\infty}^{\infty}$ with $f_n = \partial_{x_0} f_{n+1}$ for each $n \in \mathbb{Z}$.

We have an analogous result for distributional point values, again, we leave the proof of the following lemma as an exercise for the reader (see also [26]).

Lemma 4.3. *A distribution $f \in \mathcal{D}'(\mathbb{R})$ satisfies $f(x_0) = \gamma$, distributionally, if and only if there exist $n \in \mathbb{N}$ and $f_n \in \mathcal{D}'(\mathbb{R})$, continuous near x_0, such that $f = \partial_{x_0}^n f_n$ and $f_n(x_0) = \gamma$. We say that the point value is of order n.*

Observe also that if $f = \partial_{x_0}^n f_n$, and f_n is bounded near $x = x_0$, then $f(x_0)$ exists distributionally, and equals γ, if and only if $f_n(x_0) = \gamma$, distributionally.

Example 4.4. *The functions $x^\alpha e^{i/x}$, where $\alpha \in \mathbb{R}$, have regularizations $f_\alpha \in \mathcal{D}'(\mathbb{R})$ that are distributionally bounded near $x = 0$, and, in fact, $f_\alpha(0) = 0$, distributionally. Observe that if $\alpha < 0$ then f_α is unbounded near $x = 0$ in the ordinary sense. Similarly, the functions $x^\alpha e^{i/|x|^\beta}$ have regularizations $g_{\alpha,\beta} \in \mathcal{D}'(\mathbb{R})$ with $g_{\alpha,\beta}(0) = 0$, distributionally, but if $\alpha < 0$ and $\beta > 0$ is small, the order of the point value can be very large.*

Example 4.5. *The function $f(x) = |x|^i$ is bounded in the ordinary sense, and thus it defines a unique regular distribution which is distributionally bounded at $x = x_0$. It easy to see that $f(0)$ does not exist distributionally. In general the evaluation $\langle f(\varepsilon x), \phi(x)\rangle$ does not tend to a limit as $\varepsilon \to 0$ if $\phi \in \mathcal{D}(\mathbb{R})$.*

These notions have straightforward extensions to distributions defined in smooth contours of the complex plane.

4.3 Tauberian Theorem for Distributional Point Values

We start with a tauberian result for bounded analytic functions.

Theorem 4.6. *Let F be analytic and bounded in a rectangular region of the form $(a,b) \times (0,R)$. Suppose $f(x) = \lim_{y\to 0^+} F(x+iy)$ in the space $\mathcal{D}'(a,b)$. Let $x_0 \in$*

(a,b) *such that*

$$\lim_{y\to 0^+} F\left(x_0+iy\right)=\gamma\,. \tag{4.3.1}$$

Then

$$f\left(x_0\right)=\gamma\ ,\quad \textit{distributionally}. \tag{4.3.2}$$

In fact, (4.3.2) is a point value of the first order, and thus

$$\lim_{x\to x_0}\frac{1}{x-x_0}\int_{x_0}^{x} f\left(t\right)\,\mathrm{d}t=\gamma\,. \tag{4.3.3}$$

Proof. We shall first show that it is enough to prove the result if the rectangular region is the upper half-plane $\mathbb{H}=\{z\in\mathbb{C}:\Im m\, z>0\}$. Indeed, let C be a smooth simple closed curve contained in $(a,b)\times[0,R)$ such that $\mathsf{C}\cap(a,b)=[x_0-\eta,x_0+\eta]$, and which is symmetric with respect to the line $\Re e\, z=x_0$. Let φ be a conformal bijection from $\mathbb{H}$ to the region enclosed by C such that the image of the line $\Re e\, z=x_0$ is contained in $\Re e\, z=x_0$, so that, in particular, $\varphi\left(x_0\right)=x_0$. Then (4.3.1) holds if and only if $F\circ\varphi\left(x_0+iy\right)\to\gamma$ as $y\to 0^+$, while (4.3.2) and (4.3.3) hold if and only if the corresponding equations hold for a distribution given locally as $f\circ\varphi$ near $x=x_0$.

Therefore we shall assume that $a=-\infty$, and $b=R=\infty$. In this case, f belongs to H^∞, the closed subspace of $L^\infty\left(\mathbb{R}\right)$ consisting of the boundary values of bounded analytic functions on $\mathbb{H}$ ([113]); moreover, one easily verifies that H^∞ is a weak* closed subspace of L^∞, this fact will be used below. Let $f_\varepsilon\left(x\right)=f\left(x_0+\varepsilon x\right)$. Then the set $\{f_\varepsilon:\varepsilon\neq 0\}$ is weak* bounded (as a subset of the dual space $\left(L^1\left(\mathbb{R}\right)\right)'=L^\infty\left(\mathbb{R}\right)$) and, consequently, a relatively weak* compact set. Suppose that $\{\varepsilon_n\}_{n=0}^\infty$ is a sequence of non-zero numbers with $\varepsilon_n\to 0$ such that the sequence $\{f_{\varepsilon_n}\}_{n=0}^\infty$ is weak* convergent to $g\in L^\infty\left(\mathbb{R}\right)$. It will be shown that $g\equiv\gamma$. Since $g\in H^\infty$, we can write it as $g\left(x\right)=G\left(x+i0\right)$ where G is a bounded analytic function in $\mathbb{H}$, then the weak* convergence of f_{ε_n} to g implies that $F\left(x_0+\varepsilon_n z\right)$ converges to

$G(z)$ uniformly on compacts of $\mathbb{H}$, and thus $G(iy) = \gamma$ for all $y > 0$. It follows that $G \equiv \gamma$, and so $g \equiv \gamma$. Since any sequence $\{f_{\varepsilon_n}\}_{n=0}^{\infty}$ with $\varepsilon_n \to 0$ has a weak* convergent subsequence, and since that subsequence converges to the constant function γ, we conclude that $f_\varepsilon \to \gamma$ in the weak* topology of $L^\infty(\mathbb{R})$.

That $f(x_0) = \gamma$, distributionally, is now clear, because $\mathcal{D}(\mathbb{R}) \subset L^1(\mathbb{R})$.

On the other hand, (4.3.3) follows by taking $x = x_0 + \varepsilon$ and $\phi(t) = \chi_{[0,1]}(t)$, the characteristic function of the unit interval, in the limit $\lim_{\varepsilon\to 0} \langle f_\varepsilon(t), \phi(t)\rangle = \gamma \int_{-\infty}^{\infty} \phi(t)\, \mathrm{d}t$, which in view of the previous argument holds now for $\phi \in L^1(\mathbb{R})$. $\square$

We can now prove our tauberian theorem.

Theorem 4.7. *Let F be analytic in a rectangular region of the form $(a,b)\times(0,R)$. Suppose $f(x) = \lim_{y\to 0^+} F(x+iy)$ in the space $\mathcal{D}'(a,b)$. Let $x_0 \in (a,b)$ such that $\lim_{y\to 0^+} F(x_0+iy) = \gamma$. If f is distributionally bounded at $x = x_0$ then $f(x_0) = \gamma$, distributionally.*

Proof. There exists $n \in \mathbb{N}$ and a function f_n bounded in a neighborhood of x_0 such that $f = \partial_{x_0}^n f_n$; notice that $f(x_0) = \gamma$, distributionally, if and only if $f_n(x_0) = \gamma$, distributionally. But $f_n(x) = F_n(x+i0)$, as distributional boundary value, where F_n is analytic in $(a,b)\times(0,R)$; here F_n is the only angularly bounded solution of $F(z) = \partial_{x_0}^n F_n(z)$ (derivatives with respect to z). Since f_n is bounded near $x = x_0$, F_n is also bounded in a rectangular region of the form $(a_1,b_1)\times(0,R_1)$, where $x_0 \in (a_1,b_1)$. Clearly $\lim_{y\to 0^+} F_n(x_0+iy) = \gamma$, so the Theorem 4.6 yields $f_n(x_0) = \gamma$, distributionally, as required. $\square$

Observe that in general the result (4.3.3) does not follow if f is not bounded but just distributionally bounded near x_0.

The condition (4.3.1) may seem weaker than the angular convergence of $F(z)$ to γ as $z \to x_0$, however, if F is angularly bounded, which is the case if f is distributionally bounded at $x = x_0$, then angular convergence and radial convergence are equivalent. In fact [32, Thm. 13.5.4] both conditions are equivalent to the existence of an arc $\kappa : [0,1] \longrightarrow (a,b) \times [0,R)$ such that $\kappa([0,1)) \subset \{z \in \mathbb{C} : \Im m\, z \geq m\,|\Re e\, z - x_0|\}$ for some $m > 0$ and such that $\kappa(1) = x_0$, for which

$$\lim_{t \to 1^-} F(\kappa(t)) = \gamma\,. \tag{4.3.4}$$

Therefore, we may use a conformal map to obtain the following general form of the Theorem 4.7.

Theorem 4.8. *Let* C *be a smooth part of the boundary* $\partial\Omega$ *of a region* Ω *of the complex plane. Let* F *be analytic in* Ω*, and suppose that* $f \in \mathcal{D}'(\mathsf{C})$ *is the distributional boundary limit of* F*. Let* $\xi_0 \in \mathsf{C}$ *and suppose that* κ *is an arc in* Ω *that ends at* ξ_0 *and that approaches* C *angularly. If* $\lim_{t\to 1^-} F(\kappa(t)) = \gamma$ *and* f *is distributionally bounded at* $\xi = \xi_0$*, then* $f(\xi_0) = \gamma$*, distributionally.*

4.4 Application: Proof of a Hardy-Littlewood Tauberian Theorem

In this last section, we discuss an application of Theorem 4.7. Our application is a new proof of a famous tauberian theorem of Hardy and Littlewood. In fact, the version we prove here was conjectured by Littlewood in 1913 [127], but it was first proved by Ananda Rau in 1928 [5].

We begin with a lemma whose proof can be tracked down to the proof of the original first Tauber's theorem ([85, p.149], [204]).

Lemma 4.9. *Let* $\{b_n\}_{n=0}^{\infty}$ *be a sequence of complex numbers. Suppose that* $\{\lambda_n\}_{n=0}^{\infty}$ *is an increasing sequence of positive real numbers such that* $\lambda_n \to \infty$ *as* $n \to \infty$*. If*

$$b_n = O\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right), \tag{4.4.1}$$

then,
$$\sum_{n=0}^{\infty} b_n e^{-\lambda_n y} - \sum_{\lambda_n < \frac{1}{y}} b_n = O(1)\,, \quad \text{as} \ \ y \to 0^+. \tag{4.4.2}$$

Proof. Choose M such that $|b_n| \leq M\lambda_n^{-1}(\lambda_n - \lambda_{n-1})$, for every n. Then,

$$\begin{aligned}\left| \sum_{n=0}^{\infty} b_n e^{-\lambda_n y} - \sum_{\lambda_n < \frac{1}{y}} b_n \right| &\leq \sum_{\lambda_n < \frac{1}{y}} |b_n| \left(1 - e^{-\lambda_n y}\right) + \sum_{\frac{1}{y} \leq \lambda_n} |b_n|\, e^{-\lambda_n y} \\ &\leq My \sum_{\lambda_n < \frac{1}{y}} (\lambda_n - \lambda_{n-1}) + My \sum_{\frac{1}{y} \leq \lambda_n} (\lambda_n - \lambda_{n-1})\, e^{-\lambda_n y} \\ &= O(1) + My \int_{\frac{1}{y}}^{\infty} e^{-yt} \mathrm{d}t = O(1)\ , \quad y \to 0^+\ ,\end{aligned}$$

as required. □

Recall that a series $\sum_{n=0}^{\infty} c_n$ is (A, λ_n) summable to γ if $\sum_{n=0}^{\infty} c_n e^{-\lambda_n y} \to \gamma$ as $y \to 0^+$, this was defined in Section 3.3.1 (see also [85]). When $\lambda_n = n$ we obtain the notion of Abel summability, and the tauberian condition (4.4.1) becomes Littlewood's tauberian hypothesis [127, 85], that is, $nc_n = O(1)$. Then we have the ensuing Hardy-Littlewood tauberian theorem.

Theorem 4.10. *Suppose that $\{\lambda_n\}_{n=0}^{\infty}$ is an increasing sequence of non-negative real numbers such that $\lambda_n \to \infty$, as $n \to \infty$. If*
$$\sum_{n=0}^{\infty} c_n = \gamma \ \ (\mathrm{A}, \lambda_n)\,, \tag{4.4.3}$$
and $c_n = O\left(\lambda_n^{-1}(\lambda_n - \lambda_{n-1})\right)$, then $\sum_{n=0}^{\infty} c_n = \gamma$.

Proof. The plan of the proof is to associate to the series the tempered distribution $f(x) = \sum_{n=0}^{\infty} c_n e^{i\lambda_n x}$ and show that $f(0) = \gamma$, distributionally, based on this conclusion, we will deduce the convergence of the series. Let us first verify that f defines a tempered distribution; indeed from Lemma 4.9 and the assumption (4.4.3), we have that $G(x) = \sum_{\lambda_n < x} c_n$ is a bounded function, hence f is the Fourier transform of its derivative $g(x) = G'(x) = \sum_{n=0}^{\infty} c_n \delta(x - \lambda_n)$.

Now, note that $F(z) = \sum_{n=0}^{\infty} c_n e^{i\lambda_n z}$, for $\Im m\ z > 0$, is an analytic representation of f and by hypothesis $F(iy) \to \gamma$ as $y \to 0^+$. We show that $f(\varepsilon x) = O(1)$ as $\varepsilon \to 0$ in $\mathcal{S}'(\mathbb{R})$. Take $\phi \in \mathcal{S}(\mathbb{R})$. Set $\psi = \hat{\phi}$. Then $\langle f(\varepsilon x), \phi(x)\rangle = (1/\varepsilon)\left\langle g\left(x/\varepsilon\right), \psi(x)\right\rangle$, so to show that $f(\varepsilon x) = O(1)$ in $\mathcal{S}'(\mathbb{R})$ is equivalent to show that $\lambda g(\lambda x) = O\left(1\right)$ as $\lambda \to \infty$ in $\mathcal{S}'(\mathbb{R})$. But from the Lemma 4.9 once again it follows that $G(x) = \sum_{\lambda_n < x} c_n = O(1)$, hence, by Proposition 1.9, $G(\lambda x) = O(1)$ in $\mathcal{S}'(\mathbb{R})$, and therefore by differentiating $G(\lambda x)$ with respect to x, we obtain that $\lambda g(\lambda x)$ is bounded in $\mathcal{S}'(\mathbb{R})$. Therefore, by Theorem 4.7

$$f(\varepsilon x) = \gamma + o(1)\,, \quad \text{in } \mathcal{S}'(\mathbb{R})\,. \tag{4.4.4}$$

As it is easily seen, condition (4.4.4) is equivalent to

$$\lim_{\varepsilon \to 0^+} \sum_{n=0}^{\infty} c_n \phi(\varepsilon \lambda_n) = \gamma\,, \quad \text{for each} \quad \phi \in \mathcal{S}(\mathbb{R})\,. \tag{4.4.5}$$

To conclude the proof, we take in (4.4.5) suitable test functions. Let $\sigma > 0$ and let us choose the test function $\phi \in \mathcal{D}(\mathbb{R})$ such that $0 \leq \phi \leq 1$, $\phi(x) = 1$ for $x \in [0,1]$, $\operatorname{supp} \phi \subseteq [-1,2]$, ϕ is decreasing on the interval $(1,2)$, and such that $\int_1^2 \phi(x)\ \mathrm{d}x < \sigma$ where M. Then

$$\begin{aligned}\limsup_{N\to\infty} \left| \sum_{n=0}^{N} c_n - \gamma \right| &\leq \left(\limsup_{N\to\infty} \sum_{\lambda_N < \lambda_n \leq 2\lambda_N} \frac{\lambda_n - \lambda_{n-1}}{\lambda_N}\, \phi\left(\frac{\lambda_n}{\lambda_N}\right)\right) O(1) \\ &\leq \left(\int_1^2 \phi(x)\mathrm{d}x\right) O(1) < \sigma O(1)\,.\end{aligned}$$

Since σ was arbitrary, we conclude that $\sum_{n=0}^{\infty} c_n = \gamma$. □

4.5 A Fourier Transform Tauberian Condition

Theorem 4.7 may also be used to obtain Littlewood type tauberian results for distributions. The first corollary is also contained in the general theory of Vladimirov, Drozhzhinov, and Zavialov [231]

Corollary 4.11. *Let g be a tempered distribution supported on $[0,\infty)$. Suppose that*

$$\lim_{y\to 0^+} \left\langle g(x), e^{-yx} \right\rangle = \gamma. \tag{4.5.1}$$

Then, the tauberian condition

$$g(\lambda x) = O\left(\frac{1}{\lambda}\right) \quad \textit{as } \lambda \to \infty \textit{ in } \mathcal{D}'(\mathbb{R}) \tag{4.5.2}$$

implies that g has the quasiasymptotic behavior

$$g(\lambda x) = \gamma \frac{\delta(x)}{\lambda} + o\left(\frac{1}{\lambda}\right) \quad \textit{as } \lambda \to \infty \textit{ in } \mathcal{S}'(\mathbb{R})\ . \tag{4.5.3}$$

Proof. Let f be such that $\hat{f} = g$. Then (4.5.1) translates into $F(iy) \to \gamma$ as $y \to 0^+$, where $F(z) = \langle g(t), e^{izt} \rangle$ (hence $f(x) = F(x + i0)$) and (4.5.2) corresponds to the statement f distributionally bounded at $x = 0$, by Theorem 4.7, we have that $f(0) = \gamma$, distributionally. Thus, Fourier inverse transform yields (4.5.3). □

Corollary 4.12. *Let g be a tempered distribution supported on $[0,\infty)$ and $\phi \in \mathcal{E}(\mathbb{R})$. Suppose that $\langle g(x), \phi(x) \rangle = \gamma \ \ (\mathrm{A})$. Then, the tauberian condition*

$$\phi(\lambda x) g(\lambda x) = O\left(\frac{1}{\lambda}\right) \quad \textit{as } \lambda \to \infty \textit{ in } \mathcal{D}'(\mathbb{R})$$

implies that $\langle g(x), \phi(x) \rangle = \gamma \ \ (\mathrm{C})$.

Proof. Corollary 4.11 gives that

$$\phi(\lambda x) g(\lambda x) = \gamma \frac{\delta(x)}{\lambda} + o\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R})\ ,$$

which by Proposition 3.9 implies that $\langle g(x), \phi(x) \rangle = \gamma \ \ (\mathrm{C})$. □

So, we obtain our Littlewood-type tauberian characterization for distributional point values in terms of the Fourier transform.

Theorem 4.13. *Let $f \in \mathcal{S}'(\mathbb{R})$ such that* $\operatorname{supp} \hat{f} \subseteq [0,\infty)$*. The following two conditions*

$$\frac{1}{2\pi}\left\langle \hat{f}(x), e^{ix_0x}\right\rangle = \gamma \quad (\mathrm{A}) \ . \tag{4.5.4}$$

and

$$e^{i\lambda x_0 x}\hat{f}(\lambda x) = O\left(\frac{1}{\lambda}\right) \qquad \text{as } \lambda \to \infty \text{ in } \mathcal{D}'(\mathbb{R}) \tag{4.5.5}$$

are necessary and sufficient for

$$f(x_0) = \gamma \ , \quad \textit{distributionally} \ . \tag{4.5.6}$$

Proof. The necessity of (4.5.4) and (4.5.5) is clear, while the sufficiency follows from Theorem 3.27 and Corollary 4.12. □

4.6 Other Tauberian Results

Łojasiewicz introduced the definition of lateral limits of distributions at a point in [128]. Here, we present an alternative definition following the ideas of Section 4.2.

Definition 4.14. *A distribution $f \in \mathcal{D}'(\mathbb{R})$ is said to have a distributional right lateral limit at x_0 if there exist $n \in \mathbb{N}$ and $f_n \in \mathcal{D}'(\mathbb{R}\setminus x_0)$, locally bounded in an interval $(x_0, x_0+\varepsilon)$, such that $f = \partial^n_{x_0} f_n$ in $\mathcal{D}'(\mathbb{R}\setminus x_0)$ and* $\lim_{x\to x_0^+} f_n(x) = f_n(x_0^+) = \gamma_+$*. In such a case we write $f(x_0^+) = \gamma$, distributionally.*

Left lateral limits are defined in a similar fashion. We use the notation $f(x_0^-) = \gamma_-$, distributionally. We say that the distributional limit of f exists at $x = x_0$, distributionally, if both $f(x_0^{\pm}) = \gamma_{\pm}$ exist and $\gamma_+ = \gamma_- := \gamma$, in such a case we call γ the limit of the distribution at $x = x_0$. Naturally, the existence of the distributional point value at x_0 implies the existence of the distributional limit at $x = x_0$, but the converse is not true; for example $\delta(0^{\pm}) = 0$, distributionally, however, $\delta(0)$ does not exist.

The following abelian type result was shown in [55]:

Suppose that $f \in \mathcal{D}'(\mathbb{R})$ *is the boundary value of a function* F *analytic in the upper half-plane, that is* $f(x) = F(x + i0)$; *if the distributional lateral limits* $f\left(x_0^{\pm}\right) = \gamma_{\pm}$ *both exist, then* $\gamma_+ = \gamma_- = \gamma$, *and so the distributional limit* of f at $x = x_0$ *exists and equals* γ.

On the other hand, the results of [52], imply that there are distributions $f(x) = F(x + i0)$ for which one distributional lateral limit exits but not the other. As we pointed out before, the distributional point value does not have to exist in this situation.

We give below a sort of tauberian condition under which the existence of the distributional point value can be deduced, namely, if the distribution is distributionally bounded at the point, and just one lateral limit exists. Furthermore, we give a general version of this kind for analytic functions that have distributional limits on a contour. These results are used to give an interesting extension of Theorem 4.7.

We shall need the following well-known fact [11].

Lemma 4.15. *Let* F *be analytic in the half plane* $\mathbb{H}$, *and suppose that the distributional limit* $f(x) = F(x + i0)$ *exists in* $\mathcal{D}'(\mathbb{R})$. *Suppose that there exists an open, non-empty interval* I *such that* f *is equal to the constant* γ *in* I. *Then* $f = \gamma$ *and* $F = \gamma$.

Proof. In fact, it follows from the edge of the wedge theorem (see Section 1.6). $\square$

Actually using the theorem of Privalov [167, Cor 6.14], it is easy to see that if F is analytic in the half plane $\mathbb{H}$, $f(x) = F(x + i0)$ exists in $\mathcal{D}'(\mathbb{R})$, and there exists a subset $X \subset \mathbb{R}$ of non-zero measure such that the distributional point value $f(x_0)$ exists and equals γ if $x_0 \in X$, then $f = \gamma$ and $F = \gamma$.

Our first result is for *bounded* analytic functions. The proof is almost the same as that of Theorem 4.6, but we include it for completeness.

Theorem 4.16. *Let F be analytic and bounded in a rectangular region of the form $(a,b)\times(0,R)$. Suppose $f(x)=\lim_{y\to 0^+}F(x+iy)$ in the space $\mathcal{D}'(a,b)$. Let $x_0\in(a,b)$ such that the lateral limit*

$$f(x_0^+)=\gamma\,,\quad \textit{distributionally}, \tag{4.6.1}$$

exists. Then the distributional point value

$$f(x_0)=\gamma\,,\quad \textit{distributionally}, \tag{4.6.2}$$

also exists. In fact, (4.6.2) is a point value of the first order, and thus

$$\lim_{x\to x_0}\frac{1}{x-x_0}\int_{x_0}^{x}f(t)\,\mathrm{d}t=\gamma\,. \tag{4.6.3}$$

Proof. As in the proof of Theorem 4.6, we may assume that $a=-\infty$, and $b=R=\infty$. In this case, f belongs to H^∞. Let $f_\varepsilon(x)=f(x_0+\varepsilon x)$. Then the set $\{f_\varepsilon:\varepsilon\neq 0\}$ is weak* bounded (as a subset of the dual space $(L^1(\mathbb{R}))'=L^\infty(\mathbb{R})$) and, consequently, a relatively weak* compact set. If $\{\varepsilon_n\}_{n=0}^\infty$ is a sequence of positive numbers with $\varepsilon_n\to 0$ such that the sequence $\{f_{\varepsilon_n}\}_{n=0}^\infty$ is weak* convergent to $g\in L^\infty(\mathbb{R})$, then $g\equiv\gamma$, since $g\in H^\infty$, and $g(x)=\gamma$ for $x>0$. Since any sequence $\{f_{\varepsilon_n}\}_{n=0}^\infty$ with $\varepsilon_n\to 0$ has a weak* convergent subsequence, and since that subsequence converges to the constant function γ, we conclude that $f_\varepsilon\to\gamma$ in the weak* topology of $L^\infty(\mathbb{R})$. We obtain that $f(x_0)=\gamma$, distributionally, since $\mathcal{D}(\mathbb{R})\subset L^1(\mathbb{R})$. On the other hand, (4.6.3) follows by taking $x=x_0+\varepsilon$ and $\phi(t)=\chi_{[0,1]}(t)$, the characteristic function of the unit interval, in the limit $\lim_{\varepsilon\to 0}\langle f_\varepsilon(t),\phi(t)\rangle=\gamma\int_{-\infty}^{\infty}\phi(t)\,\mathrm{d}t$. □

Exactly the same argument used in the proof of Theorem 4.7, but applying Theorem 4.16 instead of Theorem 4.6, gives us the next result.

Theorem 4.17. *Let F be analytic in a rectangular region of the form $(a,b)\times(0,R)$. Suppose $f(x)=\lim_{y\to 0^+}F(x+iy)$ in the space $\mathcal{D}'(a,b)$. Let $x_0\in(a,b)$ such* *that* $f(x_0^+) = \gamma$, *distributionally. If* f *is distributionally bounded at* $x = x_0$, *then* $f(x_0) = \gamma$, *distributionally.*

We may use a conformal map to obtain the following general form of the Theorem 4.17.

Theorem 4.18. *Let* C *be a smooth part of the boundary* $\partial\Omega$ *of a region* Ω *of the complex plane. Let* F *be analytic in* Ω, *and suppose that* $f \in \mathcal{D}'(\mathsf{C})$ *is the distributional boundary limit of* F. *Let* $\xi_0 \in \mathsf{C}$ *and suppose that the distributional lateral limit* $f\left(\xi_0^+\right) = \gamma$, *distributionally, exists and* f *is distributionally bounded at* $\xi = \xi_0$, *then* $f(\xi_0) = \gamma$, *distributionally.*

We now use Theorem 4.17 to obtain an interesting generalization of Theorem 4.7.

Theorem 4.19. *Let* F *be analytic in a rectangular region of the form* $(a, b)\times(0, R)$. *Suppose* $f(x) = \lim_{y\to 0^+} F(x+iy)$ *in the space* $\mathcal{D}'(a, b)$. *Let* $x_0 \in (a, b)$ *such that the distributional limit* $\lim_{y\to 0^+} F(x_0+iy) = \gamma$ *exists in the sense of Definition 4.14. If* f *is distributionally bounded at* $x = x_0$ *then* $f(x_0) = \gamma$, *distributionally, and the ordinary limit exists:* $\lim_{y\to 0^+} F(x_0+iy) = \gamma$.

Proof. If we consider the curve C to be the union of the segments $(a, x_0]$ and $[x_0, iR)$, then the distributional lateral limit of the boundary value of F on C exists and equals γ as we approach x_0 from the right along C and so the Theorem 4.17 yields that the distributional limit from the left, which is nothing but $f\left(x_0^-\right)$ also exists and equals γ, distributionally. Then the Theorem 4.17, applied again, gives us that $f(x_0) = \gamma$, distributionally. The existence of the angular limit of $F(z)$ as $z \to x_0$ then follows, and, in particular, $\lim_{y\to 0^+} F(x_0+iy) = \gamma$. $\square$

Chapter 5

The Jump Behavior and Logarithmic Averages

5.1 Introduction

In this chapter we study several notions for pointwise jumps of distributions. We characterize them first by their structure and then by the asymptotic properties of the Fourier transform.

We also study the jump by using logarithmic averages. In the case that f is an ordinary function this is a classical subject, perhaps the place where this idea has been widely applied is in Fourier series. Let f be a function of period 2π having Fourier series,

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)\ . \tag{5.1.1}$$

Let

$$\sum_{n=1}^{\infty}\left(a_n\sin nx-b_n\cos nx\right) \tag{5.1.2}$$

be its conjugate series. A classical theorem of F. Lukács [131], [256, Thm. 8.13] states that if f is $L^1[-\pi,\pi]$ and there is a number d such that

$$\lim_{h\to 0^+}\frac{1}{h}\int_0^h\left|f(x_0+t)-f(x_0-t)-d\right|\mathrm{d}t=0\ , \tag{5.1.3}$$

then

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\left(a_n\sin nx_0-b_n\cos nx_0\right)=-\frac{d}{\pi}\ . \tag{5.1.4}$$

Relation (5.1.3) can be considered as a notion of jump at $x=x_0$ for the function f, we shall call it *symmetric Lebesgue jump behavior*, in analogy with the notion of Lebesgue point. The formula (5.1.4) for symmetric Lebesgue jump behaviors was extended later by A. Zygmund to the Abel-Poisson means of the conjugate Fourier series [256].

Recently many extensions of these results have been given [9, 54, 70, 118, 119, 120, 121, 140, 141, 248]. The study of the jump behavior and the determination of jumps by logarithmic or other types of means has become an important area because of its applications in edge detection [66, 67]. F. Móricz generalizes the Abel-Poisson version of F. Lukács result in [140, 141] by extending the notion of symmetric Lebesgue jump (5.1.3). He considered a more general notion for jump of integrable functions, namely, the existence of the limit

$$d = \lim_{h\to 0^+} \frac{1}{h} \int_0^h \left(f(x_0+t) - f(x_0-t)\right) \mathrm{d}t \ , \tag{5.1.5}$$

and he showed that

$$\lim_{r\to 1^-} \frac{1}{\log(1-r)} \sum_{n=1}^{\infty} (a_n \sin nx_0 - b_n \cos nx_0) r^n = \frac{1}{\pi}\, d \ . \tag{5.1.6}$$

It was noticed by the author and R. Estrada in [54, 218] that the jump F. Móricz considered is a particular case of a symmetric jump behavior in the sense of distributions, that is, one can define it in terms of the very well known Łojasiewicz notion of limits of distributions at points [128]. Because of that reason, we should call (5.1.5) a *first order symmetric jump.* In the cited paper the author gave the corresponding generalization of F. Móricz result to distributions in terms of logarithmic Abel-Poisson means as well.

We will consider in this chapter two notions of jumps for distributions, the *distributional jump behavior* and the *distributional symmetric jump behavior* of distributions (Section 5.2). We will give a Fourier characterizations of these notions in Section 5.3, we then proceed in Section 5.4 to study the non-tangential limits of harmonic representations under the presence of a jump behavior. We will also consider several logarithmic averages for both notions of jump. In Section 5.5, we will give formulas for the jump occurring in the jump behavior case in terms of Cesàro-logarithmic means of a decomposition of the Fourier transform; it is remarkable that these results are applicable to general tempered distributions. Next, in Section 5.6 we study the boundary behavior of analytic representations of distributions at approaching angularly from the upper and lower half-planes to a point where the distribution possesses a jump behavior; it is shown they have an asymptotic logarithmic behavior related to the jump. Then, in the same section, we analyze harmonic conjugate functions in the upper half-plane having distributional boundary values on the real axis; it turns out that they have also a logarithmic angular asymptotic behavior related to the jump. Section 5.7 is devoted to applications to Fourier series, we give formulas for the jump in terms of logarithmic averages by using Cesàro-Riesz means and Abel-Poisson means of the conjugate series; among our results, we recover (5.1.6) and a Cesàro version of (5.1.4). The last section of this chapter is dedicated to study some properties of the symmetric jump behavior of distributions, this notion is much more general than the jumps in the sense of (5.1.3) and (5.1.5); furthermore, we discuss the case of Fourier series of periodic distributions, generalizing the mentioned results from [131, 256, 140, 141, 54].

The author wants to mention that some of the results of this chapter have already appeared in publication form [216, 218].

5.2 Jump and Symmetric Jump Behaviors

In this section we explain the notions of jumps to be considered in the future. They were introduced by the author and R. Estrada in [54, 215, 216, 218, 222].

Let us define the notions of *jump behavior* and *symmetric jump behavior* of distributions at points. We begin with the jump behavior.

Definition 5.1. *A distribution* $f \in \mathcal{D}'(\mathbb{R})$ *is said to have a distributional jump behavior (or jump behavior) at* $x = x_0 \in \mathbb{R}$ *if it satisfies the following distributional* *(quasi-) asymptotic relation*

$$f\left(x_0+\varepsilon x\right)=\gamma_- H(-x)+\gamma_+ H(x)+o(1)\ , \tag{5.2.1}$$

as $\varepsilon \to 0^+$ *in* $\mathcal{D}'(\mathbb{R})$, *where* H *is the Heaviside function, i.e., the characteristic function of* $(0,\infty)$, *and* $\gamma_\pm$ *are constants. The jump (or saltus) of* f *at* $x=x_0$ *is defined then as the number* $[f]_{x=x_0}=\gamma_+-\gamma_-$.

The meaning of (5.2.1) is in the weak topology of $\mathcal{D}'(\mathbb{R})$, in the sense that for each $\phi\in\mathcal{D}(\mathbb{R})$,

$$\lim_{\varepsilon\to 0^+}\langle f(x_0+\varepsilon x),\phi(x)\rangle=\gamma_-\int_{-\infty}^{0}\phi(x)\,\mathrm{d}x+\gamma_+\int_0^\infty\phi(x)\,\mathrm{d}x\ . \tag{5.2.2}$$

Observe that when $\gamma_+=\gamma_-$ we recover the usual Łojasiewicz notion of the value of a distribution at a point [128]. It should be noticed that our notion includes the jump of ordinary functions; indeed, if a locally integrable function has a discontinuity of the first kind, that is, the right and left limits $f(x_0^\pm)$ exist, then it satisfies (5.2.2) with $\gamma_\pm=f\left(x_0^\pm\right)$. In particular, jumps of functions of local bounded variation are distributional jump behaviors. We provide more examples of classical notions for jumps in Examples 5.5 and 5.6 below.

The reader should also noticed that if f has the jump behavior (5.2.1), then, in the sense of Definition 3.56, it satisfies $f_{\mathrm{sym}}(x_0)=(\gamma_++\gamma_-)/2$.

Let us also point out the fact that if $f\in\mathcal{S}'(\mathbb{R})$, then (5.2.2) actually holds for each $\phi\in\mathcal{S}(\mathbb{R})$; in other words, the quasiasymptotic behavior (5.2.1) is valid in $\mathcal{S}'(\mathbb{R})$. Indeed, if one considers $g(x)=f(x)-([f]_{x=x_0}/2)\mathrm{sgn}(x-x_0)$, then $g(x_0)=(\gamma_++\gamma_-)/2$, distributionally; the last assertion holds for distributional point values, and so does it for f and the jump behavior. This fact is important because it allows us to apply Fourier transform to (5.2.1), as we shall do in the next section.

The jump behavior of distributions admits a structural characterization similar to the Łojasiewicz characterization of distributional point values discussed in Section 3.2 (see (3.2.5)). The proof of the following theorem follows immediately from the mentioned Łojasiewicz theorem applied to $g(x) = f(x) - ([f]_{x=x_0}/2)\operatorname{sgn}(x-x_0)$.

Theorem 5.2. *Let $f \in \mathcal{D}'(\mathbb{R})$. Then, it has the jump behavior (5.2.1) if and only if there exist $m \in \mathbb{N}$ and a function F, locally integrable on a neighborhood of x_0, such that $F^{(m)} = f$ near x_0 and*

$$\lim_{x \to x_0^{\pm}} \frac{m! F(x)}{(x - x_0)^m} = \gamma_{\pm} \ . \tag{5.2.3}$$

The minimum m such that we can find an F satisfying (5.2.3) is called *the order of the jump behavior.* Obviously, if a locally integrable function has right and left limits at $x = x_0$, then it has a distributional jump behavior of order 0. Therefore, as distributional point values, the jump behavior is actually an average notion. Arbitrary m-primitives of f admit a Peano differential of order $(m-1)$. Moreover, let F_1 be another m-primitive of f, different form F, then there exists a polynomial of degree at most $m-1$, depending on F_1, such that, as $x \to x_0$,

$$F_1(x) = p(x-x_0) + \frac{\gamma_-}{m!}(x-x_0)^m H(x_0-x) + \frac{\gamma_+}{m!}(x-x_0)^m H(x-x_0) + o(|x-x_0|^m) \, .$$

We now turn our attention to the symmetric jump behavior.

Definition 5.3. *A distribution $f \in \mathcal{D}'(\mathbb{R})$ is said to have a distributional symmetric jump behavior (or symmetric jump behavior) at $x = x_0 \in \mathbb{R}$ if the jump distribution $\psi^f_{x_0}(x) = f(x_0 + x) - f(x_0 - x)$ has jump behavior at $x = 0$. In such a case, we define the jump of f at $x = x_0$ as the number $[f]_{x=x_0} = \left[\psi^f_{x_0}\right]_{x=0}/2$.*

It is easy to see that the jump behavior of the jump distribution in Definition 5.3 must be of the form

$$\psi^f_{x_0}(\varepsilon x) = [f]_{x=x_0} \operatorname{sgn} x + o(1) \quad \text{as } \ \varepsilon \to 0^+ \quad \text{in } \ \mathcal{D}'(\mathbb{R}) \ , \tag{5.2.4}$$

where $\operatorname{sgn} x$ is the signum function.

The order of the symmetric jump is defined as the order of the jump behavior (5.2.4). We may also describe the structure of the symmetric jump behavior, by applying Theorem 5.2 to the jump distributions.

Theorem 5.4. *Let $f \in \mathcal{D}'(\mathbb{R})$. Then, it has symmetric jump behavior at $x = x_0$ if and only if there exist $m \in \mathbb{N}$ and a distribution F such that $\psi^F_{x_0}$ is locally integrable on a neighborhood of x_0, $F^{(m)} = f$ near $x = x_0$, and*

(i) if m is even

$$\lim_{h\to 0} \frac{m!\psi^F_{x_0}(h)}{h^m} = [f]_{x=x_0} \operatorname{sgn} h \ , \tag{5.2.5}$$

(ii) if m is odd

$$\lim_{h\to 0} \frac{m!\chi^F_{x_0}(h)}{h^m} = \frac{1}{2}[f]_{x=x_0} \operatorname{sgn} h \ , \tag{5.2.6}$$

where $\chi^F_{x_0}$ is the symmetric part of F about $x = x_0$ defined by (3.10.1), Section 3.10.

In the form (5.2.5), the symmetric jump behavior has been employed in classical works to study *de la Vallée Poussin generalized jumps* in terms of differentiated Fourier series. For instance, see references [255] and [256, Chap.XI].

We now discuss two examples of particular types of jump behavior related to classical functions. It is not difficult to see that both examples are particular cases of our distributional notions for jumps. Also note that the two notions for ordinary functions mentioned at the introduction are included in these two examples.

Example 5.5. *(Lebesgue jumps) Let f be a locally (Lebesgue) integrable function, then we say that f has a Lebesgue jump behavior if there are two numbers $\gamma_\pm$ such that*

$$\lim_{h\to 0^\pm} \frac{1}{h}\int_{x_0}^{x_0+h} |f(x) - \gamma_\pm| \, \mathrm{d}x = 0 \ . \tag{5.2.7}$$

We say that f has a symmetric Lebesgue jump behavior if there is a numbers $d = [f]_{x_0}$ such that

$$\lim_{h \to 0^+} \frac{1}{h} \int_0^h |f(x_0 + x) - f(x_0 - x) - d| \, \mathrm{d}x = 0 \ . \tag{5.2.8}$$

Example 5.6. *(Jump behavior of the first order) Let μ be a Radon measure. Then we say that μ has a jump behavior of the first order if there exist $\gamma_\pm$ such that*

$$\lim_{h \to 0^\pm} \frac{1}{h} \int_{x_0}^{x_0+h} \mathrm{d}\mu(x) = \gamma_\pm \ . \tag{5.2.9}$$

We say that μ has a symmetric jump behavior of the first order if there exists $d = [f]_{x=x_0}$ such that

$$\lim_{h \to 0^+} \frac{1}{h} \left(\int_{x_0}^{x_0+h} \mathrm{d}\mu(x) - \int_{x_0-h}^{x_0} \mathrm{d}\mu(x) \right) = d \ . \tag{5.2.10}$$

A particular case is obtained if $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. Moreover, the first order jump behavior and symmetric jump behavior can still be defined by an integral expression even if f is not locally (Lebesgue) integrable but just Denjoy locally integrable [76]. For instance, in such a case the existence of the jump behavior of the first order is equivalent to the existence of the limits

$$\lim_{h \to 0^\pm} \frac{1}{h} \int_{x_0}^{x_0+h} f(x) \, \mathrm{d}x = \gamma_\pm \ , \tag{5.2.11}$$

where the last integral is taken in the Denjoy sense, and similarly for the symmetric jump,

$$\lim_{h \to 0^+} (1/h) \int_0^h (f(x_0 + x) - f(x_0 - x)) \, \mathrm{d}x = d \ .$$

The notions of Lebesgue jump and symmetric jump behaviors have been widely used in Fourier series by many authors [63, 131, 256]. While the use of first order jump and symmetric jump behaviors have become popular recently [140, 141, 142, 248] for locally integrable functions.

We give two more examples.

Example 5.7. *It is worth to provide the reader with an example of jump behavior which is not included in last two cases. Consider the function*

$$f(x) = \left(\gamma_- + A\,|\,x|^{\alpha}\, e^{i/x^{\beta}}\right) H(-x) + \left(\gamma_+ + Bx^{\alpha} e^{i/x^{\beta}}\right) H(x)\ . \tag{5.2.12}$$

For any choice of the constants, one can show that there is a tempered distribution having the distributional jump behavior (5.2.1) at $x = 0$ and coinciding with f on $\mathbb{R}\setminus\{0\}$ [128]. Observe that depending on the choice of the constants α and β the function is not a function of local bounded variation. In addition, the choice of the constants can be made so that f is not locally Denjoy integrable. One may also find values for α and β such that the order of the jump behavior is arbitrarily large [128].

Example 5.8. *Note that jump behavior implies symmetric jump behavior, but the converse is not true as shown by $\delta(x)$, which has a symmetric jump 0 at $x = 0$ but does not have jump behavior at the origin.*

5.3 Characterization of Jumps by Fourier Transform

We want to characterize the jump behavior of tempered distributions by the Fourier transform. Recall that we are fixing the constants in the Fourier transform so that

$$\hat{\phi}(x) = \int_{-\infty}^{\infty} \phi(t) e^{-ixt}\mathrm{d}t\ , \tag{5.3.1}$$

for $\phi \in \mathcal{S}(\mathbb{R})$.

Suppose then that $f \in \mathcal{S}'(\mathbb{R})$ satisfies

$$f\,(x_0 + \varepsilon x) = \gamma_- H(-x) + \gamma_+ H(x) + o(1)\ \text{ as } \varepsilon \to 0^+ \ \text{ in } \mathcal{D}'(\mathbb{R}), \tag{5.3.2}$$

Hence, since it holds in $\mathcal{S}'(\mathbb{R})$, we are allowed to take Fourier transform in (5.3.2), so that it transforms into the equivalent quasiasymptotic relation

$$e^{i\lambda x_0 x} \hat{f}(\lambda x) = 2\pi d_1 \frac{\delta(x)}{\lambda} + \frac{[f]_{x=x_0}}{i}\ \text{p.v.}\left(\frac{1}{\lambda x}\right) + o\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to \infty \tag{5.3.3}$$

in $\mathcal{S}'(\mathbb{R})$, where $d_1 = (\gamma_- + \gamma_+)/2$, and p.v.$(1/x)$ is the principal value distribution given by

$$\left\langle \text{p.v.}\left(\frac{1}{x}\right), \phi(x)\right\rangle = \text{p.v.}\int_{-\infty}^{\infty} \frac{\phi(x)}{x}\,\mathrm{d}x \ , \tag{5.3.4}$$

where p.v. stands for the Cauchy principal value of the integral at the origin (Section 1.3). Notice that we have used here the formula $\widehat{H}(x) = \pi\delta(x) - i\text{p.v.}\,(1/x)$. Needless to say that (5.3.3) is interpreted in the sense of quasiasymptotics, i.e., the asymptotic formula holds after evaluation at test functions.

Therefore, if we want to study (5.3.2) is enough to study (5.3.3). In the following, we shall study the structure of the quasiasymptotic behavior

$$g(\lambda x) = \gamma\frac{\delta(x)}{\lambda} + \beta\,\text{p.v.}\left(\frac{1}{\lambda x}\right) + o\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{D}'(\mathbb{R}) \ . \tag{5.3.5}$$

Recall the definition of asymptotically homogeneous functions of degree zero, introduced in Section 3.4.1. They are measurable functions such that for each $a > 0$,

$$c(ax) = c(x) + o(1) \ , \quad x \to \infty \ .$$

We already showed in Section 3.4.1 that they satisfy $c(x) = o(\log x)$, $x \to \infty$.

We need to introduce some notation. Let $l_k(x)$ be the k-primitive of $\log|x|$ satisfying the requirements $l_k^{(j)}(0) = 0$ for $j < k$. Observe that it satisfies

$$l_k(ax) = a^k l_k(x) + \frac{(ax)^k}{k!}\log a \ , \quad a > 0 \ . \tag{5.3.6}$$

We now state and show the structural theorem for (5.3.5), which actually follows immediately from Theorem 3.16.

Theorem 5.9. *Let $g \in \mathcal{D}'(\mathbb{R})$ have the following quasiasymptotic behavior in $\mathcal{D}'(\mathbb{R})$*

$$g(\lambda x) = \gamma\frac{\delta(x)}{\lambda} + \beta\,\text{p.v.}\left(\frac{1}{\lambda x}\right) + o\left(\frac{1}{\lambda}\right) \quad \textit{as } \lambda \to \infty \ . \tag{5.3.7}$$

Then, one can find a $k \in \mathbb{N}$, a continuous function G such that $G^{(k+1)} = g$, and an asymptotically homogeneous function c of degree 0 such that,

$$G(x) = c\left(|x|\right) \frac{x^k}{k!} + \frac{\gamma}{2k!} x^k \operatorname{sgn} x + \beta l_k(x) + o\left(|x|^k\right) \quad |x| \to \infty \ , \tag{5.3.8}$$

in the ordinary sense. Moreover, $g \in \mathcal{S}'(\mathbb{R})$ and (5.3.7) holds in $\mathcal{S}'(\mathbb{R})$. Conversely (5.3.8) implies (5.3.7).

Let G be a first order primitive of g. In addition, the quasiasymptotic behavior (5.3.7) is equivalent the existence of $k \in \mathbb{N}$ such that

$$\lim_{x \to \infty} \left(G(ax) - G(-x)\right) = \alpha + \beta \log a \quad (\mathrm{C}, k) \ , \tag{5.3.9}$$

for each $a > 0$.

Proof. Apply Theorem 3.16 to $h(x) = f(x) - \beta \mathrm{p.v.}(1/x)$. □

As a corollary, we obtain a characterization of the jump behavior in terms of the summability of the Fourier transform. We state this result as a theorem.

Theorem 5.10. *Let $f \in \mathcal{S}'(\mathbb{R})$. Then, it has the jump behavior*

$$f\left(x_0 + \varepsilon x\right) = \gamma_- H(-x) + \gamma_+ H(x) + o(1) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{D}'(\mathbb{R}) \ , \tag{5.3.10}$$

if and only if for any first order primitive of $e^{ix_0 x} \hat{f}(x)$, say F, one has that there is a $k \in \mathbb{N}$ such that

$$\lim_{x \to \infty} \frac{1}{2\pi} \left(F(ax) - F(-x)\right) = \frac{\gamma_+ + \gamma_-}{2} + \frac{[f]_{x = x_0}}{2\pi i} \log a \quad (\mathrm{C}, k) \ , \tag{5.3.11}$$

for each $a > 0$.

We now consider some consequences of Theorem 5.10. For that, we use the summability kernels ϕ_a^k introduced in Section 3.5, i.e.,

$$\phi_a^k(x) = (1 + x)^k (H(-x) - H(-1 - x)) + \left(1 - \frac{x}{a}\right)^k (H(x) - H(x - a)) \ .$$

Corollary 5.11. *Let $f \in \mathcal{S}'(\mathbb{R})$ be such that $\hat{f} = \mu$ is a Radon measure. Then, f has the jump behavior (5.3.10) if and only if there exists a $k \in \mathbb{N}$ such that*

$$\lim_{x\to\infty} \frac{1}{2\pi} \int_{-x}^{ax} e^{ix_0 t} \mathrm{d}\mu(t) = \frac{\gamma_+ + \gamma_-}{2} + \frac{[f]_{x=x_0}}{2\pi i} \log a \quad (\mathrm{C}, k) \ , \tag{5.3.12}$$

for each $a > 0$, or which amounts to the same,

$$\lim_{x\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix_0 t} \phi_a^k \left(\frac{t}{x}\right) \mathrm{d}\mu(t) = \frac{\gamma_+ + \gamma_-}{2} + \frac{[f]_{x=x_0}}{2\pi i} \log a \, , \tag{5.3.13}$$

for each $a > 0$.

Corollary 5.12. *Let $f \in \mathcal{S}'(\mathbb{R})$ be a 2π-periodic distribution having Fourier series*

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \ . \tag{5.3.14}$$

Then, f has the jump behavior (5.3.10) if and only if there exists a $k \in \mathbb{N}$ such that

$$\lim_{x\to\infty} \sum_{-x<n\leq ax} c_n e^{inx_0} = \frac{\gamma_+ + \gamma_-}{2} + \frac{[f]_{x=x_0}}{2\pi i} \log a \quad (\mathrm{C}, k) \ , \quad \textit{for each } a > 0 \ , \tag{5.3.15}$$

or which amounts to the same,

$$\lim_{x\to\infty} \sum_{n=-\infty}^{\infty} \phi_a^k \left(\frac{n}{x}\right) c_n e^{inx_0} = \frac{\gamma_+ + \gamma_-}{2} + \frac{[f]_{x=x_0}}{2\pi i} \log a \, , \quad \textit{for each } a > 0 \ . \tag{5.3.16}$$

Corollary 5.13. *Let $f \in \mathcal{S}'(\mathbb{R})$ be such that $\hat{f} \in L^1_{\mathrm{loc}}(\mathbb{R})$. Then, f has the jump behavior (5.3.10) if and only if there exists a $k \in \mathbb{N}$ such that*

$$\lim_{x\to\infty} \frac{1}{2\pi} \int_{-x}^{ax} e^{ix_0 t} \hat{f}(t) \mathrm{d}t = \frac{\gamma_+ + \gamma_-}{2} + \frac{[f]_{x=x_0}}{2\pi i} \log a \quad (\mathrm{C}, k) \ , \tag{5.3.17}$$

for each $a > 0$, or which amounts to the same,

$$\lim_{x\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_a^k \left(\frac{t}{x}\right) e^{ix_0 t} \hat{f}(t) \mathrm{d}t = \frac{\gamma_+ + \gamma_-}{2} + \frac{[f]_{x=x_0}}{2\pi i} \log a \, , \tag{5.3.18}$$

for each $a > 0$.

5.4 Angular Limits of Harmonic Representations

Let $U(z)$, $\Im m\, z > 0$, be a harmonic representation of a distribution having a jump behavior at $x = x_0$. In this section we obtain the angular behavior of U at the boundary point $x = x_0$. This problem has been discussed in [54] by studying the Poisson kernel and by the author in [213] by using Fourier transform methods and the structural theorem (Theorem 5.9). Here we present the proof of the following theorem based on Theorem 3.55.

Theorem 5.14. *Let $f \in \mathcal{D}'(\mathbb{R})$ have the jump behavior*

$$f(x_0 + \varepsilon x) = \gamma_- H(-x) + \gamma_+ H(x) + o(1) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{D}'(\mathbb{R}) \ . \tag{5.4.1}$$

If U is a harmonic representation of f on the upper half-plane, then,

$$\lim_{z \to x_0,\ z \in \mathsf{l}_\vartheta} U(z) = \frac{\gamma_+ + \gamma_-}{2} + \frac{\vartheta}{\pi}[f]_{x=x_0} \ , \tag{5.4.2}$$

where l_ϑ is a ray in the upper half-plane starting at x_0 and making an angle ϑ with the ray $x = x_0$. Actually (5.4.2) holds uniformly for $|\vartheta| < \eta < \pi/2$.

Proof. Set $d_1 = (\gamma_+ + \gamma_-)/2$ and $d_2 = [f]_{x=x_0}$. Consider the distribution $g(x) = f(x) - (d_2/2)\operatorname{sgn}(x - x_0)$. Then $g(x_0) = d_1$, distributionally. On the other hand $U_1(z) = 1/2 - (1/\pi)\arg(z - x_0)$, with $0 < \arg z < \pi$, is a harmonic representation of $(1/2)\operatorname{sgn} x$ on the upper half-plane, then $U_2 = U - d_2 U_1$ is a harmonic representation of g. So, by Theorem 3.55, $U_2(z) \to d_1$ as $z \to x_0$, non-tangentially. Therefore if $z = x_0 + \varepsilon \sin\vartheta + i\varepsilon\cos\vartheta = x_0 + \exp(i(\pi/2 - \vartheta))$, we obtain

$$\lim_{\varepsilon \to 0^+} U(x_0 + \varepsilon e^{i\left(\frac{\pi}{2} - \vartheta\right)}) = d_1 + d_2 \lim_{\varepsilon \to 0^+} U_1(x_0 + \varepsilon e^{i\left(\frac{\pi}{2} - \vartheta\right)}) = d_1 + d_2 \frac{\vartheta}{\pi} \ .$$

□

We consider an example involving Fourier series.

Example 5.15. *let* $f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$, *where the series is assumed to converge in* $\mathcal{S}'(\mathbb{R})$. *Let* l_ϑ *denote the ray in the upper half-plane starting from* x_0 *and making an angle* ϑ *with the line* $x = x_0$. *Then, one has that*

$$\lim_{\xi \to x_0,\, \xi \in \mathsf{l}_\vartheta} \sum_{-\infty}^{-1} c_n e^{in\bar{\xi}} + \sum_{0}^{\infty} c_n e^{in\xi} = \frac{\gamma_+ + \gamma_-}{2} + \frac{\vartheta}{\pi}[f]_{x=x_0} \ .$$

If we write the cos *and* sin *series,* $f(x) = \sum_{n=0}^{\infty} a_n \cos(nx) + b_n \sin(nx)$, *then the last limit takes the form,*

$$\lim_{\xi \to x_0,\, \xi \in \mathsf{l}_\vartheta} \sum_{n=0}^{\infty} a_n \cos(n\xi) + b_n \sin(n\xi) = \frac{\gamma_+ + \gamma_-}{2} + \frac{\vartheta}{\pi}[f]_{x=x_0} \ ,$$

both limits hold uniformly for ϑ *in compact subsets of* $(-\pi/2, \pi/2)$. *If one takes* $\vartheta = 0$, *one obtains*

$$\lim_{r \to 1^-} \sum_{n=0}^{\infty} (a_n \cos nx_0 + b_n \sin nx_0) r^n = \frac{\gamma_+ + \gamma_-}{2} \ ,$$

which generalizes the main result from [237] obtained by G. Walter.

5.5 Jump Behavior and Logarithmic Averages in Cesàro Sense

In this section, we shall deal with tempered distributions having a jump at a point and study the logarithmic average in the Cesàro sense of the Fourier transform.

We now state and show the main theorem of this section. It will enable us to study the logarithmic average behavior of $e^{ix_0}\hat{f}(x)$ separately for any decomposition as the sum of two tempered distributions having supports in $(-\infty, 0]$ and $[0, \infty)$, respectively.

Theorem 5.16. *Let* g *have the quasiasymptotic behavior*

$$g(\lambda x) = \gamma \frac{\delta(x)}{\lambda} + \beta \,\text{p.v.}\left(\frac{1}{\lambda x}\right) + o\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to \infty \ \mathcal{S}'(\mathbb{R}) \ . \tag{5.5.1}$$

Then for any decomposition $g = g_+ + g_-$, *where* $\operatorname{supp} g_- \subseteq (-\infty, 0]$ *and* $\operatorname{supp} g_+ \subseteq [0, \infty)$, *one has that*

$$g_\pm(\lambda x) = \pm\beta \,\frac{\log \lambda}{\lambda}\, \delta(x) + o\left(\frac{\log \lambda}{\lambda}\right) \quad \text{as } \lambda \to \infty \quad \text{in } \mathcal{S}'(\mathbb{R}). \tag{5.5.2}$$

Proof. Let k , c and G be as in Theorem 5.9. Then, to a decomposition $g = g_+ + g_-$, corresponds a decomposition $G = G_+ + G_-$, with $\operatorname{supp} G_- \subseteq (-\infty, 0]$ and $\operatorname{supp} G_+ \subseteq [0, \infty)$. Hence

$$\begin{aligned} G_\pm(x) &= \beta l_k(x) + o\left(|x|^k \log|x|\right),\ x \to \pm\infty\ , \qquad (5.5.3) \\ &= \beta \frac{x^k}{k!} \log|x| + o\left(|x|^k \log|x|\right),\ x \to \pm\infty\ , \end{aligned}$$

since $c(x) = o(\log x)$ as $x \to \infty$. This implies the distributional relations

$$\begin{aligned} G_\pm(\lambda x) &= \beta l_k(\lambda x) H(\pm x) + o\left(\lambda^k \log \lambda\right) \\ &= \beta \lambda^k l_k(x) H(\pm x) + \beta \lambda^k \log \lambda \frac{x^k}{k!}\, H(\pm x) + o\left(\lambda^k \log \lambda\right) \\ &= \beta \lambda^k \log \lambda \frac{x^k}{k!}\, H(\pm x) + o\left(\lambda^k \log \lambda\right) \text{ as } \lambda \to \infty\ , \end{aligned}$$

and the last relation holds in $\mathcal{S}'(\mathbb{R})$. Therefore if we differentiate $(k+1)$-times, we obtain (5.5.2). □

Notice that (5.5.3) gives a logarithmic average in the Cesàro sense. We collect this fact in the next corollary for future reference.

Corollary 5.17. *Let g, g_+, g_-, and k be as in the last theorem, then*

$$g_\pm^{(-k-1)}(x) \sim \beta \frac{x^k}{k!} \log|x|\,, \quad x \to \pm\infty\ , \qquad (5.5.4)$$

where $g_\pm^{(-k-1)}$ are the $(k+1)$-primitives of $g_\pm$ with $\operatorname{supp} g_-^{(-k-1)} \subseteq (-\infty, 0]$ and $\operatorname{supp} g_+^{(-k-1)} \subseteq [0, \infty)$.

We now summarize our results.

Theorem 5.18. *Let $f \in \mathcal{S}'(\mathbb{R})$ have the distributional jump behavior at $x = x_0$,*

$$f\left(x_0 + \varepsilon x\right) = \gamma_- H(-x) + \gamma_+ H(x) + o(1) \ \text{ as } \varepsilon \to 0^+ \text{ in } \mathcal{D}'(\mathbb{R})\ . \qquad (5.5.5)$$

Then for any decomposition $\hat{f} = \hat{f}_+ + \hat{f}_-$*, where* $\operatorname{supp} \hat{f}_- \subseteq (-\infty, 0]$ *and* $\operatorname{supp} \hat{f}_+ \subseteq [0, \infty)$*, we have that*

$$e^{i\lambda x_0 x} \hat{f}_{\pm}(\lambda x) = \pm [f]_{x=x_0} \frac{\log \lambda}{i\lambda} \delta(x) + o\left(\frac{\log \lambda}{\lambda}\right) \quad \text{as } \lambda \to \infty \ \mathcal{S}'(\mathbb{R}) \ . \tag{5.5.6}$$

Furthermore, there exists $k \in \mathbb{N}$ *such that*

$$\left(e^{ix_0 t} \hat{f}_{\pm}(t) * t_{\pm}^k\right)(x) \sim \pm [f]_{x=x_0} \frac{|x|^k}{i} \log |x| \, , \quad |x| \to \infty \ , \tag{5.5.7}$$

in the ordinary sense.

Proof. The jump behavior implies the quasiasymptotic

$$e^{i\lambda x_0 x} \hat{f}(\lambda x) = \pi(\gamma_+ + \gamma_-) \frac{\delta(x)}{\lambda} + \frac{[f]_{x=x_0}}{i} \text{p.v.} \left(\frac{1}{\lambda x}\right) + o\left(\frac{1}{\lambda}\right) \quad \text{as} \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R}) \ ,$$

and so (5.5.6) and (5.5.7) follow from Theorem 5.16 and Corollary 5.17. □

A special case is obtained in the next corollary which follows directly from Theorem 5.18.

Corollary 5.19. *Let* $f \in \mathcal{S}'(\mathbb{R})$ *have the distributional jump behavior (5.5.5) at* $x = x_0$*. Suppose that its Fourier transform is given by a Radon measure* μ*, then there exists* $k \in \mathbb{N}$ *such that for any decomposition of* $\mu = \mu_- + \mu_+$*, as two Radon measures concentrated on* $(-\infty, 0]$ *and* $[0, \infty)$*, respectively,*

$$\lim_{x \to \infty} \frac{i}{\log x} \int_0^x e^{\pm i x_0 t} \left(1 - \frac{t}{x}\right)^k \mathrm{d}\mu_{\pm}(\pm t) = \pm [f]_{x=x_0} \ . \tag{5.5.8}$$

5.6 Logarithmic Asymptotic Behavior of Analytic and Harmonic Conjugate Functions

This section is devoted to the study of the local boundary behavior of analytic representations and harmonic conjugates to harmonic representations of distributions having a jump behavior. Recall that (Section 1.6) given $f \in \mathcal{D}'(\mathbb{R})$, we may see f

as a hyperfunction, that is, $f(x) = F(x + i0) - F(x - i0)$, where F is analytic for $\Im m\, z \neq 0$.

In the next theorem we obtain the angular behavior of $F(z)$ when z approaches a point where f has a jump behavior. We remark this is done separately when z approaches angularly the point from the upper and lower half-planes.

Given $0 < \eta \leq \pi/2$ and $x_0 \in \mathbb{R}$, we define the subset of the upper half-plane $\Delta^+_\eta(x_0)$ as the set of those z such that $\eta \leq \arg(z - x_0) \leq \pi - \eta$; similarly, we define the subset of the lower half-plane $\Delta^-_\eta(x_0)$ as the set of those z such that $\eta - \pi \leq \arg(z - x_0) \leq -\eta$.

Theorem 5.20. *Let $f \in \mathcal{D}'(\mathbb{R})$ have the distributional jump behavior*

$$f\left(x_0 + \varepsilon x\right) = \gamma_- H(-x) + \gamma_+ H(x) + o(1) \quad \textit{as } \varepsilon \to 0^+ \quad \textit{in } \mathcal{D}'(\mathbb{R}) \ . \tag{5.6.1}$$

Suppose that F is an analytic representation of f. Then for any $0 < \eta \leq \pi/2$,

$$\lim_{z \to x_0,\ z \in \Delta^{\pm}_\eta(x_0)} \frac{F(z)}{\log|z - x_0|} = -\frac{[f]_{x=x_0}}{2\pi i} \ . \tag{5.6.2}$$

Proof. Note first that if (5.6.2) holds for one analytic representation, then it holds for any analytic representation of f. In fact by the very well known edge of wedge theorem, any two such analytic representations differ by an entire function, and for entire functions (5.6.2) gives 0. Next, we see that we may assume that $f \in \mathcal{S}'(\mathbb{R})$. Indeed we can decompose $f = f_1 + f_2$ where f_2 is zero in a neighborhood of x_0 and $f_1 \in \mathcal{S}'(\mathbb{R})$. Let F_1 and F_2 be analytic representations of f_1 and f_2, respectively; then F_2 can be continued across a neighborhood of x_0 (edge of wedge theorem once again), hence $F_2(z) = F_2(x_0) + O\left(|z - x_0|\right) = o\left(|\log|z - x_0||\right)$ as $z \to x_0$. Additionally, f_1 has the same jump behavior as f. Thus, we assume that $f \in \mathcal{S}'(\mathbb{R})$. Let $\hat{f} = \hat{f}_+ + \hat{f}_-$ be a decomposition such that $\operatorname{supp} \hat{f}_- \subseteq (-\infty, 0]$ and

$\operatorname{supp} \hat{f}_+ \subseteq [0,\infty)$. Then,

$$F(z) = \begin{cases} \dfrac{1}{2\pi}\left\langle \hat{f}_+(t), e^{izt}\right\rangle , & \Im m\ z > 0 \ , \\ -\dfrac{1}{2\pi}\left\langle \hat{f}_-(t), e^{izt}\right\rangle , & \Im m\ z < 0 \ , \end{cases}$$

is an analytic representation of f (Section 1.6). Keep the number m on a compact set and $\lambda > 0$, then

$$\begin{aligned} F\left(x_0 + \frac{m}{\lambda}, \frac{\pm 1}{\lambda}\right) &= \pm\frac{1}{2\pi}\left\langle \lambda e^{i\lambda x_0 x} \hat{f}_\pm(\lambda x), e^{i(m+i)x}\right\rangle \\ &= \frac{[f]_{x=x_0}}{2\pi i}\log\lambda + o\left(\log\lambda\right) \end{aligned}$$

as $\lambda \to \infty$, where here we have used (5.5.6). $\square$

Our next goal is to study the angular behavior of harmonic conjugate functions. This is the content of the next theorem.

Theorem 5.21. *Let $f \in \mathcal{D}'(\mathbb{R})$ have the jump behavior (5.6.1) and U be a harmonic representation of f in the upper half-plane. Then if V is a harmonic conjugate to U, one has that*

$$\lim_{z\to x_0,\ z\in\Delta_\eta^+(x_0)} \frac{V(z)}{\log|z-x_0|} = \frac{1}{\pi}[f]_{x=x_0} \ , \tag{5.6.3}$$

for each $0 < \eta \leq \pi/2$.

Proof. Since harmonic conjugates to U differ by a constant, it is enough to show (5.6.3) for any particular harmonic conjugate to U.

We now show that we may work with any harmonic representation U of f we want. Suppose that U_1 and U_2 are two harmonic representations of f, then $U = U_1 - U_2$ represents the zero distribution. Then by applying the reflection principle to the real and imaginary parts of U [11, Section 4.5], [207, Section 3.4], we have that U admits a harmonic extension to a (complex) neighborhood of x_0. Consequently, if V_1 and V_2 are harmonic conjugates to U_1 and U_2, we have that $V = V_1 - V_2$ is harmonic conjugate to U, and thus it admits a harmonic extension to a (complex) neighborhood of x_0 as well. Therefore $V(z) = O(1) = o(-\log|z-x_0|)$, which shows that V_1 satisfies (5.6.3) if and only if V_2 does.

Let F be an analytic representation of f on $\Im m\ z \neq 0$. We can assume then that $U(z) = F(z) - F(\bar{z})$, $\Im m\ z > 0$. We have that $V(z) = -i\left(F(z) + F(\bar{z})\right)$, $\Im m\ z > 0$, is a harmonic conjugate to U. Therefore, an application of Theorem 5.20 yields to (5.6.3). $\square$

Example 5.22. *As an example, we discuss our results in the context of the spaces $L^p(\mathbb{R})$ with $1 < p < \infty$. Let $f \in L^p(\mathbb{R})$ and assume that it has the distributional jump behavior (5.6.1). A particular case is obtained when f has a Lebesgue jump (Example 5.5), but we remark that our assumption is much weaker. A harmonic representation of f is given by the Poisson representation, i.e., by integration against the Poisson kernel. Among all the harmonic conjugates to the Poisson representation, the natural choice is*

$$V(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Re e\ z - t}{|z-t|^2} f(t)\,\mathrm{d}t\ . \tag{5.6.4}$$

As a corollary of Theorem 5.21, we obtain the angular asymptotic behavior of this integral: it is indeed given by (5.6.3). Note that the harmonic function $V(z)$ has as boundary value a function $\tilde{f} \in L^p(\mathbb{R})$, which in fact is the Hilbert transform of f [60, 113, 206]. The asymptotic behavior of V suggests that $\tilde{f}$ has the following quasiasymptotic behavior at $x = x_0$ in $\mathcal{D}'(\mathbb{R})$,

$$\tilde{f}(x_0 + \varepsilon x) = \frac{1}{\pi}[f]_{x=x_0} \log\varepsilon + o\left(\log\left(\frac{1}{\varepsilon}\right)\right) \quad \text{as } \varepsilon \to 0^+, \tag{5.6.5}$$

which is actually the case. A proof of the last relation can be given by using the fact that the Fourier transform of $\tilde{f}$ is $-i\left(\hat{f}_+ - \hat{f}_-\right)$, for a suitable decomposition of $\hat{f}$, by using the Theorem 5.18, and then taking Fourier inverse transform. If we work on the circle, i.e., on $L^p(\mathbb{T})$, we obtain similar conclusions for the conjugate *function; we will do this in Section 5.7 but in a more general distributional setting obtaining several logarithmic asymptotic behaviors of the conjugate Fourier series.*

5.7 Logarithmic Averages of Fourier Series

We now apply our results to the Fourier series of 2π-periodic distributions. Suppose that $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$, where the series converges in $\mathcal{S}'(\mathbb{R})$. Assume also that f has the jump behavior (5.2.1). Then Theorem 5.18 implies at once that there exists $k \in \mathbb{N}$ such that

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{0\le n\le x} c_n e^{inx_0} \left(1 - \frac{n}{x}\right)^k = \frac{[f]_{x=x_0}}{2\pi i} \ , \tag{5.7.1}$$

and

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{1\le n\le x} c_{-n} e^{-inx_0} \left(1 - \frac{n}{x}\right)^k = -\frac{[f]_{x=x_0}}{2\pi i} \ , \tag{5.7.2}$$

which gives us a logarithmic average for the Cesàro-Riesz means of these two series.

The conjugate Fourier series is $\tilde{f}(x) = \sum_{n=-\infty}^{\infty} \tilde{c}_n e^{inx}$, where $\tilde{c_0} = 0$ and $\tilde{c}_n = -i \operatorname{sgn} n \, c_n$. It follows from the above relations that it has the quasiasymptotic behavior at x_0,

$$\tilde{f}\left(x_0 + \varepsilon x\right) = \frac{1}{\pi}[f]_{x=x_0} \log \varepsilon + o\left(\log\left(\frac{1}{\varepsilon}\right)\right) \text{ as } \varepsilon \to 0^+ \text{ in } \mathcal{D}'(\mathbb{R}) \ . \tag{5.7.3}$$

Moreover, since $V(z)$, $\Im m \ z > 0$, given by

$$V(z) = \sum_{n=-\infty}^{-1} \tilde{c}_n e^{i\bar{z}n} + \sum_{n=1}^{\infty} \tilde{c}_n e^{izn} \ , \tag{5.7.4}$$

is a harmonic conjugate to a harmonic representation of f, one deduces from Theorem 5.21 that for $0 < \eta \le \pi/2$

$$\lim_{z\to x_0,\ z\in\Delta^+_\eta(x_0)} \frac{1}{\log|z - x_0|} \left(\sum_{n=-\infty}^{-1} \tilde{c}_n e^{i\bar{z}n} + \sum_{n=1}^{\infty} \tilde{c}_n e^{izn} \right) = \frac{1}{\pi}[f]_{x=x_0} \ . \tag{5.7.5}$$

Hence we obtain the jump as the logarithmic angular average of the harmonic representation of the conjugate series. In particular, if we take $\eta = \pi/2$,

$$\lim_{y\to 0^+} \frac{1}{\log y} \sum_{n=1}^{\infty} \left(\tilde{c}_n e^{ix_0 n} + \tilde{c}_{-n} e^{-ix_0 n}\right) e^{-yn} = \frac{1}{\pi}[f]_{x=x_0} \ . \tag{5.7.6}$$

If we now use the sines and cosines series for f, i.e.,

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx) \ , \tag{5.7.7}$$

where $a_n = c_n + c_{-n}$, $b_n = i(c_n - c_{-n})$, then $\tilde{c}_n = \frac{1}{2}\left(-b_{|n|} - i \operatorname{sgn} n \, a_{|n|}\right)$ and

$$\tilde{f}(x) = \sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx) \ . \tag{5.7.8}$$

Therefore (5.7.6) is equivalent to

$$\lim_{r \to 1^{-1}} \frac{1}{\log(1-r)} \sum_{n=1}^{\infty} (a_n \sin nx_0 - b_n \cos nx_0)\, r^n = \frac{1}{\pi}[f]_{x=x_0} \ , \tag{5.7.9}$$

which exhibits the jump now as the Abel-Poisson logarithmic means of the conjugate Fourier series. In fact, also using the sines and cosines series expression for the conjugate series and (5.7.1)-(5.7.2), one obtains the jump as the logarithmic average of the symmetric partial sums of the conjugate series in the Cesàro-Riesz means

$$\lim_{x \to \infty} \frac{1}{\log x} \sum_{0 < n \le x} (a_n \sin nx_0 - b_n \cos nx_0) \left(1 - \frac{x}{n}\right)^k = -\frac{1}{\pi}[f]_{x=x_0} \ . \tag{5.7.10}$$

In the next section, we will obtain (5.7.9) and (5.7.10) under weaker assumptions, namely, under a symmetric jump behavior.

5.8 Symmetric Jumps and Logarithmic Averages

We conclude this chapter by analyzing the case when the distribution f has a symmetric jump behavior at $x = x_0$. We use the jump distribution

$$\psi_{x_0} := \psi_{x_0}^f(x) = f(x_0 + x) - f(x_0 - x) \ ; \tag{5.8.1}$$

so if f has a symmetric jump then

$$\psi_{x_0}(\varepsilon x) = [f]_{x=x_0} \operatorname{sgn} x + o(1) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{D}'(\mathbb{R}) \ , \tag{5.8.2}$$

We use our results from Section 5.5 and Section 5.6, applied to the jump distribution, to deduce some logarithmic averages in the case of symmetric jump behavior.

Theorem 5.23. *Suppose that $f \in \mathcal{S}'(\mathbb{R})$ has a symmetric jump at $x = x_0$. Then for any decomposition $\hat{f} = \hat{f}_- + \hat{f}_+$, where $\operatorname{supp} \hat{f}_- \subseteq (-\infty, 0]$ and $\operatorname{supp} \hat{f}_+ \subseteq [0, \infty)$, we have that*

$$e^{i\lambda x_0 x} \hat{f}_+(\lambda x) - e^{-i\lambda x_0 x} \hat{f}_-(-\lambda x) = 2\,[f]_{x=x_0} \frac{\log \lambda}{i\lambda}\,\delta(x) + o\left(\frac{\log \lambda}{\lambda}\right) \tag{5.8.3}$$

as $\lambda \to \infty$ in $\mathcal{S}'(\mathbb{R})$. Consequently, there exists k such that

$$\left(\left(e^{ix_0 t} \hat{f}_+(t) - e^{-ix_0 t} \hat{f}_-(-t)\right) * t_+^k\right)(x) \sim \frac{2}{i}\,[f]_{x=x_0}\, x^k \log x \tag{5.8.4}$$

as $x \to \infty$, in the ordinary sense.

Proof. We can apply Theorem 5.18 directly, since $\hat{\psi}_{x_0}(x) = e^{ix_0 x}\hat{f}(x) - e^{-ix_0 x}\hat{f}(-x)$, and a decomposition $\hat{f} = \hat{f}_- + \hat{f}_+$ leads to the decomposition

$$\hat{\psi}_{x_0}(x) = \left(e^{ix_0 x} \hat{f}_+(x) - e^{-ix_0 x} \hat{f}_-(-x)\right) + \left(e^{ix_o x} \hat{f}_-(x) - e^{-ix_0 x} \hat{f}_+(-x)\right).$$

$\square$

We now obtain the announced Cesàro-Riesz logarithmic version of F. Lukács Theorem.

Corollary 5.24. *Let $f \in \mathcal{S}'(\mathbb{R})$ be a 2π-periodic distribution having the following Fourier series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)\ . \tag{5.8.5}$$

If f has a symmetric jump behavior at $x = x_0$, then there is a $k \in \mathbb{N}$ such that

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{n=1}^{\infty} \left(a_n \sin nx_0 - b_n \cos nx_0\right)\left(1 - \frac{n}{x}\right)^k = -\frac{1}{\pi}\,[f]_{x=x_0}\,. \tag{5.8.6}$$

Proof. Notice that the jump distribution has Fourier series,

$$\psi_{x_0}(x) = -2 \sum_{n=1}^{\infty} \left(a_n \sin nx_0 - b_n \cos nx_0\right) \sin nx\ ,$$

then

$$\hat{\psi}_{x_0}(x) = 2\pi i \sum_{n=1}^{\infty} \left(a_n \sin nx_0 - b_n \cos nx_0\right) \left(\delta\left(x-n\right) - \delta\left(x+n\right)\right) \ .$$

Therefore one has that

$$\sum_{n=1}^{\infty} \left(a_n \sin nx_0 - b_n \cos nx_0\right) \delta(\lambda x - n) = -\frac{1}{\pi} \left[f\right]_{x=x_0} \log\lambda \, \frac{\delta(x)}{\lambda} + o\left(\frac{\log\lambda}{\lambda}\right)$$

as $\lambda \to \infty$ in $\mathcal{S}'(\mathbb{R})$, from where we deduce (5.8.6). □

We now give the radial version of Theorem 5.21 in the case of symmetric jump behavior.

Theorem 5.25. *Let $f \in \mathcal{D}'(\mathbb{R})$ have a symmetric jump behavior at $x = x_0$. Then if V is any harmonic conjugate to a harmonic representation of f on $\Im m\ z > 0$, one has that*

$$\lim_{y\to 0^+} \frac{V(x_0, y)}{\log y} = \frac{1}{\pi} \left[f\right]_{x=x_0} \ . \tag{5.8.7}$$

Proof. As is the proof of Theorem 5.20 and Theorem 5.21 we may assume that f is tempered distribution and

$$V(z) = -\frac{i}{2\pi} \left(\left\langle \hat{f}_+(t), e^{izt} \right\rangle - \left\langle \hat{f}_-(t), e^{i\bar{z}t} \right\rangle \right) \ ,$$

where $\hat{f} = \hat{f}_- + \hat{f}_+$ is any decomposition with $\operatorname{supp} \hat{f}_- \subseteq (-\infty, 0]$ and $\operatorname{supp} \hat{f}_+ \subseteq [0, \infty)$. Hence by Theorem 5.23, we obtain that

$$\begin{aligned} V(x_0, y) &= -\frac{i}{2\pi} \left\langle \hat{f}_+(t) e^{ix_0 t} - \hat{f}_-(-t) e^{-ix_0 t}, e^{-yt} \right\rangle \\ &= -\frac{i}{2\pi} \left\langle \frac{2}{i} \left[f\right]_{x=x_0} \log\left(\frac{1}{y}\right) \delta(t), e^{-yt} \right\rangle + o\left(\log\frac{1}{y}\right) \\ &= \frac{1}{\pi} \left[f\right]_{x=x_0} \log y + o\left(\log\frac{1}{y}\right) \ \text{as}\ y \to 0^+ \ , \end{aligned}$$

as required. □

In the case when f is the boundary value of an analytic function, one can get a much better result. As was obtained in [54, Thm.5], one has the angular asymptotic logarithmic behavior. We give a new proof of this fact.

Theorem 5.26. *Let F be analytic in the upper half-plane, with distributional boundary values $f(x) = F(x + i0)$. Suppose f has a distributional symmetric jump behavior at $x = x_0$. Then, for any $0 < \eta \leq \pi/2$*

$$F(z) \sim \frac{i}{\pi}[f]_{x=x_0} \log(z - x_0) \qquad \text{as } z \in \Delta^+_\eta(x_0) \to x_0 \ . \tag{5.8.8}$$

Proof. Let ψ_{x_0} be the jump distribution at $x = x_0$. Then ψ_{x_0} has a jump behavior at $x = 0$ and $[\psi_{x_0}]_{x=0} = 2[f]_{x=x_0}$. Observe that $U(z) = F(x_0 + z) - F(x_0 - \bar{z})$ is a harmonic representation of ψ_{x_0} and $V(z) = -i\left(F(x_0 + z) + F(x_0 - \bar{z})\right)$ is a harmonic conjugate. Hence, we can apply (5.4.2) and Theorem 5.21 to U and V and obtain that $F(x_0 - \bar{z}) = F(x_0 + z) + O(1)$ and so

$$F(x_0 + z) + F(x_0 - \bar{z}) = \frac{2i}{\pi}[f]_{x=x_0} \log|z| + o\left(|\log|z||\right) \ , \quad z \in \Delta^+_\eta(0) \to 0 \ ;$$

and therefore (5.8.8) follows. □

We end this section with an immediate corollary of Theorem 5.25, this is the result from [54] which generalizes F. Móricz result [140, 141], namely, we express the symmetric jump as a logarithmic average of the Abel-Poisson means of the conjugate series.

Corollary 5.27. *Let $f \in \mathcal{S}'(\mathbb{R})$ be a 2π-periodic distribution with Fourier series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right) \ . \tag{5.8.9}$$

If f has a symmetric jump behavior at $x = x_0$, then its conjugate series has the following logarithmic Abel-Poisson average value

$$\lim_{r\to 1^-} \frac{1}{\log(1-r)} \sum_{n=1}^{\infty} \left(a_n \sin nx_0 - b_n \cos nx_0\right) r^n = \frac{1}{\pi}\,[f]_{x=x_0} \ . \tag{5.8.10}$$

Chapter 6

Determination of Jumps by Differentiated Means

6.1 Introduction

We continue in this chapter our study of jumps of distributions. New types of summability means are introduced in order to find formulas for jumps, namely, *Differentiated Means*. The result of the present chapter are to be published soon in [222].

Our results are inspired in a classical result of L. Fejér ([63],[256, Vol.I, p.107]). It states that if f is a 2π-periodic function of bounded variation having Fourier series

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{6.1.1}$$

then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=-N}^{N} n c_n e^{inx_0} = \frac{1}{i\pi} \left(f(x_0^+) - f(x_0^-) \right) \ , \tag{6.1.2}$$

at every point $x = x_0$ where f has a simple discontinuity. Therefore, the limit (6.1.2) involving the differentiated Fourier series determines the jumps of the function. If one writes (6.1.1) in the cosines-sines form, i.e.,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \ , \tag{6.1.3}$$

then (6.1.2) takes the form

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} n(b_n \cos nx - a_n \sin nx) = \frac{1}{\pi} \left(f(x_0^+) - f(x_0^-) \right) \ . \tag{6.1.4}$$

Relation (6.1.4) is an example of what we call a *differentiated mean*. A. Zygmund studied a more general problem in [255] (see also [256]), under an extended notion of symmetric jump related to the notion of *de la Vallée Poussin generalized derivatives*, he obtained formulas for the jump in terms of Cesàro versions of (6.1.4).

The study of the jump behavior and the determination of jumps by different types of means has become an important area because of its applications in edge detection from spectral data [66, 67]. Results of this kind are important in applied mathematics because they have direct consequences in computational algorithms (consult references in [66]). Recently, it has attracted the attention of many authors and some generalization of classical results have been given [9, 54, 66, 67, 70, 118, 119, 120, 121, 140, 141, 143, 186, 187, 215, 218, 222, 248, 253]. We already faced some of such generalizations in Chapter 5. Basically, we could say that these generalizations go in three directions: extensions of the notion of jump, enlargement of the class of functions, and the use of different means to determine the jump.

In the present chapter we provide results of a general character. We leave the usual classes of classical functions, and obtain results for very general distributions and tempered distributions, as we have been doing in the previous chapters. The usual notions for jumps are extended to distributional notions for pointwise jumps, the jump behavior and the symmetric jump behavior, as defined in Chapter 5 (Section 5.2). The distributional jumps include those of classical functions. In order to determine the pointwise jumps of distributions, we define a new type of means, the differentiated means in the Cesàro and Riesz sense; these means are applicable to Fourier series and to the Fourier transform of tempered distributions. We then obtain formulas of type (6.1.2) in terms of the differentiated means of the Fourier transform of tempered distributions. Our results are applicable to Fourier series, we therefore generalize some of the results mentioned above. The approach we are taking has also a numerical advantage with respect to other approaches; in the case of the jump occurring in the jump behavior, our formulas only use partial part of the spectral data (either positive or negative part). For the case of symmetric jumps, we recover some results from [255, 256]. When we deal only with distributions in $\mathcal{D}'(\mathbb{R})$, thus we do not have the Fourier transform available, we can still use differentiated *Abel-Poisson* means in order to determine the jump, that is, the jump can be calculated in terms of the asymptotic behavior of partial derivatives of harmonic representations and harmonic conjugates.

6.2 Differentiated Riesz and Cesàro Means

In this section we shall define a new type of means, the *differentiated Riesz and Cesàro means*. They will be the main tool of the next section when finding formulas for jumps of distributions. We begin with the case of series.

Definition 6.1. *Let $\{\lambda_n\}_{n=0}^{\infty}$ be an increasing sequence of non-negative numbers such that $\lim_{n\to\infty} \lambda_n = \infty$. Let k and $m \in \mathbb{N}$. We say that a series $\sum_{n=0}^{\infty} c_n$ is summable to γ by the k-differentiated Riesz means of order m, relative to $\{\lambda_n\}_{n=0}^{\infty}$, if*

$$\lim_{x\to\infty} k\binom{m+k}{m} \sum_{\lambda_n<x} c_n \left(\frac{\lambda_n}{x}\right)^k \left(1-\frac{\lambda_n}{x}\right)^m = \gamma \ . \tag{6.2.1}$$

In such a case, we write

$$\text{d.m.} \sum_{n=0}^{\infty} c_n = \gamma \qquad \left(\mathrm{R}^{(k)}, \{\lambda_n\}, m\right) \ . \tag{6.2.2}$$

When $\lambda_n = n$, we simply write $\left(\mathrm{C}^{(k)}, m\right)$ for $\left(\mathrm{R}^{(k)}, \{n\}, m\right)$, and say that the series is summable by the k-differentiated Cesàro means of order m.

Notice that if $k = 0$, the means are trivial. So from now on, we assume that k is always a positive integer, while m might be equal to 0. Observe also that it is possible to take non-integral values for k and m; however, we will only use the integral case and thus we shall always take $k, m \in \mathbb{N}$. When we do not want to make reference to m, we simply write $\left(\mathrm{C}^{(k)}\right)$ or $\left(\mathrm{R}^{(k)}, \{\lambda_n\}\right)$, respectively.

The first surprising fact about our means is that these methods of summation are not *regular* [85]; that is, if $\sum_{n=0}^{\infty} c_n$ is convergent to γ, we do not necessarily have that $\sum_{n=0}^{\infty} c_n$ is $\left(\mathrm{R}^{(k)}, \{\lambda_n\}, m\right)$ summable to γ. However, our method is what

Hardy calls $\mathfrak{I}_c$ [85, p.43], it means that it sums convergent series but not necessarily to the same value of convergence. That fact is presented in the next proposition: Indeed, our method of differentiated Riesz means sums all convergent series to 0.

Proposition 6.2. *Suppose that* $\sum_{n=0}^{\infty} c_n$ *is convergent to some value* γ*, then*

$$\text{d.m.} \sum_{n=0}^{\infty} c_n = 0 \qquad \left(\mathrm{R}^{(k)}, \{\lambda_n\}, m\right) \ . \tag{6.2.3}$$

Proof. We assume that $m \geq 1$, when $m = 0$ the proof is similar. Define $s(x) = \sum_{\lambda_n < x} c_n$. We have that $s(x) \to \gamma$ as $x \to \infty$. So,

$$\begin{aligned}\sum_{\lambda_n < x} c_n \frac{\lambda_n^k}{x^k} \left(1 - \frac{\lambda_n}{x}\right)^m &= \int_0^x \left(\frac{t}{x}\right)^k \left(1 - \frac{t}{x}\right)^m \mathrm{d}s(t) \\ &= \int_0^1 \left((m+k)t - k\right) t^{k-1}(1-t)^{m-1} s(xt)\, \mathrm{d}t \ ,\end{aligned}$$

and the last term converges to

$$\gamma \left((m+k) \int_0^1 t^k (1-t)^{m-1} \mathrm{d}t - k \int_0^1 t^{k-1} (1-t)^{m-1} \mathrm{d}t \right) = 0 \ ,$$

as required. □

The fact that the differentiated Riesz means sum convergent series to 0 will be reflected in their ability to detect the jump of Fourier series.

We now generalize Definition 6.1 to distributional evaluations.

Definition 6.3. *Let* $g \in \mathcal{D}'(\mathbb{R})$ *be a distribution with support bounded on the left and let* $\phi \in \mathcal{E}(\mathbb{R})$*. We say that the evaluation* $\langle g(x), \phi(x) \rangle$ *has a value* γ *in the* k*-differentiated Cesàro sense (at order* m*) and write*

$$\text{d.m.} \langle g(x), \phi(x) \rangle = \gamma \qquad \left(\mathrm{C}^{(k)}, m\right) \ , \tag{6.2.4}$$

if

$$x^k \phi(x) g(x) = \gamma x^{k-1} + o\left(x^{k-1}\right) \qquad (\mathrm{C}, m+1) \ , \quad x \to \infty \ . \tag{6.2.5}$$

A similar definition applies if g has support bounded on the right; notice that unlike the (C) sense, where $\langle f(-x), \phi(x)\rangle = \langle f(x), \phi(-x)\rangle$ (C), in this case we have that $\text{d.m.}\, \langle f(-x), \phi(x)\rangle = -\text{d.m.}\, \langle f(x), \phi(-x)\rangle$ $\left(\mathrm{C}^{(k)}\right)$. Again, if we do not want to make reference to m, we simply write $\left(\mathrm{C}^{(k)}\right)$. Observe that one readily verifies that

$$\text{d.m.} \sum_{n=0}^{\infty} c_n = \gamma \qquad \left(\mathrm{R}^{(k)}, \{\lambda_n\}, m\right) \tag{6.2.6}$$

if and only if

$$\text{d.m.} \left\langle \sum_{n=0}^{\infty} c_n \delta(x - \lambda_n), 1 \right\rangle = \gamma \qquad \left(\mathrm{C}^{(k)}, m\right) \ . \tag{6.2.7}$$

More generally, if μ is a Radon measure concentrated on $[0, \infty)$, one writes instead of (6.2.4)

$$\text{d.m.} \int_0^{\infty} \phi(t) \mathrm{d}\mu(t) = \gamma \qquad \left(\mathrm{C}^{(k)}, m\right) \ . \tag{6.2.8}$$

Hence (6.2.8) holds if and only if

$$\lim_{x \to \infty} k \binom{m+k}{m} \int_0^x \phi(t) \left(\frac{t}{x}\right)^k \left(1 - \frac{t}{x}\right)^m \mathrm{d}\mu(t) = \gamma \ . \tag{6.2.9}$$

We want to define the k-differentiated Cesàro distributional evaluations for the case of unrestricted supports.

Lemma 6.4. *If $g \in \mathcal{E}'(\mathbb{R})$ then for any $k > 0$, $m \geq 0$ and $\phi \in \mathcal{E}(\mathbb{R})$, one has that* $\text{d.m.}\, \langle g(x), \phi(x)\rangle = 0$ $\left(\mathrm{C}^{(k)}, m\right)$.

Proof. Since $\phi(x)g(x) \in \mathcal{E}'(\mathbb{R})$, one can assume that $\phi \equiv 1$. It is enough to show the result for $m = 0$. Next, let $G \in \mathcal{D}'(\mathbb{R})$ be a distribution with support bounded at the left such that $G'(x) = x^k g(x)$, since G' vanishes in a neighborhood of infinity, then G is constant in that neighborhood of infinity, consequently, for x large enough $G(x) = o(x^k)$, as $x \to \infty$, in the ordinary sense, as required. $\square$

We can now define the k-differentiated Cesàro distributional evaluations for distributions with unrestricted support.

Definition 6.5. *Let $g \in \mathcal{D}'(\mathbb{R})$ and let $\phi \in \mathcal{E}(\mathbb{R})$. Let $g = g_1 + g_2$ be a decomposition of g where $g_1(x)$ and $g_2(-x)$ have supports bounded on the left. We say that* $\mathrm{d.m.}\,\langle g(x), \phi(x)\rangle = \gamma \ \left(\mathrm{C}^{(k)}\right)$ *if both* $\mathrm{d.m.}\,\langle g_i(x), \phi(x)\rangle = \gamma_i \ \left(\mathrm{C}^{(k)}\right)$ *exist and* $\gamma = \gamma_1 + \gamma_2$.

Observe that because of Lemma 6.4 the last definition is independent of the decomposition of f.

We also have the analog to Proposition 6.2 for distributions.

Proposition 6.6. *Let $f \in \mathcal{D}'(\mathbb{R})$ and let k be a positive integer. If $\langle g(x), \phi(x)\rangle = \gamma$ (C), for some γ, then* $\mathrm{d.m.}\,\langle g(x), \phi(x)\rangle = 0 \ \left(\mathrm{C}^{(k)}\right)$.

Proof. It is enough to assume that g has support bounded on one side, say on the left, and that $\phi \equiv 1$. The condition, together with the assumption on the support, implies that

$$g(\lambda x) = \gamma \frac{\delta(x)}{\lambda} + o\left(\frac{1}{\lambda}\right),$$

as $\lambda \to \infty$ in $\mathcal{S}'(\mathbb{R})$. Hence multiplying by $(\lambda x)^k$, we see that

$$(\lambda x)^k g(\lambda x) = o\left(\lambda^{k-1}\right) \quad \text{as} \quad \lambda \to \infty,$$

in $\mathcal{S}'(\mathbb{R})$. Hence, since the support of g is bounded on the left, we can apply Proposition 1.13 to conclude that $x^k g(x) = o\left(x^{k-1}\right)$ (C). $\square$

We were not precise in the order of summability in Proposition 6.6. If we want to obtain information about the order, then it requires a more elaborated argument.

Theorem 6.7. *Let $f \in \mathcal{D}'(\mathbb{R})$ and k be a positive integer. If $\langle g(x), \phi(x)\rangle = \gamma$ (C, m), for some γ, then* $\mathrm{d.m.}\,\langle g(x), \phi(x)\rangle = 0 \ \left(\mathrm{C}^{(k)}, n\right)$*, for $n \geq m$.*

Proof. We may assume that $\operatorname{supp} g$ is bounded at the left, $\phi \equiv 1$ and $n = m$. Let G be the $(m+1)$-primitive of g with support bounded at the left, then

$$G(x) \sim \frac{\gamma}{m!}\, x^m, \quad x \to \infty, \tag{6.2.10}$$

We now calculate the $(m+1)$-primitive of $x^k g(x)$ with support bounded at the left. In the well known formula

$$\phi h^{(m+1)} = \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} \left(\phi^{(j)} h\right)^{(m+1-j)} , \tag{6.2.11}$$

valid for $\phi \in \mathcal{E}(\mathbb{R})$ and $h \in \mathcal{D}'(\mathbb{R})$, we take $h = G$ and $\phi(x) = x^k$. This shows that

$$F(x) = \sum_{j=0}^{m+1} (-1)^j \frac{C(k,j)}{(j-1)!} \binom{m+1}{j} \int_0^x (x-t)^{j-1} t^{k-j} G(t) \mathrm{d}t ,$$

where $C(k,j) = k(k-1)\dots(k-j+1)$, is the desired $(m+1)$-primitive of $x^k g(x)$. Then, (6.2.10) implies

$$\begin{aligned} F(x) &= \frac{\gamma}{m!} \sum_{j=0}^{m+1} (-1)^j \frac{C(k,j)}{(j-1)!} \binom{m+1}{j} \int_0^x (x-t)^{j-1} t^{k-j+m} \mathrm{d}t + o(x^{m+k}) \\ &= \frac{\gamma}{(m!)^2} \int_0^x (x-t)^m t^k \frac{d^{m+1}}{dt^{m+1}} (t^m) \mathrm{d}t + o(x^{m+k}) = o(x^{m+k}) , \end{aligned}$$

as $x \to \infty$, here we have used again (6.2.11) but now with $h(x) = x^m$. $\square$

6.3 Determining the Jumps of Tempered Distributions by Differentiated Cesàro Means

In this section we determine the jump, for the jump behavior and symmetric jump behavior, of general tempered distributions. This is done in two ways, in terms of the asymptotic behavior of its Fourier transform, and in terms of differentiated Cesàro means.

Theorem 6.8. *Let $f \in \mathcal{S}'(\mathbb{R})$ have the distributional jump behavior at $x = x_0$,*

$$f(x_0 + \varepsilon x) = \gamma_- H(-x) + \gamma_+ H(x) + o(1) \quad \text{as} \ \ \varepsilon \to 0^+ . \tag{6.3.1}$$

Let k be a positive integer. Then for any decomposition $\hat{f} = \hat{f}_- + \hat{f}_+$, with $\operatorname{supp} \hat{f}_- \subseteq (-\infty, 0]$ *and* $\operatorname{supp} \hat{f}_+ \subseteq [0,\infty)$*, one has that*

$$\text{d.m.} \left\langle \hat{f}_\pm(x), e^{ix_0 x} \right\rangle = \frac{1}{i} [f]_{x=x_0} \qquad \left(\mathrm{C}^{(k)}\right) \tag{6.3.2}$$

In particular, d.m. $\left\langle \hat{f}(x), e^{ix_0x}\right\rangle = (2/i)\left[f\right]_{x=x_0} \quad \left(\mathrm{C}^{(\mathrm{k})}\right)$, *and*

$$x^k e^{i\lambda x_0 x}\hat{f}_{\pm}(\lambda x) = (\pm 1)^{k-1}\frac{1}{\lambda i}\left[f\right]_{x=x_0} x_{\pm}^{k-1} + o\left(\frac{1}{\lambda}\right) \quad \textit{as} \;\; \lambda \to \infty \ , \qquad (6.3.3)$$

where the last quasiasymptotic relation holds in the sense of weak convergence in $\mathcal{S}'(\mathbb{R})$*.*

Proof. Differentiating (6.3.1) k-times, one has that

$$f^{(k)}\left(x_0 + \varepsilon x\right) = \left[f\right]_{x=x_0}\frac{\delta^{(k-1)}(x)}{\varepsilon^k} + o\left(\frac{1}{\varepsilon^k}\right) \ , \qquad (6.3.4)$$

as $\varepsilon \to 0^+$ in $\mathcal{D}'(\mathbb{R})$. If we take Fourier transform in (6.3.4), we obtain the asymptotic behavior,

$$\left(\lambda x\right)^k e^{i\lambda x_0 x}\hat{f}(\lambda x) = \frac{1}{i}\left[f\right]_{x=x_0}\left(\lambda x\right)^{k-1} + o\left(\lambda^{k-1}\right) \quad \text{as} \;\; \lambda \to \infty \ , \qquad (6.3.5)$$

in $\mathcal{S}'(\mathbb{R})$. Therefore $x^k e^{ix_0x}\hat{f}(x)$ has quasiasymptotic behavior at infinity with respect to λ^{k-1}. The asymptotic relation (6.3.5) admits the splitting (6.3.3), due to the general structural theorem for quasiasymptotic behaviors (see Chapter 10, [212, Thm.2.6] or the decomposition theorem in [231, p.134]); and (6.3.3) yields (6.3.2), by Proposition 1.8 (Section 1.8.1). $\square$

A particular case is obtained when $\hat{f}$ is a Radon measure. Notice that this class of distributions includes the so called pseudofunctions [71].

Corollary 6.9. *Let* $f \in \mathcal{S}'(\mathbb{R})$ *have the distributional jump behavior (6.3.1). Suppose that its Fourier transform is given by a Radon measure* μ*. Then for each positive integer* k *there exists* $m \in \mathbb{N}$ *such that for any decomposition of* $\mu = \mu_- + \mu_+$ *as two Radon measures concentrated on* $(-\infty, 0]$ *and* $[0, \infty)$*, respectively, one has that*

$$\text{d.m.}\int_0^\infty e^{\pm ix_0t}\mathrm{d}\mu_{\pm}\left(\pm t\right) = \pm\frac{1}{i}\left[f\right]_{x=x_0} \qquad \left(\mathrm{C}^{(k)}, m\right) \ , \qquad (6.3.6)$$

or which amounts to the same,

$$\lim_{x\to\infty} ik\binom{m+k}{m}\int_0^x e^{\pm ix_0 t}\left(\frac{t}{x}\right)^k\left(1-\frac{t}{x}\right)^m \mathrm{d}\mu_{\pm}(\pm t) = \pm\,[f]_{x=x_0} \ . \tag{6.3.7}$$

Note that Theorem 6.8 and Corollary 6.9 provide us with formulas for the jump by only considering the spectral data of f from either the left or right side of the origin. In the case of symmetric jump behavior this is not longer possible; however, we can still recover the jump by taking symmetric means.

Theorem 6.10. *Suppose that $f \in \mathcal{S}'(\mathbb{R})$ has a symmetric jump at $x = x_0$. Let k be a positive integer. Then for any decomposition $\hat{f} = \hat{f}_- + \hat{f}_+$, where* $\operatorname{supp}\hat{f}_- \subseteq (-\infty, 0]$ *and* $\operatorname{supp}\hat{f}_+ \subseteq [0,\infty)$*, we have that*

$$\text{d.m.}\left\langle e^{ix_0x}\hat{f}_+(x) - e^{-ix_0x}\hat{f}_-(-x), 1\right\rangle = \frac{2}{i}\,[f]_{x=x_0} \qquad \left(\mathrm{C}^{(\mathrm{k})}\right) \ . \tag{6.3.8}$$

Proof. Let $\psi_{x_0} := \psi^f_{x_0}$ be the jump distribution (Section 5.2). It has the jump behavior at $x = 0$

$$\psi_{x_0}(\varepsilon x) = [f]_{x=x_0}\operatorname{sgn} x + o(1) \quad \text{as } \ \varepsilon \to 0^+ \quad \text{in } \ \mathcal{D}'(\mathbb{R}) \ ,$$

and so $[\psi_{x_0}]_{x=0} = 2\,[f]_{x=x_0}$. Since $\hat{\psi}_{x_0}(x) = e^{ix_0x}\hat{f}(x) - e^{-ix_0x}\hat{f}(-x)$, a decomposition $\hat{f} = \hat{f}_- + \hat{f}_+$ leads to the decomposition $\hat{\psi}_{x_0}(x) = \hat{\psi}_-(x) + \hat{\psi}_+(x)$ where

$$\hat{\psi}_{\pm}(x) = e^{ix_0x}\hat{f}_{\pm}(x) - e^{-ix_0x}\hat{f}_{\mp}(-x) \ ,$$

and thus Theorem 6.8 implies (6.3.8). $\square$

When $\hat{f}$ is a Radon measure, we can give formulas of type (6.3.7). Depending on the parity of k, we should use the means of a Fourier type integral or a conjugate type integral. This fact is given in the next two corollaries which follow immediately from Theorem 6.10.

Corollary 6.11. *Let $f \in \mathcal{S}'(\mathbb{R})$ have a distributional symmetric jump behavior at $x = x_0$. Suppose that its Fourier transform is a Radon measure μ. Let $2k-1$ be a positive odd integer. Then there exists $m \in \mathbb{N}$ such that*

$$\lim_{x\to\infty} \frac{i(2k-1)}{2x^{2k-1}} \binom{m+2k-1}{m} \int_{-x}^{x} t^{2k-1} e^{ix_0 t} \left(1 - \frac{|t|}{x}\right)^m \mathrm{d}\mu(t) = [f]_{x=x_0} . \quad (6.3.9)$$

Corollary 6.12. *Let $f \in \mathcal{S}'(\mathbb{R})$ have a distributional symmetric jump behavior at $x = x_0$. Suppose that its Fourier transform is a Radon measure μ. Let $2k$ be a positive even integer. Then there exists $m \in \mathbb{N}$ such that for any decomposition $\mu = \mu_- + \mu_+$, as two Radon measures concentrated on $(-\infty, 0]$ and $[0, \infty)$, respectively, one has that*

$$\lim_{x\to\infty} \frac{ik}{x^{2k}} \binom{m+2k}{m} \int_{-x}^{x} t^{2k} e^{ix_0 t} \left(1 - \frac{|t|}{x}\right)^m \mathrm{d}\sigma(t) = [f]_{x=x_0} , \quad (6.3.10)$$

where $\sigma = \mu_+ - \mu_-$.

Sometimes is possible to single out a measure σ in (6.3.10). For certain distributions one can talk about a unique Hilbert transform [60], say $\tilde{f}$, in such a case one may take $\sigma = i\hat{\tilde{f}}$. Actually, this will be done in Section 6.5 for the case of periodic distributions.

6.4 Jumps and Local Boundary Behavior of Derivatives of Harmonic and Analytic Functions

In this section, we determine the jump of a distribution in terms of the asymptotic behavior of derivatives of analytic representations (Section 1.6); we also find formulas for the jump in terms of partial derivatives of harmonic and harmonic conjugate functions. Given $0 < \eta \leq \pi/2$ and $x_0 \in \mathbb{R}$, we define the subset of the upper half-plane $\Delta_\eta^+(x_0)$ as the set of those z such that $\eta \leq \arg(z - x_0) \leq \pi - \eta$, similarly, we define the subset of the lower half-plane $\Delta_\eta^-(x_0)$ as the set of those z such that $\eta - \pi \leq \arg(z - x_0) \leq -\eta$.

We start with the jump behavior and analytic representations.

Theorem 6.13. *Let $f \in \mathcal{D}'(\mathbb{R})$ have the jump behavior at $x = x_0$*

$$f\left(x_0 + \varepsilon x\right) = \gamma_- H(-x) + \gamma_+ H(x) + o(1) \quad \text{as} \quad \varepsilon \to 0^+ \ . \tag{6.4.1}$$

Suppose that F is an analytic representation of f on $\Im m\ z \neq 0$, then for each positive integer k and $0 < \eta \leq \pi/2$, one has that

$$\lim_{z \to z_0,\, z \in \Delta_\eta^\pm(x_0)} \left(z - x_0\right)^k F^{(k)}(z) = (-1)^k \, \frac{(k-1)!}{2\pi i} \, [f]_{x=x_0} \ . \tag{6.4.2}$$

Proof. We first show that if (6.4.2) holds for one analytic representation, then it holds for any analytic representation of f. In fact by the very well known edge of the wedge theorem, any two such analytic representations differ by an entire function, and for entire functions (6.4.2) gives 0. Next, we prove that we may assume that $f \in \mathcal{S}'(\mathbb{R})$. Indeed we can decompose $f = f_1 + f_2$ where f_2 is zero in a neighborhood of x_0 and $f_1 \in \mathcal{S}'(\mathbb{R})$. Let F_1 and F_2 be analytic representations of f_1 and f_2, respectively; then F_2 can be continued across a neighborhood of x_0 (edge of the wedge theorem once again), hence $F_2(z) = F_2(x_0) + O\left(|z - x_0|\right) = O\left(1\right)$ as $z \to x_0$. Additionally, f_1 has the same jump behavior as f. Thus, we may assume that $f \in \mathcal{S}'\left(\mathbb{R}\right)$. Consider the following analytic representation [24, p.83], where $\hat{f} = \hat{f}_- + \hat{f}_+$ is a decomposition as in Theorem 6.8,

$$F(z) = \begin{cases} \dfrac{1}{2\pi}\left\langle \hat{f}_+(t), e^{izt} \right\rangle, & \Im m\ z > 0 \ , \\ -\dfrac{1}{2\pi}\left\langle \hat{f}_-(t), e^{izt} \right\rangle, & \Im m\ z < 0 \ , \end{cases}$$

Keep the number z on a compact subset of $\Delta_\eta^\pm(x_0)$, then

$$\begin{aligned} F^{(k)}\left(x_0 + \frac{z}{\lambda}\right) &= \pm\frac{i^k}{2\pi}\lambda^{k+1}\left\langle t^k e^{i\lambda x_0 t}\hat{f}_\pm\left(\lambda t\right), e^{izt}\right\rangle \\ &= \pm\frac{(\pm i)^{k-1}}{2\pi}\,[f]_{x=x_0}\,\lambda^k \int_0^\infty t^{k-1}e^{\pm izt}\mathrm{d}t + o\left(\lambda^k\right) \\ &= (-1)^k\,\frac{(k-1)!}{2\pi i}\,[f]_{x=x_0}\left(\frac{\lambda}{z}\right)^k + o\left(\lambda^k\right), \end{aligned}$$

as $\lambda \to \infty$, where we have used (6.3.3). $\square$

Next, we determine the jump, occurring in jump behavior, by finding the local boundary asymptotic behavior of partial derivatives of harmonic and harmonic conjugate functions.

Theorem 6.14. *Let $f \in \mathcal{D}'(\mathbb{R})$ have the distributional jump behavior (6.4.1) at $x = x_0$. Let U be a harmonic representation of f on $\Im m\, z > 0$. Let V be a harmonic conjugate to U. Suppose that k is a positive integer, then*

$$\frac{\partial^k U}{\partial x^k}(z) = \frac{(k-1)!}{(-1)^k \pi} [f]_{x=x_0} \Im m \frac{1}{(z-x_0)^k} + o\left(|z-x_0|^{-k}\right) \ , \tag{6.4.3}$$

and

$$\frac{\partial^k V}{\partial x^k}(z) = \frac{(k-1)!}{(-1)^{k+1} \pi} [f]_{x=x_0} \Re e \frac{1}{(z-x_0)^k} + o\left(|z-x_0|^{-k}\right) \ , \tag{6.4.4}$$

as $z \to x_0$ on any sector of the form $\Delta_+^{\eta}(x_0)$, $0 < \eta \leq \pi/2$.

Proof. Notice that, since harmonic conjugates differ from each other by a constant, we may use any specific V we want. We now show that we may work with any harmonic representation U of f. Suppose that U and U_1 are two harmonic representations of f, then $U_2 = U - U_1$ represents the zero distribution. Then by applying the reflection principle to the real and imaginary parts of U [11, Section 4.5], [207, Section 3.4], we have that U_2 admits a harmonic extension to a (complex) neighborhood of x_0. Consequently, if V and V_1 are harmonic conjugates to U and U_1, we have that $V_2 = V - V_1$ is harmonic conjugate to U_2, and thus it admits a harmonic extension to a (complex) neighborhood of x_0 as well. Therefore $\dfrac{\partial^k U_2}{\partial x^k}(z)$, $\dfrac{\partial^k V_2}{\partial x^k}(z) = O(1)$ in a neighborhood of x_0; consequently, we have that U and V satisfy (6.4.3) and (6.4.4) if and only if U_1 and V_1 do it.

Let F be an analytic representation of f. We may assume that $U(z) = F(z) - F(\bar{z})$ and $V(z) = -i\,(F(z) + F(\bar{z}))$. Notice that $\dfrac{\partial^k U}{\partial x^k}(z) = F^{(k)}(z) - F^{(k)}(\bar{z})$ and

$\dfrac{\partial^k V}{\partial x^k}(z) = -i\left(F^{(k)}(z) + F^{(k)}(\bar{z})\right)$, and then an application of (6.4.2) gives (6.4.3) and (6.4.4). □

Observe that when k is odd it is possible to recover the jump from the radial asymptotic behavior of $\dfrac{\partial^k U}{\partial x^k}$ but not from the one of $\dfrac{\partial^k V}{\partial x^k}$; similarly, when k is even we recover the jump from the radial behavior of $\dfrac{\partial^k V}{\partial x^k}$, but not from the one of $\dfrac{\partial^k U}{\partial x^k}$. This is also true for the symmetric jump behavior.

Theorem 6.15. *Let $f \in \mathcal{D}'(\mathbb{R})$ have symmetric jump at $x = x_0$. Let k be a positive integer. Suppose that U is a harmonic representation of f on $\Im m\, z > 0$ and V is a harmonic conjugate to U. Then,*

$$\lim_{y \to 0^+} y^{2k-1} \frac{\partial^{2k-1} U}{\partial x^{2k-1}}\,(x_0 + iy) = (-1)^{k+1} \frac{(2k-2)!}{\pi}\,[f]_{x=x_0}\ , \tag{6.4.5}$$

and

$$\lim_{y \to 0^+} y^{2k} \frac{\partial^{2k} V}{\partial x^{2k}}\,(x_0 + iy) = (-1)^{k+1} \frac{(2k-1)!}{\pi}\,[f]_{x=x_0}\ . \tag{6.4.6}$$

Proof. We apply our results to the jump distribution $\psi_{x_0} := \psi^f_{x_0}$ (Section 5.2). Let U be a harmonic representation of f and V be a harmonic conjugate. We have that $U\,(x_0 + z) - U\,(x_0 - \bar{z})$ and $V\,(x_0 + z) + V\,(x_0 - \bar{z})$ are a harmonic representation and a harmonic conjugate for ψ_{x_0}. The result now follows from Theorem 6.14 and the fact $[\psi_{x_0}]_{x=0} = 2\,[f]_{x=x_0}$. □

We remark that for distributions the radial behavior of its harmonic representations can be considered as Abel-Poisson means, while the radial behavior of harmonic conjugate functions can be considered as conjugate Abel-Poisson means; hence, one can say that Theorem 6.15 gives the jump in terms of *differentiated Abel-Poisson means.* We will apply this useful observation to Fourier series in the next section. We also want to point out that Theorem 6.13 and Theorem 6.14 are much stronger than Theorem 6.15, and in the context of Fourier series, as we shall see, can be used to express the jump as differentiated Abel-Poisson means of only a partial part of the spectrum.

If we assume that f is the boundary value of an analytic function on the upper half-plane, we can get a better result than Theorem 6.15. This is the content of the next theorem.

Theorem 6.16. *Let F be analytic in the upper half-plane, with distributional boundary values $f(x) = F(x + i0)$. Suppose f has a distributional symmetric jump behavior at $x = x_0$. Then, for any $0 < \eta \leq \pi/2$*

$$F^{(k)}(z) \sim \frac{(k-1)![f]_{x=x_0}}{(-1)^k i\pi (z-x_0)^k} \qquad \textit{as} \;\; z \in \Delta_\eta^+(x_0) \to x_0 \ . \tag{6.4.7}$$

Proof. Let ψ_{x_0} be the jump distribution at $x = x_0$. Then ψ_{x_0} has a jump behavior at $x = 0$ and $[\psi_{x_0}]_{x=0} = 2[f]_{x=x_0}$. Observe that $U(z) = F(x_0 + z) - F(x_0 - \bar{z})$ is a harmonic representation of ψ_{x_0} and $V(z) = -i\left(F(x_0 + z) + F(x_0 - \bar{z})\right)$ is a harmonic conjugate. Hence, we can apply Theorem 6.14 to U and V to obtain that

$$F^{(k)}(x_0 + z) = (-1)^k F(x_0 - \bar{z}) + 2(-1)^k \frac{(k-1)!}{\pi} [f]_{x=x_0} \Im m\, \frac{1}{z^k} + o\left(|z|^{-k}\right)$$

and

$$F^{(k)}(x_0 + z) = (-1)^{k+1} F(x_0 - \bar{z}) + 2(-1)^k \frac{(k-1)!}{i\pi} [f]_{x=x_0} \Re e\, \frac{1}{z^k} + o\left(|z|^{-k}\right)$$

as $z \in \Delta_\eta^+(0) \to 0$; and therefore (6.4.7) follows. □

6.5 Applications to Fourier Series

This section is dedicated to applications of our results to Fourier series. We determine the jump of 2π-periodic distributions in terms of differentiated Cesàro-Riesz and Abel-Poisson means.

Throughout this section f is a 2π-periodic distribution with Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \ , \tag{6.5.1}$$

where the series converges in $\mathcal{S}'(\mathbb{R})$.

6.5.1 Jump Behavior and Fourier Series

Notice that the Fourier transform of f is given by

$$\hat{f}(x) = 2\pi \sum_{n=-\infty}^{\infty} c_n \delta(x-n) \ , \tag{6.5.2}$$

hence, as an immediate corollary of Theorem 6.8, we obtain,

Theorem 6.17. *If f has a jump behavior at $x = x_0$, with jump $[f]_{x=x_0}$, then for each positive integer k we have that*

$$\text{d.m.} \sum_{n=0}^{\infty} c_n e^{ix_0 n} = \frac{1}{2\pi i} [f]_{x=x_0} \qquad \left(\mathrm{C}^{(k)}\right) \ , \tag{6.5.3}$$

and

$$\text{d.m.} \sum_{n=1}^{\infty} c_{-n} e^{-ix_0 n} = -\frac{1}{2\pi i} [f]_{x=x_0} \qquad \left(\mathrm{C}^{(k)}\right) \ . \tag{6.5.4}$$

Notice that, as we have previously remarked, in our formulas we only need either the positive or the negative part of the spectral data of f, having an advantage over other approaches where the complete spectral data of f is used.

We now interpret Theorem 6.13 in the context of Fourier series; again notice that only one part of the spectrum is used. Observe that

$$F(z) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} c_n e^{izn}, & \Im m\ z > 0 \ , \\ \displaystyle -\sum_{n=-\infty}^{1} c_n e^{izn}, & \Im m\ z < 0 \ , \end{cases} \tag{6.5.5}$$

is an analytic representation of f, from where we have immediately.

Theorem 6.18. *If f has a jump behavior at $x = x_0$, with jump $[f]_{x=x_0}$, then for each positive integer k we have that for $0 < \eta \le \pi/2$,*

$$\lim_{z\to x_0,\ z\in\Delta_\eta^+(x_0)} (z-x_0)^k \sum_{n=0}^{\infty} n^k c_n e^{inz} = -\frac{(k-1)!}{2\pi(-i)^{k+1}} [f]_{x=x_0} \ , \tag{6.5.6}$$

and

$$\lim_{z\to x_0,\ z\in\Delta_\eta^-(x_0)} (z-x_0)^k \sum_{n=-\infty}^{1} n^k c_n e^{inz} = \frac{(k-1)!}{2\pi(-i)^{k+1}} [f]_{x=x_0} \ . \tag{6.5.7}$$

Remark 6.19. We remark that we may also consider non-harmonic series and obtain analog results. Indeed, suppose that $\{\lambda_n\}_{n=0}^{\infty}$ is an increasing sequence such that $0 \leq \lambda_0$ and $\lim_{n\to\infty} \lambda_n = \infty$, let

$$g(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\lambda_{|n|}x} \ , \tag{6.5.8}$$

convergent in $\mathcal{S}'(\mathbb{R})$. Then if g has a distributional jump behavior at $x = x_0$, we have that for each positive integer k

$$\text{d.m.} \sum_{n=0}^{\infty} c_n e^{ix_0\lambda_n} = \frac{1}{2\pi i}\left[g\right]_{x=x_0} \qquad \left(\mathrm{R}^{(k)}, \{\lambda_n\}\right) \ , \tag{6.5.9}$$

$$\text{d.m.} \sum_{n=1}^{\infty} c_{-n} e^{ix_0\lambda_n} = -\frac{1}{2\pi i}\left[g\right]_{x=x_0} \qquad \left(\mathrm{R}^{(k)}, \{\lambda_n\}\right) \ , \tag{6.5.10}$$

$$\lim_{z\to x_0,\ z\in\Delta_\eta^{+}(x_0)} (z-x_0)^k \sum_{n=0}^{\infty} \lambda_n^k c_n e^{i\lambda_n z} = -\frac{(k-1)!}{2\pi(-i)^{k+1}}\left[g\right]_{x=x_0} \ , \tag{6.5.11}$$

and

$$\lim_{z\to x_0,\ z\in\Delta_\eta^{-}(x_0)} (z-x_0)^k \sum_{n=1}^{\infty} \lambda_n^k c_{-n} e^{i\lambda_n z} = \frac{(k-1)!}{2\pi\,(-i)^{k+1}}\left[g\right]_{x=x_0} \ . \tag{6.5.12}$$

6.5.2 Symmetric Jump Behavior and Fourier Series

As usual, we define the conjugate distribution of f as

$$\tilde{f}(x) = \sum_{n=-\infty}^{\infty} \tilde{c}_n e^{inx} \ , \tag{6.5.13}$$

with $\tilde{c}_n = -i \operatorname{sgn} n\, c_n$, $\tilde{c}_0 = 0$. Notice that $\tilde{f}$ is the Hilbert transform of f [60]. Since we will use symmetric means, it is convenient to use the sines and cosines series for f, i.e.,

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} \left(a_n \cos nx + b_n \sin nx\right) \ , \tag{6.5.14}$$

where $a_n = c_n + c_{-n}$, $b_n = i(c_n - c_{-n})$, then $\tilde{c}_n = \left(-b_{|n|} - i\,\mathrm{sgn}\, n a_{|n|}\right)/2$ and

$$\tilde{f}(x) = \sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx) \ . \tag{6.5.15}$$

We obtain from Theorem 6.10.

Theorem 6.20. *Let k be a positive integer. If f has a symmetric jump at $x = x_0$, then*

$$\text{d.m.} \sum_{n=1}^{\infty} (a_n \sin nx_0 - b_n \cos nx_0) = -\frac{1}{\pi} [f]_{x=x_0} \qquad \left(\mathrm{C}^{(k)}\right) \ . \tag{6.5.16}$$

Proof. Observe that $\hat{f} = \hat{f}_- + \hat{f}_+$, where

$$\hat{f}_+(x) = a_0 \pi\, \delta(x) + \pi \sum_{n=1}^{\infty} (a_n - ib_n)\, \delta(x-n) \ ,$$

and

$$\hat{f}_-(x) = \pi \sum_{n=1}^{\infty} (a_n + ib_n)\, \delta(x+n) \ .$$

Thus, an easy calculation gives that $e^{ix_0 x} \hat{f}_+(x) - e^{-ix_0 x} \hat{f}_-(-x)$ is equal to

$$a_0 \pi\, \delta(x) + 2\pi i \sum_{n=1}^{\infty} (a_n \sin nx_0 - b_n \cos nx_0)\, \delta(x-n) \ ,$$

and therefore (6.5.16) is a direct consequence of Theorem 6.10. □

Relation (6.5.16) can also be written in terms of the Fourier coefficients $\{c_n\}$ and $\{\tilde{c}_n\}$. By direct computation, or by applying Corollaries 6.11 and 6.12, one obtains the following corollary.

Corollary 6.21. *Let k be a positive integer. If f has a symmetric jump at $x = x_0$, then*

$$\lim_{x \to \infty} (2k-1) \binom{m+2k-1}{m} \sum_{-x<n<x} c_n e^{ix_0 n} \left(\frac{n}{x}\right)^{2k-1} \left(1 - \frac{|n|}{x}\right)^m = \frac{1}{i\pi} [f]_{x=x_0} \ , \tag{6.5.17}$$

and

$$\lim_{x\to\infty} 2k\binom{m+2k}{m}\sum_{-x<n<x}\tilde{c}_n e^{ix_0 n}\left(\frac{n}{x}\right)^{2k}\left(1-\frac{|n|}{x}\right)^m = -\frac{1}{\pi}\,[f]_{x=x_0}\ . \tag{6.5.18}$$

We now express the jump in terms of differentiated Abel-Poisson means.

Theorem 6.22. *If f has symmetric jump behavior at $x=x_0$, then for any positive k we have that*

$$\sum_{n=1}^{\infty}(a_n\sin nx_0 - b_n\cos nx_0)\,n^k r^n \sim -\frac{(k-1)!\,[f]_{x=x_0}}{\pi(1-r)^k}\ , \tag{6.5.19}$$

as $r\to 1^-$.

Proof. Notice that

$$U(z)=\frac{a_0}{2}+\frac{1}{2}\sum_{n=1}^{\infty}(a_n-ib_n)\,e^{izn}+\frac{1}{2}\sum_{n=1}^{\infty}(a_n+ib_n)\,e^{-i\bar{z}n}$$

and

$$V(z)=-\frac{1}{2}\sum_{n=1}^{\infty}(ia_n+b_n)\,e^{izn}+\frac{1}{2}\sum_{n=1}^{\infty}(ia_n-b_n)\,e^{-i\bar{z}n}$$

are a harmonic representation of f and a harmonic conjugate. If k is odd, we obtain that

$$\frac{\partial^k U}{\partial x^k}(x_0+iy)=i^{k+1}\sum_{n=1}^{\infty}(a_n\sin nx_0-b_n\cos nx_0)\,n^k e^{-ny}\ ,$$

on the other hand if k is even, we have that

$$\frac{\partial^k V}{\partial x^k}(x_0+iy)=i^{k}\sum_{n=1}^{\infty}(a_n\sin nx_0-b_n\cos nx_0)\,n^k e^{-ny}\ .$$

So, in any case we obtain from Theorem 6.15 that for each positive integer

$$\lim_{y\to 0^+} y^k\sum_{n=1}^{\infty}(a_n\sin nx_0-b_n\cos nx_0)\,n^k e^{-ny} = -\frac{(k-1)!}{\pi}\,[f]_{x=x_0}\ .$$

□

We end this section with a direct corollary of Theorem 6.16.

Corollary 6.23. *Suppose that the 2π-periodic distribution f is the boundary value of analytic function, i.e., its Fourier series expansion is of the form*

$$f(x) = \sum_{n=0}^{\infty} c_n e^{ixn} \ . \tag{6.5.20}$$

If f has symmetric jump behavior at $x = x_0$, then for any positive integer k and $0 < \eta \leq \pi/2$, one has that

$$\sum_{n=0}^{\infty} n^k c_n e^{izn} \sim (-1)^k \frac{(k-1)![f]_{x=x_0}}{\pi i^{k+1}(z-x_0)^k} \qquad \text{as} \ \ z \in \Delta_\eta^+(x_0) \to x_0 \ . \tag{6.5.21}$$

6.6 A Characterization of Differentiated Cesàro Means

In this section we provide a characterization of the summability method by differentiated Cesàro means in terms of the Cesàro behavior of the sequence $\left\{n^k c_n\right\}_{n=1}^{\infty}$. This equivalence is stated in the next theorem. The proof adapts an argument from the proof of [85, Thm.58, p.113] to our context; G. Hardy attributes the main argument to A.E. Ingham [100]. One may also adapt M. Riesz's original proof of the equivalence between the $(\mathrm{R}, \{n\})$ and (C) methods of summation [94, 172].

Theorem 6.24. *Let $\{c_n\}_{n=0}^{\infty}$ be a sequence of complex numbers. Let k be a positive integer. Then*

$$\text{d.m.} \sum_{n=0}^{\infty} c_n = \gamma \qquad \left(\mathrm{C}^{(k)}, m\right) \tag{6.6.1}$$

if and only if

$$n^k c_n = \gamma n^{k-1} + o\left(n^{k-1}\right) \qquad (\mathrm{C}, m+1) \ . \tag{6.6.2}$$

Proof. Set $a_n = n^k c_n - \gamma n^{k-1}$, since

$$\lim_{x\to\infty} \binom{m+k}{m} \frac{k}{x^k} \sum_{0<j<x} j^{k-1} \left(1 - \frac{j}{x}\right)^m = k \binom{m+k}{m} \int_0^1 t^{k-1}(1-t)^m \mathrm{d}t$$
$$= 1 \ ,$$

we have that (6.6.1) holds if and only if

$$T_m(x) := \sum_{0<j<x} a_j (x-j)^m = o\left(x^{m+k}\right) \ , \qquad x \to \infty \ . \tag{6.6.3}$$

Set

$$A_{m+1}(n) = \sum_{j=0}^{n} \binom{m+j}{m} a_{n-j} \ . \tag{6.6.4}$$

Observe that relation (6.6.2) is equivalent to

$$A_{m+1}(n) = o(n^{m+k}) \ , \qquad n \to \infty \ . \tag{6.6.5}$$

Therefore, we shall show that (6.6.3) and (6.6.5) are equivalent.

Assume first that $A_{m+1}(n) = o\left(n^{m+k}\right)$. Set $x = n + \vartheta$, where n is an integer and $0 \le \vartheta < 1$. Since $T_m(x) = \sum_{j=0}^{n} (n-j+\vartheta)^m a_j$, we have that for $|z| < 1$

$$\begin{aligned} \sum_{n=0}^{\infty} T_m(x) z^n &= \sum_{n=0}^{\infty} (n+\vartheta)^m z^n \quad \sum_{n=0}^{\infty} a_n z^n \\ &= (1-z)^{m+1} \sum_{n=0}^{\infty} (n+\vartheta)^m z^n \quad \sum_{n=0}^{\infty} A_{m+1}(n) z^n \ . \end{aligned}$$

Now, it is easy to see [85, p.113] that

$$(1-z)^{m+1} \sum_{n=0}^{\infty} (n+\vartheta)^m z^n = \sum_{j=0}^{m} c_j(\vartheta) z^j \ ,$$

where the coefficients $c_j(\vartheta)$ are polynomials in ϑ of degree m. Thus,

$$T_m(x) = \sum_{j=0}^{m} c_j(\vartheta) A_{m+1}(n-j) = o\left(x^{m+k}\right) \ , \qquad x \to \infty \ .$$

We now assume that $T_m(x) = o(x^{m+k})$. We take $m+1$ numbers $0 < \vartheta_0 < \vartheta_1 < \ldots, < \vartheta_m$. The equation

$$\binom{n+m}{m} = \sum_{j=0}^{m} b_j (n+\vartheta_j)^m$$

can be written as a system of $m+1$ equations with non-zero determinant, then it has unique solutions $b_0, \ldots, b_m$. Hence, we obtain

$$A_{m+1}(n) = \sum_{j=0}^{n} \binom{n-j+m}{m} a_j = \sum_{j=0}^{m} b_j T_m(n+\vartheta_j) = o\left(n^{m+k}\right) \ ,$$

as $n \to \infty$, as required. $\square$

It is convenient to spell out what (6.6.2) says. Recall the definition of the Cesàro means [85, 256] of a sequence $\{b_n\}_{n=0}^{\infty}$. Given $l \geq 0$, the Cesàro mean of order l of the sequence (not to be confused with the means of a series) are

$$C_l\left\{b_j;n\right\} := \frac{l!}{n^l}\sum_{j=0}^{n}\binom{j+l-1}{l-1}b_{n-j}\ .$$

Notice that

$$\begin{aligned}\sum_{j=0}^{n}\binom{m+j}{m}(n-j)^{k-1} &\sim \frac{1}{m!}\sum_{j=1}^{n}j^m(n-j)^{k-1}\\ &\sim \frac{(k-1)!}{(m+k)!}\,n^{m+k}\ , \quad n\to\infty\ .\end{aligned}$$

Therefore if we define

$$\begin{aligned}C_m^{(k)}\left\{c_j\,;n\right\} &:= \frac{(m+k)!}{(k-1)!\,n^{m+k}}\sum_{j=1}^{n}\binom{n-j+m}{m}j^k c_j \qquad (6.6.6)\\ &= \frac{k}{m+1}\binom{m+k}{m}\frac{C_{m+1}\left\{j^k c_j\ ;\ n\right\}}{n^{k-1}}\ ,\end{aligned}$$

we have then that (6.6.2) means

$$\lim_{n\to\infty} C_m^{(k)}\left\{c_j\,;n\right\} = \gamma\ . \qquad (6.6.7)$$

So alternatively, we could use (6.6.7) to define the k-differentiated Cesàro means instead of the means originally used in Definition 6.1. This also justifies the switch of notation from $\left(\mathrm{R}^{(k)},\{n\}\right)$ to $\left(\mathrm{C}^{(k)}\right)$ in Definition 6.1.

Theorem 6.24 has an interesting distributional consequence which is presented in the next corollary. We denote the integral part of a number x by $[x]$. Given a sequence $\{a_n\}_{n=0}^{\infty}$, we denote by $a_{[x]}$ the piecewise constant function equal to a_n for $n \leq x < n+1$.

Corollary 6.25. *Let* $\{a_n\}_{n=0}^{\infty}$ *be a sequence of complex numbers and let* k *be a non-negative integer. Then,*

$$\sum_{n=0}^{\infty} a_n \delta(x-n) = \gamma x^k + o\left(x^k\right) \qquad (\mathrm{C}, m) \ , \quad x \to \infty \ , \tag{6.6.8}$$

if and only if

$$a_n = \gamma n^k + o\left(n^k\right) \qquad (\mathrm{C}, m) \ , \quad n \to \infty \ , \tag{6.6.9}$$

and, in turn, if and only if

$$a_{[x]} = \gamma x^k + o\left(x^k\right) \qquad (\mathrm{C}, m) \ , \quad x \to \infty \ . \tag{6.6.10}$$

On combining Theorem 6.17 and Theorem 6.24, we obtain new formulas for the jump of Fourier series occurring in the jump behaviors.

Corollary 6.26. *Let* f *be a* 2π*-periodic distribution having Fourier series*

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} \ .$$

If f *has a jump behavior at* $x = x_0$*, then*

$$\lim_{n\to\infty} n c_n e^{inx_0} = \frac{1}{2\pi i}[f]_{x=x_0} \qquad (\mathrm{C}) \ , \tag{6.6.11}$$

and

$$\lim_{n\to\infty} n c_{-n} e^{-inx_0} = -\frac{1}{2\pi i}[f]_{x=x_0} \qquad (\mathrm{C}) \ . \tag{6.6.12}$$

We end this chapter with a corollary that can be tracked down to the work of A. Zygmund [81, 255], of course he stated it in a very different form; at that time distribution theory did not even exist! The proof follows immediately from Theorem 6.20 and Theorem 6.24.

Corollary 6.27. *Let* f *be a* 2π*-periodic distribution having Fourier series*

$$\sum_{n=0}^{\infty} \left(a_n \cos nx + b_n \sin x\right) \ .$$

If f *has a symmetric jump behavior at* $x = x_0$*, then*

$$\lim_{n\to\infty} n\left(b_n \cos nx_0 - a_n \sin nx_0\right) = \frac{1}{\pi}[f]_{x=x_0} \qquad (\mathrm{C}) \ . \tag{6.6.13}$$

Chapter 7
Distributionally Regulated Functions

7.1 Introduction

In [128], Łojasiewicz introduced and studied the class of distributions that have a distributional value at *every* point. As he showed, these distributions deserve to be called "functions" since the function given by its values is a well-defined measurable function, and the correspondence between the distributions with values at every point and the function of its values is a bijection. Although there is a notion, that of *regular* distribution, that appears to apply exactly to those distributions that correspond to functions, it is fair to say that the distributions introduced by Łojasiewicz, even if not "regular," are objects that one would call "functions."

The aim of this chapter is to introduce and study a generalization of the class of Łojasiewicz functions, namely the distributionally regulated functions, which are those distributions that have a distributional lateral limit at every point without having Dirac delta functions or derivatives at any point, i.e., they have jump behavior everywhere. We also consider the related class of distributionally regulated functions with delta functions, which are those distributions that have a distributional lateral limit at every point; we show that in this case the set of points where there are delta functions is countable at the most.

If f is a distributionally regulated function (without delta functions), with lateral limits $f(x^+)$ and $f(x^-)$ at each $x \in \mathbb{R}$ then we introduce the function

$$\widetilde{f}(x) = \frac{f(x^+) + f(x^-)}{2}. \tag{7.1.1}$$

The function $\widetilde{f}$ is a well-defined measurable function, and the correspondence $f \leftrightarrow \widetilde{f}$ is one-to-one and onto. Therefore, it is justified to identify the distribution

f and the function $\widetilde{f}$, and call f a "function." When f is a distributionally regulated function with delta functions, then $\widetilde{f}$ captures the ordinary function part of f, and $f-\widetilde{f}$ is a singular distribution that consists of sums of Dirac delta functions and derivatives on some countable at the most set. The distributionally regulated functions also generalize the classical regulated functions, which are those functions that have ordinary lateral limits at every point [36]. The classical regulated functions play a role in many areas of mathematics such as conformal mapping theory [167], in the description of curves by their radius of curvature [53] and the application of these ideas to the study of crystals [247], and in the study of theories of integration more general than the Lebesgue integral, a subject that has received increased attention in recent years [10, 76]. Actually, Łojasiewicz proved that there is a descriptive integral that can be defined for distributions that have a value at every point, and as it is easy to see, this integral is also defined for distributionally regulated functions. For this integral one has

$$\langle f(x), \phi(x)\rangle = \int_{-\infty}^{\infty} \widetilde{f}(x)\, \phi(x)\, \mathrm{d}x\,, \tag{7.1.2}$$

for any test function $\phi \in \mathcal{D}(\mathbb{R})$.

The chapter is organized as follows. In Section 7.2 we give some preliminary notions on lateral limits of distributions at points. Distributionally regulated functions are defined in Section 7.3. The next section introduces the $\phi-$transform, a function of two variables $F(x,y)$, $x \in \mathbb{R}$, $y>0$, that satisfies $F(x,0^{+}) = f(x)$ distributionally and that allows us to study the local behavior of a distribution f. In sections 7.4 and 7.5 we consider the pointwise boundary behavior of $F(x,y)$ as (x,y) approaches the point $(x_0,0)$ in the cases when the distributional value $f(x_0)$ exists and when just the distributional limits $f\left(x_0^{\pm}\right)$ exist. We give several formulas for the distributional jumps of f in terms of the $\phi-$transform and related functions; these formulas complement those from Sections 5.4, 5.6, and 6.4, (Chapters 5 and 6), which were given in terms of the boundary behavior of harmonic representations and harmonic conjugates. Our formulas apply to distributions with arbitrary support and are given not only in terms of conjugate harmonic functions but in terms of more general solutions of partial differential equations, as follows from the results of Section 7.8.

In Section 7.6 we show that the set of singular points of a distributionally regulated function, namely where the lateral limits do not coincide, or where there are delta functions, is countable at the most; this result is easily proved for classical regulated functions, but a new proof is required in this case. Actually, our arguments also apply to show that the set of points where a distribution may have non-equal lateral limits is countable at most; the result holds for general distributions not necessarily being distributionally regulated functions. The last fact about the size of the set of singular points is used in Section 7.7 to prove the one-to-one correspondence between the functions and the distributions. In Section 7.8 we show that the $\phi-$transform is many times the solution of a partial differential equation, such as the Laplace equation or the heat equation, and therefore our results become results on the boundary behavior of solutions of partial differential equations. Finally, in Section 7.9 we provide two characterizations of the Fourier transform of tempered distributionally regulated functions. The results of this chapter have already been published in [215].

7.2 Limits and Lateral Limits at a Point

In Section 4.6 we introduced lateral limits of distributions at points. Definition 4.14 differs from Łojasiewicz original definition, but both are equivalent. We now discuss Łojasiewicz original approach [128], which will be more convenient for the purposes of this chapter. We say that $f \in \mathcal{D}'(\mathbb{R})$ *has a distributional limit* γ at the point $x = x_0$ if $f(x_0 + \varepsilon x) = \gamma + o(1)$ as $\epsilon \to 0$ in $\mathcal{D}'(\mathbb{R} \setminus \{0\})$. In this case we write

$$\lim_{x\to x_0} f(x) = \gamma\ , \quad \text{distributionally.} \tag{7.2.1}$$

Observe (7.2.1) means that

$$\lim_{\varepsilon\to 0} \langle f(x_0+\varepsilon x), \phi(x)\rangle = \gamma \int_{-\infty}^{\infty} \phi(x)\,\mathrm{d}x \ \ , \ \ \phi \in \mathcal{D}(\mathbb{R}\setminus\{0\})\,. \tag{7.2.2}$$

Notice that the distributional limit $\lim_{x\to x_0} f(x)$ can be defined for $f \in \mathcal{D}'(\mathbb{R}\setminus\{x_0\})$. If the distributional point value $f(x_0)$ exists then the distributional limit $\lim_{x\to x_0} f(x)$ exists and equals $f(x_0)$. On the other hand, if $\lim_{x\to x_0} f(x) = \gamma$, distributionally, then [128] there exist constants $a_0, \ldots, a_n$ such that $f(x) = f_0(x) + \sum_{j=0}^{n} a_j \delta^{(j)}(x-x_0)$, where the distributional point value $f_0(x_0) = \gamma$, distributional

We may also consider lateral limits. We say that the *distributional lateral limit* $f(x_0^+) = \gamma_+$ exists if $f(x_0^+) = \lim_{\varepsilon\to 0^+} f(x_0+\varepsilon x)$ in $\mathcal{D}'(0,\infty)$, that is,

$$\lim_{\varepsilon\to 0^+} \langle f(x_0+\varepsilon x), \phi(x)\rangle = f(x_0^+) \int_{0}^{\infty} \phi(x)\,\mathrm{d}x \ \ , \ \ \phi \in \mathcal{D}(0,\infty)\,. \tag{7.2.3}$$

We write $f(x_0^+) = \gamma_+$, distributionally. Similar definitions apply to $f(x_0^-)$. Notice that the distributional limit $\lim_{x\to x_0} f(x)$ exists if and only if the distributional lateral limits $f(x_0^-)$ and $f(x_0^+)$ exist and coincide. If both lateral limits exist, then the jump is defined as the number $[f]_{x=x_0} = \gamma_+ - \gamma_-$.

Suppose that $f(x_0^\pm) = \gamma_\pm$, then there exists f_0 and constants such that $f(x) = f_0(x) + \sum_{j=0}^{n} a_j \delta^{(j)}(x-x_0)$, and f_0 has the jump behavior

$$f_0(x_0+\varepsilon x) = \gamma_- H(-x) + \gamma_+ H(x) + o(1) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{D}'(\mathbb{R})\ . \tag{7.2.4}$$

If no delta and its derivatives occur, that is, $f = f_0$, then we actually obtain that f has jump behavior at $x = x_0$. We will indistinctly used the phrases *f has jump behavior at $x = x_0$* and *the distributional lateral limits exist and f has no delta functions at $x = x_0$*.

7.3 Regulated Functions

In his pioneering work, Łojasiewicz [128] introduced and studied the distributions that have a distributional point value at *every* point. He proved that if one considers the *function* having those distributional values as values, then this function is measurable and in a very precise sense, the distribution corresponds to the function. It is common usage to call a distribution "regular" if it arises from a locally Lebesgue integrable function. The functions studied by Łojasiewicz are more general instances of what one should call "regular" distributions, namely those arising from a function by integration. However, in general, the functions that arise from the distributional point values are many times not locally integrable in the sense of Lebesgue; sometimes they are locally integrable with respect to more general integration processes such as the Denjoy-Perron-Henstock integral, as the function $f_1(x) = x^{-1}\sin x^{-1}$, $x \neq 0$, $f_1(0) = 0$, but sometimes they are not, as the function $f_2(x) = x^{-2}\sin x^{-1}$, $x \neq 0$, $f_2(0) = 0$.

In this section we shall study a somewhat bigger class, that of the distributionally regulated functions. The definition is as follows.

Definition 7.1. *A distribution $f \in \mathcal{D}'(\mathbb{R})$ is called a distributionally regulated function if at each point $x_0 \in \mathbb{R}$ both distributional lateral limits $f\left(x_0^{\pm}\right)$ exist and f has no Dirac delta functions at $x = x_0$. We say that f is a distributionally regulated function with delta functions if at each point $x_0 \in \mathbb{R}$ both distributional lateral limits $f\left(x_0^{\pm}\right)$ exist.*

It will follow from our study that a distribution that is a distributionally regulated function actually corresponds to an actual *function,* the function given by the distributional point value $f(x_0)$, which is defined whenever $f\left(x_0^{+}\right) = f\left(x_0^{-}\right)$, an equation that holds for all x_0 except for those of an exceptional set that is countable at the most.

On the other hand a distributionally regulated function with delta functions is a distribution, and the name "function" is used in the way the name function is used for the Dirac delta function.

Sometimes we shall refer to distributionally regulated functions as "distributionally regulated functions without delta functions."

The distributionally regulated functions that have no distributional jump at any point are the functions studied in [128], and therefore we shall call them Łojasiewicz functions.

Our definitions were given for a distribution $f \in \mathcal{D}'(\mathbb{R})$, defined over the whole real line. However, one can consider any of these notions over finite intervals in the obvious way, namely, a distribution is, say, a distributionally regulated function over the interval (a, b) if its distributional lateral limits exist at each point, and no delta functions are present.

It is worth to point out that the classical regulated functions are those classical functions that have lateral limits at every point. They are precisely the uniform limits of step functions [36]. Observe that the classical analogue of the Łojasiewicz functions are the continuous functions.

7.4 The $\phi-$transform

Our main tool to study the local behavior of distributions is the $\phi-$transform, a function of two variables that we now define.

Let $\phi \in \mathcal{D}(\mathbb{R})$ be a fixed test function that satisfies

$$\int_{-\infty}^{\infty} \phi(x) \, \mathrm{d}x = 1 . \tag{7.4.1}$$

If $f \in \mathcal{D}'(\mathbb{R})$ we introduce the function of two variables $F = F_\phi \{f\}$ by the formula

$$F(x, y) = \langle f(x + y\xi), \phi(\xi) \rangle \ , \quad x \in \mathbb{R}, \quad y > 0 , \tag{7.4.2}$$

the distributional evaluation being taken with respect to the variable ξ. We call F the $\phi-$transform of f.

The $\phi-$transform can also be defined if ϕ does not belong to $\mathcal{D}(\mathbb{R})$ as long as we consider only distributions f of a more restricted class. Indeed, we can consider the case when $\phi \in \mathcal{A}(\mathbb{R})$ and $f \in \mathcal{A}'(\mathbb{R})$ for any suitable space of test functions $\mathcal{A}(\mathbb{R})$, such as $\mathcal{S}(\mathbb{R})$, $\mathcal{K}(\mathbb{R})$, or $\mathcal{E}(\mathbb{R})$. Observe that we assume (7.4.1) in every case.

Our first result shows that $f(x)$ is the *distributional* boundary value of $F(x,y)$ as $y \to 0$.

Theorem 7.2. *If $f \in \mathcal{D}'(\mathbb{R})$ and F is its $\phi-$transform defined by (7.4.2) then*

$$\lim_{y\to 0} F(x,y) = f(x) \ , \tag{7.4.3}$$

distributionally in the space $\mathcal{D}'(\mathbb{R})$, that is,

$$\lim_{y\to 0} \langle F(x,y), \psi(x)\rangle = \langle f(x), \psi(x)\rangle \quad , \quad \forall \psi \in \mathcal{D}(\mathbb{R}) \ . \tag{7.4.4}$$

Proof. If $\psi \in \mathcal{D}(\mathbb{R})$ then

$$\langle F(x,y), \psi(x)\rangle = \langle \Psi(y\xi), \phi(\xi)\rangle \ , \tag{7.4.5}$$

where

$$\Psi(z) = \langle f(x), \psi(x-z)\rangle \ , \tag{7.4.6}$$

is a smooth function of z. Therefore, $\Psi(0)$ exists in the ordinary sense and consequently in the distributional sense of Łojasiewicz. Hence,

$$\lim_{y\to 0} \langle \Psi(y\xi), \phi(\xi)\rangle = \Psi(0) = \langle f(x), \psi(x)\rangle \ , \tag{7.4.7}$$

and (7.4.4) follows. $\square$

The result will also hold when $f \in \mathcal{E}'(\mathbb{R})$ and $\phi \in \mathcal{E}(\mathbb{R})$ if $\phi \in L^1(\mathbb{R})$. In that case (7.4.7) follows from the Lebesgue dominated convergence theorem, since Ψ

would belong to $\mathcal{D}(\mathbb{R})$. Another case when $f(x)$ is the distributional boundary value of $F(x,y)$ as $y \to 0$ is if

$$f(x) = O\left(|x|^{\beta}\right) \quad (\mathrm{C}) \ , \quad \text{as } |x| \to \infty \,, \tag{7.4.8}$$

$$\phi(x) = O(|x|^{\alpha}) \ , \quad \text{strongly as } |x| \to \infty \,, \tag{7.4.9}$$

and

$$\alpha < -1 \ , \quad \alpha + \beta < -1 \,, \tag{7.4.10}$$

as follows from [54, Theorem 1]. Actually, we will show a multidimensional version of such result later in Section 12.3. Recall that (7.4.9) means that it holds after differentiation, i.e., $\phi^{(k)}(x) = O(|x|^{\alpha-k})$, for all $k \in \mathbb{N}$. It is true in particular if $f \in \mathcal{S}'(\mathbb{R})$ and $\phi \in \mathcal{S}(\mathbb{R})$.

For future reference, we say that if $f \in \mathcal{D}'(\mathbb{R})$ and $\phi \in \mathcal{D}(\mathbb{R})$ we are in Case I. If (7.4.8), (7.4.9), and (7.4.10) are satisfied, we say that we are in Case II. When $f \in \mathcal{S}'(\mathbb{R})$ and $\phi \in \mathcal{S}(\mathbb{R})$ we say that we are in Case III. Most of our results will hold in any of these three cases. However, the results are usually false when we just assume that $f \in \mathcal{E}'(\mathbb{R})$ and $\phi \in \mathcal{E}(\mathbb{R})$.

Theorem 7.3. *Suppose*

$$f(x_0) = \gamma \,, \tag{7.4.11}$$

distributionally. In any of the cases I, II, or III, we have

$$\lim_{(x,y)\to(x_0,0)} F(x,y) = \gamma \,, \tag{7.4.12}$$

in any sector $y \geq m\,|x - x_0|$ for any $m > 0$.

Proof. Let us show that if $|x_1| \leq 1/m$ then $\lim_{\varepsilon \to 0^+} F(x_0 + \varepsilon x_1, \varepsilon) = \gamma$. Indeed, if $\phi \in \mathcal{D}(\mathbb{R})$, then

$$\begin{aligned} F(x_0 + \varepsilon x_1, \varepsilon) &= \langle f(x_0 + \varepsilon x_1 + \varepsilon \xi), \phi(\xi) \rangle \\ &= \langle f(x_0 + \varepsilon \omega), \phi(\omega - x_1) \rangle \\ &= \langle f(x_0 + \varepsilon \omega), \phi_{x_1}(\omega) \rangle \ , \end{aligned}$$

where $\phi_{x_1}(\omega) = \phi(\omega - x_1)$ also belongs to $\mathcal{D}(\mathbb{R})$ and $\int_{-\infty}^{\infty} \phi_{x_1}(\omega)\, \mathrm{d}\omega = 1$. Thus (7.4.12) follows. The limit is uniform with respect to x_1 for $|x_1| \leq 1/m$ since $\{\phi_{x_1} : |x_1| \leq 1/m\}$ is a compact set in $\mathcal{D}(\mathbb{R})$. The proof in cases II and III is similar. □

Angular convergence of $F(x, y)$ to $\gamma = f(x_0)$ is obtained when the distributional point value exists. On the other hand, the *radial* limit, $\lim_{y \to 0^+} F(x_0, y)$ exists under weaker hypothesis, namely, under symmetric point values (Section 3.10).

Theorem 7.4. *Suppose case I, II, or III holds, and the test function ϕ is even. Let $\chi_{x_0}^f(s) = (f(x_0 + s) + f(x_0 - s))/2$, that is, the even part of f about the point $x = x_0$. If*

$$f_{\text{sym}}(x_0) = \gamma\,, \quad \textit{distributionally}\,, \tag{7.4.13}$$

i.e., $\chi_{x_0}^f(0) = \gamma$, distributionally, then

$$\lim_{y \to 0^+} F(x_0, y) = \gamma\,. \tag{7.4.14}$$

Proof. The fact that ϕ is even yields

$$\begin{aligned} \lim_{y \to 0^+} F(x_0, y) &= \lim_{y \to 0^+} \langle f(x_0 + y\xi), \phi(\xi) \rangle \\ &= \lim_{y \to 0^+} \langle f(x_0 + y\xi), (\phi(\xi) + \phi(-\xi))/2 \rangle \\ &= \lim_{y \to 0^+} \langle \chi_{x_0}(y\xi), \phi(\xi) \rangle \\ &= \gamma\,, \end{aligned}$$

as required. $\square$

Remark 7.5. *The above result does not hold if $f \in \mathcal{E}'(\mathbb{R})$ and $\phi \in \mathcal{E}(\mathbb{R})$. Indeed, if*

$$\phi(x) = \frac{3\sin x^3}{\pi x}\,, \tag{7.4.15}$$

then $\phi \in \mathcal{E}(\mathbb{R})$ and $\int_{-\infty}^{\infty} \phi(x)\,\mathrm{d}x = 1$. If $f(x) = \delta(x)$, then

$$F(x,y) = \left(\frac{3}{\pi x}\right)\sin\left(\frac{x}{y}\right)^3. \tag{7.4.16}$$

If $x_0 \neq 0$ then $f(x_0) = 0$ but not even the radial limit $\lim_{y\to 0^+} F(x_0,y)$ exists.

Suppose now that the distribution $f \in \mathcal{D}'(\mathbb{R})$ has lateral distributional limits $f\left(x_0^{\pm}\right) = \gamma_{\pm}$ as $x \to x_0$ from the right and from the left, respectively, and no delta functions at $x = x_0$. This means that f has the following jump behavior: for each $\psi \in \mathcal{D}(\mathbb{R})$,

$$\lim_{\varepsilon\to 0^+} \langle f(x_0+\varepsilon\xi), \psi(\xi)\rangle = \gamma_- \int_{-\infty}^{0} \psi(\xi)\,\mathrm{d}\xi + \gamma_+ \int_{0}^{\infty} \psi(\xi)\,\mathrm{d}\xi\,. \tag{7.4.17}$$

Then we have the ensuing result.

Theorem 7.6. *Suppose case I, II, or III holds and f satisfies (7.4.17). Then for each $\vartheta \in (0,\pi)$ there exits $\alpha = \alpha(\vartheta) \in [0,1]$ such that*

$$\lim_{\substack{(x,y)\to(x_0,0)\\ (x,y)\in \mathsf{l}_\vartheta}} F(x,y) = \alpha(\vartheta)\,\gamma_+ + (1-\alpha(\vartheta))\,\gamma_-\,, \tag{7.4.18}$$

where l_ϑ is the line $y = \tan\vartheta\,(x-x_0)$.

In cases II or III, $\lim_{\vartheta\to 0}\alpha(\vartheta) = 1$, $\lim_{\vartheta\to\pi}\alpha(\vartheta) = 0$. In case I actually there exist $\vartheta_0, \vartheta_1 \in (0,\pi)$ such that $\alpha(\vartheta) = 1$ for $\vartheta \le \vartheta_0$ while $\alpha(\vartheta) = 0$ for $\vartheta \ge \vartheta_1$.

When ϕ is even then $\alpha(\pi/2) = 1/2$.

Proof. The limit of $F(x,y)$ as $(x,y) \to (x_0,0)$ along l_ϑ is given as

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \langle f(x_0+\varepsilon\cos\vartheta+\varepsilon\sin\vartheta\,\xi), \phi(\xi)\rangle &= \lim_{\varepsilon\to 0^+} \langle f(x_0+\varepsilon\omega), \phi_\vartheta(\omega)\rangle \\ &= \gamma_- \int_{-\infty}^{0} \phi_\vartheta(\omega)\,\mathrm{d}\omega + \gamma_+ \int_{0}^{\infty} \phi_\vartheta(\xi)\,\mathrm{d}\omega\,,\end{aligned}$$

where

$$\phi_{\vartheta}(\omega)=\frac{1}{\sin\vartheta}\phi\left(\frac{\omega-\cos\vartheta}{\sin\vartheta}\right). \tag{7.4.19}$$

The result follows by taking

$$\alpha(\vartheta)=\int_{0}^{\infty}\phi_{\vartheta}(\omega)\,\mathrm{d}\omega=\int_{-\cot\vartheta}^{\infty}\phi(\omega)\,\mathrm{d}\omega\,, \tag{7.4.20}$$

which has the stated properties. □

Remark 7.7. *If $f\left(x_0^{\pm}\right)=\gamma_{\pm}$ exist distributionally, then*

$$f(x)=f_0(x)+\sum_{j=0}^{m}c_j\delta^{(j)}(x-x_0)$$

where f_0 has no delta functions at $x=x_0$. It follows that

$$F(x,y)=F_0(x,y)+\sum_{j=0}^{m}\frac{c_j}{y^{j+1}}\phi^{(j)}\left(\frac{x_0-x}{y}\right). \tag{7.4.21}$$

Therefore (7.4.18) is still valid for the finite part *of the limit:*

$$\text{F.p.}\lim_{\substack{(x,y)\to(x_0,0)\\(x,y)\in\mathsf{l}_{\vartheta}}}F(x,y)=\alpha(\vartheta)\gamma_{+}+(1-\alpha(\vartheta))\gamma_{-}\,. \tag{7.4.22}$$

Remark 7.8. *If ϕ is even and $f\left(x_0^{\pm}\right)=\gamma_{\pm}$ exist distributionally while f has no delta functions at $x=x_0$ then (7.4.18) shows that the radial limit $\lim_{y\to0^+}F(x_0,y)$ exists and equals $(\gamma_{+}+\gamma_{-})/2$. However, Theorem 7.4 is a stronger result, since the lateral limits may not exist if $\chi_{x_0}^{f}(s)$ has the distributional limit γ at $s=0$. More generally, if*

$$\lim_{s\to0^+}\chi_{x_0}^{f}(s)=\gamma\,, \tag{7.4.23}$$

distributionally, then

$$\text{F.p.}\lim_{y\to0^+}F(x_0,y)=\gamma\,. \tag{7.4.24}$$

Remark 7.9. *If f is a distributionally regulated function with delta functions then the finite part limit* $\text{F.p.}\lim_{y\to0^+}F(x,y)$ *exists for each $x\in\mathbb{R}$, and equals $\left(f\left(x_0^{+}\right)+f\left(x_0^{-}\right)\right)/2$. It will follow from the results of Section 7.6 that the set* *of points where the limit is not an ordinary limit is countable at the most. If f is a distributionally regulated function without delta functions then the limit is an ordinary limit for each $x \in \mathbb{R}$. On the other hand, if f is a distributionally regulated function without delta functions then $\lim_{(x,y)\to(x_0,0),(x,y)\in \mathsf{l}} F(x,y)$ exists for each non-horizontal line l, the set of points where the limit is not independent of l is countable at the most, while if f is a Łojasiewicz function then the limit is independent of l for each $x_0 \in \mathbb{R}$.*

7.5 Determination of Jumps by the $\phi-$transform

Suppose $f \in \mathcal{D}'(\mathbb{R})$ is such that the lateral limits $f\left(x_0^{\pm}\right) = \gamma_{\pm}$ exist distributionally. In this section we consider certain formulas for the jump $d = [f]_{x=x_0} = \gamma_+ - \gamma_-$ in terms of the radial limits of some functions related to $F(x,y)$.

Let us start with the case when f does not have delta functions at $x = x_0$. Observe that sometimes we shall use the notation $F_{,x}$ or $F_{,y}$ for the partial derivatives $\partial F/\partial x$ and $\partial F/\partial y$, respectively.

Theorem 7.10. *Let f be a distribution and ϕ a test function that satisfies (7.4.1). Suppose case I, II, or III holds. Suppose the distributional lateral limits $f\left(x_0^{\pm}\right) = \gamma_{\pm}$ exist and f has no delta functions at $x = x_0$. Let $d = \gamma_+ - \gamma_-$ be the jump of f at $x = x_0$ and let $\nu = \phi(0)$. Then*

$$\lim_{y\to 0^+} yF_{,x}(x_0, y) = \nu d\,. \tag{7.5.1}$$

Proof. The hypotheses yield the jump behavior

$$f(x_0 + \varepsilon x) = \gamma_+ H(x) + \gamma_- H(-x) + o(1) \quad \text{as } \varepsilon \to 0^+ \tag{7.5.2}$$

in the space $\mathcal{D}'(\mathbb{R})$, where H is the Heaviside function. Since distributional expansions can be differentiated, we obtain the quasiasymptotic behavior

$$f'(x_0 + \varepsilon x) = \frac{d}{\varepsilon}\delta(x) + o\left(\frac{1}{\varepsilon}\right) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{D}'(\mathbb{R}). \tag{7.5.3}$$

Observe now that $F_{,x}$ is precisely the $\phi-$representation of $f'(x)$. Thus (7.5.3) yields

$$F_{,x}(x_0, y) = \frac{d\phi(0)}{y} + o\left(\frac{1}{y}\right), \quad y \to 0^+, \tag{7.5.4}$$

and (7.5.1) follows. $\square$

If we just assume that the distributional lateral limits $f\left(x_0^{\pm}\right) = \gamma_{\pm}$ exist, then f may have delta functions at $x = x_0$ and thus the formula (7.5.1) can be modified by using the finite part of the limit:

$$\text{F.p.} \lim_{y \to 0^+} yF_{,x}(x_0, y) = \nu d. \tag{7.5.5}$$

Actually, to obtain (7.5.5) and in particular (7.5.1) there is no need to assume that the distributional lateral limits $f\left(x_0^{\pm}\right)$ exist; it is enough to suppose that the jump distribution

$$\psi_{x_0}(s) := \psi_{x_0}^f(s) = f(x_0 + s) - f(x_0 - s), \tag{7.5.6}$$

has a distributional limit as $s \to 0$.

Theorem 7.11. *Let f be a distribution and ϕ a test function that satisfies (7.4.1). Suppose case I, II, or III holds. Suppose*

$$\psi_{x_0}\left(0^+\right) = d, \tag{7.5.7}$$

distributionally. If ϕ is even then

$$\text{F.p.} \lim_{y \to 0^+} y\frac{\partial F}{\partial x}(x_0, y) = \nu d. \tag{7.5.8}$$

When $\psi_{x_0}(s)$ does not have delta functions at $s = 0$ then (7.5.8) is an ordinary limit.

Proof. Indeed, the result follows by applying (7.5.5) or (7.5.1) to $\Psi(x,y)$, the $\phi-$representation of $\psi_{x_0}(x)$ and by observing that

$$\begin{aligned}
F_{,x}(x_0,y) &= \langle f'(x_0+y\xi), \phi(\xi)\rangle \\
&= \langle f'(x_0+y\xi), (\phi(\xi)+\phi(-\xi))/2\rangle \\
&= \langle (f'(x_0+y\xi)-f'(x_0-y\xi))/2, \phi(\xi)\rangle \\
&= \frac{1}{2}\left\langle \psi'_{x_0}(y\xi), \phi(\xi)\right\rangle \\
&= \frac{1}{2}\Psi_{,x}(0,y)\ ,
\end{aligned}$$

since $\psi_{x_0}(0^+) = -\psi_{x_0}(0^-) = d$, and hence $[\psi_{x_0}]_{x=0} = 2d$. □

Another formula for the jump is given in terms of logarithmic averages. Observe that in case II, that is $f(x) = O\left(|x|^\beta\right)$ (C), as $|x| \to \infty$, and $\phi(x) = O(|x|^\alpha)$ strongly as $|x| \to \infty$, we need to assume not only that $\alpha < -1$ and $\alpha+\beta < -1$, but also that $\beta < 0$.

Theorem 7.12. *Let f be a distribution and ϕ a test function that satisfies (7.4.1). Suppose case I or case II with $\beta < 0$ holds. If $\psi_{x_0}(0^+) = d$, then*

$$\text{F.p.} \lim_{y\to 0^+} \frac{1}{\log y}\left\langle f(x_0+y\xi), \frac{\phi(\xi)-\phi(0)}{\xi}\right\rangle = \nu d\,. \tag{7.5.9}$$

Proof. Observe that the condition $\beta < 0$, or case I, guarantee that the Cesàro evaluation $\langle f(x_0+y\xi), \rho(\xi)\rangle$, where $\rho(\xi) = (\phi(\xi)-\phi(0))/\xi$ is well-defined. Notice also that if $f\left(x_0^{\pm}\right) = \gamma_\pm$ exist and f has no delta functions at $x = x_0$ then one may argue that $\langle f(x_0+y\xi), \rho(\xi)\rangle$ approaches $\gamma_- \int_{-\infty}^0 \rho(\xi)\ \mathrm{d}\xi + \gamma_+ \int_0^\infty \rho(\xi)\ \mathrm{d}\xi$ as $y \to 0^+$; however, both integrals diverge: $\left|\int_{-\infty}^0 \rho(\xi)\ \mathrm{d}\xi\right| = \left|\int_0^\infty \rho(\xi)\ \mathrm{d}\xi\right| = \infty$.

On the other hand,

$$\begin{aligned}\frac{\partial}{\partial y}\left\langle f\left(x_0+y\xi\right),\frac{\phi\left(\xi\right)-\phi\left(0\right)}{\xi}\right\rangle &= \left\langle \xi f'\left(x_0+y\xi\right),\frac{\phi\left(\xi\right)-\phi\left(0\right)}{\xi}\right\rangle \\ &= \left\langle f'\left(x_0+y\xi\right),\phi\left(\xi\right)-\phi\left(0\right)\right\rangle \\ &= \left\langle f'\left(x_0+y\xi\right),\phi\left(\xi\right)\right\rangle \\ &= \frac{\partial F}{\partial x}\left(x_0,y\right)\,.\end{aligned}$$

Thus we may use L'Hôpital rule to obtain

$$\begin{aligned}\text{F.p.}\lim_{y\to 0^+}\frac{1}{\log y}\left\langle f\left(x_0+y\xi\right),\frac{\phi\left(\xi\right)-\phi\left(0\right)}{\xi}\right\rangle &= \text{F.p.}\lim_{y\to 0^+} y\frac{\partial F}{\partial x}\left(x_0,y\right) \\ &= \nu d\,,\end{aligned}$$

as required. □

Remark 7.13. *The function* $\widetilde{F}\left(x,y\right)=\left\langle f\left(x+y\xi\right),\left(\phi\left(\xi\right)-\phi\left(0\right)\right)/\xi\right\rangle$ *is a type of "conjugate" function to the* $\phi-$*transform* $F\left(x,y\right)$. *Actually if* $\phi\left(x\right)=\pi^{-1}(1+x^2)^{-1}$ *then* $F\left(x,y\right)$ *is a harmonic function and* $\widetilde{F}\left(x,y\right)$ *is precisely its harmonic conjugate.*

Example 7.14. *Let us consider the distributional behavior of the distribution* f_α, $\alpha>0$, *given by the nonharmonic series*

$$f_\alpha\left(x\right)=\sum_{n=1}^{\infty}\frac{\sin n^\alpha x}{n}\,, \tag{7.5.10}$$

as $x\to 0$. *Observe that* $f_\alpha\left(x\right)=O\left(\left|x\right|^{-\infty}\right)$ (C) *as* $\left|x\right|\to\infty$. *Let us consider the conjugate function* $\widetilde{F}\left(x,y\right)$ *with* $\phi\left(x\right)=\pi^{-1}(1+x^2)^{-1}$ *as in the remark above. Then*

$$\widetilde{F}\left(x,y\right)=\sum_{n=1}^{\infty}\frac{e^{-n^\alpha y}\cos n^\alpha x}{n}\,, \tag{7.5.11}$$

and thus $\widetilde{F}\left(0,y\right)\sim(1/\alpha)\ln y$, *since* $\sum_{n^\alpha\le N}1/n\sim(1/\alpha)\ln N$ *as* $N\to\infty$, *and it follows that* $\nu d=1/\alpha$, *or* $d=\pi/\alpha$, *since* $\phi\left(0\right)=1/\pi$. *Therefore, since* f_α *is odd,* *we obtain the distributional lateral limits*

$$f_\alpha\left(0^+\right)=\frac{\pi}{2\alpha}\,,\quad f_\alpha\left(0^-\right)=\frac{-\pi}{2\alpha}\,. \tag{7.5.12}$$

Observe that this is easy to see for $\alpha=1$ *from the well-known formula*

$$f_1\left(x\right)=\frac{\pi-x}{2}\,,\quad 0<x<\pi\,, \tag{7.5.13}$$

and for $\alpha=1/2$ *from the formula*

$$f_{1/2}\left(x\right)=\pi+\sum_{j=0}^{\infty}\frac{\left(-1\right)^j\zeta\left(1/2-j\right)x^{2j+1}}{\left(2j+1\right)!}\,,\quad x>0\,, \tag{7.5.14}$$

obtained by Boersma [20] when solving a problem proposed by Glasser [69]; see also [45]. It is not hard to see that if $\alpha>1$ *then (7.5.12) are not ordinary limits, since* f_α *is unbounded as* $x\to0$.

7.6 The Number of Singularities

In this section we show that if f is a distributionally regulated function, with or without delta functions, then the distributional point value $f\left(x\right)$ exists for all x save for those of an exceptional set which is countable at the most. Actually, the result holds without assuming that f is distributionally regulated, that is, we will show that for a general distribution the set where the lateral limits exist but the distributional point value do not exist is countable at most.

The corresponding result for ordinary regulated functions is well-known, and actually very easy to prove. Indeed, if $f\left(x\right)$ is a regulated function in some interval I then for any $\lambda>0$ the set $\mathfrak{S}_\lambda$ consisting of the points x where $\left|f\left(x^+\right)-f\left(x^-\right)\right|\geq\lambda$ is discreet in I, since at an accumulation point of $\mathfrak{S}_\lambda$ at least one of the lateral limits cannot exist. Thus $\mathfrak{S}_\lambda$ is countable at the most, and hence so is $\mathfrak{S}=\bigcup_{\lambda>0}\mathfrak{S}_\lambda=\bigcup_{n=1}^{\infty}\mathfrak{S}_{1/n}$. When f is a regulated function of bounded variation, then one can even bound the $n_\lambda\left(K\right)$, the number of elements of $\mathfrak{S}_\lambda\cap K$ for any compact interval K by $n_\lambda\leq V/\lambda$, where V is the total variation of f over K.

This argument does not work if f is distributionally regulated, since in that case the set $\mathfrak{S}_\lambda$ could have limit points, as the next example shows.

Example 7.15. *Let us consider the function f with support in $[0,\infty)$ with derivative*

$$f'(x) = \sum_{n=1}^{\infty} (-1)^n n^q \delta\left(x - \frac{1}{n}\right) \qquad (C) \ , \tag{7.6.1}$$

where $q \in \mathbb{R}$. Then f is a distributionally regulated function, constant in all the intervals $(1/(n+1), 1/n)$ for $n \in \mathbb{N}$, and in $(-\infty, 0)$ where it vanishes. The set of points where f has a non-zero jump is exactly $\mathfrak{S} = \{1/n : n \in \mathbb{N}\}$. In particular, $0 \notin \mathfrak{S}$, since the function has the distributional point value $f(0) = 0$. If $q > 0$ then $\mathfrak{S}_\lambda = \mathfrak{S}$ for $\lambda \leq 1$, and thus 0 is an accumulation point of $\mathfrak{S}_\lambda$. Actually, we may replace the sequence $\{(-1)^n n^q\}_{n=1}^{\infty}$ by any distributionally small sequence $\{c_n\}_{n=1}^{\infty}$, that is, a sequence with the property that $\sum_{n=1}^{\infty} c_n \delta(x-n)$ belongs to $\mathcal{K}'(\mathbb{R})$ [61, Section 5.4] and still obtain that $f(0) = 0$. Indeed,

$$\begin{aligned}
\langle f'(\varepsilon x), \phi(x) \rangle &= \left\langle \sum_{n=1}^{\infty} c_n \delta(\varepsilon x - 1/n), \phi(x) \right\rangle \\
&= \sum_{n=1}^{\infty} \frac{c_n}{\varepsilon} \phi\left(\frac{1}{\varepsilon n}\right) \\
&= \sum_{n=1}^{\infty} n c_n \tau(\varepsilon n) \\
&= o(\varepsilon^{\infty}) \qquad \textit{as} \ \ \varepsilon \to 0^+ \ ,
\end{aligned}$$

where $\tau(x) = (1/x)\phi(1/x)$ belongs to $\mathcal{K}(\mathbb{R})$ if $\phi \in \mathcal{D}(\mathbb{R})$, and where all series are considered in the Cesàro sense. Hence f is "distributionally smooth" at $x = 0$ since it follows that $f^{(m)}(0) = 0 \ \forall m \geq 0$.

We have the following result on the number of jump singularities of an arbitrary distribution.

Theorem 7.16. *Let $f \in \mathcal{D}'(\mathbb{R})$. Let*

$$\mathfrak{S} = \{x \in \mathbb{R} : \textit{the lateral limits exist but } f(x) \textit{ does not exist distributionally}\} .$$

Then $\mathfrak{S}$ is countable at the most.

Proof. Let us consider first the set $\mathfrak{S}_0$ of those elements of $\mathfrak{S}$ where f does not have delta functions. Then if $x_0 \in \mathfrak{S}_0$ it follows that $f\left(x_0^+\right) \neq f\left(x_0^-\right)$. Let $\phi \in \mathcal{D}(\mathbb{R})$ that satisfies (7.4.1), and let $F(x,y)$ be the $\phi-$representation of f. There exists $\theta \in (0, \pi/2)$ such that

$$\lim_{x \to x_0^\pm} F(x, \tan\theta\,|x - x_0|) = f\left(x_0^\pm\right) , \quad \forall x_0 \in \mathbb{R} . \tag{7.6.2}$$

Let $\mathsf{U}_0 = \{(r, \infty) : r \in \mathbb{Q}\} \cup \{(-\infty, r) : r \in \mathbb{Q}\}$ and let $\mathsf{U} = \{(I_+, I_-) \in \mathsf{U}_0 \times \mathsf{U}_0 : I_+ \cap I_- = \emptyset\}$. If $x_0 \in \mathfrak{S}_0$ then there exists $(I_+, I_-) \in \mathsf{U}$ and $n \in \mathbb{N}$ such that

$$F(x, \tan\theta\,(x - x_0)) \in I_+ \quad \text{for} \quad x_0 < x < x_0 + 1/n , \tag{7.6.3}$$

$$F(x, \tan\theta\,(x_0 - x)) \in I_- \quad \text{for} \quad x_0 - 1/n < x < x_0 . \tag{7.6.4}$$

For fixed $(I_+, I_-) \in \mathsf{U}$ and fixed $n \in \mathbb{N}$ the family of intervals $(x_0 - 1/n, x_0 + 1/n)$, where $x_0 \in \mathfrak{S}_0$ satisfies (7.6.3) and (7.6.4) is pairwise disjoint and, consequently, there is an at most countable number of such intervals. Hence

$$\mathfrak{S}_0 = \bigcup_{(I_+, I_-) \in \mathsf{U}} \bigcup_{n=1}^{\infty} \{x_0 \in \mathbb{R} : x_0 \text{ satisfies (7.6.3) and (7.6.4)}\} , \tag{7.6.5}$$

is also countable at the most.

The analysis at points where f has delta functions of a given order follows by integrating f a suitable number of times. Indeed, let $\mathfrak{S}_N$ be the set of points of $\mathfrak{S}$ where f has no delta function of order greater than N. Let F be a primitive of f of order $N+1$, i.e., $F^{(N+1)}(x) = f(x)$. Then $\mathfrak{S}_N \setminus \mathfrak{S}_{N-1}$ is exactly the set of points where F has a jump but no delta functions; hence $\mathfrak{S}_N \setminus \mathfrak{S}_{N-1}$ is countable at the most, and thus so is $\mathfrak{S}_N$. It follows that $\mathfrak{S}$ is countable at the most. $\square$

We immediately obtain that distributionally regulated functions have distributionally point values except perhaps for a countable set.

Theorem 7.17. *Let $f \in \mathcal{D}'(\mathbb{R})$ be distributionally regulated, with or without delta functions. Let*

$$\mathfrak{S} = \{x \in \mathbb{R} : f(x) \text{ does not exist distributionally}\} . \tag{7.6.6}$$

Then $\mathfrak{S}$ is countable at the most.

7.7 One-to-one Correspondence

We now show that if $f \in \mathcal{D}'(\mathbb{R})$ is distributionally regulated then the correspondence $f \leftrightarrow \tilde{f}$ is one-to-one, where $\tilde{f}(x) = (f(x_-) + f(x_+))/2$.

Theorem 7.18. *Let $f \in \mathcal{D}'(\mathbb{R})$ be distributionally regulated. If $\tilde{f}(x) = 0$, for all value of x except perhaps for a countable set, then $f \equiv 0$.*

Proof. Notice that, by Theorem 7.17, the distributional point value of f exists except for set which is countable at most. Next, since f is distributionally regulated, then it is distributionally bounded everywhere, hence its primitive has distributional point values everywhere. Łojasiewicz showed in [128, p.31] that these two facts together with the hypothesis $\tilde{f}(x) = 0$, except perhaps on a countable set, imply that $f \equiv 0$. □

7.8 Boundary Behavior of Solutions of Partial Differential Equations

The results of the previous sections apply to general distributions and test functions. When the test function ϕ is of certain special forms, however, we have that the $\phi-$transform becomes a particular solution of a partial differential equation, and those results become results on the boundary behavior of solutions of partial differential equations.

Suppose first that $\phi = \phi_1$ where

$$\phi_1(x) = \frac{p(x)}{q(x)}, \tag{7.8.1}$$

p and q are polynomials, $\alpha = \deg q - \deg p \geq 2$, q does not have real zeros, and $\int_{-\infty}^{\infty} \phi_1(x)\, \mathrm{d}x = 1$. Let

$$q(x) = \sum_{k=0}^{n} a_k x^k. \tag{7.8.2}$$

Then if $f \in \mathcal{D}'(\mathbb{R})$ satisfies the estimate $f(x) = O\left(|x|^\beta\right)$ (C), $|x| \to \infty$, where $\alpha + \beta < -1$, then the $\phi-$transform

$$F_1(x,y) = \langle f(x + y\xi), \phi_1(\xi)\rangle, \quad x \in \mathbb{R},\ y > 0, \tag{7.8.3}$$

is a solution of the partial differential equation

$$\sum_{k=0}^{n} a_{n-k} \frac{\partial^n F}{\partial x^k \partial y^{n-k}} = 0, \tag{7.8.4}$$

with $F(x, 0^+) = f(x)$ distributionally, since

$$\begin{aligned}
\sum_{k=0}^{n} a_{n-k} \frac{\partial^n F}{\partial x^k \partial y^{n-k}} &= \sum_{k=0}^{n} a_{n-k} \left\langle f^{(n)}(x + y\xi)\, \xi^{n-k}, \phi_1(\xi)\right\rangle \\
&= \left\langle f^{(n)}(x + y\xi)\, q(\xi), \phi_1(\xi)\right\rangle \\
&= \left\langle f^{(n)}(x + y\xi), p(\xi)\right\rangle \\
&= 0.
\end{aligned}$$

In the particular case when $q(x) = x^2 + 1$, $p(x) = 1/\pi$, we obtain

$$\phi_2(x) = \frac{1}{\pi(x^2 + 1)}, \tag{7.8.5}$$

and $F_2(x,y)$ is the Poisson "integral" of f, which in case $f(x) = O\left(|x|^\beta\right)$ (C), $|x| \to \infty$, for some $\beta < 1$, is the harmonic function with $F_2(x, 0^+) = f(x)$ distributionally that satisfies $F_2(x,y) = O\left(|x|^\beta\right)$ (C), $|x| \to \infty$, for each fixed $y > 0$. Observe that

$$F_2(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)\, \mathrm{d}\xi}{(x - \xi)^2 + y^2}, \tag{7.8.6}$$

if f is locally integrable.

Let us now take $\phi = \varphi_\nu$ where its Fourier transform is given by

$$\widehat{\varphi_\nu}\,(u) = e^{-u^\nu}, \tag{7.8.7}$$

where $\nu = 2p$ is an even positive integer. Alternatively, φ_ν is the only solution in $\mathcal{S}(\mathbb{R})$ of the ordinary differential equation

$$\varphi^{(\nu-1)}\,(\xi) = (-1)^p\,\frac{\xi}{\nu}\varphi\,(\xi)\ , \tag{7.8.8}$$

with $\int_{-\infty}^{\infty}\varphi\,(\xi)\;\mathrm{d}\xi = 1$. Then if $f \in \mathcal{S}'\,(\mathbb{R})\,,$ and F is the $\phi-$transform corresponding to φ_ν, the function

$$G_\nu\,(x,t) = F\left(x, t^{1/\nu}\right)\ , \quad x \in \mathbb{R},\, t > 0\,, \tag{7.8.9}$$

is a solution of the initial value problem

$$\frac{\partial G}{\partial t} = (-1)^{p-1}\,\frac{\partial^\nu G}{\partial x^\nu}\,, \tag{7.8.10}$$
$$G\left(x, 0^+\right) = f\,(x)\ ,\ \text{distributionally.}$$

In particular, if $\nu = 2$, then

$$\widehat{\varphi_\nu}\,(u) = e^{-u^2}, \quad \varphi_\nu\,(\xi) = \frac{1}{2\sqrt{\pi}}e^{-\xi^2/4}, \tag{7.8.11}$$

and $G_2\,(x,t)$ is the solution of the heat equation $G_{,t} = G_{,xx}$ that satisfies the initial condition $G\,(x,0^+) = f\,(x)$, distributionally, and with $G\,(x,t) \in \mathcal{S}'\,(\mathbb{R})$ for each fixed $t > 0$. If f is a locally integrable function then $G_2\,(x,t)$ takes the familiar form

$$G_2\,(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} f\,(\xi)\,e^{-\frac{(\xi-x)^2}{4t}}\mathrm{d}\xi\ . \tag{7.8.12}$$

If the distributional value $f\,(x_0) = \gamma$ exists, then $F_1\,(x,y)\,,$ and in particular $F_2\,(x,y)\,,$ satisfies that $F_1\,(x,y) \to \gamma$ as $(x,y) \to (x_0,0)$ in any sector $y \geq m\,|x - x_0|$ for $m > 0$. Also $G_\nu\,(x,t) \to \gamma$ in any region of the type $t \geq m\,(x - x_0)^\nu$ for

$m > 0$. Actually, if $\chi_{x_0}(s) = (f(x_0+s) + f(x_0-s))/2$, and the distributional value $\chi_{x_0}(0) = \gamma$ exists, then $F_1(x_0, y) \to \gamma$ as $y \to 0^+$ and $G_\nu(x_0, t) \to \gamma$ as $t \to 0^+$. If instead of the existence of the distributional value one just has the existence of the distributional limit $f\left(x_0^{\pm}\right) = \gamma$, then the *finite part* of the limit of $F_1(x, y)$ as $(x, y) \to (x_0, 0)$ in any sector $y \geq m|x - x_0|$ exist and equals γ; similarly, one obtains the existence of the finite part of the limits in the other cases.

Example 7.19. *It is interesting to observe that if f is almost periodic or periodic, then*

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\alpha_n x}, \tag{7.8.13}$$

where $\alpha_n \to \pm\infty$ as $n \to \pm\infty$. It follows that

$$F(x, y) = \sum_{n=-\infty}^{\infty} c_n e^{i\alpha_n x} \hat{\phi}(-\alpha_n y) \ , \tag{7.8.14}$$

so that in particular

$$F_2(x, y) = \sum_{n=-\infty}^{\infty} c_n e^{i\alpha_n x} e^{-|\alpha_n| y} = \sum_{n=-\infty}^{\infty} c_n e^{i\alpha_n x} r^{|\alpha_n|}, \tag{7.8.15}$$

where $r = e^{-y} \to 1^-$ as $y \to 0^+$. The study of the behavior of the $\phi-$transform in this case becomes the study of the series (7.8.13) in the Abel sense. Also

$$G_\nu(x, t) = \sum_{n=-\infty}^{\infty} c_n e^{i\alpha_n x} e^{-|\alpha_n|^\nu t}. \tag{7.8.16}$$

The problem of finding the (ordinary) jumps of a Fourier series was first solved by Fejér [63] in terms of the partial sums of the differentiated series, and was later consider by Zygmund [256, 9.11, Chapter III, pg. 108] in terms of the differentiated Abel-Poisson means of the Fourier series. A different formula using logarithmic means was given by Lukács [131], [256, Thm. 8.13]. We considered in Chapters 5 and 6 extensions of such results; in particular formulas were given in terms of the boundary asymptotic behavior of analytic, harmonic, and harmonic conjugates functions. The Theorems 7.11 and 7.12 provide very general results of the Fejér and Lukács type, respectively, for a general test function ϕ (which provides many different types of summability means, such as (7.8.15) or (7.8.16)) and not only for Fourier series, but also for nonharmonic series and actually for *any* distribution, these summability means can be related to the boundary behavior of solutions to partial differential equations, as we have seen in the present section.

7.9 The Fourier Transform of Regulated Functions

In this section we shall characterize the Fourier transform of distributionally regulated functions, with or without delta functions. We first start by reformulating Theorem 5.10, notice that the next theorem shows that if we merely assume the existence of the limits (5.3.11), they are forced to be of the form $\alpha + \beta \log a$.

Theorem 7.20. *Let $f \in \mathcal{S}'(\mathbb{R})$. If $x_0 \in \mathbb{R}$ then the distributional lateral limits $f\left(x_0^{\pm}\right) = \gamma_{\pm}$ exist and f has no Dirac delta function at $x = x_0$ if and only if there exists k such that whenever $g(u)$ is a primitive of $\hat{f}(u)\, e^{iux_0}$ then the Cesàro limit*

$$\lim_{u\to\infty} \left(g(au) - g(-u)\right) = I_{x_0}(a) \qquad (\mathrm{C}, k)\ , \tag{7.9.1}$$

exists $\forall a > 0$. If this is the case then

$$I_{x_0}(a) = \pi\left(\gamma_+ + \gamma_-\right) - i\left(\gamma_+ - \gamma_-\right)\log a\,. \tag{7.9.2}$$

Proof. Half of the statement is the content of Theorem 5.10. Conversely, suppose that $I_{x_0}(a)$ exists for each $a > 0$. Clearly $I_{x_0}(a)$ is a measurable function of a. Then an easy computation shows that $I_{x_0}(a)$ satisfies the functional equation

$$I_{x_0}(ab) = I_{x_0}(a) + I_{x_0}(b) - I_{x_0}(1)\ . \tag{7.9.3}$$

While this functional equation has many solutions, constructed using a suitable Hamel basis, an analysis that can be traced back to Sierpinski shows that the only *measurable* solutions are

$$I_{x_0}(a) = I_{x_0}(1) + \beta \log a\,, \tag{7.9.4}$$

for some constant β. So, the result follows from Theorem 5.10 again. $\square$

We obtain the following characterization of the Fourier transforms of distributionally regulated functions.

Theorem 7.21. *Let $f \in \mathcal{S}'(\mathbb{R})$. The distribution f is a distributionally regulated function with delta functions if and only if $\forall x_0 \in \mathbb{R}$, the distribution $\hat{f}(u)\, e^{iux_0}$ admits the decomposition*

$$\hat{f}(u)\, e^{iux_0} = p_{x_0}(u) + g'_{x_0}(u)\ , \tag{7.9.5}$$

where $p_{x_0}(u)$ is a polynomial and where for some k

$$\lim_{u\to\infty} \left(g(au) - g(-u)\right) = I_{x_0}(a) \qquad (\mathrm{C}, k)\ , \tag{7.9.6}$$

exists $\forall a > 0$. The distribution f is a distributionally regulated function (without delta functions) if $p_{x_0}(u) = 0$ for each $x_0 \in \mathbb{R}$; if also $I_{x_0}(a)$ is a constant function of a for each $x_0 \in \mathbb{R}$ then f is a Łojasiewicz function.

In any case, the set of points x_0 where $p_{x_0}(u) \neq 0$ is countable, as is countable the set of points x_0 where $I_{x_0}(a)$ is not a constant function of a.

We now give another characterization of distributions having lateral limits based on a decomposition in terms of boundary limits of analytic functions from the upper and lower half planes. Observe that only principal value Cesàro evaluations are needed in the following theorem.

Theorem 7.22. *Let $f \in \mathcal{S}'(\mathbb{R})$. Let $x_0 \in \mathbb{R}$. Then the distributional lateral limits $f\left(x_0^{\pm}\right) = \gamma_{\pm}$ exist and f has no Dirac delta function at $x = x_0$ if and only if*

$$\hat{f}(u)\, e^{iux_0} = H_{x_0}(u + i0) + H_{x_0}(u - i0)\ , \tag{7.9.7}$$

where $H_{x_0}(z)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$, the distributional boundary distributions $H_{x_0}(u \pm i0)$ belong to $\mathcal{S}'(\mathbb{R})$ and the principal value Cesàro evaluations

$$\text{p.v.} \left\langle H_{x_0}(u \pm i0), 1 \right\rangle = \nu_\pm \qquad (\mathrm{C}) , \tag{7.9.8}$$

both exist. In this case $\nu_\pm = \pi\gamma_\mp$.

Proof. If the distributional lateral limits $f\left(x_0^\pm\right) = \gamma_\pm$ exist and f has no Dirac delta function at $x = x_0$ we can write $f = f_+ + f_-$ where $f_\pm$ do not have delta functions at $x = x_0$, $\operatorname{supp} f_+ \subset [x_0, \infty)$, $\operatorname{supp} f_- \subset (-\infty, x_0]$, $f_+\left(x_0^+\right) = \gamma_+$, and $f_-\left(x_0^-\right) = \gamma_-$. Then we define

$$H_{x_0}(z) = \begin{cases} \left\langle f_+(x), e^{-iz(x-x_0)} \right\rangle , & \Im m\, z < 0 , \\ \\ \left\langle f_-(x), e^{-iz(x-x_0)} \right\rangle , & \Im m\, z > 0 , \end{cases} \tag{7.9.9}$$

so that $H_{x_0}(u \pm i0) = e^{ix_0u} \hat{f}_\mp(u)$, and consequently

$$\text{p.v.} \left\langle H_{x_0}(u \pm i0), 1 \right\rangle = \pi\gamma_\mp \quad (\mathrm{C}) . \tag{7.9.10}$$

Conversely, if (7.9.7) holds, then $f = f_+ + f_-$ where

$$f_\pm(x) = \mathcal{F}^{-1}\left\{ e^{-iux_0} H_{x_0}(u \mp i0), x \right\} . \tag{7.9.11}$$

But this implies that $\operatorname{supp} f_+ \subset [x_0, \infty)$, while $\operatorname{supp} f_- \subset (-\infty, x_0]$. Then (7.9.8) yields that the even parts of $f_\pm$ have the distributional values $\gamma_\pm/2$ at $x = x_0$. But since the distributions $f_\pm$ vanish on one side of x_0, it follows that the distributional lateral limits exist and no delta function is present. □

We immediately obtain the ensuing result.

Theorem 7.23. *Let $f \in \mathcal{S}'(\mathbb{R})$. The distribution f is a distributionally regulated function with delta functions if and only if $\forall x_0 \in \mathbb{R}$, the distribution $\hat{f}(u) e^{iux_0}$ admits the decomposition*

$$\hat{f}(u) e^{iux_0} = p_{x_0}(u) + H_{x_0}(u + i0) + H_{x_0}(u - i0) , \tag{7.9.12}$$

where $p_{x_0}(u)$ *is a polynomial and where* $H_{x_0}(z)$ *is analytic for* $z \in \mathbb{C} \setminus \mathbb{R}$, *the distributional boundary distributions* $H_{x_0}(u \pm i0)$ *belong to* $\mathcal{S}'(\mathbb{R})$ *and the principal value Cesàro evaluations*

$$\text{p.v.} \langle H_{x_0}(u \pm i0), 1 \rangle = \nu_{\pm} \qquad (\mathrm{C}) , \tag{7.9.13}$$

both exist. The distribution f *is a distributionally regulated function (without delta functions) if* $p_{x_0}(u) = 0$ *for each* $x_0 \in \mathbb{R}$; *if also* $\nu_+ = \nu_-$ *for each* $x_0 \in \mathbb{R}$ *then* f *is a Łojasiewicz function.*

In any case the set of points $x_0 \in \mathbb{R}$ *where* $p_{x_0}(u) \neq 0$ *is countable, as is countable the set of points where* $\nu_+ \neq \nu_-$.

One can use these ideas to prove that if the distributional lateral limits of a distribution that is the boundary value of an analytic function from the upper or lower half plane exist, then they must coincide [50].

Chapter 8

Order of Summability in Fourier Inversion Problems

8.1 Introduction

In the chapter we study the order of summability in the pointwise Fourier inversion formula for tempered distributions found in Chapter 3 and its implications in the local behavior of distributions. We show that the order of summability and the order of the point value are intimately related. We also analyze the order of summability in other Fourier inverse problems such as the ones considered in Chapter 5.

Recall the characterization of distributional point values of Fourier series: If $f \in \mathcal{D}'(\mathbb{R})$ is 2π-periodic with Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$, then $f(x_0) = \gamma$, distributionally, if and only if there exists $k \in \mathbb{N}$ such that

$$\lim_{x\to\infty} \sum_{-x<n\leq ax} c_n e^{inx_0} = \gamma \qquad (\mathrm{C}, k) \ ,$$

for each $a > 0$.

We shall notice that this result is merely existential, in the sense that it does not provide information about k more than its existence. It is therefore interesting to ask about the relation of k and the local properties of f. For instances, if $f(x)$ is continuous near $x = x_0$, then Fejér's theorem [62, 256] actually tells us that it can be taken to be at least $k = 1$. On the other hand, a careful review of the work of G. Walter [236] shows that a similar relation holds for distributions, at least for the summability of the series in the principal value sense. Another indication that such a relation should exist has been recently provided by F. González Vieli in [72, 74], where a the multidimensional pointwise Fourier transform for some particular classes of tempered distributions is investigated using Bochner-Riesz means.

In the general case, Theorem 3.21 provides a full characterization of the distributional point values of tempered distributions. However, Theorem 3.21 has a gap, namely, it does not establish a connection between the order of summability of the Fourier inversion formula and the order of the point values (see Section 8.2 for the definition of the latter). Our aim is to establish a relation between these two orders. Among other results, we show that if a tempered distribution, with certain restrictions of growth at ∞, has a point value of order k, then the special value of the Fourier inversion formula is summable $(C, k+1)$ to the value. In the case of Fourier series, these restrictions of growth do not appear, hence we generalized the result from [236]. Furthermore, we also investigate the opposite problem, that is, given the order of summability we estimate the order of the point value. We will also analyze exactly the same order problem in the formulas for jumps given in Chapter 5; observe that this information is valuable from a numerical point of view. Indeed, the formulas for jumps can be used as numerical detectors for edges of functions and distributions, but this only can be done as long as we give precise information about the order of summability at which they hold.

The chapter is organized as follows. In Section 8.2, we define a notion of order for distributional point values; it is slightly more restrictive than the one introduced by Łojasiewicz in [128], but it is more adequate for our framework with tempered distributions and Fourier transform. In Section 8.3, we extend the definitions of Cesàro limits and distributional evaluations in order to include fractional orders. Section 8.4 is dedicated to the study of the order of summability of the Fourier inversion formula upon the knowledge of the order of the point value, we show that for certain tempered distributions having a point value of order k at a point, the special value of the Fourier inversion formula is summable (C,β) to the point value for any $\beta > k$; then, we apply this result to cases of interest; at the end of the section we calculate a bound for the order of summability of the Fourier inversion formula in the general case. Next, in Section 8.5, we study the opposite problem, namely, we estimate the order of the point value having the order of Cesàro summability of pointwise Fourier inversion formula. Section 8.6 is dedicated to the study of symmetric distributional point values; that is, we investigate order problems in the solution of the Hardy-Littlewood (C) summability for tempered distributions, on the way we recover and extend the classical results for Fourier series [89, 81, 255]. Finally, we study jumps of distributions and find the order in the various formulas for the jump originally found in [216, 218] and already studied in Chapter 5.

8.2 Definition of Order of Point Values

In this section we shall define the order of distributional point values for tempered distributions. Recall the structural average characterization of distributional point values given in Section 3.2. It was shown by Łojasiewicz [128] that the existence of the distributional point value $f(x_0) = \gamma$ is equivalent to the existence of $n \in \mathbb{N}$, and a primitive of order n of f, that is $F^{(n)} = f$, which is continuous in a neighborhood of x_0 and satisfies

$$\lim_{x \to x_0} \frac{n! F(x)}{(x - x_0)^n} = \gamma \; . \tag{8.2.1}$$

If $f \in \mathcal{S}'(\mathbb{R})$, then n can be taken such that the function F is locally integrable and of at most polynomial growth. Łojasiewicz himself defined a notion of order for distributional point values, but it is convenient to provide a reformulation of *the order of the value* more suitable for tempered distribution and our purposes. For the sake of convenience, we should adopt a little variant of Łojasiewicz's original definition which differs from that given in [128].

Definition 8.1. *Let f be a tempered distribution. We say that f has a (distributional) point value γ at $x = x_0$ in $\mathcal{S}'(\mathbb{R})$ of order n, and write $f(x_0) = \gamma$ in $\mathcal{S}'(\mathbb{R})$ with order n, if n is the minimum integer such that there exists a locally bounded measurable function F of polynomial growth at infinity such that $F^{(n)} = f$ and F satisfies (8.2.1).*

A similar definition has been also adopted in [242, Sect.8.3, Def.8.1] for studying distributional point values of tempered distributions in relation with orthogonal wavelet expansions and multiresolution analysis approximations for spaces of tempered distributions.

8.3 Cesàro Limits: Fractional Orders

Recall that given a distribution $f \in \mathcal{D}'(\mathbb{R})$, with support bounded on the left, we denote its β-primitive by the convolution

$$f^{(-\beta)} = f * \frac{x_+^{\beta-1}}{\Gamma(\beta)} \ .$$

Since we will frequently use fractional primitives in long calculations, its convenient to introduce some additional notation. Thus, we also denote the β-primitive by

$$I_\beta \left\{ f(t); x \right\} := f^{(-\beta)}(x) \ ,$$

so that when f is locally integrable,

$$I_\beta \left\{ f(t); x \right\} = \frac{1}{\Gamma(\beta)} \int_0^x f(t) \, (x-t)^{\beta-1} \, \mathrm{d}t \ . \tag{8.3.1}$$

In Section 1.8.2 we defined Cesàro limits of distributions for only integral orders, we should now extend the definition in order to allow fractional orders.

Definition 8.2. *Let $f \in \mathcal{D}'(\mathbb{R})$ and $\beta \geq 0$. We say that f has a limit ℓ at infinity in the Cesàro sense of order β (in the (C, β) sense) and write*

$$\lim_{x \to \infty} f(x) = \ell \qquad (\mathrm{C}, \beta) \ , \tag{8.3.2}$$

if for a decomposition $f = f_- + f_+$ as sum of two distributions with supports bounded on the right and left, respectively, one has that the β-primitive of f_+ is an ordinary function (locally integrable) for large arguments and satisfies the ordinary asymptotic relation

$$f_+^{(-\beta)}(x) = \frac{\ell\, x^\beta}{\Gamma(\beta+1)} + o\left(x^\beta\right) \ , \quad \text{as } x \to \infty \ .$$

As usual, if we do not want to make reference to the order β in (8.3.2), we simply write

$$\lim_{x\to\infty} f(x) = \ell \qquad \text{(C)} \ .$$

We must check that the definition does not depend on the decomposition $f = f_- + f_+$; this fact follows immediately from the next proposition.

Proposition 8.3. *Suppose that f has compact support. If $\beta \geq 0$ and $\alpha > -1$, then $f^{(-\beta)}(x) = o(x^{\beta+\alpha})$, $x \to \infty$; in particular, $\lim_{x\to\infty} f(x) = 0 \quad (\mathrm{C}, \beta)$ for each $\beta \geq 0$.*

Proof. If β is an non-negative integer, the conclusion is obvious. Assume $\beta > 0$ is not a positive integer. We show that $f^{(-\beta)}$ is locally integrable for large arguments and $f^{-\beta}(x) = o(x^{\beta+\alpha})$, $x \to \infty$. Let k be a positive integer such that $f^{(-k)}$ is continuous over the whole real line. Then $f^{(-k)} = P + F$, where $P(x) = \sum_{j=0}^{k-1} a_j (x_+^j / j!)$, for some constants, and F is continuous on certain compact interval, say $[a, b]$, and 0 off $[a, b]$. We have that $f = P^{(k)} + F^{(k)}$. Note first that

$$\begin{aligned} P^{(k)} * \frac{x^{\beta-1}}{\Gamma(\beta)} &= \sum_{j=0}^{k-1} a_j \delta^{(k-1-j)} * \frac{x_+^{(\beta-1)}}{\Gamma(\beta)} = \sum_{j=0}^{k-1} a_j \frac{x_+^{(\beta+j-k)}}{\Gamma(\beta+1+j-k)} \\ &= O\left(x^{\beta-1}\right) = o\left(x^{\beta+\alpha}\right), \quad x \to \infty. \end{aligned}$$

So, it is enough to show that

$$F^{(k)} * \left(x^{\beta-1}/\Gamma(\beta)\right) = F * \left(x^{\beta-k-1}/\Gamma(\beta-k)\right)$$

is locally integrable for large arguments and satisfies an estimate $o(x^{\beta+\alpha})$ as $x \to \infty$. Indeed, we show it is locally integrable on $(b+1,\infty)$. If $\phi \in \mathcal{D}(\mathbb{R})$ is so that $\operatorname{supp}\phi \subseteq (b+1,\infty)$, then $\operatorname{supp}\phi * F(-t) \subseteq [1,\infty)$, hence,

$$\begin{aligned}\left\langle F * x^{\beta-k-1}, \phi(x)\right\rangle &= \left\langle x^{\beta-k-1}, \left(F(-t)*\phi\right)(x)\right\rangle \\ &= \int_1^\infty x^{\beta-k-1}\left(\int_{-\infty}^\infty F(t-x)\phi(t)\mathrm{d}t\right)\mathrm{d}x \\ &= \int_{-\infty}^\infty \left(\int_1^\infty t^{\beta-k-1}F(x-t)\mathrm{d}t\right)\phi(x)\mathrm{d}x \ .\end{aligned}$$

On the other hand if $x > b+1$, we obtain, as $x \to \infty$,

$$\int_1^\infty t^{\beta-k-1}F(x-t)\mathrm{d}t = \int_a^b (x-t)^{\beta-k-1}F(t)\mathrm{d}t = O\left(x^{\beta-1-k}\right) = o\left(x^{\beta+\alpha}\right) \ .$$

□

Therefore, our definition of Cesàro behavior has the following expected property.

Corollary 8.4. *If f has Cesàro limit at infinity of order β, then it has Cesàro limit of order $\tilde{\beta} > \beta$.*

We can also define Cesàro distributional evaluations of fractional orders by taking $m = \beta$ in Definition 3.4. Observe that if μ is a Radon measure supported on $[0,\infty)$ then $\langle \mu(t), \phi(t)\rangle = \gamma \quad (\mathrm{C},\beta)$ if and only if $\int_0^\infty \phi(t)\mathrm{d}\mu(t) = \gamma \quad (\mathrm{C},\beta)$. In particular, the considerations in Example 3.5 are still applicable to fractional orders.

We now want to discuss fractional orders for distributional *evaluations in the e.v. Cesàro sense*, they were introduced in Definition 3.18 only for positive integral orders.

Definition 8.5. *Let $g \in \mathcal{D}'(\mathbb{R})$, $\phi \in \mathcal{E}(\mathbb{R})$ and $\beta \geq 0$. We say that the evaluation $\langle g(x), \phi(x)\rangle$ exists in the* e.v. *Cesàro sense (of order β), and write*

$$\text{e.v.}\ \langle g(x), \phi(x)\rangle = \gamma \qquad (\mathrm{C},\beta) \ , \tag{8.3.3}$$

if for some primitive G of $g\phi$ and $\forall a>0$ we have

$$\lim_{x\to\infty}(G(ax)-G(-x))=\gamma \qquad (\mathrm{C},\beta)\ .$$

For series, measures and integrals, we shall adopt a similar notation to the one introduced in Section 3.5.

The last definition allows us to make sense out of the Fourier inversion formula for fractional orders of summability; indeed from Theorem 3.21 we obtain that $f\in\mathcal{S}'(\mathbb{R})$ has a distributional point value γ at $x=x_0$ if and only if

$$\mathrm{e.v.}\left\langle \hat{f}(x), e^{ix_0x}\right\rangle = 2\pi\gamma \qquad (\mathrm{C},\beta)\ , \tag{8.3.4}$$

for some sufficiently large β. As we mentioned at the Introduction, this result does not say anything about the relationship between the order of summability of this inversion formula and the order of the distributional point value; this will be the main subject of Section 8.4 and Section 8.5 in the present chapter.

8.4 Order of Summability

In this section we obtain a bound for the order of summability of the Fourier inversion formula for tempered distributions in the general case. We also analyze two particular cases, the case of Fourier series and the case of distributions with compact support and in both cases we obtain the expected result: if the distribution has a value of order k, then the order of summability of the Fourier inversion formula is at least $k+1$.

We will use indistinctly the notations $\hat{f}$, $\mathcal{F}(f)$ and $\mathcal{F}\left\{f(t);x\right\}$ to denote the Fourier transform of f.

Suppose that $f\in\mathcal{S}'(\mathbb{R})$ is so that $\hat{f}\in L^1_{\mathrm{loc}}(\mathbb{R})$. Denote by θ_A the characteristic function of a set A. Then note that (8.3.4) holds if and only if

$$\lim_{x\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(t)e^{ix_0t}\phi_a^{\beta}\left(\frac{t}{x}\right)\mathrm{d}t = f(x_0)\ , \tag{8.4.1}$$

where ϕ_a^β is the summability kernel given by

$$\phi_a^\beta(t) = (1+t)^\beta \theta_{[-1,0]}(t) + \left(1 - \frac{t}{a}\right)^\beta \theta_{[0,a]}(t) \ . \tag{8.4.2}$$

Indeed, this follows directly from Definition 8.5. Observe that we may consider (8.4.2) as the summability kernels of *asymmetric* (C, β) means. Notice also that if (8.4.1) holds for some β, then it holds for any $\tilde{\beta} \geq \beta$. We shall need some properties of these kernels, they are stated in the next lemma.

Lemma 8.6. *Suppose that* $0 < \beta \leq 1$. *Then,*

$$\left|\hat{\phi}_a^\beta(t)\right| \leq \frac{2 + 3\beta\,(1 + a^{-1})}{t^{\beta+1}} \quad , \ t > 1 \ .$$

Moreover, $\int_{-\infty}^{\infty} \hat{\phi}_a^\beta(t)\mathrm{d}t = 2\pi$.

Proof. Suppose the inequality is satisfied, then $\hat{\phi}_a^\beta \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, so the very well known classical result [17, p.62] implies that the Fourier inversion formula holds pointwise in this case, and thus we have $\int_{-\infty}^{\infty} \hat{\phi}_a^\beta(t)\mathrm{d}t = 2\pi\phi_a^\beta(0) = 2\pi$. Let us now show the inequality.

$$\begin{aligned}
\left|\hat{\phi}_a^\beta(t)\right| &= \left|\int_0^1 (1-u)^\beta (e^{itu} + ae^{-iatu})\mathrm{d}u\right| \\
&= \frac{\beta}{t}\left|\int_0^1 (1-u)^{\beta-1} (e^{-iatu} - e^{itu})\mathrm{d}u\right| \\
&= \frac{\beta}{t^{\beta+1}}\left|\int_0^t u^{\beta-1} (e^{-iat}e^{iau} - e^{it}e^{-iu})\mathrm{d}u\right| \\
&\leq \frac{2}{t^{\beta+1}} + \frac{\beta}{t^{\beta+1}}\left|\int_1^t u^{\beta-1} (e^{-iat}e^{iau} - e^{it}e^{-iu})\mathrm{d}u\right| \\
&\leq \frac{2}{t^{\beta+1}} + \frac{\beta}{t^{\beta+1}}(a^{-1}+1)\left(1 + t^{\beta-1} + (1-\beta)\int_1^\infty u^{\beta-2}\mathrm{d}u\right) \ ,
\end{aligned}$$

where in the last step we have used integration by parts. □

The explicit value of the constant term in the bound from the last lemma is unimportant, however, we will use the fact that this estimate holds uniformly for a on compact subsets of $(0, \infty)$.

We start to study the pointwise Fourier inversion formula. We first show a proposition concerning the $L^2(\mathbb{R})$ case. The proof of the following proposition is similar to that of [206, Thm.13], but we include it for the sake of completeness.

Proposition 8.7. *Suppose that $g \in L^2(\mathbb{R})$. If g is continuous at x_0, then we have for any $\beta > 0$,*

$$\frac{1}{2\pi} \text{ e.v.} \left\langle \hat{g}(t), e^{ix_0 t} \right\rangle = g(x_0) \qquad (\mathrm{C}, \beta) \ ,$$

or, which amounts to the same,

$$\lim_{x\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(t) e^{ix_0 t} \phi_a^{\beta}\left(\frac{t}{x}\right) \mathrm{d}t = g(x_0) \ , \tag{8.4.3}$$

uniformly for a on compact subsets of $(0, \infty)$.

Proof. By considering $g(x+x_0)$, we may assume that $x_0 = 0$. We may also assume that $0 < \beta \leq 1$, because if it holds for those values of β, then it holds for any $\beta > 0$.

We have that

$$\int_{-\infty}^{\infty} \hat{g}(t) \phi_a^{\beta}\left(\frac{t}{x}\right) \mathrm{d}t = x \int_{-\infty}^{\infty} g(t) \hat{\phi}_a^{\beta}(xt) \, \mathrm{d}t \ .$$

Therefore (8.4.3) holds if and only if

$$\lim_{x\to\infty} \int_{-\infty}^{\infty} g(t) K_a^{\beta}(t, x) \, \mathrm{d}t = g(0) \ ,$$

where $K_a^{\beta}(t,x) = x\hat{\phi}_a^{\beta}(xt)/(2\pi)$. Now, because of Lemma 8.6 and the boundedness of $\hat{\phi}_a^{\beta}$, the kernel $K_a^{\beta}(t,x)$ satisfies the following properties

$$\int_{-\infty}^{\infty} K_a^{\beta}(t,x) \mathrm{d}t = 1, \ \left|K_a^{\beta}(t,x)\right| \leq Mx \ , \ \left|K_a^{\beta}(t,x)\right| \leq \frac{N}{x^{\beta} t^{\beta+1}} \ , \tag{8.4.4}$$

for some positives constants M and N, and the last inequality being valid for $x\,|t| \geq 1$. The estimates are satisfied uniformly for a on compact sets. Pick $\sigma > 0$

such that if $|t| < \sigma$ then $|g(t) - g(0)| < \varepsilon$; keep $x^{-1} < \min\{\varepsilon, \sigma\}$, then

$$\begin{aligned}
&\left| \int_{-\infty}^{\infty} (g(t) - g(0))\, K_a^\beta(t,x) \mathrm{d}t \right| \\
&\le \varepsilon \int_{-\frac{1}{x}}^{\frac{1}{x}} \left| K_a^\beta(t,x) \right| \mathrm{d}t + \int_{|t| \ge \frac{1}{x}} |g(t) - g(0)| \left| K_a^\beta(t,x) \right| \mathrm{d}t \\
&\le 2\varepsilon(M + N\beta^{-1}) + \frac{N}{x^\beta} \int_{|t|\ge\sigma} \frac{|g(t) - g(0)|}{t^{\beta+1}} \,\mathrm{d}t
\end{aligned}$$

hence,

$$\limsup_{x\to\infty} \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(t) \phi_a \left(\frac{t}{x} \right) \mathrm{d}t - g(0) \right| \le 2\varepsilon(M + N\beta^{-1}) \ ,$$

since ε is arbitrary, this completes the proof. □

Remark 8.8. *Proposition 8.7 still holds if one assumes that x_0 is a Lebesgue point of g instead of the continuity at x_0. This proposition is also true for kernels ϕ other than ϕ_a^β; in fact, the proposition is valid if $K(t,x) = x\phi(xt)/(2\pi)$ satisfies (8.4.4), that is $K(t,x)$ satisfies $\int_{-\infty}^{\infty} K(t,x)\mathrm{d}t = 1$, $|K(t,x)| \le Mx$ for $|t|\,x < B$ and $|K(t,x)| < Nx^{-\alpha}t^{-\alpha-1}$, for some positive constants B, M, N and α. For other related results, the reader can consult Titchmarsh's book [206, Chap.1].*

In order to make further progress, we need two formulas. They are stated in the next two lemmas.

Lemma 8.9. *Let $h \in \mathcal{D}'(\mathbb{R})$ and $m, k \in \mathbb{N}$. Suppose that $m \ge k$, then*

$$x^k h^{(m)}(x) = \sum_{j=0}^{k} (-1)^j \frac{k!}{(k-j)!} \binom{m}{j} \frac{d^{m-j}}{dx^{m-j}} \left(x^{k-j} h(x) \right) \ .$$

Proof. It follows directly form the very well known formula [26, Lemm.1.3], valid if $\varphi \in C^\infty(\mathbb{R})$,

$$\varphi h^{(m)} = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \frac{d^{m-j}}{dx^{m-j}} \left(\varphi^{(j)} h \right)$$

applied to $\varphi(x) = x^k$. □

Lemma 8.10. *Let h be a locally integrable function supported on $[0,\infty)$. For any positive number β and positive integer k*

$$I_\beta\left\{t^k h(t);x\right\} = \sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{\Gamma(\beta+j)}{\Gamma(\beta)}x^{k-j}h^{(-\beta-j)}(x)\ .$$

Proof. We proceed by induction over k. For $k=1$,

$$\begin{aligned}
I_\beta\left\{th(t);x\right\} &= \frac{1}{\Gamma(\beta)}\int_0^x (x-t)^{\beta-1}th(t)\mathrm{d}t \\
&= xh^{(-\beta)}(x) - \frac{1}{\Gamma(\beta)}\int_0^x\left(\int_0^t (x-u)^{\beta-1}h(u)\mathrm{d}u\right)\mathrm{d}t \\
&= xh^{(-\beta)}(x) - \frac{1}{\Gamma(\beta)}\int_0^x (x-u)^{\beta}h(u)\mathrm{d}u \\
&= xh^{(-\beta)}(x) - \beta h^{(-\beta-1)}(x)\ .
\end{aligned}$$

If the formula true for k, then

$$\begin{aligned}
I_\beta\left\{t^{k+1}h(t);x\right\} =& \sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{\Gamma(\beta+j)}{\Gamma(\beta)}x^{k-j}I_{\beta+j}\left\{th(t);x\right\} \\
=& \sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{\Gamma(\beta+j)}{\Gamma(\beta)}x^{k+1-j}h^{(-\beta-j)}(x) \\
&- \sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{\Gamma(\beta+j)}{\Gamma(\beta)}(\beta+j)x^{k-j}h^{(-\beta-j-1)}(x) \\
=& \sum_{j=0}^{k+1}(-1)^j\binom{k+1}{j}\frac{\Gamma(\beta+j)}{\Gamma(\beta)}x^{k+1-j}h^{(-\beta-j)}(x)\ .
\end{aligned}$$

□

We begin to analyze the case of tempered distributions, by first imposing some strong restrictions to the behavior of the distribution at infinity.

Theorem 8.11. *Let $f\in\mathcal{S}'(\mathbb{R})$. Suppose that there exists an $m\in\mathbb{N}$ such that every m-primitive of f is a locally integrable function for large arguments and satisfies an estimate $O\left(|x|^{m-1}\right)$, as $x\to\infty$. If f has a distributional point value $f(x_0)=\gamma$ at x_0 in $\mathcal{S}'(\mathbb{R})$, whose order is n, then*

$$\frac{1}{2\pi}\ \mathrm{e.v.}\left\langle \hat{f}(x), e^{ix_0x}\right\rangle = \gamma \qquad (\mathrm{C},\beta)\ ,$$

for any $\beta > k = \max\{m, n\}$.

Proof. We can assume that $x_0 = 0$. Take h, a k-primitive of f, such that h is a locally bounded measurable function and $h(x) = O\left(|x|^{k-1}\right)$, as $|x| \to \infty$, and $h(x) = \gamma x^k/k! + o\left(|x|^k\right)$ as $x \to 0$. Set $g(x) = h(x)/x^k$, then $g \in L^2(\mathbb{R})$ and g is continuous at 0 with $g(0) = \gamma/k!$. Consider $\hat{g} \in L^2(\mathbb{R})$. Then,

$$(\hat{g})^{(k)}(x) = (-i)^k \mathcal{F}\left\{t^k g(t); x\right\} = (-i)^k \mathcal{F}\{h(t); x\} = (-i)^k \hat{h}(x) \ .$$

Thus,

$$\hat{f}(x) = \mathcal{F}\left\{h^{(k)}(t); x\right\} = i^k x^k \hat{h}(x) = (-1)^k x^k (\hat{g})^{(k)}(x) \ . \tag{8.4.5}$$

We now look at a k-primitive of $\hat{f}$. Indeed, by (8.4.5) and Lemma 8.9

$$F(x) = \sum_{j=0}^{k} (-1)^{k-j} \frac{k!}{(k-j)!} \binom{k}{j} I_j\left\{t^{k-j}\hat{g}(t)(t); x\right\} \tag{8.4.6}$$

is a k-primitive of $\hat{f}$. Let $\beta > k$ and $a > 0$. Set $\tilde{\beta} = \beta - k$. To show the theorem, one should prove that

$$F_1(x) := \frac{1}{a^{k-1}} F(ax) + (-1)^k F(-x) = \frac{2\pi\gamma x^{k-1}}{(k-1)!} + o\left(x^{k-1}\right) \qquad (\mathrm{C}, \beta - k + 1)$$

as $x \to \infty$. Therefore, we have to show that

$$I_{\tilde{\beta}+1}\{F_1(t); x\} = \frac{1}{\Gamma(\tilde{\beta}+1)} \int_0^x (x-t)^{\tilde{\beta}} F_1(t) \mathrm{d}t \tag{8.4.7}$$

$$= \frac{2\pi\gamma x^{\beta}}{\Gamma(\beta+1)} + o\left(x^{\beta}\right) \ , \quad \text{as } x \to \infty \ .$$

Notice that

$$a^{1-k} I_j\left\{t^{k-j}\hat{g}(t); ax\right\} + (-1)^k I_j\left\{t^{k-j}\hat{g}(at); -x\right\}$$
$$= I_j\left\{t^{k-j}(a\hat{g}(t) + \hat{g}(-t)); x\right\} \ ,$$

So, setting $g_1(t) := a\hat{g}(at) + \hat{g}(-t)$ for $t \geq 0$ and $g_1(t) := 0$ for $t < 0$ we obtain from Lemma 8.9 and (8.4.6)

$$\begin{aligned} F_1(x) &= \sum_{j=0}^{k} (-1)^{k-j} \frac{k!}{(k-j)!} \binom{k}{j} I_j \left\{ t^{k-j} g_1(t); x \right\} \\ &= (-1)^k I_k \left\{ t^k g_1^{(k)}(t); x \right\} \quad , \quad \text{for } x > 0 \ , \end{aligned}$$

then, by Lemma 8.10, for $x > 0$

$$\begin{aligned} I_{\tilde{\beta}+1} \left\{ F_1(t); x \right\} &= (-1)^k I_{\beta+1} \left\{ t^k g_1^{(k)}(t); x \right\} \\ &= \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{\Gamma(\beta+1+j)}{\Gamma(\beta+1)} x^{k-j} g_1^{(-\tilde{\beta}-1-j)}(x) \ , \end{aligned}$$

but

$$g_1^{(-\tilde{\beta}-1-j)}(x) = \frac{x^{\tilde{\beta}+j}}{\Gamma(\tilde{\beta}+1+j)} \int_{-\infty}^{\infty} \hat{g}(t) \phi_a^{\tilde{\beta}+j} \left(\frac{t}{x} \right) \mathrm{d}t \sim \frac{2\pi\gamma x^{\tilde{\beta}+j}}{k!\Gamma(\tilde{\beta}+1+j)} \ ,$$

as $x \to \infty$, where the last asymptotic relation holds in view of Proposition 8.7, the continuity of g at 0, and the fact $g(0) = \gamma/k!$. Therefore,

$$\begin{aligned} I_{\tilde{\beta}+1} \left\{ F_1(t); x \right\} &= \frac{2\pi\gamma\, x^{\beta}}{k!\Gamma(\beta+1)} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{\Gamma(\beta+1+j)}{\Gamma(\beta-k+1+j)} + o(x^{\beta}) \\ &= \frac{2\pi\gamma\, x^{\beta}}{k!\Gamma(\beta+1)} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \left. \frac{d^k}{dt^k} \left(t^{\beta+j} \right) \right|_{t=1} + o(x^{\beta}) \\ &= \frac{2\pi\gamma\, x^{\beta}}{k!\Gamma(\beta+1)} \left. \frac{d^k}{dt^k} \left(t^{\beta} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} t^j \right) \right|_{t=1} + o(x^{\beta}) \\ &= \frac{2\pi\gamma\, x^{\beta}}{\Gamma(\beta+1)} \left(\frac{1}{k!} \left. \frac{d^k}{dt^k} \left(t^{\beta}(t-1)^k \right) \right|_{t=1} \right) + o\left(x^{\beta}\right) \\ &= \frac{2\pi\gamma\, x^{\beta}}{\Gamma(\beta+1)} + o\left(x^{\beta}\right) \quad \text{as } x \to \infty \ ; \end{aligned}$$

hence, we have established (8.4.7), as required. □

Remark 8.12. *It follows from the proof of the last theorem and Proposition 8.7 that (8.4.7) holds uniformly for a in compact subsets of $(0, \infty)$.*

The next corollary follows directly from equation (8.4.6).

Corollary 8.13. *Under the hypothesis of Theorem 8.11, then $\hat{f}$ is the k^{th} derivative of a locally integrable function.*

Although it imposes conditions on the behavior at infinity of the tempered distribution, we may apply Theorem 8.11 to several cases of special interest. The next two corollaries follow directly from Theorem 8.11 (for the direct application of Theorem 8.11 in Corollary 8.15 one should argue that it is enough to assume $c_0 = 0$).

Corollary 8.14. *Let f be a distribution with compact support. If $f(x_0) = \gamma$ in $\mathcal{S}'(\mathbb{R})$ with order k, then for each $a > 0$ and $\beta > k$,*

$$\lim_{x\to\infty} \frac{1}{2\pi} \int_{-x}^{ax} \hat{f}(t) e^{ix_0 t} \mathrm{d}t = \gamma \qquad (\mathrm{C}, \beta) \ ,$$

or which is the same

$$\lim_{x\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_a^{\beta}\left(\frac{t}{x}\right) \hat{f}(t) e^{ix_0 t} \mathrm{d}t = \gamma \ . \tag{8.4.8}$$

Moreover, relation (8.4.8) holds uniformly for a in compact subsets of $(0, \infty)$.

Corollary 8.15. *Let $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ixn}$ be a 2π-periodic distribution. Suppose that $f(x_0) = \gamma$ in $\mathcal{S}'(\mathbb{R})$ with order $k \geq 1$. Then for each $a > 0$ and $\beta > k$,*

$$\lim_{x\to\infty} \sum_{-x \leq n \leq ax} c_n e^{ix_0 n} = \gamma \qquad (\mathrm{C}, \beta) \ ,$$

or equivalently

$$\lim_{x\to\infty} \sum_{-x \leq n \leq ax} \phi_a^{\beta}\left(\frac{n}{x}\right) c_n e^{ix_0 n} = \gamma \ . \tag{8.4.9}$$

Moreover, relation (8.4.9) holds uniformly for a in compact subsets of $(0, \infty)$.

As a particular case of Corollary 8.15, we obtain almost everywhere summability of order $\beta > 1$ for Denjoy integrable functions [76, 94]. This result extends that of Privalov (see [94, p.573]) which only considers the symmetric series. The reader should notice that Privalov theorem is included in the much stronger result of Marcinkiewicz [135], [256, Chap.XI, Thm.5.4].

Corollary 8.16. *Let f be a 2π-periodic function which is Denjoy integrable on $[-\pi,\pi]$. Let $\beta>1$. If its Fourier series is $\sum_{n=-\infty}^{\infty} c_n e^{ixn}$, then we have for almost every x_0*

$$\lim_{x\to\infty} \sum_{-x\le n\le ax} \phi_a^\beta\left(\frac{n}{x}\right) c_n e^{ix_0 n} = f(x_0)\ , \quad \textit{for all } a>0\ .$$

We now aboard the case of general behavior at infinity. For that, we need the following two lemmas.

Lemma 8.17. *Let $g\in L^2(\mathbb{R})$. Suppose that $x_0\notin \operatorname{supp} g$, then,*

$$\lim_{x\to\infty}\int_{-x}^{ax} \hat{g}(t)e^{ix_0t}\mathrm{d}t = 0\ ,$$

uniformly for a in compact subsets of $(0,\infty)$.

Proof. The proof is trivial, just apply Parseval's relation and then use Riemann-Lebesgue lemma. □

Lemma 8.18. *Let $f\in\mathcal{S}'(\mathbb{R})$. Suppose that $x_0\notin\operatorname{supp} f$ and that*

$$f(x)=O(|x|^\alpha) \qquad (\mathrm{C})\ ,\ \text{ as }\ |x|\to\infty\ ,$$

for some $\alpha>-1$. Let m be the minimum integer such that any m-primitive of f is locally bounded and $O(|x|^{m+\alpha})$ as $|x|\to\infty$. Then

$$\text{e.v.}\left\langle f(x), e^{ix_0x}\right\rangle = 0 \qquad (\mathrm{C},k)\ ,$$

where $k=[m+\alpha+\frac{1}{2}]+1$ ($[\,\cdot\,]$ stands for the integral part of a number).

Proof. The proof is completely analogous to that of Theorem 8.11. We may assume that $x_0=0$. Let h be an m-primitive of f such that h is 0 in a neighborhood of 0 and

$$h(x)=O(|x|^{m+\alpha}) \quad \text{as }\ |x|\to\infty.$$

Set $g(x) = h(x)/x^k$, then g satisfies the hypothesis of Lemma 8.17. Define $G(x) = \int_0^x \hat{g}(t)dt$; by Lemma 8.9, the following function is a $(k+1)$-primitive of $\hat{f}$

$$F(x) = \sum_{j=0}^{k} (-1)^{k-j} \frac{k!}{(k-j)!} \binom{k+1}{j} I_j \left\{ t^{k-j} G(t); x \right\} \ .$$

Since

$$\frac{1}{a^k} I_j \left\{ t^{k-j} G(t); ax \right\} - (-1)^k I_j \left\{ t^{k-j} G(t); -x \right\} = I_j \left\{ t^{k-j} \int_{-t}^{at} \hat{g}(u) \mathrm{d}u; x \right\} \ ,$$

we can use Lemma 8.17 to conclude

$$\begin{aligned} \frac{1}{a^k} F(x) - (-1)^k F(-x) &= \sum_{j=0}^{k} (-1)^{k-j} \frac{k!}{(k-j)!} \binom{k+1}{j} I_j \left\{ o(t^{k-j}); x \right\} \\ &= o\left(x^k\right) \quad \text{as } x \to \infty \ , \end{aligned}$$

uniformly for a on compact subsets of $(0, \infty)$. $\square$

We now combine Theorem 8.11 and and Lemma 8.18 to obtain a bound for the order of summability of the Fourier inversion formula of a general tempered distribution. We remark that every tempered distribution satisfies an estimate of type (8.4.10).

Theorem 8.19. *Let $f \in \mathcal{S}'(\mathbb{R})$ have the behavior at infinity*

$$f(x) = O(|x|^\alpha) \qquad (\mathrm{C}, m) \ , \quad \textit{as } |x| \to \infty \ . \tag{8.4.10}$$

If $f(x_0) = \gamma$ in $\mathcal{S}'(\mathbb{R})$ with order n, then

$$\frac{1}{2\pi} \text{e.v.} \left\langle \hat{f}(x), e^{ix_0 x} \right\rangle = \gamma \qquad (\mathrm{C}, k+1),$$

where $k = \max\left\{m, n, [n + \alpha + \frac{1}{2}], [m + \alpha + \frac{1}{2}]\right\}$.

8.5 Order of Point Value

In this section we show that if e.v. $\left\langle \hat{f}(x), e^{ixx_0} \right\rangle = 2\pi\gamma$ (C, k), then $f(x_0) = \gamma$, distributionally, and the order of the point value in $\mathcal{S}'(\mathbb{R})$ is less or equal to $k+2$.

We begin with a particular case which has its inspiration in Riemann's theorems on the formal integration of trigonometrical series [256, Chap.IX, p. 319].

Recall the definition of asymptotically homogeneous functions given in Section 3.4.1, they are a fundamental tool in the study of distributional evaluations in the e.v Cesàro sense.

Theorem 8.20. *Let f be an element of $\mathcal{S}'(\mathbb{R})$. Suppose that*

$$\frac{1}{2\pi}\,\text{e.v.}\left\langle \hat{f}(x), e^{ix_0x}\right\rangle = \gamma \qquad (\mathrm{C},0)\ ,$$

then, $f(x_0)=\gamma$, distributionally; moreover if F_1 and F_2 are any primitives of order 1 and 2 respectively, then F_1 is locally integrable and F_2 possesses a Peano second order differential at x_0, with γ as the second order term, i.e., F_2 is differentiable at x_0 and as $x\to x_0$

$$F_2(x)=F_2(x_0)+F_2'(x_0)(x-x_0)+\frac{\gamma}{2}(x-x_0)^2+o\left((x-x_0)^2\right)\ .$$

Hence, the point value is at most of order 2 in $\mathcal{S}'(\mathbb{R})$.

Proof. We may assume that $x_0=0$. We also can assume that $0\notin\operatorname{supp}\hat{f}$ and that $\hat{f}$ is the derivative of a locally integrable function. Indeed, otherwise we express $\hat{f}=\hat{f}_2+\hat{f}_1$, where $\hat{f}_2$ is the derivative of a distribution with compact support, $0\notin\operatorname{supp}\hat{f}_1$ and $\hat{f}_1$ is the first order derivative of a locally integrable function. Observe that f_2 is a C^∞-function and $2\pi f_2(0)=\left\langle \hat{f}_2(x),1\right\rangle=0$; consequently, f_1 satisfies the hypothesis of the present theorem and f satisfies the conclusions of the theorem if and only if f_1 does.

The hypothesis implies that if G is a primitive of f, then for each $a>0$,

$$G(ax)-G(-x)=2\pi\gamma+o(1)\text{ as }x\to\infty\ .$$

We choose G such that $0 \notin \operatorname{supp} G$. Set $c = G - \pi\gamma$, then c is asymptotically homogeneous of degree 0, and

$$G(x) = \pi\gamma \operatorname{sgn} x + c\left(|x|\right) + o(1), \quad \text{as } |x| \to \infty. \tag{8.5.1}$$

Therefore, $x^{-1}G(x) \in L^2(\mathbb{R})$ and $x^{-2}G(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, since $c(x) = o(\log x)$. Set,

$$h(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixt}\frac{G(t)}{t^2}\,\mathrm{d}t\ ,$$

then h is continuous and $h(x) = o(1)$ as $|x| \to \infty$. We now relate h to f, note that $h'' = -\mathcal{F}^{-1}(G)$, so $ixh''(x) = f(x)$. In addition, we have that $h'(x) = i\mathcal{F}^{-1}\left\{t^{-1}G(t); x\right\} \in L^2(\mathbb{R})$. Let F_2 be the following second order primitive of f,

$$F_2(x) = ixh(x) - \frac{1}{\pi}\int_{-\infty}^{\infty} e^{ixt}\frac{G(t)}{t^3}\,\mathrm{d}t\ .$$

Clearly, $F_1(x) = F_2'(x) = ixh'(x) - ih(x)$, which shows that every first order primitive of f is locally integrable. We now show that F_2 is differentiable at 0,

$$\frac{F_2(x) - F_2(0)}{x} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{G(t)}{t^2}\left(\frac{itxe^{ixt} - 2e^{ixt} + 2}{tx}\right)\mathrm{d}t\ , \tag{8.5.2}$$

we can apply Lebesgue Dominated Convergence Theorem in (8.5.2) to conclude that

$$F_2'(0) = -\frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{G(t)}{t^2}\,\mathrm{d}t\ .$$

We now calculate the Peano second order differential of F_2 at 0.

$$\Delta^2(x) = \frac{F_2(x) - F_2(0) - xF_2'(0)}{x^2} = \frac{x}{2\pi}\int_{-\infty}^{\infty} G(t)K(xt)\,\mathrm{d}t\ , \tag{8.5.3}$$

where $K(t) = t^{-3}\left(ite^{it} - 2e^{it} + 2 + it\right)$. Note that $(1 + |\log(t)|)K(t)$ belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Changing variables in (8.5.3) and applying in combination with

(8.5.1), we obtain that as $x \to 0$

$$\begin{aligned}\Delta^2(x) &= \frac{\operatorname{sgn} x}{2\pi}\int_{-\infty}^{\infty} G\left(\frac{t}{x}\right) K(t)\,\mathrm{d}t \\ &= \frac{\gamma}{2}\int_{-\infty}^{\infty} \operatorname{sgn} t\, K(t)\,\mathrm{d}t + \frac{1}{2\pi}\operatorname{sgn}(x) c(|x|^{-1})\int_{-\infty}^{\infty} K(t)\,\mathrm{d}t + o(1) \\ &= \frac{\gamma}{2} + o(1)\ ,\end{aligned}$$

since $\int_0^\infty (K(t)+K(-t)\mathrm{d}t) = 0$ and $\int_0^\infty (K(t)-K(-t))\,\mathrm{d}t = 1$. This completes the proof. □

We now use Theorem 8.20 to attack the general problem.

Theorem 8.21. *Let $f \in \mathcal{S}'(\mathbb{R})$. Suppose that*

$$\frac{1}{2\pi}\ \text{e. v.}\left\langle \hat{f}(x), e^{ix_0x}\right\rangle = \gamma \qquad (\mathrm{C}, k)\ ;$$

then, $f(x_0) = \gamma$, distributionally, f is the derivative of order $k+1$ of a locally integrable function and the order of $f(x_0) = \gamma$ is less or equal to $k+2$.

Proof. As in the proof of the last theorem, we can assume that $x_0 = 0$, $0 \notin \operatorname{supp} \hat{f}$ and $\hat{f}$ is the derivative of order $k+1$ of a locally integrable function.

By our assumptions, we can choose G, a locally integrable function, such that $G^{k+1} = \hat{f}$, $0 \notin \operatorname{supp}\ G$, and for each $a > 0$,

$$a^{-k}G(ax) + (-1)^{k+1}G(-x) = \frac{2\pi\gamma}{k!}x^k + o\left(x^k\right) \quad \text{as } x \to \infty\ .$$

Let h be the following tempered distribution

$$h(x) = -ix\mathcal{F}^{-1}\left\{t^{-k}G(t); x\right\} = \mathcal{F}^{-1}\left\{\left(t^{-k}G(t)\right)'; x\right\}\ ,$$

note that h satisfies the hypothesis of Theorem 8.20. Therefore, there is a locally integrable primitive h_1 of h such that $h_1(\varepsilon x) = \gamma\varepsilon x/k! + o(\varepsilon)$ as $\varepsilon \to 0$ in $\mathcal{S}'$. Set $h_2(x) = \int_0^x h_1(t)\mathrm{d}t$, then, by Theorem 8.20,

$$h_2(x) = \frac{\gamma}{2k!}\,x^2 + o\left(x^2\right) \quad \text{as } x \to 0\ , \tag{8.5.4}$$

since $h_2'(0)$ is equal to the distributional point value of h_1 at 0 and $h_1(0) = 0$ in $\mathcal{D}'$. We now relate h to f. We show that

$$F_{k+1}(x) = \sum_{j=0}^{k} \frac{(-1)^{k-j}k!}{(k-j)!}\binom{k+1}{j} I_j\left\{t^{k-j}h_1(t);x\right\} \tag{8.5.5}$$

$$-\sum_{j=0}^{k} \frac{(-1)^{k-j}k!}{(k-j-1)!}\binom{k+1}{j} I_{j+1}\left\{t^{k-j-1}h_1(t);x\right\}$$

is a $(k+1)$-primitive of f. Differentiating (8.5.5) $(k+1)$ times, we obtain,

$$\begin{aligned}
F_{k+1}^{(k+1)}(x) &= \sum_{j=0}^{k} \frac{(-1)^{k-j}k!}{(k-j)!}\binom{k+1}{j}\frac{d^{k+1-j}}{dx^{k+1-j}}\left(x^{k-j}h_1(x)\right. \\
&\quad \left. -(k-j)\int_0^x t^{k-j-1}h_1(t)\mathrm{d}t\right) \\
&= \sum_{j=0}^{k} \frac{(-1)^{k-j}k!}{(k-j)!}\binom{k+1}{j}\frac{d^{k-j}}{dx^{k-j}}\left(x^{k-j}h(x)\right) \\
&= -i\sum_{j=0}^{k} \frac{(-1)^{k-j}k!}{(k-j)!}\binom{k+1}{j}\frac{d^{k-j}}{dx^{k-j}}\left(x^{k+1-j}\mathcal{F}^{-1}\left\{G(t)/t^k;x\right\}\right) \\
&= \sum_{j=0}^{k} \frac{(-i)^{k-j}k!}{(k-j)!}\binom{k+1}{j}\frac{d^{k-j}}{dx^{k-j}}\left(\mathcal{F}^{-1}\left\{\left(G(t)/t^k\right)^{(k+1-j)};x\right\}\right) \\
&= \mathcal{F}^{-1}\left\{\sum_{j=0}^{k+1}\binom{k+1}{j}\frac{d^j}{dt^j}(t^k)\left(G(t)/t^k\right)^{(k+1-j)};x\right\} \\
&= \mathcal{F}^{-1}\left\{G^{(k+1)}(t);x\right\} = \mathcal{F}^{-1}\left\{\hat{f}(t);x\right\} = f(x)
\end{aligned}$$

Therefore, F_{k+1} is a primitive of order $k+1$ of f. Since h_1 is locally integrable, so is F_{k+1}. We integrate (8.5.5) to obtain a continuous $(k+2)$-primitive of f, given by

$$\begin{aligned}
F_{k+2}(x) &= \sum_{j=0}^{k} \frac{(-1)^{k-j}k!}{(k-j)!}\binom{k+1}{j} I_j\left\{t^{k-j}h_2(t) - (k-j)\int_0^t s^{k-j-1}h_2(s)\mathrm{d}s;x\right\} \\
&+ \sum_{j=0}^{k} \frac{(-1)^{k-j}k!}{(k-j-1)!}\binom{k+1}{j} I_{j+1}\left\{(k-j-1)\int_0^t s^{k-j-2}h_2(s)\mathrm{d}s - t^{k-j-1}h_2(t);x\right\}.
\end{aligned}$$

By using (8.5.4), we can conclude that

$$\begin{aligned}
F_{k+2}(x) &= \frac{\gamma}{2k!}\sum_{j=0}^{k}\frac{(-1)^{k-j}k!}{(k-j)!}\binom{k+1}{j}\frac{2I_j\left\{t^{k+2-j};x\right\}}{(k+2-j)(k+1-j)}+o\left(x^{k+2}\right)\\
&= \frac{\gamma}{k!}x^{k+2}\sum_{j=0}^{k}\frac{(-1)^{k-j}k!}{(k-j)!}\binom{k+1}{j}\frac{(k-j)!}{(k+2)!}+o\left(x^{k+2}\right)\\
&= \frac{\gamma}{k!(k+2)}x^{k+2}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{1}{(k+1-j)}+o\left(x^{k+2}\right)\\
&= \frac{\gamma}{k!(k+2)}x^{k+2}(-1)^k\int_0^1(t-1)^k\mathrm{d}t+o(x^{k+2})\\
&= \frac{\gamma}{(k+2)!}x^{k+2}+o\left(x^{k+2}\right)\ \text{ as } x\to 0,
\end{aligned}$$

this shows that $f(0)=\gamma$ in $\mathcal{S}'(\mathbb{R})$ with the order at most $k+2$. □

8.6 Order of Symmetric Point Values

We shall study in this section the order of summability in the solution of the Hardy-Littlewood (C) summability problem for tempered distributions (Section 3.11). Recall the notion of symmetric point values (Section 3.10), they are studied by means of the symmetric part of a distribution about $x=x_0$, that is, the distribution

$$\chi_{x_0}^f=\frac{f(x_0+x)+f(x_0-x)}{2}\ .$$

So, we have that $f_{\mathrm{sym}}(x_0)=\gamma$ if and only if $\chi_{x_0}^f(0)=\gamma$, distributionally. In the same manner as for distributional point values, we define the *order of symmetric point values.*

Definition 8.22. *Let $f\in\mathcal{D}'(\mathbb{R})$. We say that f has a (distributional) symmetric point value γ at $x=x_0$ in $\mathcal{S}'(\mathbb{R})$ of order n, and write $f_{\mathrm{sym}}(x_0)=\gamma$ in $\mathcal{S}'(\mathbb{R})$ with order n, if $\chi_{x_0}^f\in\mathcal{S}'(\mathbb{R})$ and $\chi_{x_0}^f(0)=\gamma$ in $\mathcal{S}'(\mathbb{R})$ of order n.*

Alternatively, we could have defined the order of the symmetric point value as the minimum integer n such that the conclusion of Theorem 3.57 is satisfied, equation (3.10.4), and $F(x_0+x)+(-1)^n F(x_0-x)$ is locally integrable of at most polynomial growth.

Most of the results for symmetric point values can be obtained from those of distributional point values. Let us discuss an example in which we show how to obtain Theorem 3.67 and Corollary 3.68 by applying the corresponding results for distributional point values.

Example 8.23. *Let $f \in \mathcal{D}'(\mathbb{R})$ have a distributional point value γ at $x = x_0$. Let U be a harmonic representation of f on the upper half-plane. We showed in Theorem 3.55 that*

$$\lim_{z\to x_0} U(z) = \gamma, \quad \textit{non-tangentially from the upper half-plane.}$$

We can use this result applied to the symmetric distribution to obtain a radial version of this result in the case of symmetric point values. Indeed, suppose now that $f_{\mathrm{sym}}(x_0) = \gamma$, distributionally. If U is a harmonic representation of f. Then $U_1(z) = (U(x_0+z)+U(x_0-\bar{z}))/2$ is a harmonic representation of $\chi^f_{x_0}$, hence $U_1(z) = \gamma + o(1)$ as z approaches 0 from the upper half-plane in a non-tangential manner. Therefore,

$$\lim_{y\to 0^+} U(x_0+iy) = \lim_{y\to 0^+} U_1(iy) = \gamma\ .$$

In particular, if f is a 2π-periodic distribution with sines and cosines series $f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$, then

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx_0 + b_n \sin nx_0) = \gamma \qquad (\mathrm{A})\ .$$

Our main goal in this section is to study the order of summability in Theorem 3.64. Let us first discuss a known case, namely Fourier series [256, Chap.XI].

Example 8.24. *Suppose that f is a 2π-periodic distribution with sines and cosines Fourier series $f(x) = a_0/2 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$. We proved in Corollary*

3.65 that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx_0 + b_n \sin nx_0) = \gamma \qquad \text{(C)}$$

if and only if $f_{\mathrm{sym}}(x_0) = \gamma$, *distributionally. In [81, 255], using the language of de la Vallée Poussin derivatives the order of summability is estimated upon knowledge of the order of the point value; indeed, A. Zygmund showed that if the order of the point value is* k, *then the order of summability can be actually taken* β, *for any* $\beta > k$. *The opposite problem was first investigated in [89] by assuming that* f *is a function. The general case is stated in [256, Chap.XI, Thm.2.1] and establishes that if the series is summable* (C, m), *then the symmetric point value is of order at least* $m + 2$; *such a result goes back to A. Plessner, as attributed in [256].*

In order to study symmetric point values in terms of summability of the Fourier transform, we need to choose the correct notion of summability. As follows from the results of Section 3.11, the right notion is that of *principal value* distributional evaluations in the (C) sense. We now proceed to define the order of summability for that method of summability.

Definition 8.25. *Let* $g \in \mathcal{D}'(\mathbb{R})$, $\phi \in \mathcal{E}(\mathbb{R})$ *and* $\beta \geq 0$. *We say that the evaluation* $\langle g(x), \phi(x) \rangle$ *exists in the* p.v. *Cesàro sense (at order* β*), and write*

$$\mathrm{p.v.}\, \langle g(x), \phi(x) \rangle = \gamma \qquad (\mathrm{C}, \beta) \ , \tag{8.6.1}$$

if for some primitive G *of* $g\phi$ *we have*

$$\lim_{x \to \infty} (G(x) - G(-x)) = \gamma \qquad (\mathrm{C}, \beta) \ .$$

If (8.6.1) exits, we also say that the principal value of the evaluation exists in the (C, β) *sense.*

We easily obtain a version of Theorem 8.11 for symmetric point values.

Theorem 8.26. *Let $f \in \mathcal{D}'(\mathbb{R})$. Suppose that there exists an $m \in \mathbb{N}$, such that every m-primitive of $\chi^f_{x_0}$ is a locally bounded measurable function for large arguments and satisfies an estimate $O\left(|x|^{m-1}\right)$, $x \to \infty$. If $f_{\text{sym}}(x_0) = \gamma$ in $\mathcal{S}'(\mathbb{R})$ with order n, then*

$$\frac{1}{2\pi}\,\text{p.v.}\left\langle \hat{\chi}^f_{x_0}(x), 1\right\rangle = \gamma \qquad (\mathrm{C}, \beta)\ ,$$

for any $\beta > k = \max\{m, n\}$. When $f \in \mathcal{S}'(\mathbb{R})$, we obtain

$$\frac{1}{2\pi}\,\text{p.v.}\left\langle \hat{f}(x), e^{ix_0x}\right\rangle = \gamma \qquad (\mathrm{C}, \beta)\ ,$$

for any $\beta > k = \max\{m, n\}$.

Proof. Our hypotheses imply that $\chi^f_{x_0} \in \mathcal{S}'(\mathbb{R})$, thus we can apply Theorem 8.11 to $\chi^f_{x_0}$. Since, $\chi^f_{x_0}(0) = \gamma$ in $\mathcal{S}'(\mathbb{R})$ with order n, then

$$\text{e.v.}\left\langle \hat{\chi}^f_{x_0}(x), 1\right\rangle = 2\pi\gamma \qquad (\mathrm{C}, \beta)\ ,$$

for any $\beta > k = \max\{m, n\}$, in particular the last relation holds in the p.v. sense. If we assume that $f \in \mathcal{S}'(\mathbb{R})$, then

$$\hat{\chi}^f_{x_0}(x) = \frac{1}{2}\left(e^{ix_0x}\hat{f}(x) + e^{-ix_0x}\hat{f}(-x)\right)\ ,$$

so, if F is first order primitive of $e^{ix_0x}\hat{f}(x)$, then $G(x) = (F(x) - F(-x))/2$ is a first order primitive of $\hat{\chi}^f_{x_0}(x)$, and hence

$$\lim_{x\to\infty}(G(x) - G(-x)) = \lim_{x\to\infty}(F(x) - F(-x)) = 2\pi\gamma \qquad (\mathrm{C}, \beta)\ .$$

$\square$

When f has compact support we obtain the following result.

Corollary 8.27. *Let f be a distribution with compact support. If $f_{\text{sym}}(x_0) = \gamma$ in $\mathcal{S}'(\mathbb{R})$ with order k, then for any $\beta > k$,*

$$\frac{1}{2\pi}\text{p.v.}\int_{-\infty}^{\infty}\hat{f}(t)e^{ix_0t}\mathrm{d}t = \gamma \qquad (\mathrm{C}, \beta)\ ,$$

or which is the same

$$\lim_{x\to\infty} \frac{1}{2\pi} \int_{-x}^{x} \left(1 - \frac{|t|}{x}\right)^{\beta} \hat{f}(t) e^{ix_0 t} \mathrm{d}t = \gamma \ .$$

For Fourier series, we obtain the result of A. Zygmund [255] mentioned in Example 8.24. Obviously, our language differers from that of Zygmund's original statement.

Corollary 8.28. *Let f be a 2π-periodic distribution, with cosines and sines Fourier series*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \ .$$

Suppose that $f_{\mathrm{sym}}(x_0) = \gamma$ in $\mathcal{S}'(\mathbb{R})$ with order $k \geq 0$. Then for any $\beta > k$,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx_0 + b_n \sin nx_0) = \gamma \qquad (\mathrm{C}, \beta) \ ,$$

or equivalently

$$\lim_{x\to\infty} \frac{a_0}{2} + \sum_{0<n<x} \left(1 - \frac{|n|}{x}\right)^{\beta} (a_n \cos nx_0 + b_n \sin nx_0) = \gamma \ .$$

Proof. If $k \geq 1$, we can assume that $a_0 = 0$ and proceed to apply Theorem 8.26. For $k = 0$, then f is a bounded 2π-periodic function which is continuous at x_0, and hence the conclusion follows from the classical result [93, 256]. □

As in the proof of Theorem 8.26, one can apply the result for distributional point values, Theorem 8.19, to the distribution $\chi^{f}_{x_0}$ to easily obtain the next bound for the order of summability in the case of the principal value of Fourier inversion formula for general tempered distributions.

Theorem 8.29. *Let $f \in \mathcal{D}'(\mathbb{R})$. Suppose that*

$$\chi^{f}_{x_0}(x) = O(|x|^{\alpha}) \qquad (\mathrm{C}, m) \ , \quad as \ |x| \to \infty \ .$$

If $f_{\mathrm{sym}}(x_0) = \gamma$ in $\mathcal{S}'(\mathbb{R})$ with order n, then

$$\frac{1}{2\pi} \mathrm{p.v.} \left\langle \hat{\chi}^{f}_{x_0}(x), 1 \right\rangle = \gamma \qquad (\mathrm{C}, k+1) \ ,$$

where $k = \max\left\{m, n, [n+\alpha+\frac{1}{2}], [m+\alpha+\frac{1}{2}]\right\}$. *If we assume* $f \in \mathcal{S}'(\mathbb{R})$, *then we obtain*

$$\frac{1}{2\pi}\text{p.v.}\left\langle \hat{f}(x), e^{ix_0 x}\right\rangle = \gamma \qquad (\mathrm{C}, k+1)\ .$$

Finally, we estimate the order of the symmetric point value in terms of the order of summability of the principal value Fourier inversion formula. We need the following lemma.

Lemma 8.30. *Let* $g \in \mathcal{D}'(\mathbb{R})$ *be an even distribution, that is,* $g(-x) = g(x)$, *then*

$$\text{e.v.}\left\langle g(x), 1\right\rangle = \gamma \qquad (\mathrm{C}, \beta) \tag{8.6.2}$$

if and only if

$$\text{p.v.}\left\langle g(x), 1\right\rangle = \gamma \qquad (\mathrm{C}, \beta)\ . \tag{8.6.3}$$

In fact the above relations are equivalent to

$$\lim_{x\to\infty} G(x) = \frac{\gamma}{2} \qquad (\mathrm{C}, \beta)\ , \tag{8.6.4}$$

where G *is the unique odd first order primitive of* g.

Proof. That (8.6.3) and (8.6.4) are equivalent is clear. Relation (8.6.2) obviously implies (8.6.3). We now show that (8.6.4) implies (8.6.2). Let G be the odd first order primitive of g, so we have that $G(x) = \gamma/2 + o(1)$ (C, β) as $x \to \infty$, hence we also have that $G(ax) = \gamma/2 + o(1)$ (C, β) as $x \to \infty$, and thus for each $a > 0$

$$\lim_{x\to\infty}\left(G(ax) - G(-x)\right) = 2\lim_{x\to\infty}\left(G(ax) + G(x)\right) = \gamma \qquad (\mathrm{C}, \beta)\ .$$

$\square$

Therefore, on combining Lemma 8.30 and Theorem 8.21, we immediately obtain the following result. Notice that, as a corollary, we obtain the classical result of Plessner [256, ChapXI, Thm.2.1] for Fourier series.

Theorem 8.31. *Let $f \in \mathcal{S}'(\mathbb{R})$. Suppose that*

$$\frac{1}{2\pi}\ \mathrm{p.\,v.}\left\langle \hat{f}(x), e^{ix_0x}\right\rangle = \gamma \qquad (\mathrm{C}, k)\ ,$$

then, $f_{\mathrm{sym}}(x_0)$, distributionally, $\chi^f_{x_0}$ is the derivative of order $k+1$ of a locally integrable function and the order of $f_{\mathrm{sym}}(x_0)$ is less or equal to $k+2$.

The solution of the (C) symmetric problem for "trigonometric integrals" of distributions given in Section 3.10 is recovered by the methods of this chapter. It extends Hardy-Littlewood-Plessner characterization [89, 256] of (symmetric) (C) summability at a point from Fourier series to general tempered distributions.

Theorem 8.32. *Let $f \in \mathcal{D}'(\mathbb{R})$. Suppose that $\chi^f_{x_0} \in \mathcal{S}'(\mathbb{R})$. Then*

$$\frac{1}{2\pi}\ \mathrm{p.\,v.}\left\langle \hat{\chi}^f_{x_0}, 1\right\rangle = \gamma \qquad (\mathrm{C}) \tag{8.6.5}$$

if and only if $f_{\mathrm{sym}}(x_0) = \gamma$, distributionally. If in addition $f \in \mathcal{S}'(\mathbb{R})$, then (8.6.5) is the same as

$$\frac{1}{2\pi}\ \mathrm{p.\,v.}\left\langle \hat{f}(x), e^{ix_0x}\right\rangle = \gamma \qquad (\mathrm{C})\ .$$

8.7 The Order of Jumps and Symmetric Jumps

In this last section we shall study the order of summability in several characterizations and formulas that we have already obtained in Chapter 5 for the jump behavior and symmetric jump behavior of distributions (see also [215, 216, 218, 222]). Let us define the order of jump and symmetric jump behaviors.

Suppose that $f \in \mathcal{D}'(\mathbb{R})$ has the following jump behavior at $x = x_0$,

$$f\left(x_0 + \varepsilon x\right) = \gamma_- H(-x) + \gamma_+ H(x) + o(1) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{D}'(\mathbb{R})\ . \tag{8.7.1}$$

By Theorem 5.2, f has the jump behavior (8.7.1) if and only if there exist $n \in \mathbb{N}$ and a function F, locally integrable on a neighborhood of x_0, such that $F^{(n)} = f$ near x_0 and

$$\lim_{x \to x_0^{\pm}} \frac{n!F(x)}{(x - x_0)^n} = \gamma_{\pm}\ . \tag{8.7.2}$$

If $f \in \mathcal{S}'(\mathbb{R})$, then n can be chosen so that F is locally integrable of polynomial growth. So we can define the order of the jump behavior in $\mathcal{S}'(\mathbb{R})$ of a tempered distribution.

Definition 8.33. *Let $f \in \mathcal{S}'(\mathbb{R})$. Suppose that f has jump behavior at x_0. The order of the jump behavior in $\mathcal{S}'(\mathbb{R})$ is defined as the minimum integer n such that there exists a locally bounded measurable function F of at most polynomial growth at infinity satisfying $F^{(n)} = f$ and (8.7.2).*

Recall the definition of the jump distribution of f at $x = x_0$, it is given by

$$\psi^f_{x_0}(x) = f(x_0 + x) - f(x_0 - x) \ .$$

We defined in Section 5.2 the symmetric jump in terms of the jump behavior of $\psi^f_{x_0}$ at $x = x_0$.

Definition 8.34. *A distribution $f \in \mathcal{D}'(\mathbb{R})$ is said to have a symmetric jump behavior at $x = x_0$ in $\mathcal{S}'(\mathbb{R})$ of order n if $\psi^f_{x_0} \in \mathcal{S}'(\mathbb{R})$ and $\psi^f_{x_0}$ has jump behavior at $x = 0$ of order n.*

Notice that a distribution f has jump behavior (8.7.1) at $x = x_0$ if and only if it has symmetric point value and symmetric jump behavior at $x = x_0$ and $f_{\text{sym}}(x_0) = (\gamma_- + \gamma_+)/2$ and $[f]_{x=x_0} = \gamma_+ - \gamma_-$.

We now add information about the order of summability to the characterization of the jump behavior given in Section 5.3 (see also [215, 216].

Theorem 8.35. *Let $f \in \mathcal{S}'(\mathbb{R})$ have the distributional jump behavior (8.7.1) at $x = x_0$ of order n. Suppose that there exists an $m \in \mathbb{N}$, such that every m-primitive of f is a locally integrable function for large arguments and satisfies an estimate $O\left(|x|^{m-1}\right)$, $x \to \infty$. Let F be a fist order primitive of $e^{ix_0x}\hat{f}$, then if $\beta > k = \max\{m, n\}$,*

$$\frac{1}{2\pi} \lim_{x\to\infty} (F(ax) - F(-x)) = f_{\text{sym}}(x_0) + \frac{[f]_{x=x_0}}{2\pi i} \log a \qquad (\mathrm{C}, \beta) \ ,$$

uniformly for a in compact subsets of $(0,\infty)$.

Proof. Define the distribution

$$v=-\theta_{[-1,0]}+\theta_{[0,1]}\ ,$$

where $theta_A$ denotes the characteristic function of a set A. Then the distribution

$$h(x)=f(x)-\frac{1}{2}[f]_{x=x_0}\,v(x-x_0)$$

satisfies the hypothesis of Theorem 8.11 and $h(x_0)=f_{\text{sym}}(x_0)$ in $\mathcal{S}'(\mathbb{R})$ with order n. Therefore,

$$\text{e.v.}\left\langle \hat{h}(x),e^{ix_0x}\right\rangle=2\pi f_{\text{sym}}(x_0)\qquad (\mathrm{C},\beta),$$

whenever $\beta>k=\max\{m,n\}$. Observe that

$$e^{ix_0x}\mathcal{F}\left\{v(t-x_0);x\right\}=\hat{v}(x)=\frac{2-2\cos x}{ix}.$$

Let G be a first order primitive of $e^{ix_0x}\hat{h}(x)$, hence

$$F(x)=G(x)+\frac{[f]_{x=x_0}}{i}\int_0^x\frac{1-\cos t}{t}\,\mathrm{d}t$$

satisfies $F'(x)=e^{ix_0x}\hat{f}(x)$. Then, we obtain as $x\to\infty$

$$\begin{aligned}F(ax)-F(-x)&=G(ax)-G(-x)+\frac{[f]_{x=x_0}}{i}\int_{-x}^{ax}\frac{1-\cos t}{t}\,\mathrm{d}t\\&=2\pi f_{\text{sym}}(x_0)+\frac{[f]_{x=x_0}}{i}\int_{x}^{ax}\frac{1-\cos t}{t}\,\mathrm{d}t+o(1)\\&=2\pi f_{\text{sym}}(x_0)+\frac{[f]_{x=x_0}}{i}\log a+o(1)\qquad(\mathrm{C},\beta)\ .\end{aligned}$$

□

We obtain immediately form Theorem 8.35 the corresponding results for compactly supported distributions and Fourier series. Here we only state the result for Fourier series and leave the statement for compactly supported distributions to the reader.

Corollary 8.36. *Let* $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ixn}$ *be a* 2π*-periodic distribution. Suppose that* f *has the distributional jump behavior (8.7.1) at* $x = x_0$ *in* $\mathcal{S}'(\mathbb{R})$ *with order* $k \geq 1$*. Then for each* $a > 0$ *and* $\beta > k$*,*

$$\lim_{x\to\infty} \sum_{-x\leq n\leq ax} c_n e^{ix_0 n} = f_{\mathrm{sym}}(x_0) + \frac{[f]_{x=x_0}}{2\pi i}\log a \qquad (\mathrm{C}, \beta) \ ,$$

or equivalently

$$\lim_{x\to\infty} \sum_{-x\leq n\leq ax} \phi_a^{\beta}\left(\frac{n}{x}\right) c_n e^{ix_0 n} = f_{\mathrm{sym}}(x_0) + \frac{[f]_{x=x_0}}{2\pi i}\log a \ . \tag{8.7.3}$$

Moreover, relation (8.7.3) holds uniformly for a *in compact subsets of* $(0,\infty)$*.*

Using the same procedure as in the proof of Theorem 8.35, we obtain from Theorem 8.19 and Theorem 8.21.

Theorem 8.37. *Let* $f \in \mathcal{S}'(\mathbb{R})$ *have the distributional jump behavior (8.7.1) at* $x = x_0$ *of order* n*. Suppose that*

$$f(x) = O(|x|^{\alpha}) \qquad (\mathrm{C}, m) \ , \quad as \ |x| \to \infty \ .$$

Let F *be a first order primitive of* $e^{ix_0x}\hat{f}(x)$*. Then we have, uniformly for* a *in compact subset of* $(0,\infty)$*,*

$$\frac{1}{2\pi}\lim_{x\to\infty}(F(ax) - F(-x)) = f_{\mathrm{sym}}(x_0) + \frac{[f]_{x=x_0}}{2\pi i}\log a \qquad (\mathrm{C}, k+1) \ ,$$

where $k = \max\left\{m, n, [n+\alpha+\frac{1}{2}], [m+\alpha+\frac{1}{2}]\right\}$*.*

Theorem 8.38. *Let* $f \in \mathcal{S}'(\mathbb{R})$*. Let* F *be a first order primitive of* f*. Suppose that for some constants* d_1 *and* d_2

$$\frac{1}{2\pi}\lim_{x\to\infty}(F(ax) - F(-x)) = d_1 + d_2\log a \qquad (\mathrm{C}, k) \ ,$$

for a *in a subset of positive measure of the interval* $(0,\infty)$*. Then,* f *has the distributional jump behavior (8.7.1) at* x_0 *with constants* $\gamma_{\pm} = d_1 \pm i\pi d_2$*,* f *is the derivative of order* $k+1$ *of a locally integrable function and the order of the jump behavior is less or equal to* $k+2$*.*

It is possible to formulate analogous results for the symmetric jump behavior in terms of the jump distribution; however, we choose only to do it for the case of Fourier series.

Theorem 8.39. *Let f be a 2π-distribution with Fourier series*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \ .$$

Suppose that f has a symmetric jump behavior at $x = x_0$ of order $k \geq 1$. Then for any $\beta > k$

$$\lim_{x\to\infty} \sum_{x<n\leq ax} (a_n \sin nx_0 - b_n \cos nx_0) = -\frac{[f]_{x=x_0}}{\pi} \log a \qquad (\mathrm{C}, \beta) \ ,$$

uniformly for a in compact subsets of $[1,\infty)$.

Proof. The jump distribution has Fourier transform

$$\psi_{x_0}^{f}(x) = -2\sum_{n=1}^{\infty} (a_n \sin nx_0 - b_n \cos nx_0) \sin nx \ ,$$

it has Fourier transform

$$\hat{\psi}_{x_0}^{f}(x) = 2\pi i \sum_{n=1}^{\infty} (a_n \sin nx_0 - b_n \cos nx_0) \left(\delta\left(x-n\right) - \delta\left(x+n\right)\right) \ .$$

Therefore,

$$\Psi(x) = 2\pi i \sum_{1\leq n<|x|} (a_n \sin nx_0 - b_n \cos nx_0)$$

is a first order primitive of the $\hat{\psi}_{x_0}^{f}$. Since it has a jump behavior at $x = 0$ with jump $2[f]_{x=x_0}$, Theorem 8.35 implies the result. □

Reasoning as in Theorem 8.39, we can prove using Theorem 8.38 the following result.

Theorem 8.40. *Let f be a 2π-distribution with Fourier series*

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty}(a_n \cos nx + b_n \sin nx) \ .$$

Suppose that

$$\lim_{x\to\infty}\sum_{x<n\leq ax}(a_n\sin nx_0-b_n\cos nx_0)=d\log a \qquad (\mathrm{C},k)\ ,$$

for a in a subset of positive measure of the interval $[1,\infty)$. Then, f has the distributional symmetric jump behavior at x_0 with jump $[f]_{x=x_0}=-\pi d$, $\psi^f_{x_0}$ is the derivative of order $k+1$ of a locally integrable function and the order of the jump behavior is less or equal to $k+2$.

We may use Theorems 8.39, Theorem 8.40 and Corollary 8.28 to characterize the distributional jump behavior of a 2π-periodic distribution from its cosines and sines Fourier series and its conjugate series.

Theorem 8.41. *Let f be a 2π-distribution with Fourier series*

$$f(x)=\frac{a_0}{2}+\sum_{n=0}^{\infty}(a_n\cos nx+b_n\sin nx)\ .$$

Then f has distributional jump behavior at $x=x_0$ if and only if there exists $\beta\geq 0$ such that for some constants d_1 and d_2

$$\frac{a_0}{2}+\sum_{n=0}^{\infty}(a_n\cos nx_0+b_n\sin nx_0)=d_1 \qquad (\mathrm{C},\beta)\ ,$$

and

$$\lim_{x\to\infty}\sum_{x<n\leq ax}(a_n\sin nx_0-b_n\cos nx_0)=d_2\log a \qquad (\mathrm{C},\beta)\ ,$$

for a in a subset of positive measure of the interval $[1,\infty)$. In such case $f_{\mathrm{sym}}(x_0)=d_1$ and $[f]_{x=x_0}=-\pi d_2$.

The last results we want to comment are in relation with the classical formula of F. Lukács for the jump of a function [131, 140, 141, 218]. Indeed, exactly the same arguments given in Section 5.5 but now in combination with the information about the order from Theorem 8.35, Corollary 8.36 and Theorem 8.37 yield the following series of results.

Theorem 8.42. *Let $f \in \mathcal{S}'(\mathbb{R})$ have the distributional jump behavior at $x = x_0$ of order n. Suppose that there exists an $m \in \mathbb{N}$ such that every m-primitive of f is a locally integrable function for large arguments and satisfies an estimate $O\left(|x|^{m-1}\right)$, as $x \to \infty$. Then for any decomposition $\hat{f} = \hat{f}_- + \hat{f}_+$, where* $\operatorname{supp} \hat{f}_- \subseteq (-\infty, 0]$ *and* $\operatorname{supp} \hat{f}_+ \subseteq [0, \infty)$*, and for any $\beta > \max\{n, m\}$, we have that the following convolutions are locally bounded functions and*

$$\left(e^{\pm i x_0 t} \hat{f}_\pm(\pm t) * t_+^\beta\right)(x) \sim \pm [f]_{x=x_0} \frac{|x|^\beta}{i} \log x \ , \qquad x \to \infty \ ,$$

in the ordinary sense.

Theorem 8.43. *Let $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ixn}$ be a 2π-periodic distribution. Suppose it has distributional jump behavior at $x = x_0$ of order $k \geq 1$. Then for any $\beta > k$*

$$\lim_{x \to \infty} \frac{1}{\log x} \sum_{0 \leq n \leq x} c_{\pm n} e^{\pm i n x_0} \left(1 - \frac{n}{x}\right)^\beta = \pm \frac{[f]_{x=x_0}}{2\pi i} \ .$$

Theorem 8.44. *Let $f \in \mathcal{S}'(\mathbb{R})$ have the distributional jump behavior at $x = x_0$ of order n. Suppose that*

$$f(x) = O(|x|^\alpha) \qquad (\mathrm{C}, m) \ , \quad \textit{as } |x| \to \infty \ .$$

Then for any decomposition $\hat{f} = \hat{f}_- + \hat{f}_+$, where $\operatorname{supp} \hat{f}_- \subseteq (-\infty, 0]$ *and* $\operatorname{supp} \hat{f}_+ \subseteq [0, \infty)$*. We have that the following convolutions are locally bounded functions and*

$$\left(e^{\pm i x_0 t} \hat{f}_\pm(\pm t) * t_+^{k+1}\right)(x) \sim \pm [f]_{x=x_0} \frac{|x|^{k+1}}{i} \log x \ , \quad \textit{as } x \to \infty \ ,$$

in the ordinary sense, where $k = \max\left\{m, n, [n + \alpha + \frac{1}{2}], [m + \alpha + \frac{1}{2}]\right\}$.

For the case of symmetric jumps of Fourier series we have the following result.

Theorem 8.45. *Let $f \in \mathcal{S}'(\mathbb{R})$ be a 2π-periodic distribution having the following Fourier series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \ .$$

If f has a symmetric jump behavior at $x = x_0$ of order $k \geq 1$, then for any $\beta > k$

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{n=1}^{\infty} (a_n \sin nx_0 - b_n \cos nx_0) \left(1 - \frac{n}{x}\right)^{\beta} = -\frac{1}{\pi} \, [f]_{x=x_0} \, .$$

Chapter 9

Extensions of Tauber's Second Tauberian Theorem

9.1 Introduction

We now continue our investigations about tauberian type results which were started in Chapter 4. A new feature to be introduced is the use of one-sided tauberian conditions.

Tauberian Theory was initiated in 1897 by two simple theorems of Tauber for the converse of Abel's theorem [204, 115]. The present chapter is dedicated to provide extensions of Tauber's second theorem in several directions.

Let us state Tauber's original theorems.

Theorem 9.1. *(Tauber's first theorem) If* $\sum_{n=0}^{\infty} c_n = \gamma$ (A) *and*

$$c_n = o\left(\frac{1}{n}\right) , \quad n \to \infty , \tag{9.1.1}$$

then $\sum_{n=0}^{\infty} c_n$ *converges to* γ*.*

Theorem 9.2. *(Tauber's second theorem) If* $\sum_{n=0}^{\infty} c_n = \gamma$ (A) *and*

$$\sum_{n=1}^{N} nc_n = o(N) , \quad N \to \infty , \tag{9.1.2}$$

then $\sum_{n=0}^{\infty} c_n$ *converges to* γ*.*

A version of Tauber's second theorem for Stieltjes integrals appeared in [245].

Tauber's theorems are very simple to show [204, 85]. In 1910, Littlewood [127] gave his celebrated extension of Tauber's first theorem, he substituted the tauberian condition (9.1.1) by the weaker one $c_n = O\left(n^{-1}\right)$ and obtained the same conclusion of convergence as in Theorem 9.1; actually, it can be shown that the hypotheses imply the (C, β) summability for any $\beta > -1$ [86]. It turns out that Littlewood's theorem is much deeper and difficult to prove than Theorem 9.1. Two years later [86], Hardy and Littlewood conjectured that the condition $nc_n > -K$ would be enough to ensure the convergence; indeed, they provided a proof later in [87].

Extensions of Theorem 9.2 are also known. It is natural to ask whether the replacement of (9.1.2) by a big O condition would lead to convergence; unfortunately, it does not suffice (see [171] for example). Nevertheless, one gets $(\mathrm{C}, 1)$ summability as shown in the next theorem of O. Szász [200] (see also [168, 171, 201]), where even less is assumed.

Theorem 9.3. *(Szász [200]) Suppose that $\sum_{n=0}^{\infty} c_n = \gamma$ (A). Then the tauberian condition*

$$\sum_{n=1}^{N} nc_n > -KN \ , \tag{9.1.3}$$

for some $K > 0$, implies that $\sum_{n=0}^{\infty} c_n = \gamma$ $(\mathrm{C}, 1)$.

We will actually show (see Corollary 9.39 below) that if a two-sided condition is assumed instead of (9.1.3), then the series is summable (C, β), for all $\beta > 0$. Versions of Theorem 9.3 for Dirichlet series can be found in [200] and [28, Section 3.8].

It should be noticed that Theorem 9.3 includes the Hardy-Littlewood's theorem quoted above, it may also be used to give direct proofs of other classical tauberians . As a motivation for further extensions of Theorem 9.3, let us discuss how to deduce the results from [85, pp.153–155] as corollaries.

Corollary 9.4. *(Hardy and Littlewood) Suppose that $\sum_{n=0}^{\infty} c_n = \gamma$ (A). The tauberian condition $\sum_{n=0}^{N} c_n = O(1)$ implies the $(\mathrm{C}, 1)$ summability of the series to γ.*

Proof. Indeed the tauberian hypothesis implies (9.1.3); for

$$\sum_{n=0}^{N} nc_n = N \sum_{n=0}^{N} c_n - \sum_{n=0}^{N-1} \left(\sum_{j=0}^{n} c_j \right) = O(N) \ .$$

□

It also implies the following result.

Corollary 9.5. *(Hardy and Littlewood) Suppose that* $F(r) = \sum_{n=0}^{\infty} c_n r^n \sim \gamma/(1-r)$, $r \to 1^-$. *If* $c_n = O(1)$, *then* $s_N = \sum_{n=0}^{N} c_n \sim \gamma N$, $N \to \infty$.

Proof. Define $b_0 = c_0$, $b_n = c_n - c_{n-1}$ for $n \geq 1$. Then as $r \to 1^-$,

$$(1-r)\sum_{n=0}^{\infty} c_n r^n = \sum_{n=0}^{\infty} b_n r^n \to \gamma ,$$

the hypothesis $c_n = O(1)$ implies

$$\sum_{n=0}^{N} n b_n = N c_N - \sum_{n=0}^{N-1} c_n = O(N) .$$

So we conclude that $\sum_{n=0}^{\infty} b_n = \gamma$ $(\mathrm{C}, 1)$, or which is the same, as $N \to \infty$,

$$\frac{1}{N}\sum_{n=0}^{N}\sum_{j=0}^{n} b_n = \frac{1}{N}\sum_{n=0}^{N} c_n \to \gamma .$$

$\square$

Finally, the one-sided Littlewood's theorem.

Corollary 9.6. *(Littlewood [127], Hardy and Littlewood [87]) If* $\sum_{n=0}^{\infty} c_n = \gamma$ (A) *and* $nc_n > -K$, *for some constant* $K > 0$, *then* $\sum_{n=0}^{\infty} c_n = \gamma$.

Proof. The condition implies (9.1.3) and hence $\sum_{n=0}^{\infty} c_n = \gamma$ $(\mathrm{C}, 1)$. Since $nc_n \geq -K$, Hardy's tauberian theorem for $(\mathrm{C}, 1)$ summability, that is, Corollary 3.32 (see also [85, p.121]), which is much more elementary than the present theorem, implies the convergence. $\square$

The classical tauberian theorems for power series have stimulated the creation of many interesting methods and theories in order to obtain extensions and easier proofs for them. Among the classical ones, one could mention those of N. Wiener [246] and J. Karamata [109, 110]. Other important ones come from the theory of generalized functions. In [229], Vladimirov obtained a multidimensional extension of Hardy-Littlewood type theorems for measures under positivity tauberian conditions. Later on, the results of [229] were generalized to include tempered distributions, resulting in a powerful multidimensional tauberian theory for the Laplace transform [37, 231] (see also [38]). Distributional tauberian theorems for other integral transforms are investigated in [139, 159, 160]. Other related results are found in [149, 157].

In Chapter 4 we were able to deduce Littlewood's tauberian theorem [127] from the tauberian theorem for distributional point values; actually the method of Section 4.4 recovered the more general version for Dirichlet series proved first by Ananda Rau [5]. A similar approach, but with a more comprehensive character, will be taken in this chapter.

The structure of the chapter is as follows. Section 9.2 is devoted to the study of Cesàro limits and summability in the context of Schwartz distributions; we define one-sided Cesàro boundedness of fractional order, then we provide several technical tauberian theorems which will establish the link between results for generalized functions and Stieltjes integrals. The main part of the chapter is Section 9.3. There, we first show a theorem for distributional point values which generalizes Theorem 9.3; moreover, our theorem is capable to recover Theorem 9.3, and it is applicable to much more situations. Finally, we generalize [201, Thm.B] from series to Stieltjes integrals and use this new result to give proofs of some classical tauberians of Hardy-Littlewood and O. Szász for Dirichlet series.

9.2 Tauberian Theorems for (C) Summability

In this section we show tauberian theorems for (C) summability of distributions and measures related to Theorem 9.2. We first study Cesàro boundedness. Next, a convexity theorem is shown. Finally, we present the tauberian theorems.

9.2.1 Cesàro Boundedness: Fractional Orders

We defined in Section 8.3 Cesàro limits of fractional order; we now extend these ideas to boundedness. In the case of integral orders, it coincides with the definition from [49]. We will also define one-sided boundedness.

Recall that given $f \in \mathcal{D}'(\mathbb{R})$, with support bounded at the left, its *β-primitive* is given by the convolution [230]

$$f^{(-\beta)} = f * \frac{x_+^{\beta-1}}{\Gamma(\beta)} \ . \tag{9.2.1}$$

Definition 9.7. *Let $f \in \mathcal{D}'(\mathbb{R})$, and $\beta \geq 0$. We say that f is bounded at infinity in the Cesàro sense of order β (in the (C, β) sense), and write*

$$f(x) = O(1) \quad (\mathrm{C}, \beta), \ \text{as } x \to \infty \ , \tag{9.2.2}$$

if for any decomposition $f = f_- + f_+$ as sum of two distributions with supports bounded on the right and left, respectively, one has that the β-primitive of f_+ is an ordinary function (locally integrable) for large arguments and satisfies the ordinary order relation

$$f_+^{(-\beta)}(x) = O\left(x^{\beta}\right), \quad x \to \infty \ , \tag{9.2.3}$$

in the ordinary sense. A similar definition applies for the little o-symbol.

Observe that, because of Proposition 8.3, we can always assume in Definition 9.7 that $f = f_+$, if needed. Notice also that Definition 9.7 is consistent with Definition 8.2, since

$$\lim_{x \to \infty} f(x) = \ell \quad (\mathrm{C}, \beta)$$

if and only if

$$f(x) - \ell H(x) = o(1) \quad (\mathrm{C}, \beta) \ , \quad x \to \infty \ .$$

We now define one-sided boundedness. Recall that a positive distribution is nothing else than a positive Radon measure.

Definition 9.8. *Let $f \in \mathcal{D}'(\mathbb{R})$, $\beta \in \mathbb{R}$, and $\alpha \in \mathbb{R} \setminus \{-1, -2, \dots\}$. We say that f is bounded from below (or left bounded) near infinity by $O_L(x^\alpha)$ in the Cesàro sense of order β, and write*

$$f(x) = O_L(x^\alpha) \quad (\mathrm{C}, \beta), \text{ as } x \to \infty \ , \tag{9.2.4}$$

if there exists a decomposition $f = f_- + f_+$, as sum of two distributions with supports bounded on the right and left, respectively, a constant $K > 0$, and an interval (a, ∞) such that $f_+^{(-\beta)} + K x_+^{\alpha+\beta}$ is a positive distribution on (a, ∞). A similar definition applies for right boundedness, in such a case we employ the symbol $O_R(x^\alpha)$.

Our definitions of Cesàro behavior have the following expected property.

Proposition 9.9. *If f is Cesàro bounded (has Cesàro limit, or is one-sided bounded) at infinity of order β, then it it Cesàro bounded (has Cesàro limit, or is one-sided bounded by $O(x^\alpha)$, respectively) at infinity of order $\tilde{\beta} > \beta$.*

Proof. Proposition 8.3 implies the case of limits and boundedness; for one-sided boundedness, it follows easily from the definition. □

When we do not want to make reference to the order β in (C, β), we simply write (C). We will often drop $x \to \infty$ from the notation. Note that if $f(x) = O(1)$, $x \to \infty$, then $f_+ \in \mathcal{S}'(\mathbb{R})$ (here $f = f_- + f_+$ as in Definition 9.7). In addition, it should be noticed that both $f(x) = O_L(1)$ and $f(x) = O_R(1)$, in the (C, β) sense, imply $f(x) = O(1)$ (C, β) (prove it!).

We will need the following observation concerning to numerical series in the future. Given a sequence $\{b_n\}_{n=0}^{\infty}$ and $\beta > 0$, write

$$b_N = O(N) \quad (\mathrm{C}, \beta) \ ,$$

if the Cesàro means of order β of the sequence (not to be confuse with the Cesàro means of a series) are $O(N)$, that is,

$$\sum_{n=0}^{N} \binom{N-n+\beta-1}{\beta-1} b_n = O(N^{\beta+1}) \ .$$

Likewise, we define the symbols O_R and O_L in the Cesàro sense for sequences. Following Ingham's method [100], we obtain the following useful equivalence.

Lemma 9.10. *Let $\beta \geq 0$. The conditions*

$$\sum_{n=0}^{N} c_n = O(N) \quad (\mathrm{C}, \beta) \tag{9.2.5}$$

and

$$\sum_{n=0}^{\infty} c_n \delta(x-n) = O(1) \quad (\mathrm{C}, \beta+1) \ , \quad \textit{as } x \to \infty \ , \tag{9.2.6}$$

are equivalent. The same holds for the symbols O_R and O_L.

Proof. Repeating the arguments from [100] (see also Theorem 6.24 in Chapter 6 or [222, Section 7]), with the obvious modifications, one is led to the equivalence between (9.2.5) and the relation

$$\sum_{n<x} (x-n)^{\beta} c_n = O(x^{\beta+1}), \quad (\text{resp. } O_R \text{ and } O_L),$$

which turns out to be the meaning of (9.2.6). □

9.2.2 A Convexity (Tauberian) Theorem

We now show a convexity theorem for the Cesàro limits of distributions. It generalizes [85, Thm.70].

Theorem 9.11. *Let $f \in \mathcal{D}'(\mathbb{R})$. Suppose that $\lim_{x\to\infty} f(x) = \gamma$ (C, β_2), for some $\beta_2 > 0$. If $f(x) = O_L(1)$ (C, β_1), then $\lim_{x\to\infty} f(x) = \gamma$ (C, β), for any $\beta \geq \beta_1 + 1$. The same conclusion holds if we replace $O_L(1)$ by $O_R(1)$. If now $f(x) = O(1)$ (C, β_1), as $x \to \infty$, then $\lim_{x\to\infty} f(x) = \gamma$ (C, β), for any $\beta > \beta_1$.*

Theorem 9.11 follows immediately from the following theorem. Notice that it extends results on asymptotics of derivatives from [115, p.34–37] and [85, Thm.112].

For the first part we give a proof with distributional flavor, following the method from [37, Lemma 3]. Given a Radon measure, we denote by s_μ a function of local bounded variation such that $\mu = \mathrm{d}s_\mu$; in the sense that μ is given by an Stieltjes integral and $s_\mu' = \mu$. If μ has support on $[0,\infty)$, then $s_\mu(0) = 0$. The variation measure $|\mu|$ associated to μ is also denoted by $|\mathrm{d}s_\mu|$.

Theorem 9.12. *Let μ be a Radon measure supported in $[0,\infty)$ and $\alpha > -1$. Suppose that for some $\beta_1 > 1$*

$$\int_0^x (x-t)^{\beta_1-1}\mathrm{d}s_\mu(t) \sim \frac{\gamma\Gamma(\beta_1)\Gamma(\alpha+1)}{\Gamma(\beta_1+\alpha+1)}x^{\alpha+\beta_1}\ , \quad x\to\infty\ . \tag{9.2.7}$$

If the one-sided condition,

$$Cx_+^\alpha + \mu \ \ \textit{is a positive measure,} \tag{9.2.8}$$

is satisfied for some constant C, then for any $\beta \geq 1$

$$\int_0^x \left(1-\frac{t}{x}\right)^{\beta-1}\mathrm{d}s_\mu(t) \sim \frac{\gamma\Gamma(\beta)\Gamma(\alpha+1)}{\Gamma(\beta+\alpha+1)}x^{\alpha+1}\ ,\ x\to\infty\ . \tag{9.2.9}$$

If in addition μ is absolutely continuous with respect to the Lebesgue measure, and the two sided condition

$$F_\mu(x) = O\left(x^\alpha\right)\ ,\ x\to\infty\ , \tag{9.2.10}$$

is satisfied, where $F_\mu \in L^1_{\mathrm{loc}}(\mathbb{R})$ is so that $\mathrm{d}s_\mu(t) = F_\mu(t)\mathrm{d}t$, then (9.2.9) holds whenever $\beta > \max\{-\alpha, 0\}$.

Proof. Let us show the first part of the theorem. By adding Cx_+^α to μ, we may assume that $C=0$, and so we are assuming that μ is a positive measure. Next, we show that we may assume that $\beta_1 \in \mathbb{N}$; indeed, if we convolve (9.2.7) with $x_+^{[\beta_1]-\beta_1}$, we obtain the same relation for $[\beta_1]+1$. It follows that

$$\mu(\lambda x) = \gamma\lambda^\alpha x_+^\alpha + o\left(\lambda^\alpha\right)\ ,\ \lambda\to\infty \tag{9.2.11}$$

in $\mathcal{D}'(\mathbb{R})$; we want to show that we may take $\beta_1 = 1$ in (9.2.9), the rest follows trivially. Let $\sigma > 0$. Pick $\phi \in \mathcal{D}'(\mathbb{R})$ with the properties $0 \leq \phi \leq 1$ $\operatorname{supp}\phi \subseteq [-1, 1+\sigma]$ and $\phi(x) = 1$ on $[0,1]$. Then, from (9.2.11)

$$\begin{aligned}
&\limsup_{\lambda\to\infty}\left(\frac{1}{\lambda^{\alpha+1}}\int_0^\lambda \mathrm{d}s_\mu(t) - \frac{\gamma}{\alpha+1}\right)\\
&\leq \lim_{\lambda\to\infty}\frac{1}{\lambda^{\alpha+1}}\int_0^\infty \phi\left(\frac{t}{\lambda}\right)\mathrm{d}s_\mu(t) - \frac{\gamma}{\alpha+1}\\
&= \lim_{\lambda\to\infty}\frac{1}{\lambda^{\alpha}}\langle\mu(\lambda t), \phi(t)\rangle - \frac{\gamma}{\alpha+1}\\
&= \gamma\int_0^{1+\sigma} t^\alpha\phi(t)\mathrm{d}t - \frac{\gamma}{\alpha+1} \leq \gamma\int_1^{1+\sigma} t^\alpha \mathrm{d}t\ .
\end{aligned}$$

Similarly, choosing the test function with the properties $0 \leq \phi \leq 1$, $\operatorname{supp}\phi \subseteq [-1,1]$ and $\phi(x) = 1$ on $[0, 1-\sigma]$, we come to the conclusion

$$\liminf_{\lambda\to\infty}\left(\frac{1}{\lambda^{\alpha+1}}\int_0^\lambda \mathrm{d}s_\mu(t) - \frac{\gamma}{\alpha+1}\right) \geq -\gamma\int_{1-\sigma}^1 t^\alpha\mathrm{d}t\ .$$

Since σ is arbitrary, we have that

$$\int_0^\lambda \mathrm{d}s_\mu(t) \sim \gamma\frac{\lambda^{\alpha+1}}{\alpha+1}\ ,\ \lambda\to\infty\ .$$

This completes the proof of the first part.

For the second part, write $F := F_\mu$. We assume that $|F(x)| \leq Mx^\alpha$ for some constant M and x large enough. Moreover, it is clear that we can assume this condition to hold for all x, by Proposition 8.3. Denoting $F*H$ by$F_1(x) = \int_0^x F(t)\mathrm{d}t$, we obtain from the first part that $F_1(x) \sim \gamma x^{\alpha+1}/(\alpha+1)$, $x\to\infty$. We also have that if $0 < r < 1$

$$|F_1(rx) - F_1(x)| \leq M\int_{rx}^x t^\alpha\mathrm{d}t = \frac{M}{\alpha+1}(1-r)^{\alpha+1}x^{\alpha+1}\ .$$

Hence, if $\max\{-\alpha, 0\} < \beta < 1$

$$
\begin{aligned}
&\int_0^x F(t)\left(1-\frac{t}{x}\right)^{\beta-1}\mathrm{d}t = \lim_{r\to 1^-}\int_0^{rx} F(t)\left(1-\frac{t}{x}\right)^{\beta-1}\mathrm{d}t \\
&= \lim_{r\to 1^-}\left(F_1(rx)(1-r)^{\beta-1} + \frac{\beta-1}{x}\int_0^{rx} F_1(t)\left(1-\frac{t}{x}\right)^{\beta-2}\mathrm{d}t\right) \\
&= \lim_{r\to 1^-}\left((1-r)^{\alpha+\beta}\frac{F_1(rx)-F_1(x)}{(1-r)^{\alpha+1}} + F_1(x)(1-r)^{\beta-1}\right. \\
&\qquad\left. + \frac{\beta-1}{x}\int_0^{rx} F_1(t)\left(1-\frac{t}{x}\right)^{\beta-2}\mathrm{d}t\right) \\
&= F_1(x) + \lim_{r\to 1^-}\frac{\beta-1}{x}\int_0^{rx}\left(F_1(t)-F_1(x)\right)\left(1-\frac{t}{x}\right)^{\beta-2}\mathrm{d}t \\
&= (\beta-1)\int_0^1\left(F_1(xt)-F_1(x)\right)(1-t)^{\beta-2}\mathrm{d}t + \gamma\frac{x^{\alpha+1}}{\alpha+1} + o(x^{\alpha+1}) \\
&= x^{\alpha+1}\left((\beta-1)\int_0^1\frac{F_1(xt)-F_1(x)}{x^{\alpha+1}(1-t)}(1-t)^{\beta-1}\mathrm{d}t + \frac{\gamma}{\alpha+1} + o(1)\right) \\
&= \gamma\frac{\beta-1}{\alpha+1}x^{\alpha+1}\left(\int_0^1 (t^{\alpha+1}-1)(1-t)^{\beta-2} + \frac{1}{\beta-1} + o(1)\right) \\
&= \gamma\frac{\beta-1}{\alpha+1}x^{\alpha+1}\left(\frac{\Gamma(\beta-1)\Gamma(\alpha+2)}{\Gamma(\beta+\alpha+1)} - \frac{1}{\beta-1} + \frac{1}{\beta-1} + o(1)\right) \\
&= \gamma\frac{\Gamma(\beta)\Gamma(\alpha+1)}{\Gamma(\beta+\alpha+1)}x^{\alpha+1} + o\left(x^{\alpha+1}\right)\ ,\ x\to\infty\ .
\end{aligned}
$$

□

9.2.3 Tauberian Theorems for (C) Summability

We now analyze Tauber's second type conditions. For that, we need the following formula, here we use the Laplace transform, so given $g \in \mathcal{S}'(\mathbb{R})$, with support bounded at the left, its Laplace transform is $\mathcal{L}\{g; z\} := \langle g(t), e^{-zt}\rangle$, for $\Re e\ z > 0$.

Lemma 9.13. *Suppose that $f \in \mathcal{D}'(\mathbb{R})$ has support bounded at the left. Then*

$$(xf)^{(-\beta)} = xf^{(-\beta)} - \beta f^{(-\beta-1)}\ . \tag{9.2.12}$$

Proof. We first assume that $f \in \mathcal{S}'(\mathbb{R})$. We make use of the injectivity of the Laplace transform. Set $F(z) = \mathcal{L}\{f(t); z\}$. Then,

$$\mathcal{L}\left\{tf^{(-\beta)}(t); z\right\} = -\frac{d}{dz}\left(\mathcal{L}\left\{f^{(-\beta)}(t); z\right\}\right) = -\frac{d}{dz}\left(F(z)\mathcal{L}\left\{\frac{t^{(\beta-1)}}{\Gamma(\beta)}; z\right\}\right)$$
$$= \beta\frac{F(z)}{z^{\beta+1}} - \frac{F'(z)}{z^{\beta}} = \mathcal{L}\left\{\beta f^{(-\beta-1)} + (tf)^{(-\beta)}; z\right\} \ ,$$

which shows (9.2.12). In the general case we take a sequence $\{f_n\}_{n=0}^{\infty}$, with each f_n being tempered and having support on some fixed interval $[a, \infty)$, such that $f_n \to f$ in $\mathcal{D}'(\mathbb{R})$; then (9.2.12) is satisfied for each f_n. Thus, the continuity of the fractional integration operator [230], on $\mathcal{D}'[a, \infty)$, shows (9.2.12) for f, after passing to the limit. $\square$

We now connect Tauber's second type conditions with Cesàro boundedness.

Lemma 9.14. *Let $f \in \mathcal{D}'(\mathbb{R})$. Suppose that $f(x) = O(1)\ (\mathrm{C}, \beta_2)$, as $x \to \infty$, for some $\beta_2 \geq 0$. Then the condition $xf'(x) = O(1)\ (\mathrm{C}, \beta_1 + 1)$ holds if and only if $f(x) = O(1)\ (\mathrm{C}, \beta_1)$.*

Proof. Assume that f has support bounded on the left. We can assume that β_2 has the form $\beta_2 = \beta_1 + k$, for some $k \in \mathbb{N}$. Let $g = xf'$, then, by Lemma 9.13,

$$xf^{(-\beta_1-k+1)}(x) = \beta_2 f^{(-\beta_1-k)}(x) + g^{(-\beta_1-k)}(x) = g^{(-\beta_1-k)}(x) + O\left(x^{\beta_1+k}\right) \ ,$$

$x \to \infty$; then $f(x) = O(1)\ (\mathrm{C}, \beta_1 + k - 1)$ if and only if $g(x) = O(1)\ (\mathrm{C}, \beta_1 + k)$, $x \to \infty$. A recursive argument shows that $f(x) = O(1)\ (\mathrm{C}, \beta_1)$ if and only if $g(x) = O(1)\ (\mathrm{C}, \beta_1 + 1)$. $\square$

The same proof applies for one-sided boundedness.

Lemma 9.15. *Let $f \in \mathcal{D}'(\mathbb{R})$. Suppose that $f(x) = O(1)\ (\mathrm{C}, \beta_2)$, as $x \to \infty$, for some $\beta_2 \geq 0$. Then the condition $xf'(x) = O_L(1)\ (\mathrm{C}, \beta_1 + 1)$ holds if and only if $f(x) = O_L(1)\ (\mathrm{C}, \beta_1)$. The same is true if $O_L(1)$ is replaced by $O_R(1)$.*

So, we immediately obtain from Theorem 9.11

Theorem 9.16. *Let $f \in \mathcal{D}'(\mathbb{R})$. Suppose that* $\lim_{x\to\infty} f(x) = \gamma$ (C, β_2) *for some* $\beta_2 \geq 0$. *The tauberian condition* $xf'(x) = O(1)$ $(\mathrm{C}, \beta_1 + 1)$, *for some* $\beta_1 \geq 0$; *implies that* $\lim_{x\to\infty} f(x) = \gamma$ (C, β) *for all* $\beta > \beta_1$.

Proof. Indeed, we obtain, by Lemma 9.14, $f(x) = O(1)$ (C, β_1), $x \to \infty$; hence, an application of Theorem 9.11 gives the result. □

Theorem 9.17. *Let $f \in \mathcal{D}'(\mathbb{R})$. Suppose that* $\lim_{x\to\infty} f(x) = \gamma$ (C, β_2) *for some* $\beta_2 \geq 0$. *The tauberian condition* $xf'(x) = O_L(1)$ $(\mathrm{C}, \beta_1 + 1)$, *for some* $\beta_1 \geq 0$; *implies that* $\lim_{x\to\infty} f(x) = \gamma$ (C, β), *for all* $\beta \geq \beta_1 + 1$. *The same holds if we replace* $O_L(1)$ *by* $O_R(1)$.

Proof. From Lemma 9.15, we have $f(x) = O_L(1)$ (C, β_1), $x \to \infty$; hence, again, we can an apply Theorem 9.11. □

We also analyze a little o condition. It generalizes [85, Thm.65] to distributions.

Theorem 9.18. *Let $f \in \mathcal{D}'(\mathbb{R})$. Suppose that* $\lim_{x\to\infty} f(x) = \gamma$ (C, β_2). *If* $\beta_2 > \beta_1 \geq 0$, *a necessary and sufficient condition for the limit to hold* (C, β_1) *is* $xf'(x) = o(1)$ $(\mathrm{C}, \beta_1 + 1)$.

Proof. We retain the notation from the proof of Lemma 9.14. If for some $k > 0$, $\lim_{x\to\infty} f(x) = \gamma$ $(\mathrm{C}, k + \beta_1)$, then the relation

$$\begin{aligned} xf^{(-\beta_1-k+1)}(x) &= g^{(-\beta_1-k)}(x) + (\beta_1 + k) f^{(-\beta_1-k)}(x) \\ &= g^{(-\beta_1-k)}(x) + \frac{\gamma x^{\beta_1+k}}{\Gamma(\beta_1 + k)} + o\left(x^{\beta_1+k}\right) \end{aligned}$$

shows the equivalence at level $k - 1$. A recursive argument proves that the equivalence should hold for $k = 1$. □

We may state our results in terms of (C) summability of distributional evaluations. We obtain the next series of corollaries directly from Theorem 9.16, Theorem 9.17, and Theorem 9.18.

Corollary 9.19. *Let $g \in \mathcal{D}'(\mathbb{R})$ and $\phi \in \mathcal{E}(\mathbb{R})$. Suppose that* $\operatorname{supp} g$ *is bounded at the left and*

$$\langle g(x), \phi(x) \rangle = \gamma \quad (\mathrm{C}) \ . \tag{9.2.13}$$

If $x\phi(x)g(x) = O(1)$ $(\mathrm{C}, \beta_1 + 1)$, as $x \to \infty$, for some $\beta_1 \geq 0$, then

$$\langle g(x), \phi(x) \rangle = \gamma \quad (\mathrm{C}, \beta) \ , \tag{9.2.14}$$

for all $\beta > \beta_1$.

Corollary 9.20. *Let $g \in \mathcal{D}'(\mathbb{R})$ and $\phi \in \mathcal{E}(\mathbb{R})$. Suppose that* $\operatorname{supp} g$ *is bounded at the left and*

$$\langle g(x), \phi(x) \rangle = \gamma \quad (\mathrm{C}) \ . \tag{9.2.15}$$

If $x\phi(x)g(x) = O_L(1)$ $(\mathrm{C}, \beta_1 + 1)$, as $x \to \infty$, for some $\beta_1 \geq 0$, then

$$\langle g(x), \phi(x) \rangle = \gamma \quad (\mathrm{C}, \beta) \ , \tag{9.2.16}$$

for all $\beta \geq \beta_1 + 1$.

Corollary 9.21. *Let $g \in \mathcal{D}'(\mathbb{R})$ and $\phi \in \mathcal{E}(\mathbb{R})$. Suppose that* $\operatorname{supp} g$ *is bounded at the left and*

$$\langle g(x), \phi(x) \rangle = \gamma \quad (\mathrm{C}) \ . \tag{9.2.17}$$

Given $\beta \geq 0$, a necessary and sufficient condition for (9.2.17) to hold (C, β) is $x\phi(x)g(x) = o(1)$ $(\mathrm{C}, \beta + 1)$, as $x \to \infty$.

9.3 Tauber's Second Type Theorems for Point Values and (A) Summability

We now analyze tauberian problems related to Abel summability.

9.3.1 Tauberian Theorem for Distributional Point Values

We are ready to show the main theorem of this chapter.

Theorem 9.22. *Let F be analytic in a rectangular region of the form $(a,b)\times(0,R)$. Suppose $f(x) = F(x+i0)$ in $\mathcal{D}'(a,b)$. Let $x_0 \in (a,b)$ such that $F(x_0+iy) \to \gamma$ as $y \to 0^+$. The tauberian condition $f'(x_0+\varepsilon x) = O(\varepsilon^{-1})$ as $\varepsilon \to 0^+$ in $\mathcal{D}'(a,b)$, implies that $f(x_0) = \gamma$, distributionally.*

Proof. Clearly, by translating, we can assume that $x_0 = 0$. We first show that it may be assumed $f \in \mathcal{S}'(\mathbb{R})$ and F is the Fourier-Laplace representation. Let C be a smooth simple curve contained in $(a,b)\times[0,R)$ such that $\mathsf{C}\cap(a,b) = [x_0-\sigma, x_0+\sigma]$, for some small σ, and which is symmetric with respect to the imaginary axis. Let τ be a conformal bijection [32, 167] between the upper half-plane and the region enclosed by C such that the image of the imaginary axis is contained on the imaginary axis and τ extends to a C^∞-diffeomorphism from $\mathbb{R}$ to $\mathsf{C}\setminus(\mathsf{C}\cap i\mathbb{R}_+)$. Then, $F\circ\tau(iy) \to \gamma$ as $y \to 0^+$ and $f(\tau(\varepsilon x)) = O(\varepsilon^{-1})$ as $\varepsilon \to 0^+$ in $\mathcal{D}'(\mathbb{R})$ if and only if $F(iy) \to \gamma$ and $f(\varepsilon x) = O(\varepsilon^{-1})$ in $\mathcal{D}'(\mathbb{R})$. Moreover [128], $f\circ\tau(0) = \gamma$ if and only if $f(0) = \gamma$, distributionally. In addition $F\circ\tau$ is bounded away an open half-disk about the origin, hence it is the Fourier-Laplace analytic representation of $f\circ\tau$. So, we can therefore assume that $f \in \mathcal{S}'(\mathbb{R})$ and

$$F(z) = \frac{1}{2\pi}\left\langle \hat{f}(t), e^{izt}\right\rangle \ .$$

Our aim is to show that f is distributionally bounded at $x = 0$. Indeed, if one established this fact then $f(0) = \gamma$,distributionally, by Theorem 4.7. The condition $f'(\varepsilon x) = O(\varepsilon^{-1})$ still holds in $\mathcal{S}'(\mathbb{R})$. If we integrate this condition [227], we obtain from the definition of primitive in $\mathcal{S}'(\mathbb{R})$ that there exists a function c, continuous on $(0,\infty)$, such that

$$f(\varepsilon x) = c(\varepsilon) + O(1) \ ,$$

as $\varepsilon \to 0^+$ in $\mathcal{S}'(\mathbb{R})$, in the sense that for each $\phi \in \mathcal{S}(\mathbb{R})$

$$\langle f(\varepsilon x), \phi(x)\rangle = c(\varepsilon)\int_{-\infty}^{\infty} \phi(x)\mathrm{d}x + O(1) \ ,$$

as $\varepsilon \to 0^+$. Fourier transforming the last relation, we have that

$$\hat{f}(\lambda x) = 2\pi c\left(\lambda^{-1}\right)\frac{\delta(x)}{\lambda} + O\left(\frac{1}{\lambda}\right) \ ,$$

as $\lambda \to \infty$ in $\mathcal{S}'(\mathbb{R})$. Evaluating at e^{-x}, we obtain, as $y \to 0^+$,

$$O(1) = F(iy) = \frac{1}{2\pi}\left\langle \hat{f}(t), e^{-yt}\right\rangle = \frac{1}{2\pi y}\left\langle \hat{f}\left(y^{-1}t\right), e^{-t}\right\rangle = c(y) + O(1) \ .$$

Hence, c is bounded near the origin, and thus $f(\varepsilon x) = c(\varepsilon) + O(1) = O(1)$ as $\varepsilon \to 0^+$ in $\mathcal{S}'(\mathbb{R})$, as required. $\square$

So, we obtain the following tauberian theorem in terms of the Laplace transform.

Theorem 9.23. *Let $G \in \mathcal{D}'(\mathbb{R})$ have support bounded at the left. Necessary and sufficient conditions for*

$$\lim_{\lambda\to\infty} G(\lambda x) = \gamma \quad \textit{in } \mathcal{D}'(\mathbb{R}) \ , \tag{9.3.1}$$

are

$$\lim_{y\to 0^+} y\mathcal{L}\left\{G; y\right\} = \lim_{y\to 0^+} \mathcal{L}\left\{G'; y\right\} = \gamma, \tag{9.3.2}$$

and

$$\lambda x G'(\lambda x) = O(1) \quad \textit{as } \lambda \to \infty \textit{ in } \mathcal{D}'(\mathbb{R}). \tag{9.3.3}$$

Proof. Either (9.3.1) or (9.3.3) imply that G is a tempered distribution and hence its Laplace transform is well defined for $\Re e \ z > 0$. The necessity is clear. Now, the condition (9.3.3) translates into $f'(\varepsilon x) = O(\varepsilon^{-1})$ in $\mathcal{S}'(\mathbb{R})$, where $\hat{f} = 2\pi G'$. Relation (9.3.15) gives $F(iy) = \mathcal{L}\left\{G'; y\right\} \to \gamma$ as $y \to 0^+$, for the Fourier-Laplace representation of f, hence by Theorem 9.22, $f(0) = \gamma$ in $\mathcal{S}'(\mathbb{R})$. Hence, taking Fourier inverse transform, we conclude that $G'(\lambda x) \sim \lambda^{-1}\gamma\delta(x)$ as $\lambda \to \infty$ in $\mathcal{S}'(\mathbb{R})$, which implies (9.3.1). $\square$

9.3.2 Tauberians for Abel Limitability

Let us define Abel limitability for distributions. Recall that (Section 1.5) $g \in \mathcal{D}'(\mathbb{R})$, it is called Laplace transformable [180] on the strip $a < \Re e\, z < b$ if $e^{-\xi t} g(t)$ is a tempered distribution for $a < \xi < b$; in such a case its Laplace transform is well defined on that strip.

Definition 9.24. *Let $f \in \mathcal{D}'(\mathbb{R})$. We say that f has a limit γ at infinity in the Abel sense, and write*

$$\lim_{x\to\infty} f(x) = \gamma \quad (\mathrm{A}) \ , \tag{9.3.4}$$

if there exists a distribution f_+ with support bounded at the left such that f_+ coincides with f on an open interval (a, ∞), f_+ is Laplace transformable for $\Re e\, z > 0$, and

$$\lim_{y\to 0^+} y\mathcal{L}\{f_+; y\} = \gamma \ . \tag{9.3.5}$$

Notice that Definition 9.24 is independent on the choice of f_+, because every compactly supported distribution satisfies (9.3.5) with $\gamma = 0$. The case of locally integrable functions is of interest, it is analyzed in the next example.

Example 9.25. *If $f \in L^1_{\mathrm{loc}}[0, \infty)$ is such that the improper integral*

$$\mathcal{L}\{f; y\} = \int_0^\infty f(t) e^{-ty} \mathrm{d}t, \quad \textit{converges for each } y > 0 \ , \tag{9.3.6}$$

and

$$\lim_{y\to 0^+} y\mathcal{L}\{f; y\} = \gamma \ , \tag{9.3.7}$$

then f has γ as an Abel limit in the sense of Definition 9.24. However, the Abel limit of f, in the sense of Definition 9.24, exists under weaker assumptions, namely under the existence of the Laplace transform as integrals in the Cesàro sense, i.e.,

$$\mathcal{L}\{f; y\} = \int_0^\infty f(t) e^{-ty} \mathrm{d}t \quad (\mathrm{C}) \ , \quad \textit{exists for each } y > 0 \ , \tag{9.3.8}$$

and (9.3.7). Conversely, the reader can verify that the existence of the Abel limit, interpreted as in Definition 9.24, of a locally integrable function is equivalent to (9.3.8) and (9.3.7).

Observe that (9.3.5) is precisely (9.3.2). Therefore, using the well known equivalence between Cesàro behavior and parametric (quasiasymptotic) behavior (Proposition 1.13), we may reformulate Theorem 9.23

Corollary 9.26. *Let $f \in \mathcal{D}'(\mathbb{R})$. Necessary and sufficient conditions for*

$$\lim_{\lambda\to\infty} f(x) = \gamma \quad (\mathrm{C}) \tag{9.3.9}$$

are

$$\lim_{x\to\infty} f(x) = \gamma \quad (\mathrm{A}) \quad \textit{and} \quad xf'(x) = O(1) \quad (\mathrm{C})\,, \ \textit{as } x \to \infty\,. \tag{9.3.10}$$

We now combine Corollary 9.26 with the results from Section 9.2.3 to obtain more precise information about the Cesàro order in (9.3.9).

Theorem 9.27. *Let $f \in \mathcal{D}'(\mathbb{R})$. Suppose that* $\lim_{x\to\infty} f(x) = \gamma$ (A)*. The tauberian condition $xf'(x) = O(1)$ $(\mathrm{C}, \beta_1 + 1)$, as $x \to \infty$, implies* $\lim_{x\to\infty} f(x) = \gamma$ (C, β)*, for all $\beta > \beta_1$.*

Proof. It follows directly form Corollary 9.26 and Theorem 9.16. □

We can also consider a one-sided tauberian condition.

Theorem 9.28. *Let $f \in \mathcal{D}'(\mathbb{R})$. Suppose that* $\lim_{x\to\infty} f(x) = \gamma$ (A)*. Let $\beta_1 \geq 1$. The one-sided tauberian condition $xf'(x) = O_L(1)$ (C, β_1), as $x \to \infty$, implies* $\lim_{x\to\infty} f(x) = \gamma$ (C, β)*, for all $\beta \geq \beta_1$.*

Proof. We may assume that f has support bounded at the left. If $xf'(x) = O(1)$ (C) is established, we could apply first Theorem 9.17, and then Corollary 9.26 to obtain the desired conclusion. Because of Lemma 9.13, $xf'(x) = O(1)$ (C) is satisfied if $f(x) = O(1)$ (C); so, let us show the two-sided boundedness of $f(x)$. By adding a term of the form $KH(x)$ and a compactly supported distribution to $f(x)$, we may assume that there exists $k \in \mathbb{N}$ such that $g^{(-k)} \in L^{\infty}_{\mathrm{loc}}(\mathbb{R})$, and $g^{(-k)}(x) \geq 0$, for x large enough, where $g(x) = xf'(x)$. Actually k can be chosen so that $f^{(-k)} \in L^{\infty}_{\mathrm{loc}}(\mathbb{R})$. Furthermore, by adding a suitable compactly supported bounded function we can additionally assume that $g^{(-k)}(x) \geq 0$, for all x, and both $g^{(-k)}$ and $f^{(-k)}$ vanish in a neighborhood of the origin. Using Lemma 9.13, we have that

$$\frac{g^{(-k)}(x)}{x^{k+1}} = \left(\frac{f^{(-k)}(x)}{x^k}\right)' \ ;$$

therefore, $f^{(-k)}$ is a non-negative function. Finally, using the non-negativity of $f^{(-k)}$, we have that

$$\begin{aligned} f^{(-k-1)}(x) &= \int_0^x f^{(-k)}(t)\mathrm{d}t \leq e\int_0^x f^{(-k)}(t)e^{-\frac{t}{x}}\mathrm{d}t \\ &= ex^k \mathcal{L}\left\{f; \frac{1}{x}\right\} = e\gamma x^{k+1} + o(x^{k+1}) = O(x^{k+1}) \ , \end{aligned}$$

hence $f(x) = O(1)$ $(\mathrm{C}, k+1)$. □

We obtain from Theorem 9.28 an extension of a classical important result of O. Szász [200, Thm.1].

Theorem 9.29. *Let $f \in L^1_{\mathrm{loc}}[0,\infty)$. Suppose that $\lim_{x\to\infty} f(x) = \gamma$ (A) in the sense that it satisfies (9.3.8) and (9.3.7). Then, the one-sided tauberian condition*

$$xf(x) - \int_0^x f(t)\mathrm{d}t \geq -Kx, \quad x > a \ , \tag{9.3.11}$$

for some positive constants K and a, implies that

$$f^{(-1)}(x) = \int_0^x f(t)\mathrm{d}t \sim \gamma x, \quad x \to \infty \ . \tag{9.3.12}$$

Proof. Note that (9.3.11) exactly means that $xf'(x) = O_L(1)$ $(\mathrm{C}, 1)$ and (9.3.12) that $\lim_{x\to\infty} f(x) = \gamma$ $(\mathrm{C}, 1)$. So, Theorem 9.28 yields (9.3.12). □

In particular.

Corollary 9.30. *(Szász [200, Thm.1]) Let* $f \in L^1_{\mathrm{loc}}[0,\infty)$ *satisfy (9.3.6) and (9.3.7). The one-sided tauberian condition (9.3.11) implies (9.3.12).*

Remark 9.31. *If* $\beta \geq 0$*, we might replace (9.3.11) in Theorem 9.29 and Corollary 9.30 by*

$$xf(x) - \int_0^x f(t)\mathrm{d}t = O_L(x) \quad (\mathrm{C},\beta), \quad \text{as } x \to \infty \ ,$$

then same arguments apply to conclude $\lim_{x\to\infty} f(x) = \gamma\ (\mathrm{C},\beta+1)$.

We can use Theorem 9.18 to obtain a Tauber type characterization of (C,β) limits; the next result follows easily from Corollary 9.26 and Theorem 9.18.

Theorem 9.32. *Let* $f \in \mathcal{D}'(\mathbb{R})$ *and* $\beta \geq 0$*. Necessary and sufficient conditions for* $\lim_{x\to\infty} f(x) = \gamma\ (\mathrm{C},\beta)$ *are* $\lim_{x\to\infty} f(x) = \gamma\ (\mathrm{A})$ *and* $xf'(x) = o(1)\ (\mathrm{C},\beta+1)$.

9.3.3 Tauberians for Abel Summability of Distributions

Let $g \in \mathcal{D}'(\mathbb{R})$ with support bounded at the left and $\phi \in \mathcal{E}(\mathbb{R})$. We defined in Chapter 3 Abel summability of distributional evaluation as follows:

$$\langle g(x), \phi(x)\rangle = \gamma \quad (\mathrm{A})\ . \tag{9.3.13}$$

if $e^{-yx}\phi(x)g(x) \in \mathcal{S}'(\mathbb{R})$, for every $y > 0$, and

$$\lim_{y\to 0^+} \left\langle \phi(t)g(t), e^{-yt}\right\rangle = \gamma\ . \tag{9.3.14}$$

Notice that (9.3.13) holds if and only if $\lim_{x\to\infty} G(x) = \gamma$ (A), where G is the first order primitive of ϕg with support bounded at the left, that is, $G = (\phi g) * H$ (here H is the Heaviside function). So, our theorems from Section 9.3.2 give at once the following results.

Theorem 9.33. *Let* $g \in \mathcal{D}'(\mathbb{R})$ *with support bounded at the left and* $\phi \in \mathcal{E}(\mathbb{R})$*. Suppose that*

$$\langle g(x), \phi(x)\rangle = \gamma \quad (\mathrm{A})\ . \tag{9.3.15}$$

The tauberian condition $xg(x)\phi(x) = O_L(1)$ $(\mathrm{C}, \beta_1 + 1)$*, as* $x \to \infty$*, for* $\beta_1 \geq 0$*, implies*

$$\langle g(x), \phi(x) \rangle = \gamma \quad (\mathrm{C}, \beta) \ , \tag{9.3.16}$$

for all $\beta \geq \beta_1 + 1$*. While the stronger tauberian condition* $xg(x)\phi(x) = O(1)$ $(\mathrm{C}, \beta_1 + 1)$ *implies that (9.3.16) holds for all* $\beta > \beta_1$*.*

Theorem 9.34. *Let* $g \in \mathcal{D}'(\mathbb{R})$ *with support bounded at the left and* $\phi \in \mathcal{E}(\mathbb{R})$*. Necessary and sufficient conditions for (9.3.16) are* $\langle g(x), \phi(x) \rangle = \gamma$ (A) *and* $xg(x)\phi(x) = o(1)$ $(\mathrm{C}, \beta + 1)$ *as* $x \to \infty$*.*

The case when $g = \hat{f}$ and $\phi(x) = e^{ix_0x}$ is interesting, it provides the order of summability in the pointwise Fourier inversion formula for Łojasewicz point values (Chapter 3). This is the content of the next corollary.

Corollary 9.35. *Let* $f \in \mathcal{S}'(\mathbb{R})$ *be such that* $\operatorname{supp} \hat{f}$ *is bounded at the left and*

$$\frac{1}{2\pi} \left\langle \hat{f}(x), e^{ix_0x} \right\rangle = \gamma \quad (\mathrm{A}) \ . \tag{9.3.17}$$

Then $xe^{ix_0x}\hat{f}(x) = O_L(1)$ $(\mathrm{C}, \beta_1 + 1)$*, for some* $\beta_1 \geq 0$*, implies that* $f(x_0) = \gamma$*, distributionally. Moreover, the pointwise Fourier inversion formula holds* (C, β) *for any* $\beta \geq \beta_1 + 1$*, that is*

$$\frac{1}{2\pi} \left\langle \hat{f}(x), e^{ix_0x} \right\rangle = \gamma \quad (\mathrm{C}, \beta) \ . \tag{9.3.18}$$

Moreover, the stronger tauberian condition $xe^{ix_0x}\hat{f}(x) = O(1)$ $(\mathrm{C}, \beta_1 + 1)$ *implies that (9.3.18) holds for all* $\beta > \beta_1$*.*

9.3.4 Tauberians for Series and Stieltjes Integrals

The cases of Stieltjes integrals and series is also of importance. We obtain directly from Theorem 9.33 the following corollary.

Corollary 9.36. *Let s be a function of local bounded variation such that $s(x)=0$ for $x\leq 0$. Suppose that the improper integral*

$$\mathcal{L}\left\{\mathrm{d}s;y\right\}=\int_{0}^{\infty}e^{-yx}\mathrm{d}s(x)\quad(\mathrm{C})\ ,\quad \textit{exists for each } y>0\ , \tag{9.3.19}$$

and that

$$\lim_{y\to 0^{+}}\mathcal{L}\left\{\mathrm{d}s;y\right\}=\gamma. \tag{9.3.20}$$

Let $\beta_1\geq 0$. Then, the tauberian condition

$$\int_{0}^{x}t\mathrm{d}s(t)=O_{L}(x)\quad(\mathrm{C},\beta_1)\ , \tag{9.3.21}$$

implies that for all $\beta\geq\beta_1+1$

$$\lim_{x\to\infty}s(x)=\gamma\quad(\mathrm{C},\beta)\ . \tag{9.3.22}$$

Moreover, if we replace $O_L(x)$ by $O(x)$ in (9.3.21), we conclude that (9.3.22) holds for all $\beta>\beta_1$.

Observe that in particular Corollary 9.36 holds if we replace (9.3.19) by the stronger assumption of the existence of the improper integrals $\int_0^\infty e^{-yx}\mathrm{d}s(x)=\lim_{t\to\infty}\int_0^t e^{-yx}\mathrm{d}s(x)$, for each $y>0$.

Let $\lambda_n\nearrow\infty$ be an increasing sequence of non-negative real numbers. Recall that we write $\sum_{n=0}^{\infty}c_n=\gamma$ $(\mathrm{A},\{\lambda_n\})$ if the Dirichlet series $F(z)=\sum_{n=0}^{\infty}c_n e^{-z\lambda_n}=\gamma$ converges on $\Re e z>0$ and $\lim_{y\to 0^+}F(y)=\gamma$.

Corollary 9.37. *Suppose that $\sum_{n=0}^{\infty}a_n=\gamma$ $(\mathrm{A},\{\lambda_n\})$. The tauberian condition $\sum_{\lambda_n<x}c_n=O_L(x)$ (C,β_1), for some $\beta_1\geq 0$, implies that $\sum_{n=0}^{\infty}c_n=\gamma$ $(\mathrm{R},\{\lambda_n\},\beta)$, for all $\beta\geq\beta_1+1$. The stronger tauberian condition $\sum_{\lambda_n<x}c_n=O(x)$ (C,β_1) implies the $(\mathrm{R},\{\lambda_n\},\beta)$ summability of the series to γ, for all $\beta>\beta_1$.*

Furthermore, we can formulate a much stronger version of Corollary 9.37.

Corollary 9.38. *Suppose that*

$$F(y) = \sum_{n=0}^{\infty} c_n e^{-y\lambda_n} \quad (\mathrm{R}, \{\lambda_n\}) \ , \quad \textit{exists for each } y > 0 \ , \tag{9.3.23}$$

and

$$\lim_{y \to 0^+} F(y) = \gamma \ . \tag{9.3.24}$$

The tauberian condition $\sum_{\lambda_n < x} c_n = O_L(x)$ (C, β_1)*, for some* $\beta_1 \geq 0$*, implies that* $\sum_{n=0}^{\infty} c_n = \gamma$ $(\mathrm{R}, \{\lambda_n\}, \beta)$*, for all* $\beta \geq \beta_1 + 1$*. The stronger tauberian condition* $\sum_{\lambda_n < x} c_n = O(x)$ (C, β_1) *implies the* $(\mathrm{R}, \{\lambda_n\}, \beta)$ *summability of the series to* γ*, for all* $\beta > \beta_1$*.*

We now obtain a general form of Theorem 9.3 stated at the introduction; it is a particular case of Corollary 9.37.

Corollary 9.39. *Suppose that* $\sum_{n=0}^{\infty} c_n = \gamma$ (A)*. The one-sided tauberian condition* $\sum_{n=0}^{N} c_n = O_L(N)$ (C, β_1)*, for some* $\beta_1 \geq 0$*, implies that* $\sum_{n=0}^{\infty} c_n = \gamma$ (C, β)*, for all* $\beta \geq \beta_1 + 1$*. The stronger tauberian condition* $\sum_{n=0}^{N} c_n = O(N)$ (C, β) *implies the* (C, β) *summability of the series to* γ*, for all* $\beta > \beta_1$*.*

If we specialize Corollary 9.38 to power series, we have.

Corollary 9.40. *Suppose that*

$$F(r) = \sum_{n=0}^{\infty} c_n r^n \quad (\mathrm{C}) \ , \quad \textit{exists for each } 0 \leq r < 1 \ , \tag{9.3.25}$$

and

$$\lim_{r \to 1^-} F(r) = \gamma \ . \tag{9.3.26}$$

The tauberian condition $\sum_{n=0}^{N} c_n = O_L(N)$ (C, β_1)*, for some* $\beta_1 \geq 0$*, implies that* $\sum_{n=0}^{\infty} c_n = \gamma$ (C, β)*, for all* $\beta \geq \beta_1 + 1$*. The stronger tauberian condition* $\sum_{n=0}^{N} c_n = O(N)$ (C, β) *implies the* (C, β) *summability of the series to* γ*, for all* $\beta > \beta_1$*.*

9.4 Applications: Tauberian Conditions for Convergence

This section is devoted to applications of the distributional method in classical tauberians for series and Dirichlet series. Let $f \in \mathcal{D}'(\mathbb{R})$ have support bounded at the left, we have that $\lim_{x\to\infty} f(x) = \gamma$ (C) if and only if its derivative has the following quasiasymptotic behavior

$$f'(\lambda x) = \gamma \frac{\delta(x)}{\lambda} + o\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{D}'(\mathbb{R}) \ . \tag{9.4.1}$$

Let $1 < \sigma < 2$. Throughout this section $\phi_\sigma \in \mathcal{D}'(\mathbb{R})$ is a fixed test function with the following properties: $0 \leq \phi_\sigma \leq 1$, $\phi_\sigma(x) = 1$ for $x \in [0,1]$, and $\operatorname{supp} \phi_\sigma \subseteq [-1, \sigma]$,. We first extend a Theorem of O. Szász [201] (see also [171]) from series to Stieltjes integrals.

Theorem 9.41. *Let s be a function of local bounded variation such that $s(x) = 0$ for $x \leq 0$. Suppose that* $\lim_{x\to\infty} s(x) = \gamma$ (A)*. Then, the tauberian conditions*

$$\int_0^x t \mathrm{d}s(t) = O_L(x) \quad (\mathrm{C}, \beta) \ , \tag{9.4.2}$$

for some $\beta \geq 0$, and

$$\lim_{\sigma\to 1^+} \limsup_{x\to\infty} \frac{1}{x} \int_x^{\sigma x} t \left|\mathrm{d}s\right|(t) = 0 \ , \tag{9.4.3}$$

imply that $\lim_{x\to\infty} s(x) = \gamma$.

Proof. Theorem 9.28 and (9.4.2) imply that $\lim_{x\to\infty} s(x) = \gamma$ (C). Then s' has the quasiasymptotic behavior (9.4.1), evaluating the quasiasymptotic at ϕ_σ, we obtain

$$\begin{aligned} \limsup_{\lambda\to\infty} |s(\lambda) - \gamma| &\leq \limsup_{\lambda\to\infty} \int_\lambda^{\sigma\lambda} \phi_\sigma\left(\frac{t}{\lambda}\right) \left|\mathrm{d}s\right|(t) \\ &\leq \limsup_{\lambda\to\infty} \frac{1}{\lambda} \int_\lambda^{\sigma\lambda} t \left|\mathrm{d}s\right|(t) \end{aligned}$$

Since σ is arbitrary, we obtain the convergence from (9.4.3). $\square$

We recover the result of Szász mentioned above.

Corollary 9.42. *(Szász, [201, Thm.1]). Suppose that* $\sum_{n=0}^{\infty} c_n = \gamma$ (A)*. The tauberian conditions*

$$V_N = \frac{1}{N}\sum_{n=0}^{N} n\left|c_n\right| = O(1) \ , \tag{9.4.4}$$

and

$$V_m - V_n \to 0, \quad \textit{as } \frac{m}{n} \to 1^+ \text{ and } n \to \infty \ , \tag{9.4.5}$$

imply the convergence of the series to γ*.*

Proof. We show that (9.4.5) implies (9.4.3). Indeed,

$$\begin{aligned} \frac{1}{x}\sum_{x<n\leq\sigma x} n\left|c_n\right| &= \frac{[\sigma x]-[x]}{x}V_{[\sigma x]} + \frac{[x]}{x}(V_{[\sigma x]} - V_{[x]}) \\ &< \frac{\sigma x - x - 1}{x}O(1) + (V_{[\sigma x]} - V_{[x]}) \ , \end{aligned}$$

and the last expression tends to 0 as $x \to \infty$ and $\sigma \to 1^+$. □

The next tauberian theorem for Dirichlet series belongs to Hardy and Littlewood [87] (see also [199] and [200, Thm.6]).

Theorem 9.43. *(Hardy-Littlewood). Suppose that* $\sum_{n=0}^{\infty} c_n = \gamma$ $(\mathrm{A}, \{\lambda_n\})$*. The tauberian condition*

$$\sum_{n=1}^{\infty}\left(\frac{\lambda_n}{\lambda_n - \lambda_{n-1}}\right)^{p-1}\left|c_n\right|^p < \infty \ , \tag{9.4.6}$$

where $1 \leq p < \infty$*, implies the convergence of the series to* γ*.*

Proof. The case $p = 1$ is trivial, we assume $1 < p < \infty$. Let $q = p/(p-1)$. Hölder's inequality implies (9.4.2), with $\beta = 0$, for $s(x) = \sum_{\lambda_n \leq x} c_n$. So, Corollary 9.37 implies the $(\mathrm{R}, \{\lambda_n\}, 1)$ summability. Then $\sum_{n=0}^{\infty} c_n\delta(x-\lambda_n)$ has the quasiasymptotic behavior (9.4.1), evaluating at ϕ_σ and using Hölder's inequality, we obtain

$$
\begin{aligned}
\limsup_{N\to\infty}\left|\sum_{n=0}^{N} c_n-\gamma\right| &\leq \limsup_{N\to\infty} \sum_{\lambda_N<\lambda_n\leq\sigma\lambda_N} \phi_\sigma\left(\frac{\lambda_n}{\lambda_N}\right)|c_n| \\
&\leq \limsup_{N\to\infty}\left(\sum_{\lambda_N<\lambda_n\leq\sigma\lambda_N}\frac{\lambda_n-\lambda_{n-1}}{\lambda_n}\left|\phi_\sigma\left(\frac{\lambda_n}{\lambda_N}\right)\right|^q\right)^{\frac{1}{q}} O(1) \\
&\leq \limsup_{N\to\infty}\left(\sum_{\lambda_N<\lambda_n\leq\sigma\lambda_N}\frac{\lambda_n-\lambda_{n-1}}{\lambda_n}\right)^{\frac{1}{q}} O(1) \\
&\leq (\sigma-1)^{\frac{1}{q}} O(1)\ ,
\end{aligned}
$$

taking $\sigma\to 1^+$, we obtain the result. $\square$

We end this section proving a theorem of Szász [198, 199, 200] (the case for power series was first discovered by Hardy and Littlewood).

Theorem 9.44. *(Szász, [200]). Suppose that* $\sum_{n=0}^{\infty} c_n=\gamma$ $(\mathrm{A},\{\lambda_n\})$. *The tauberian condition*

$$
\sum_{n=1}^{N}\lambda_n^p\left(\lambda_n-\lambda_{n-1}\right)^{1-p}|c_n|^p=O(\lambda_N)\ , \tag{9.4.7}
$$

for some $1<p<\infty$, *implies the convergence of the series to* γ.

Proof. Let $q=p/(p-1)$. Again, Hölder's inequality implies (9.4.2), with $\beta=0$, for $s(x)=\sum_{\lambda_n\leq x} c_n$. So, Corollary 9.37 implies that $\sum_{n=0}^{\infty} c_n\delta(x-\lambda_n)$ has the quasiasymptotic behavior (9.4.1), evaluating at ϕ_σ and using Hölder's inequality, we obtain

$$
\begin{aligned}
\limsup_{N\to\infty}\left|\sum_{n=0}^{N} c_n-\gamma\right| &\leq \limsup_{N\to\infty} \sum_{\lambda_N<\lambda_n\leq\sigma\lambda_N} \phi_\sigma\left(\frac{\lambda_n}{\lambda_N}\right)|c_n| \\
&\leq \left(\limsup_{N\to\infty}\lambda_N^{\frac{1}{p}}\left(\sum_{\lambda_N<\lambda_n\leq\sigma\lambda_N}\frac{\lambda_n-\lambda_{n-1}}{\lambda_n^q}\left|\phi_\sigma\left(\frac{\lambda_n}{\lambda_N}\right)\right|^q\right)^{\frac{1}{q}}\right) O(1) \\
&\leq \limsup_{N\to\infty}\left(\frac{1}{\lambda_N}\sum_{\lambda_N<\lambda_n\leq\sigma\lambda_N}(\lambda_n-\lambda_{n-1})\right)^{\frac{1}{q}} O(1) \\
&= (\sigma-1)^{\frac{1}{q}} O(1)\ .
\end{aligned}
$$

Since σ is arbitrary, we obtain the convergence. $\square$

Chapter 10
The Structure of Quasiasymptotics

10.1 Introduction

The quasiasymptotic behavior has been a fundamental concept throughout our investigations of local properties of distributions. It is a very convenient notion to describe the local behavior of a distribution around a point, or its asymptotic behavior at infinity. One gains generality by considering quasiasymptotics rather than ordinary asymptotics of functions because they are directly applicable to the nature of a distribution; moreover, one might say that every distribution shows, in one way or another, quasiasymptotic properties. Despite its generality, the concept is extremely useful in practice; in fact, it has an evident advantage over the asymptotics of ordinary function: its flexibility under analytical manipulations such as differentiation or integral transformations. So far, we have only considered some particular cases of the quasiasymptotic behavior, mainly in connection with distributional point values and jump behaviors, we now analyze general quasiasymptotic properties of distributions.

In this chapter we make a comprehensive study of quasiasymptotic properties of distributions in one variable. The exposition is based on a recent series of papers by the author [212, 213, 227], where some open structural problems were undertook and solved.

The concept of the quasiasymptotic behavior of distributions was introduced by B. I. Zavialov for tempered distributions in [249] as a result of his investigations in Quantum Field Theory, and further developed by him, Vladimirov and Drozhzhinov [231]. Later this concept was slightly reformulated in [151, 152] for distributions of one variable.

The quasiasymptotic behavior has found many applications in mathematics and mathematical physics. As previously mentioned, it was created as a response to theoretical questions in mathematical physics, where it has been effectively applied [231, 233, 234, 249]. Later on, it had its main developments within the study of integral transforms on spaces of distributions [61, 160, 231, 232]. It is remarkable the predominant role that tauberian and abelian type results have had in the theory [37, 38, 40, 41, 160, 231, 232]. The asymptotic notions for distributions are also very appropriate for the study of asymptotics of solutions to convolution equations, integral equations, and partial differential equations [41, 60, 231, 235]. It has also important connections with problems in Fourier analysis, as we have been seen in the previous chapters.

Since its introduction, the study of the structure of the quasiasymptotics has deserved a special place [43, 54, 128, 151, 152, 150, 153, 156, 160, 192, 216, 231]. S. Łojasiewicz introduced the value of a distribution at a point, and he provided the corresponding structural theorem for it (Section 3.2 above). V. S. Vladimirov, Yu. N. Drozhzhinov and B. I. Zavialov gave a complete structural theorem for quasiasymptotics at infinity of tempered distributions with support on cones. S. Pilipović gave partial structural theorems for one dimensional quasiasymptotics at the origin and infinity. However, a complete structural theorem for quasiasymptotics has remained as an open question for long time. The importance and necessity of a solution to such an open problem has been pointed out in several articles [43, 156, 192, 213, 216]. Experience has shown that the structure of quasiasymptotics plays a very important theoretical role in the application of the notion to other contexts, this makes the solution of the structural problem a critical issue in generalized asymptotic analysis.

The principal aims of this chapter are to provide a solution for this structural open question in the one dimensional case, and then discuss some of its consequences and generalizations. Our presentation is intended to be complete and self-contained. For the sake of coherence, we comment some well known results of preliminary character in Sections 10.2, 10.3, and 10.4; though our approach and proofs may differ from the original sources. Therein, we also make some bibliographical remarks which may be useful for the reader.

Section 10.5 is chief part of the chapter. We characterize there the quasiasymptotic behavior by means of structural theorems; the cases at finite points and infinity are both studied. Our exposition follows the lines of the author's articles [212, 227]. Our analysis is based on the properties of the parametric coefficients resulting after performing several integrations of the quasiasymptotic behavior, then we single out the asymptotic properties of such coefficients. The key points for our structural theorems will be then the notions of *asymptotically and associate asymptotically homogeneous functions* with respect to slowly varying functions, they are actually the parametric coefficients in the integration of quasiasymptotics; such classes of functions are suitable and natural extensions of those introduced in Section 3.4.1. Observe that the same sort of ideas have been previously applied in Section 3.4 in the context of summability of the Fourier transform (see also [47, 216]) and deeply depended in the analysis given in Section 3.4.1; however, the problem we are about to study is much more difficult and technical.

In Section 10.6 we study the structural properties of quasiasymptotic boundedness with respect to regularly varying functions. We follow the author's paper [213]. The technique of integration and asymptotic analysis of parametric coefficients is employed once again. The parametric coefficients of integration will fit into the concept of *asymptotically homogeneously bounded functions* with respect to slowly varying functions, which will be introduced in Section 10.6.1. Using asymptotically homogeneously bounded functions, we obtain the structural theorems for quasiasymptotic boundedness in Section 10.6.2.

Further quasiasymptotic properties of distributions are discussed in Section 10.7. We apply the structural theorems to study problems in what the author denominates *quasiasymptotic extension* problems. We study three of such problems.

We shall study in Section 10.7.1 the asymptotic properties of extensions to $\mathbb{R}$ of distributions initially defined on $\mathbb{R} \setminus \{0\}$ (or just $(0, \infty)$) and having a prescribed asymptotic behavior at either the origin or infinity; here we follow the approach from [212, 213, 228], and complement with some new results. Notice that the latter problem is important from a mathematical physics perspective, since it is of relevance to renormalization procedures; indeed, the problem of renormalization in quantum field theory is nothing but a problem of this nature [21, 125, 178, 233, 234, 249]. It also has much relevance to the study of singular integral equations on spaces of distributions [60].

In Section 10.7.2, we show that if a tempered distribution has quasiasymptotic behavior or is quasiasymptotic bounded at point in the space $\mathcal{D}'(\mathbb{R})$, then the same quasiasymptotic property is preserved in the space $\mathcal{S}'(\mathbb{R})$. Observe that we have made extensive use of this fact for distributional point values and jump behavior in the past chapters; in fact, this property was of vital importance because it allows one to apply Fourier transform and translate local properties of tempered distributions into asymptotics of the Fourier transform at infinity. A similar problem is studied in Section 10.7.3, but this time at infinity; we show that the quasiasymptotic behavior holds in smaller spaces than $\mathcal{S}'(\mathbb{R})$, namely on some spaces of the type $\mathcal{K}'_\beta(\mathbb{R})$, consequently, this fact provides conditions over test functions which allows one to evaluate them at quasiasymptotics, these test functions are in bigger spaces than $\mathcal{S}(\mathbb{R})$ and, as shown in Chapter 7 (Section 7.8), they can be related to partial differential equations. We also consider similar problems for quasiasymptotic boundedness with respect to regularly varying functions.

10.2 Comments on the Quasiasymptotic Behavior

We would like to make some comments about the definition of the quasiasymptotic behavior, other known facts, the problems to be considered in the rest of our discussion, and references to the literature.

In Section 1.8.1, Definitions 1.3 and 1.5, we defined the quasiasymptotic behavior for $f \in \mathcal{D}'(\mathbb{R})$ as an asymptotic relation of the form

$$f(hx) \sim \rho(h) g(x) \quad \text{as } h \to 0^{+}\ ,\ \text{or } h \to \infty\ , \tag{10.2.1}$$

in the distributional sense, that is, holding after evaluation at each test function

$$\langle f(hx), \phi(x) \rangle \sim \rho(h) \langle g(x), \phi(x) \rangle\ , \quad \text{for each } \phi \in \mathcal{D}(\mathbb{R})\ . \tag{10.2.2}$$

Our assumption is that that ρ is defined, positive and measurable near 0 (resp. ∞). It follows from the definition itself that if g is assumed to be non-zero, then ρ and g in (10.2.1) cannot have an arbitrary form [61, 160, 231]; indeed, ρ must be a regularly varying function (Section 1.7) and g must be a homogeneous distribution [61] having degree of homogeneity equal to the index of regular variation of ρ. We will reproduce the proof of this fact. It should be mentioned that a more general result can be found in [58] (the so-called asymptotic separation of variables).

Lemma 10.1. *Suppose that ρ is a function defined, positive and measurable near 0 (resp. ∞). If (10.2.2) holds at the origin (resp. at ∞) and $g \neq 0$, then ρ is a regularly varying function at the origin (resp. at ∞) and g is a homogeneous* *distribution having degree of homogeneity equal to the index of regular variation of ρ.*

Proof. We show the assertion at the origin, the one at infinity is completely analogous. Select ϕ such that $\langle g(x), \phi(x) \rangle = 1$, then, for each $a > 0$,

$$\lim_{\varepsilon \to 0^+} \frac{\rho(a\varepsilon)}{\rho(\varepsilon)} = \lim_{\varepsilon \to 0^+} \frac{\langle f(a\varepsilon x), \phi(x) \rangle}{\rho(\varepsilon)} = \lim_{\varepsilon \to 0^+} \frac{\frac{1}{a} \left\langle f(\varepsilon x), \phi\left(\frac{x}{a}\right) \right\rangle}{\rho(\varepsilon)} = \frac{1}{a} \left\langle g(x), \phi\left(\frac{x}{a}\right) \right\rangle ,$$

denoting the last term of the equation by $\tau(a)$, the continuity of the dilation gives that $\tau(a)$ does not vanish in a neighborhood of $a = 1$, which in particular has non-zero Lebesgue measure. It implies (Section 1.7) that $\tau(a) = a^\alpha$, for some $\alpha > 0$ and each $a > 0$. Therefore, ρ is regularly varying with index α. On the other hand,

$$\langle g(ax), \varphi(x) \rangle = \lim_{\varepsilon \to 0^+} \frac{\rho(a\varepsilon)}{\rho(\varepsilon)} \frac{\langle f(a\varepsilon x), \varphi(x) \rangle}{\rho(a\varepsilon)} = a^\alpha \langle g(x), \varphi(x) \rangle .$$

$\square$

Obviously, Lemma 10.1 holds if one considers the quasiasymptotic behavior in $\mathcal{A}'$, the dual of a space of functions in which the dilation is a continuous operation onto itself.

Since any regularly varying function ρ can be written as $\rho(h) = h^\alpha L(h)$, where L is a slowly varying function, we may only talk about slowly varying functions in the rest of our discussion. In order to introduce some language, we reformulate our definitions in terms of slowly varying functions.

Definition 10.2. *An distribution $f \in \mathcal{D}'(\mathbb{R})$ is said to have quasiasymptotic behavior of degree α at $x = x_0$ with respect to the slowly varying function L if there exists $g \in \mathcal{D}'(\mathbb{R})$ such that*

$$f(x_0 + \varepsilon x) \sim \varepsilon^\alpha L(\varepsilon) g(x) \quad \textit{as } \varepsilon \to 0^+ \textit{ in } \mathcal{D}'(\mathbb{R}) . \tag{10.2.3}$$

Definition 10.3. *An distribution $f \in \mathcal{D}'(\mathbb{R})$ has quasiasymptotic behavior of degree α at infinity in $\mathcal{D}'(\mathbb{R})$ with respect to the slowly varying function L if there exists $g \in \mathcal{D}'(\mathbb{R})$ such that*

$$f(\lambda x) \sim \lambda^{\alpha} L(\lambda) g(x) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{D}'(\mathbb{R}) \, . \tag{10.2.4}$$

In the sense of (10.2.1), relation (10.2.3), resp. (10.2.4), is the most general asymptotic behavior that a distribution can have at small scale, resp. large scale.

The same considerations discussed in Section 1.8.1 apply for Definitions 10.2 and 10.3, that is, Definition 10.2 is of local character and hence it is meaningful when f is just defined in a neighborhood of $x = x_0$, by shifting in most cases is enough to consider $x_0 = 0$ in (10.2.3), and the quasiasymptotic (10.2.4) is not a local property when $\alpha \leq -1$. We may talk about Definitions 10.2 and 10.3 in other spaces of generalized functions constructed as dual spaces of suitable spaces of functions. Finally, the notation (1.8.7) will also be widely employed in the sequel.

Our aim is now to characterize the structure of the quasiasymptotic behavior, that is, to describe it by asymptotics, in the ordinary sense, of primitives of the distribution. This will be done in Section 10.5.

We now want to make some comments about the previous known properties of the structure of the quasiasymptotics available in the literature, this is valuable for the reader since many important techniques can be found in the references.

We start with quasiasymptotics at infinity. The complete structural theorems for distributions from $\mathcal{S}'[0, \infty)$ can be found in [231]. Such results will be reproduce in Section 10.4 below. In addition, in page 134 of the cited book, one finds a decomposition theorem, which basically implies the structural theorem when the degree of the quasiasymptotic behavior is not a negative integer and with no restrictions on the support of the distribution. The details about how this is implied by the decomposition theorem can be found in [151]. Therefore, in the case at infinity the only unknown structural theorem was for negative integral degrees. The results that we studied in Section 3.4.2 are a particular case of such an open question, they were obtained by the author and R. Estrada in [216]. The general case was recently obtained by the author in [212] and will be discussed in Section 10.5 below.

In the case at the origin, only partial results were known under restrictions on the degree of the quasiasymptotic behavior ($\alpha > -1$) and boundedness of L [153]. The reader can also consult [156, 160, 192] for more about these structural results. The general case was recently obtained in [227] by the author and S. Pilipović, it will be also the subject of Sections 10.5.

10.3 Remarks on Slowly Varying Functions: Estimates and Integrals

In this section we collect some results about slowly varying functions to be used in the future. Some facts were already discussed in Section 1.7, but the subsequent work demands us more detailed information about slowly varying functions; we are particularly interested in some reductions and estimates that will be crucial for some future arguments.

10.3.1 Estimates and Reductions

Let us assume that L is a slowly varying function at the origin. Similar considerations are applicable for slowly varying functions at infinity. Our first obvious observation is that for the quasiasymptotic behavior only the behavior of L near 0 plays a role in (10.2.3), and so we may impose to L any behavior we want in intervals of the form $[A, \infty)$. Moreover, if $\tilde{L}$ is any measurable function which satisfies

$$\lim_{x \to 0^+} \frac{\tilde{L}(x)}{L(x)} = 1 \ ,$$

we may replace L by $\tilde{L}$ in any statement about quasiasymptotics without loosing generality in the original statement. Recall the representation formula for slowly varying functions (Section 1.7): L is slowly varying at the origin if and only if there exist measurable functions u and w defined on some interval $(0,B]$, u being bounded and having a finite limit at 0 and w being continuous on $[0,B]$ with $w(0)=0$, such that

$$L(x)=\exp\left(u(x)+\int_x^B \frac{w(t)}{t}\mathrm{d}t\right), \quad x\in(0,B]\ .$$

Since we are looking for suitable modifications of L, our first remark is that, when dealing with quasiasymptotics, we can always assume that L is defined in the whole $(0,\infty)$ and L is everywhere non-negative (or even positive). This is shown by extending u and w to $(0,\infty)$ in any way we want.

A direct consequence of the representation formula is the following useful bound. Given any fixed $\sigma>0$ and $M>1$, there exists $0<\tilde{B}\leq B$ such that

$$\frac{1}{M}\min\left\{x^{-\sigma},x^{\sigma}\right\}<\frac{L(\varepsilon x)}{L(\varepsilon)}<M\max\left\{x^{-\sigma},x^{\sigma}\right\}\ , \tag{10.3.1}$$

for $\varepsilon x<\tilde{B}$ and $\varepsilon<\tilde{B}$. Furthermore, given any $A>0$, there exists $\tilde{A}$ such that (10.3.1) holds if $x<A$ and $\varepsilon<\tilde{A}$; for instance, take $\tilde{A}=\min\left\{\tilde{B},(\tilde{B}/A)\right\}$. This result is known as Potter's theorem [15, p.25], and will be of vital importance in our investigations of the structural properties for quasiasymptotics. Potter's estimate (10.3.1) also holds for slowly varying functions at infinity, with the obvious modifications in the parameters.

Sometimes is useful to modify L away the origin (resp. infinity) such that (10.3.1) holds globally in the following sense. Given a fixed $\sigma>0$, then by modifying u and w, we can assume, when it is convenient, that $B=1$, u is bounded on all over $(0,\infty)$ and $|w(x)|<\sigma$, $x\in(0,\infty)$. In particular, we obtain the estimate (10.3.1) $\forall x,\varepsilon\in(0,\infty)$.

As an application of the ideas just discussed, we give a proof a result which will have some importance in the future. An alternative constructive proof can be found in Seneta's book [183], based on the nice construction of Adamović [1, 2].

Lemma 10.4. *Let L be slowly varying at the origin (resp. at infinity). There exists another slowly varying function $\tilde{L} \in C^{\infty}(0,\infty)$ at the origin (resp. at infinity) such that*

$$L(x) \sim \tilde{L}(x) \ , \quad \textit{and} \ \ \tilde{L}^{(n)}(x) = o\left(\frac{L(x)}{x^n}\right) \ ,$$

for each $n \in \mathbb{N}$. Moreover, $\tilde{L}$ can be chosen so that it vanishes in a neighborhood of infinity (resp. the origin).

Proof. Observe that only the behavior of L near the origin (resp. infinity) plays a role in the statement, so we can modify it on irrelevant intervals such that it satisfies (10.3.1) for all $x, \varepsilon \in (0,\infty)$. We can also assume that $L(x) = 0$ for $x > 1$ (resp. $x < 1$). Under our assumptions, we can use Lebesgue's dominated convergence theorem in

$$\int_0^{\infty} \left(\frac{L\left(\varepsilon x\right)}{L(\varepsilon)} - 1\right) \phi(x) \mathrm{d}x \ ,$$

for $\phi \in \mathcal{D}(\mathbb{R})$, to deduce that (resp. the same relation with $\varepsilon = \lambda \to \infty$)

$$L(\varepsilon x) H(x) = L(\varepsilon) H(x) + o\left(L(\varepsilon)\right) \qquad \text{as } \varepsilon \to 0^{+} \text{ in } \mathcal{D}'(\mathbb{R}) \ .$$

Take now $\tilde{L}(x) = \int_0^{\infty} L(xt)\phi(t)\mathrm{d}t$, where $\phi \in \mathcal{D}(0,\infty)$ with $\int_0^{\infty} \phi(t)\mathrm{d}t = 1$; it satisfies all the requirements. □

We may also impose more conditions on w to obtain more reasonable assumptions on L. For example, in the case of slowly varying functions at the origin the assumption $t^{-1}w(t) \in L^1[1,\infty)$ implies

$$\tilde{M} < L(x) < M, \ x > 1,$$

for some positive constants $\tilde{M}$ and M.

We finally recall a well known fact [15, 183]: As soon as $L(ax) \sim L(x)$ holds for each $a > 0$ on a set of positive measure, it automatically holds uniformly for a in compact subsets of $(0, \infty)$.

10.3.2 Asymptotics of Some Integrals

We now want to discuss the quasiasymptotic behavior in relation to ordinary asymptotics of functions, such results are very well known [160, 231], but we include them here for the sake of completeness. The next lemma is due to Aljančić, Bojanić, and Tomić [3] (see also [183, Section 2.3]).

Lemma 10.5. *Let L be slowly varying at at infinity defined on $I = (A, \infty)$, $A > 0$ (resp. the origin and $I = (0, A)$). If $x^{\sigma} g \in L^1(I)$ (resp. $x^{-\sigma} g$), for some $\sigma > 0$, then*

$$\int_I L(\lambda x) g(x) \mathrm{d}x \sim L(\lambda) \int_I g(x) \mathrm{d}x \ , \quad \lambda \to \infty \ \ (\textit{resp. } \lambda \to 0^+) \ . \tag{10.3.2}$$

Additionally, if $L, g \in L^1_{\mathrm{loc}}(0, \infty)$ and $g(x) = O(x^{\alpha})$, $x \to 0^+$, for some $\alpha > -1$, (resp. $\alpha < -1$, $x \to \infty$), then

$$\int_{\lambda^{-1} I} L(\lambda x) g(x) \mathrm{d}x \sim L(\lambda) \int_I g(x) \mathrm{d}x \ , \quad \lambda \to \infty \ \ (\textit{resp. } \lambda \to 0^+) \ . \tag{10.3.3}$$

Proof. We only give the proof of the assertion at infinity, the one at the origin follows from the change of variables $x \leftrightarrow x^{-1}$. Find $M, B > 0$ such that if $x > A$ and $\lambda > B$, then

$$\frac{L(\lambda x)}{L(\lambda)} |g(x)| \leq M x^{\sigma} |g(x)| \in L^1(A, \infty) \ . \tag{10.3.4}$$

Therefore, (10.3.2) follows from the Lebesgue dominated convergence theorem if $I = (A, \infty)$.

For the case $\lambda^{-1}I = ((A/\lambda), \infty)$, we may assume that $-1 < \alpha < 0$ and $\sigma < \alpha+1$, then there exist $M_1, B_1 > 0$ and $A_1 > A$ such that

$$\frac{L(\lambda x)}{L(\lambda)} |g(x)| \leq M_1 \max\left\{x^{-\sigma}, x^{\sigma}\right\} |g(x)| \in L^1(0,\infty) \ . \tag{10.3.5}$$

for $\lambda > B_1$, and $\lambda x > A_1$. Write

$$\int_{\frac{A}{\lambda}}^{\infty} \left(\frac{L(\lambda x)}{L(\lambda)} - 1\right) g(x)\mathrm{d}x = I_1(\lambda) - I_2(\lambda) + I_3(\lambda) \ ,$$

where $I_1(\lambda) = \int_{A_1/\lambda}^{\infty} ((L(\lambda x)/L(\lambda)) - 1)\, g(x)\mathrm{d}x$, $I_2(\lambda) = \int_{A/\lambda}^{A_1/\lambda} g(x)\mathrm{d}x$, and $I_3(\lambda) = \int_{A/\lambda}^{A_1/\lambda}(L(\lambda x)/L(\lambda))g(x)\mathrm{d}x$. Because of (10.3.5), we can apply Lebesgue dominated convergence theorem to conclude that $I_1(\lambda) = o(1)$, $\lambda \to \infty$. That $I_2(\lambda) = o(1)$ follows easily from the assumption over g. Finally,

$$|I_3(\lambda)| \leq \frac{1}{\lambda L(\lambda)} \int_{A}^{A_1} \left|L(x) g\left(\frac{x}{\lambda}\right)\right| \mathrm{d}x = O\left(\frac{1}{\lambda^{1+\alpha} L(\lambda)}\right) = o(1) \ , \quad \lambda \to \infty \ .$$

□

From Lemma 10.5, we immediately obtain the next corollaries.

Corollary 10.6. *Let $f \in L^1_{\mathrm{loc}}(x_0, \infty)$ and $\alpha > -1$. If L is slowly varying at the origin, and*

$$f(x) \sim C(x - x_0)^{\alpha} L(x - x_0) \ , \quad x \to x_0^+ \ , \tag{10.3.6}$$

for $C \in \mathbb{R}$, then $f \in L^1_{\mathrm{loc}}[x_0, \infty)$ and

$$f(x_0 + \varepsilon x) H(x) \sim C \varepsilon^{\alpha} L(\varepsilon) x_+^{\alpha} \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{D}'(\mathbb{R}) \ . \tag{10.3.7}$$

Proof. By shifting, we may assume that $x_0 = 0$. That $f \in L^1_{\mathrm{loc}}[0, \infty)$ follows from the estimate $f(x) = O(x^{\alpha} L(x)) = O(x^{\alpha - \sigma})$, $x \to 0^+$, where σ is chosen so that $\alpha - \sigma > -1$. Next, decompose $f = C x^{\alpha} \tilde{L} + G$, where G vanishes near the origin, $\operatorname{supp} \tilde{L} \subseteq (0, B]$, $B > 0$, and $\tilde{L}(x) \sim L(x)$, $x \to 0^+$. Then, obviously,

$G(\varepsilon x) = o(\varepsilon^\alpha L(\varepsilon))$, and because of (10.3.2) of Lemma 10.5, if $\phi \in \mathcal{D}'(\mathbb{R})$

$$\begin{aligned}\int_0^\infty f(\varepsilon x)\phi(x)\mathrm{d}x &= C\varepsilon^\alpha \int_0^{\frac{1}{\varepsilon}} x^\alpha \phi(x)\tilde{L}(\varepsilon x)\mathrm{d}x + o(\varepsilon^\alpha L(\varepsilon))\\ &= C\varepsilon^\alpha \int_{\operatorname{supp}\phi} x^\alpha \phi(x)\tilde{L}(\varepsilon x)\mathrm{d}x + o(\varepsilon^\alpha L(\varepsilon))\\ &\sim C\varepsilon^\alpha \tilde{L}(\varepsilon)\int_0^\infty x^\alpha\phi(x)\mathrm{d}x = C\varepsilon^\alpha L(\varepsilon)\left\langle x_+^\alpha, \phi(x)\right\rangle \ ,\end{aligned}$$

as $\varepsilon \to 0^+$. □

Corollary 10.7. *Let $f \in L^1[0,\infty)$ and $\alpha > -1$. If L is slowly varying at infinity, and*

$$f(x) \sim Cx^\alpha L(x)\ , \quad x \to \infty\ , \tag{10.3.8}$$

for $C \in \mathbb{R}$, then $f \in \mathcal{S}'[0,\infty)$ and

$$f(\lambda x)H(x) \sim C\lambda^\alpha L(\lambda)x_+^\alpha \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R})\ . \tag{10.3.9}$$

Proof. The proof is similar to that of Corollary 10.6. First notice that f has tempered growth, so it is a tempered distribution. We decompose $f = Cx^\alpha\tilde{L}+G$, where G vanishes near infinity, $\operatorname{supp}\tilde{L} \subseteq [B,\infty)$, $B > 0$, and $\tilde{L}(x) \sim L(x)$, $x \to \infty$. We have that $G(\lambda x) = O(\lambda^{-1}) = o(\lambda^\alpha L(\lambda))$ because it has compact support and satisfies the moment asymptotic expansion (see (1.8.11) in Section 1.8.1). So, the rest follows by (10.3.3) of Lemma 10.5 applied to $\tilde{L}$ and $x^\alpha\phi$, where $\phi \in \mathcal{S}(\mathbb{R})$. □

10.4 Structural Theorems in $\mathcal{D}'[0,\infty)$ and $\mathcal{S}'[0,\infty)$

In this section we show the structural theorem for the quasiasymptotic behavior of distributions in $\mathcal{D}'[0,\infty)$ and $\mathcal{S}'[0,\infty)$. This case is much simpler than the general one of unrestricted support, which we postpone for Sections 10.5.3 and 10.5.5. We follow [231] for the proofs; actually, they are essentially the same as the proof of Proposition 1.8 previously discussed in Section 1.8.

Proposition 10.8. *Let L be slowly varying at the origin. A distribution $f \in \mathcal{D}'[0,\infty)$ has quasiasymptotic behavior*

$$f(\varepsilon x) \sim CL(\varepsilon)\frac{(\varepsilon x)_+^{\alpha}}{\Gamma(\alpha+1)} \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{D}'(\mathbb{R}) \tag{10.4.1}$$

if and only if there exists a non-negative integer $m > -\alpha - 1$ such that $f^{(-m)}$ is an ordinary function (locally integrable) in a neighborhood of the origin and

$$f^{(-m)}(x) \sim C\frac{x^{\alpha+m}L(x)}{\Gamma(\alpha+m+1)}\ , \quad x \to 0^+\ . \tag{10.4.2}$$

Proof. The converse follows directly from Corollary 10.6, and then differentiating m-times the quasiasymptotic relation obtained. The Banach-Steinhaus theorem, the quasiasymptotic behavior (10.4.1) and the definition of convergence in $\mathcal{D}'[0,\infty)$ imply that there exists n, sufficiently large, such that the evaluation of f at $\phi_n(t) := (1-t)^n(H(t)-H(t-1))$ makes sense and (10.4.1) holds when evaluated at ϕ_n. Put $m = n+1$, then, as $x \to 0^+$,

$$\begin{aligned} f^{(-m)}(x) =& \frac{1}{(m-1)!}\left\langle f(t), (x-t)^{m-1}(H(t)-H(t-1))\right\rangle \\ =& \frac{x^{m-1}}{(m-1)!}\left\langle f(t), \phi_n\left(\frac{t}{x}\right)\right\rangle \\ =& \frac{x^m}{(m-1)!}\left\langle f(xt), \phi_n\left(t\right)\right\rangle \\ \sim& \frac{Cx^{m+\alpha}L(x)}{(m-1)!\Gamma(\alpha+1)} \text{ F.p.} \int_0^1 t^{\alpha}(1-t)^{m-1}\mathrm{d}t \\ =& \frac{Cx^{m+\alpha}L(x)}{\Gamma(\alpha+m+1)}\ . \end{aligned}$$

□

Proposition 10.9. *A distribution $f \in \mathcal{D}'[0,\infty)$ has quasiasymptotic behavior*

$$f(\lambda x) \sim CL(\lambda)\frac{(\lambda x)_+^{\alpha}}{\Gamma(\alpha+1)} \quad \text{as } \lambda \to \infty \text{ in } \mathcal{D}'(\mathbb{R}) \tag{10.4.3}$$

if and only if $f \in \mathcal{S}'[0,\infty)$ and there exists a non-negative integer $m > -\alpha - 1$ such that $f^{(-m)} \in L^1_{\text{loc}}[0,\infty)$ and

$$f^{(-m)}(x) \sim C\frac{x^{\alpha+m}}{\Gamma(\alpha+m+1)}\ , \quad x \to \infty\ , \tag{10.4.4}$$

in the ordinary sense. Moreover, the quasiasymptotic behavior (10.4.3) holds actually in $\mathcal{S}'[0,\infty)$.

Proof. The converse follows directly from Corollary 10.7 and differentiating m-times; furthermore Corollary 10.7 also implies that (10.4.3) holds actually in the space $\mathcal{S}'[0,\infty)$. The other part is established exactly in the same way as in the proof of Proposition 10.8. $\square$

We now discuss some results about the convolution.

Theorem 10.10. *Let* $f,g\in\mathcal{D}'[0,\infty)$. *Suppose that*

$$f(\varepsilon x)\sim C_1L_1(\varepsilon)\frac{(\varepsilon x)_+^{\alpha}}{\Gamma(\alpha+1)}\quad as\ \varepsilon\to 0^+\ in\ \mathcal{D}'(\mathbb{R}) \tag{10.4.5}$$

and

$$g(\varepsilon x)\sim C_2L_2(\varepsilon)\frac{(\varepsilon x)_+^{\nu}}{\Gamma(\nu+1)}\quad as\ \varepsilon\to 0^+\ in\ \mathcal{D}'(\mathbb{R})\ . \tag{10.4.6}$$

Then,

$$(f*g)(\varepsilon x)\sim C_1C_2L_1(\varepsilon)L_2(\varepsilon)\frac{(\varepsilon x)_+^{\alpha+\nu+1}}{\Gamma(\alpha+\nu+2)}\quad as\ \varepsilon\to 0^+\ in\ \mathcal{D}'(\mathbb{R})\ . \tag{10.4.7}$$

Proof. The proof is simple. Consider $f\otimes g\in\mathcal{D}'(\mathbb{R}^2)$. Then by Proposition 10.8 there exist $n>-\alpha-1$ and $m>-\nu-1$ such that $f^{(-n)}$ and $g^{(-m)}$ are locally integrable in a neighborhood of the origin,

$$\lim_{x\to 0^+}\frac{\Gamma(\alpha+n+1)f^{(-n)}(x)}{x^{\alpha+n}L_1(x)}=C_1$$

and

$$\lim_{y\to 0^+}\frac{\Gamma(\nu+m+1)g^{(-m)}(y)}{y^{\nu+m}L_2(y)}=C_2,$$

hence for each $\phi\in\mathcal{D}(\mathbb{R}^2)$,

$$\begin{aligned}\langle f\otimes g(\varepsilon x,\epsilon y),\phi(x,y)\rangle &= \frac{(-1)^{n+m}}{\varepsilon^{n+m}}\int\int f^{(-n)}(\varepsilon x)g^{(-m)}(\varepsilon y)\frac{\partial^{n+m}\phi}{\partial x^n\partial y^m}(x,y)\mathrm{d}x\mathrm{d}y\\ &\sim \varepsilon^{\alpha+\nu}L_1(\varepsilon)L_2(\varepsilon)\left\langle\frac{C_1x_+^{\alpha}}{\Gamma(\alpha+1)}\otimes\frac{C_2y_+^{\alpha}}{\Gamma(\nu+1)},\phi(x,y)\right\rangle\ .\end{aligned}$$

So (10.4.7) follows then from the definition of convolution and the last relation. $\square$

The same property of the convolution holds for the quasiasymptotic behavior at infinity, and the proof is identically the same as the one of Theorem 10.10. In [231], this assertion at ∞ is shown by means of tauberian theory; see Lemma 1, Chapter 4, Section 11.1 in [231].

Theorem 10.11. *Let $f, g \in \mathcal{S}'[0,\infty)$. Suppose that*

$$f(\lambda x) \sim C_1 L_1(\lambda) \frac{(\lambda x)_+^{\alpha}}{\Gamma(\alpha+1)} \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R}) \tag{10.4.8}$$

and

$$g(\lambda x) \sim C_2 L_2(\lambda) \frac{(\lambda x)_+^{\nu}}{\Gamma(\nu+1)} \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R})\ . \tag{10.4.9}$$

Then,

$$(f * g)(\lambda x) \sim C_1 C_2 L_1(\lambda) L_2(\lambda) \frac{(\lambda x)_+^{\alpha+\nu+1}}{\Gamma(\alpha+\nu+2)} \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R})\ . \tag{10.4.10}$$

10.5 Structural Theorems for Quasiasymptotics: General Case

We now proceed to undertake the general structural study of the quasiasymptotic behavior. We shall introduce two classes of functions having regular variational asymptotic properties, the class of *asymptotically homogeneous functions* and the class of *associate asymptotically homogeneous functions of degree 0*. These functions extend those discussed in Section 3.4.1. We will later derive the announced structural theorems from the fundamental properties of such functions.

The technique to be employed here is based on the asymptotic analysis of the parametric coefficients resulting after performing several integrations of the quasiasymptotic behavior, these coefficients are naturally connected with the classes of asymptotically and associate asymptotically homogeneous functions. The technique of integration of distributional asymptotic relations goes back to the classical work of Łojasiewicz [128] (see also [47, 153, 216]). Later on, the properties of the parametric coefficients were singled out and recognized as asymptotically and associate asymptotically homogeneous functions by the author in [212, 213, 227].

10.5.1 Asymptotically Homogeneous Functions

We study some properties of asymptotically homogeneous functions which will be applied later to the structural study of the quasiasymptotic behavior. Let us proceed to define this class of functions.

Definition 10.12. *A function b is said to be asymptotically homogeneous of degree α at the origin (respectively at infinity) with respect to the slowly varying function at the origin L, if it is measurable and defined in some interval $(0, A)$ (respectively on (A, ∞)), $A > 0$, and for each $a > 0$,*

$$b(ax) = a^{\alpha} b(x) + o(L(x)) \ , \quad x \to 0^+ \quad (\textit{resp. } x \to \infty) \ . \tag{10.5.1}$$

Obviously, asymptotically homogeneous functions at the origin and at infinity are connected by the change of variables $x \leftrightarrow x^{-1}$; therefore, most of the properties of the class of asymptotically homogeneous functions at infinity can be obtained from those of the corresponding class at origin.

Let us now obtain a crucial property of these functions. Observe that no uniformity with respect to a is assumed in Definition 10.12; however, the definition itself forces (10.5.1) to hold uniformly for a on compact subsets. Indeed, we will show this fact by using a classical argument of H. Korevaar, T. van Aardenne Ehrenfest, and N. G. de Bruijn [114, 15, 183, 227].

Lemma 10.13. *Let b be an asymptotically homogeneous function of degree α with respect to L. Then, the relation*

$$b(ax) = a^{\alpha} b(x) + o(L(x)) \ , \tag{10.5.2}$$

holds uniformly for a in compact subsets of $(0, \infty)$.

Proof. We show the assertion at the origin, the case at infinity can be obtained by the change of variables $x \leftrightarrow x^{-1}$. So assume that b is asymptotically homogeneous of degree α at the origin with respect to L. We may assume that b is defined on $(0,1]$. We rather work with the functions $c(x) = e^{\alpha x} b\left(e^{-x}\right)$ and $s(x) = L\left(e^{-x}\right)$, hence c and s are defined in $[0,\infty)$. By using a linear transformation between an arbitrary compact subinterval of $[0,\infty)$ and $[0,1]$, it is enough to show that

$$c(h+x) - c(x) = o(e^{\alpha x} s(x)) \ , \quad x \to \infty \ , \tag{10.5.3}$$

uniformly for $h \in [0,1]$. Suppose that (10.5.3) is false. Then, there exist $0 < \varepsilon < 1$, a sequence $\{h_m\}_{m=1}^{\infty} \in [0,1]^{\mathbb{N}}$, and an increasing sequence of real numbers $\{x_m\}_{m=1}^{\infty}$, $x_m \to \infty, \ m \to \infty$, such that

$$\left|c\left(h_m + x_m\right) - c\left(x_m\right)\right| \geq \varepsilon e^{\alpha x_m} s\left(x_m\right), \ m \in \mathbb{N} \ . \tag{10.5.4}$$

Define, for $n \in \mathbb{N}$,

$$A_n = \left\{h \in [0,2] : \left|c\left(h + x_m\right) - c\left(x_m\right)\right| < \frac{\varepsilon}{3} e^{\alpha x_m} s\left(x_m\right), m \geq n\right\},$$

$$B_n = \left\{h \in [0,2] : \left|c\left(h + x_m + h_m\right) - c\left(h_m + x_m\right)\right| < \frac{\varepsilon}{3} e^{\alpha x_m} s\left(x_m + h_m\right), m \geq n\right\} \ .$$

Note that

$$[0,2] = \bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} B_n \ ,$$

so we can select N such that $\mu(A_n)$, $\mu(B_n) > \frac{3}{2}$ (here $\mu(\cdot)$, and only here, stands for Lebesgue measure), for all $n \geq N$. For each $n \in \mathbb{N}$, put $C_n = \{h_n\} + B_n$. Then, we have $\mu(C_n) > \frac{3}{2}, n \geq N$, and $C_n, A_n \subseteq [0,3]$. It follows that $A_n \bigcap C_n \neq \emptyset, n > N$. For each $n \geq N$, select $u_n \in A_n \bigcap C_n$. In particular, we have $u_n - h_n \in B_n$, and hence,

$$\left|c\left(u_n + x_n\right) - c\left(x_n\right)\right| < \frac{\varepsilon}{3} e^{\alpha x_n} s\left(x_n\right) \ ,$$

$$\left|c\left(u_n + x_n\right) - c\left(x_n + h_n\right)\right| < \frac{\varepsilon}{3} e^{\alpha x_n} s\left(x_n + h_n\right)$$

which implies that for all $n \geq N$,

$$|c(x_n + h_n) - c(x_n)| < \frac{\varepsilon}{3} e^{\alpha x_n} \left(s(x_n) + s(x_n + h_n) \right) \ .$$

Using that $s(x+h) - s(x) = o(s(x))$, $x \to \infty$, uniformly for h on compact subsets of $(0, \infty)$, we have that for all n sufficiently large, $s(x_n + h_n) \leq 2s(x_n)$, which implies that for n big enough

$$|c(x_n + h_n) - c(x_n)| < \varepsilon e^{\alpha x_n} s(x_n) \ ,$$

in contradiction to (10.5.4), Therefore, (10.5.3) must hold uniformly for $h \in [0, 1]$.

□

Corollary 10.14. *If b is asymptotically homogeneous at the origin (resp. at infinity) with respect to a slowly varying function, then b is locally bounded in some interval of the form $(0, B)$ (resp. (B, ∞)).*

Proof. It follows directly from Lemma 10.13; indeed, let us only discuss the case at infinity. Let $B > 1$ be such that

$$|b(ax) - a^{\alpha} b(x)| < L(x)$$

for all $B \leq x$, $a \in [1, 2]$ and $L \in L^{\infty}_{\text{loc}}[B, \infty)$. It is enough to show that b is bounded on each interval of the form $x \in [B, 2^n B]$, $n \in \mathbb{N}$. Let M_n be a bound for L on $x \in [B, 2^n B]$, $n \in \mathbb{N}$. So, we have $|b(x)| < 2^{|\alpha|} b(B) + M_n$ for $x \in [B, 2B]$, $|b(x)| < 2^{|\alpha|} b(2B) + M_n < 2^{2|\alpha|} b(B) + 2^{|\alpha|} M_n + M_n$ for $x \in [2B, 4B]$, and so. Therefore,

$$|b(x)| < 2^{n|\alpha|} b(B) + M_n \frac{2^{n|\alpha|} - 2^{|\alpha|}}{2^{|\alpha|} - 1} \ , \quad \forall\, x \in [B, 2^n B] \ .$$

□

It is interesting to observe that is not absolutely necessary to ask (10.5.1) for all $a > 0$; indeed, it is enough to assume that it initially holds for a merely in a set of positive measure.

Proposition 10.15. *Suppose that (10.5.1) holds merely for each $a \in \mathfrak{B}$, a set of positive Lebesgue measure, then (10.5.1) remains valid for all $a > 0$.*

Proof. We may assume that $\mathfrak{B}$ is the maximal set where (10.5.1) holds. Let us show that $\mathfrak{B}$ is multiplicative subgroup of $\mathbb{R}_+$. If $a, a' \in \mathfrak{B}$, then

$$b(aa'x) = b(a'x) + o(L(a'x)) = b(a'x) + o(L(x)) = b(x) + o(L(x)) \ ,$$

and so $aa' \in \mathfrak{B}$. On the other hand

$$\begin{aligned} b(a^{-1}x) &= b(x) + (b(a^{-1}x) - b(a(x/a)) \\ &= b(x) + (b(a^{-1}x) - b(x/a) - o(L(x/a))) \\ &= b(x) + o(L(x)) \ . \end{aligned}$$

Therefore, $\mathfrak{B}$ is a subgroup. Since its measure is positive, it follows from the well known theorem of Steinhaus (see [15, p.2], [148], the original source is [196, Théorème VII]) that it contains an open interval and hence $\mathfrak{B} = (0, \infty)$. $\square$

We now obtain the behavior of asymptotically homogeneous functions of non-zero degree.

Theorem 10.16. *Suppose that b is asymptotically homogeneous at the origin (resp. at infinity) with respect to the slowly varying function L. Then*

(i) If $\alpha > 0$ (resp. $\alpha < 0$ for the case at infinity), then

$$b(x) = o(L(x)) \ , \quad x \to 0^+ \ (\textit{resp. } x \to \infty) \ . \tag{10.5.5}$$

(ii) If $\alpha < 0$ (resp. $\alpha > 0$), then there exists a constant γ such that

$$b(x) = \gamma x^\alpha + o\left(L(x)\right) \ , \quad x \to 0^+ \ (\textit{resp. } x \to \infty) \ . \tag{10.5.6}$$

Proof. We show only the assertion at the origin, the case at infinity follows again from a change of variables.

Let us first show (i). Assume that $\alpha > 0$. Let $0 < \eta$ be any arbitrary number. We keep $\varepsilon < 2^\alpha - 1$. Let $x_0 > 0$ such that

$$\left| b\left(\frac{x}{2}\right) - 2^{-\alpha} b(x) \right| \leq \eta L(x) \quad \text{and} \quad |L(2x) - L(x)| \leq \eta L(x), \; 0 < x < x_0 \; . \quad (10.5.7)$$

We may assume that b and L are bounded on $[x_0, 2x_0]$. So, let

$$M = \sup\left\{ \frac{|b(x)|}{L(x)} : \frac{1}{2} x_0 \leq x \leq x_0 \right\} \; .$$

Take $x \in [x_0/2, x_0]$. From (10.5.7) it follows that

$$\left| \frac{b(x/2^n)}{L(x/2^n)} \right| \leq 2^{-\alpha n} \frac{|b(x)|}{L(x/2^n)} + \eta \sum_{j=0}^{n-1} 2^{-\alpha(n-1-j)} \frac{L(x/2^j)}{L(x/2^n)} \; .$$

Thus, with $t = x/2^n$, and $t \in [x_0/2^{n+1}, x_0/2^n]$,

$$\left| \frac{b(t)}{L(t)} \right| \leq 2^{-n\alpha} M \frac{L(2^n t)}{L(t)} + \eta \sum_{j=0}^{n-1} 2^{-j\alpha} \frac{L(2^{j+1} t)}{L(t)} \; .$$

By this and

$$L(2^{j+1} t)/L(2^j t) \leq (1+\eta), \; j = 0, \ldots, n-1,$$

we have that if $t \in \left[2^{-(n+1)} x_0, 2^{-n} x_0\right]$, then

$$\left| \frac{b(t)}{L(t)} \right| \leq M \left(\frac{1+\eta}{2^\alpha}\right)^n + \eta(1+\eta) \sum_{j=0}^{\infty} \left(\frac{1+\eta}{2^\alpha}\right)^j = M \left(\frac{1+\eta}{2^\alpha}\right)^n + \eta(1+\eta) \frac{2^\alpha}{2^\alpha - 1 - \eta} \; .$$

Let us prove that for every $\varepsilon > 0$ there exists a positive σ such that $|b(t)/L(t)| < \varepsilon, \; t \in (0, \sigma)$. First, we have to take η, small enough, such that

$$\eta(1+\eta) \frac{2^\alpha}{2^\alpha - 1 - \eta} < \frac{\varepsilon}{2}$$

and $n_0 \in \mathbb{N}$ such that

$$M \left(\frac{1+\eta}{2^\alpha}\right)^n < \frac{\varepsilon}{2} \; , \quad n \geq n_0 \; .$$

Then, it follows that $|b(t)/L(t)| < \varepsilon,\ t \in (0, \sigma)$, if we take $\sigma = x_0/2^{n_0}$. This completes the first part of the proof.

We now show (ii). Assume that $\alpha < 0$. We rather work with $c(x) = e^{\alpha x} b(e^{-x})$ and $s(x) = L(e^{-x})$. Then c satisfies

$$c(h + x) - c(x) = o\left(e^{\alpha x} s(x)\right), \quad x \to \infty\ ,$$

uniformly for $h \in [0, 1]$. Given $\varepsilon > 0$, we can find $x_0 > 0$ such that for all $x > x_0$ and $h \in [0, 1]$,

$$|c(x + h) - c(x)| \leq \varepsilon e^{\alpha x} s(x) \ \text{ and } \ |s(h + x) - s(x)| \leq \left(e^{-\frac{\alpha}{2}} - 1\right) s(x)\ .$$

So we have that

$$\begin{aligned}
|c(h + n + x) - c(x)| &\leq |c(h + n + x) - c(n + x)| + |c(n + x) - c(x)| \\
&\leq \varepsilon e^{\alpha(n+x)} s(n + x) + \sum_{j=0}^{n-1} |c(j + 1 + x) - c(j + x)| \\
&\leq \varepsilon e^{\alpha x} \sum_{j=0}^{n} e^{\alpha j} s(j + x) \\
&\leq \varepsilon e^{\alpha x} s(x) \frac{1}{1 - e^{\frac{\alpha}{2}}}\ ,
\end{aligned}$$

where the last estimate follows from $s(x + j) \leq s(x) e^{-\alpha j/2}$. Since $s(x) = o\left(e^{-\alpha x}\right)$ as $x \to \infty$, it shows that there exists $\gamma \in \mathbb{R}$ such that

$$\lim_{x \to \infty} c(x) = \gamma\ .$$

Moreover, the estimate shows that

$$c(x) = \gamma + o\left(e^{\alpha x} s(x)\right),\ x \to \infty\ ,$$

thus, changing the variables back, we have obtained,

$$b(x) = \gamma x^{\alpha} + o\left(L(x)\right)\ ,\ x \to 0^{+}.$$

□

Remark 10.17. *Notice that the converse of Theorem 10.16 is trivially true, that is, Theorem 10.16 is a full characterization of asymptotically homogeneous, with non-zero degree, with respect to slowly varying functions.*

Asymptotically homogeneous functions of degree zero have a more complex asymptotic behavior. For example if $L \equiv 1$, any asymptotically homogeneous functions function is the logarithm of a slowly varying function. Instead of attempting to find their behavior in the classical sense, we will study their distributional behavior. A representation formula for them will be obtained in Section 10.5.4 (Theorems 10.39 and 10.60). The next lemma roughly estimate the growth properties of asymptotically homogeneous functions of degree zero.

Lemma 10.18. *Let b be asymptotically homogeneous of degree 0 at the origin (respectively at infinity) with respect to the slowly varying function L. If $\sigma < 0$ (resp. $\sigma > 0$) then,*

$$b(x) = o\left(x^{\sigma}\right), \quad x \to 0^{+} \text{ (resp. } x \to \infty) \text{ .}$$

In particular, $b(x)\left(L(x)\right)^{-1}$ is integrable near the origin (resp. locally integrable near ∞).

Proof. We know that $L(x) = o\left(x^{\sigma}\right)$. Then for each $a > 0$, $b(ax) = b(x) + o\left(x^{\sigma}\right)$ and this implies that $x^{-\sigma}b(x)$ is asymptotically homogeneous of degree $-\sigma > 0$ with respect to the constant slowly varying function $L \equiv 1$. From Theorem 10.16, it follows that $b(x) = o\left(x^{\sigma}\right)$. □

We now describe the behavior of asymptotically homogeneous functions of degree zero at the origin. The next two theorems will be very important in Section 10.5.3.

Theorem 10.19. *Let b be asymptotically homogeneous of degree zero at the origin with respect to the slowly varying function L. Suppose that b is integrable on $(0, A]$.*

Then

$$b(\varepsilon x)(H(x) - H(\varepsilon x - A)) = b(\varepsilon)H(x) + o(L(\varepsilon)) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{D}'(\mathbb{R}) \ , \tag{10.5.8}$$

where H is the Heaviside function.

Proof. Let $\phi \in \mathcal{D}(\mathbb{R})$. Find B such that $\operatorname{supp}\phi \subseteq [-B, B]$, then there exists $\varepsilon_\phi < 1$ such that

$$\langle b(\varepsilon x), \phi(x)\rangle = \int_0^{\frac{1}{\varepsilon}} b(\varepsilon x)\phi(x)dx = \int_0^B b(\varepsilon x)\phi(x)\mathrm{d}x, \ \varepsilon < \varepsilon_\phi \ . \tag{10.5.9}$$

Replacing $\phi(x)$ by $B\phi(Bx)$ and ε_ϕ by $B\varepsilon_\phi$, we may assume that $B = 1$. Our aim is to show that for some $\varepsilon_0 < 1$,

$$\frac{b(\varepsilon x) - b(\varepsilon)}{L(\varepsilon)}, \ x \in (0, 1], \ \varepsilon < \varepsilon_0$$

is dominated by an integrable function in $(0, 1]$ for the use of the Lebesgue theorem. For this goal, we assume that L satisfies the following estimate,

$$\frac{L(\varepsilon x)}{L(\varepsilon)} \le Mx^{-\frac{1}{2}} \ , \quad x \in (0, 1], \varepsilon \in (0, \varepsilon_\phi) \ . \tag{10.5.10}$$

By Lemma 10.13, there exists $0 < \varepsilon_0 < \varepsilon_\phi$ such that

$$|b(\varepsilon x) - b(\varepsilon)| < L(\varepsilon), \ x \in [1/2, 1], \ \varepsilon < \varepsilon_0 \ .$$

We keep $\varepsilon < \varepsilon_0$ and $x \in [2^{-n-1}, 2^{-n}]$. Then

$$\begin{aligned} |b(\varepsilon x) - b(\varepsilon)| &\le |b(2\varepsilon x) - b((2x\varepsilon)/2)| + |b(2\varepsilon x) - b(\varepsilon)| \\ &\le L(2\varepsilon x) + |b(2\varepsilon x) - b(\varepsilon)| \\ &\le \sum_{i=1}^{n} L\left(2^i \varepsilon x\right) + L(\varepsilon) \\ &\le \sum_{i=1}^{n} (2^i x)^{-1/2} L(\varepsilon) + L(\varepsilon). \end{aligned}$$

It follows from (10.5.10) that if $\varepsilon < \varepsilon_0$ and $x \leq 1$, then

$$\left| \frac{b(\varepsilon x) - b(\varepsilon)}{L(\varepsilon)} \right| \leq M_1 x^{-\frac{1}{2}} + 1,$$

where $M_1 = M(\sqrt{2}+1)$. Therefore we can apply Lebesgue's dominated convergence theorem to deduce (10.5.8). $\square$

We also have a similar result at infinity, this fact is stated in the next theorem. Since its a corollary of Theorem 10.37 , we omit its proof and refer the reader to Section 10.5.4.

Theorem 10.20. *Let b be asymptotically homogeneous of degree zero at the infinity with respect to the slowly varying function L. Suppose that b is locally integrable on $[A, \infty)$. Then*

$$b(\lambda x) H(\lambda x - A) = b(\lambda) H(x) + o(L(\lambda)) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R}) \ . \qquad (10.5.11)$$

Remark 10.21. *The results of this section were obtained by the author and S. Pilipović in [227]. The author recently learned from [183, Section 2.4] and [15, Chap.3] that some of them could have been also obtained from properties of a class of functions introduced by R. Bojanić and J. Karamata in [22], but, at the time we wrote [227], we were unaware of the existence of such results. The functions introduced by R. Bojanić and J. Karamata are measurable functions defined in some interval of the form $[A, \infty)$, $A > 0$, satisfying*

$$c(ax) = c(x) + \tau(a) x^{\alpha} L(x) + o(x^{\alpha} L(x)) \ , \quad x \to \infty \ . \qquad (10.5.12)$$

for each $a > 0$. Now, if b is asymptotically homogeneous at infinity of degree α with respect to L, then $c(x) = b(x)/x^{\alpha}$ satisfies (10.5.12) with $\tau(a) = 0$ and α replaced by $-\alpha$. The class of functions satisfying (10.5.12) has been extensively studied [8, 15, 65, 84, 115, 183]; the associated theory is usually referred as second-order theory of regular variation or de Haan theory [15, 84].

10.5.2 Relation Between Asymptotically Homogeneous Functions and Quasiasymptotics

We introduced asymptotically homogeneous functions in order to study the structure of the quasiasymptotics for Schwartz distributions. The next proposition provides the intrinsic link between quasiasymptotics and asymptotically homogeneous functions.

Proposition 10.22. *Let $f \in \mathcal{D}'(\mathbb{R})$ have quasiasymptotic behavior in $\mathcal{D}'(\mathbb{R})$*

$$f(\lambda x) = L(\lambda)g(\lambda x) + o(\lambda^{\alpha}L(\lambda)) \quad \text{as } \lambda \to \infty \ (\text{resp. } \lambda \to 0^{+}) \ , \tag{10.5.13}$$

where L is a slowly varying function and g is a homogeneous distribution of degree $\alpha \in \mathbb{R}$. Let $n \in \mathbb{N}$. Suppose that g admits a primitive of order n, that is, $G_n \in \mathcal{D}'(\mathbb{R})$ and $G_n^{(n)} = g$, which is homogeneous of degree $n+\alpha$. Then, for any given F_n, an n-primitive of f in $\mathcal{D}'(\mathbb{R})$, there exist functions $b_0, \dots, b_{n-1}$, continuous on $(0,\infty)$, such that

$$F_n(\lambda x) = L(\lambda)G_n(\lambda x) + \sum_{j=0}^{n-1} \lambda^{\alpha+n} b_j(\lambda) \frac{x^{n-1-j}}{(n-1-j)!} + o\left(\lambda^{\alpha+n}L(\lambda)\right) \tag{10.5.14}$$

as $\lambda \to \infty$ (resp. $\lambda \to 0^{+}$) in $\mathcal{D}'(\mathbb{R})$, where each b_j is asymptotically homogeneous of degree $-\alpha - j - 1$.

Proof. Recall that any $\phi \in \mathcal{D}(\mathbb{R})$ is of the form

$$\phi = C_{\phi}\phi_0 + \theta', \text{ where } C_{\phi} = \int_{-\infty}^{\infty} \phi(t)\mathrm{d}t, \ \theta \in \mathcal{D}(\mathbb{R}) \tag{10.5.15}$$

and $\phi_0 \in \mathcal{D}(\mathbb{R})$ is chosen so that $\int_{-\infty}^{\infty} \phi_0(t)\mathrm{d}t = 1$. The evaluations of primitives F_1 of f and G_1 of g on ϕ are given by

$$\langle F_1, \phi\rangle = C_{\phi}\langle F_1, \phi_0\rangle - \langle f, \theta\rangle \ \text{ and } \ \langle G_1, \phi\rangle = C_{\phi}\langle G_1, \phi_0\rangle - \langle g, \theta\rangle \ .$$

This implies

$$\left\langle \frac{F_1(\lambda x)}{\lambda^{\alpha+1}L(\lambda)}, \phi(x) \right\rangle = C_{\phi} \left\langle \frac{F_1(\lambda x)}{\lambda^{\alpha+1}L(\lambda)}, \phi_0(x) \right\rangle - \left\langle \frac{f(\lambda x)}{\lambda^{\alpha}L(\lambda)}, \theta(x) \right\rangle \ , \tag{10.5.16}$$

and

$$\left\langle \frac{G_1(\lambda x)}{\lambda^{\alpha+1}L(\lambda)}, \phi(x) \right\rangle = C_\phi \left\langle \frac{G_1(\lambda x)}{\lambda^{\alpha+1}L(\lambda)}, \phi_0(x) \right\rangle - \left\langle \frac{g(\lambda x)}{\lambda^{\alpha}L(\lambda)}, \theta(x) \right\rangle \ . \tag{10.5.17}$$

With $c_0(\lambda) = \langle (F_1 - G_1)(\lambda x), \phi_0(x) \rangle,\ \lambda \in (0,\infty)$, from (10.2.1), it follows

$$F_1\left(\lambda x\right) = L(\lambda)G_1(\lambda x) + c_0(\lambda) + o\left(\lambda^{\alpha+1}L(\lambda)\right) \quad \text{in } \mathcal{D}'(\mathbb{R}) \ . \tag{10.5.18}$$

So relation (10.5.14) follows by induction from (10.5.18) and (10.2.1).

We shall now concentrate in showing the property of the b_j's. We set $F_m = F_n^{(n-m)}$ and $G_m = G_n^{(n-m)}, m \in \{1, \dots, n\}$. By differentiating relation (10.5.14) $(n-m)$-times, it follows that

$$F_m(\lambda x) = L(\lambda)G_m(\lambda x) + \sum_{j=0}^{m-1} \lambda^{\alpha+m} b_j(\lambda) \frac{x^{m-1-j}}{(m-1-j)!} + o\left(\lambda^{\alpha+m}L(\lambda)\right) \tag{10.5.19}$$

in $\mathcal{D}'(\mathbb{R})$. Choose $\phi \in \mathcal{D}(\mathbb{R})$ such that $\int_{-\infty}^{\infty} \phi(x)x^j \mathrm{d}x = 0$ for $j = 1, \dots, m-1$, and $\int_{-\infty}^{\infty} \phi(x)\mathrm{d}x = 1$. Then evaluating (10.5.19) at ϕ, we have that

$$\begin{aligned}
&(a\lambda)^{\alpha+m} b_{m-1}(a\lambda) + L(a\lambda)\left\langle G_m(a\lambda x), \phi(x)\right\rangle + o\left(\lambda^{\alpha+m}L(\lambda)\right) \\
&= \left\langle F_m(a\lambda x), \phi(x) \right\rangle \\
&= \frac{1}{a}\left\langle F_m(\lambda x), \phi\left(\frac{x}{a}\right)\right\rangle \\
&= \lambda^{\alpha+m} b_{m-1}(\lambda) + L(\lambda)\left\langle G_m(a\lambda x), \phi(x)\right\rangle + o\left(\lambda^{\alpha+m}L(\lambda)\right) \ ,
\end{aligned}$$

and so, with $j = m-1 \in \{0, \dots, n-1\}$,

$$b_j(a\lambda) = a^{-\alpha-j-1} b_j(\lambda) + o\left(L(\lambda)\right) \ ,$$

for each $a > 0$. □

10.5.3 Structural Theorems for Some Cases

We now derive structural theorems for quasiasymptotics in some cases with the aid of asymptotically homogeneous functions (Theorems 10.16, 10.19 and 10.20) and Proposition 10.22.

Theorem 10.23. *Let $f \in \mathcal{D}'(\mathbb{R})$ have quasiasymptotic behavior at infinity in $\mathcal{D}'(\mathbb{R})$*

$$f(\lambda x) = C_{-}L(\lambda)\frac{(\lambda x)_{-}^{\alpha}}{\Gamma(\alpha+1)} + C_{+}L(\lambda)\frac{(\lambda x)_{+}^{\alpha}}{\Gamma(\alpha+1)} + o\left(\lambda^{\alpha}L(\lambda)\right) \quad \text{as } \lambda \to \infty\ , \tag{10.5.20}$$

where $\alpha \notin \mathbb{Z}_{-}$. Then there exist a non-negative integer $m > -\alpha - 1$ and an m-primitive F of f such that $F \in L^{1}_{\text{loc}}(\mathbb{R})$ and

$$\lim_{x\to\pm\infty} \frac{\Gamma(\alpha+m+1)F(x)}{x^{m}\left|x\right|^{\alpha} L\left(\left|x\right|\right)} = C_{\pm}\ . \tag{10.5.21}$$

Conversely, if these conditions hold, then (by differentiation) (10.5.20) follows. Moreover, it follows that f is tempered and (10.5.20) holds in the space $\mathcal{S}'(\mathbb{R})$.

Proof. The converse follows from Corollary 10.7 and then m differentiations. The last claim is implied by Corollary 10.7. We now focus in showing 10.5.21. On combining Proposition 10.22 and Theorem 10.16, one obtains that for each $n \in \mathbb{N}$ and F_n, an arbitrary n-primitive of f, there exist constants $\gamma_0, \ldots, \gamma_{n-1}$ such that, in the sense of convergence in $\mathcal{D}'(\mathbb{R})$,

$$F_n(\lambda x) = \sum_{j=0}^{n-1} \gamma_j \frac{(\lambda x)^j}{j!} + C_{-}\frac{(-1)^n L(\lambda)(\lambda x)_{-}^{\alpha+n}}{\Gamma(\alpha+n+1)} + C_{+}\frac{L(\lambda)(\lambda x)_{+}^{\alpha+n}}{\Gamma(\alpha+n+1)} + o\left(\lambda^{\alpha+n}L(\lambda)\right) \tag{10.5.22}$$

It follows from the convergence $\mathcal{D}'(\mathbb{R})$ that there is $m \in \mathbb{N}$, sufficiently large, such that any m-primitive of f is continuous and (10.5.22) holds (with $n = m$) uniformly for $x \in [-1, 1]$. Pick a specific m-primitive of f, say F_m, then from (10.5.22) there is a polynomial p of degree at most $m - 1$ such that

$$F_m(\lambda x) = p(\lambda x) + C_{-}L(\lambda)\frac{(-1)^m(\lambda x)_{-}^{\alpha+m}}{\Gamma(\alpha+m+1)} + C_{+}L(\lambda)\frac{(\lambda x)_{+}^{\alpha+m}}{\Gamma(\alpha+m+1)} + o\left(\lambda^{\alpha+m}L(\lambda)\right)\ ,$$

uniformly for $x \in [-1, 1]$. Then setting $F = F_m - p$, $x = 1, -1$ and replacing λ by x, relation (10.5.21) follows at once. $\square$

Let us make some comment about Theorem 10.23.

Remark 10.24. *It should be observed that (10.5.21) holds for every m-primitive of f, provided that $\alpha > -1$. In fact, since $p_{m-1}(x) = o(x^{m+\alpha}L(x)) \to 0,\ x \to \infty$, whenever $\alpha > -1$, we have that in such a case the polynomial is irrelevant in the proof of the last Theorem.*

Remark 10.25. *When $\alpha < -1$, there is one and only one m-primitive F of f satisfying (10.5.21). Indeed, if F_1 is another m-primitive satisfying (10.5.21), then $F_1 = F + p$, where p is a polynomial of degree at most $m-1$; then, $p(x) = o(x^{m+\alpha}L(x)) = o(x^{m-1})$, and the latter implies that p is identically zero.*

Remark 10.26. *The proof of Theorem 10.23 gives that m can be selected so that $F \in C(\mathbb{R})$; but this fact actually follows directly by one integration of (10.5.21).*

Remark 10.27. *We obtain at once the decomposition theorem from [231, p.134].*

We also have the analog to Theorem 10.23 at the origin. The proof is identically the same as the one of Theorem 10.23; we therefore omit it.

Theorem 10.28. *Let $f \in \mathcal{D}'(\mathbb{R})$ have quasiasymptotic behavior at the origin in $\mathcal{D}'(\mathbb{R})$*

$$f(\varepsilon x) = C_{-}L(\varepsilon)\frac{(\varepsilon x)_{-}^{\alpha}}{\Gamma(\alpha+1)} + C_{+}L(\varepsilon)\frac{(\varepsilon x)_{+}^{\alpha}}{\Gamma(\alpha+1)} + o\left(\varepsilon^{\alpha}L(\varepsilon)\right) \quad \text{as } \varepsilon \to 0^{+}\ , \tag{10.5.23}$$

where $\alpha \notin \mathbb{Z}_{-}$. Then there exist a non-negative integer $m > -\alpha - 1$ and an m-primitive F of f such that F is locally integrable near the origin and

$$\lim_{x \to 0^{\pm}} \frac{\Gamma(\alpha+m+1)F(x)}{x^{m}\left|x\right|^{\alpha}L\left(\left|x\right|\right)} = C_{\pm}\ . \tag{10.5.24}$$

Conversely, if these conditions hold, then (by differentiation) (10.5.23) follows.

Remark 10.29. *Theorem 10.28 gives at once the structure of quasiasymptotics at finite points, it is obtained by translation.*

Remark 10.30. *If $\alpha > -1$, then the m-primitive satisfying (10.5.24) is unique; however, if $\alpha < -1$, then (10.5.24) is valid for more than one m-primitive, but in general not for all.*

We now give a second application of asymptotically homogeneous functions, we will study the quasiasymptotic behavior $f(\lambda x) = \gamma L(\lambda)\delta(\lambda x) + o(\lambda^{-1}L(\lambda))$. We postpone the general case of negative integral degrees for Section 10.5.5, after the introduction of associate asymptotically homogeneous function in Section 10.5.4.

Proposition 10.31. *Let $f \in \mathcal{D}'(\mathbb{R})$ have quasiasymptotic behavior at infinity*

$$f(\lambda x) = \gamma \frac{L(\lambda)}{\lambda}\,\delta(x) + o\left(\frac{L(\lambda)}{\lambda}\right) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{D}'(\mathbb{R})\ . \tag{10.5.25}$$

Then, there exist $m \in \mathbb{N}$, a function b, being asymptotically homogeneous function of degree 0 with respect to L, and an $(m+1)$-primitive F of f such that $F \in L^1_{\mathrm{loc}}(\mathbb{R})$ and

$$F(x) = \gamma L(|x|)\frac{x^m}{2m!}\,\operatorname{sgn} x + c\,(|x|)\,\frac{x^m}{m!} + o\left(|x|^m\, L(|x|)\right)\ , \quad x \to \infty\ . \tag{10.5.26}$$

Conversely, if (10.5.26) holds, then (10.5.25) follows by differentiation. Moreover, (10.5.25) is valid in the space $\mathcal{S}'(\mathbb{R})$.

Proof. The existence of m, b, and F satisfying (10.5.26) follows from the weak convergence of (10.5.25), Proposition 10.22 and Theorem 10.16, as in the proof of Theorem 10.23. The converse is shown by applying Theorem 10.20 and differentiating $(m+1)$-times. $\square$

Likewise, one shows.

Proposition 10.32. *Let $f \in \mathcal{D}'(\mathbb{R})$ have quasiasymptotic behavior at the origin*

$$f(\varepsilon x) = \gamma \frac{L(\varepsilon)}{\varepsilon}\,\delta(x) + o\left(\frac{L(\varepsilon)}{\varepsilon}\right) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{D}'(\mathbb{R})\ . \tag{10.5.27}$$

Then, there exist $m \in \mathbb{N}$, a function b, being asymptotically homogeneous function of degree 0 *with respect to L, and an $(m+1)$-primitive F of f such that F is locally integrable near the origin and*

$$F(x) = \gamma L(|x|) \frac{x^m}{2m!} \operatorname{sgn} x + c\left(|x|\right) \frac{x^m}{m!} + o\left(|x|^m L(|x|)\right) \ , \quad x \to 0 \ . \tag{10.5.28}$$

Conversely, if (10.5.28) holds, then (10.5.27) follows by differentiation.

10.5.4 Associate Asymptotically Homogeneous Functions

We now introduce the main tool for the study of structural properties of quasi-asymptotics of negative integral degree. What makes impossible the application of Proposition 10.22 to the -1 degree case is the fact that, in general, the primitives of a homogeneous distribution of degree -1 are not homogeneous. In Section 10.5.5, the technique of integrating the quasiasymptotic and studying the coefficients of integration is employed again; moreover, the main coefficient of this integration will fit into the context of *associate asymptotically homogeneous functions*, which we now proceed to define.

Definition 10.33. *A function b is said to be associate asymptotically homogeneous of degree 0 at the origin (resp. at infinity) with respect to the slowly varying function L, if it is measurable and defined in some interval $(0, A)$ (resp. (A, ∞)), $A > 0$, and there exists a constant β such that for each $a > 0$,*

$$b(ax) = b(x) + \beta L(x) \log a + o(L(x)) \ , \quad x \to 0^+ \ (\textit{resp. } x \to \infty) \ . \tag{10.5.29}$$

We may use the same argument employed in the proof of Lemma 10.13 to show uniform convergence of (10.5.29). Furthermore, the same argument of Proposition 10.15 lead to a proof of the following claim: if one just assumes (10.5.29) for a is a set of positive measure then it should hold for each $a > 0$. We leave the details to the reader.

Lemma 10.34. *Suppose that (10.5.29) holds merely for a in a set of positive Lebesgue measure, then it holds for each $a > 0$. Moreover, relation (10.5.29) holds uniformly for a in compact subsets of $(0, \infty)$.*

We shall study the distributional asymptotic properties of this class of functions in detail. We first roughly estimate the behavior of associate asymptotically homogeneous functions of degree 0.

Lemma 10.35. *Let b be associate asymptotically homogeneous of degree 0 at the origin (resp. at infinity) with respect to L, then for each $\sigma < 0$ (resp. $\sigma > 0$),*

$$b(x) = o(x^{\sigma}) \ , \quad x \to 0^{+} \ (\textit{resp. } x \to \infty) \ . \tag{10.5.30}$$

Hence, b is integrable near the origin (resp. locally integrable near infinity).

Proof. We know that $L(x) = o(x^{\sigma})$, for each $\sigma > 0$ [183]. Hence $b(ax) = b(x)+o(x^{\sigma})$ and thus $x^{-\sigma}b(x)$ is asymptotically homogeneous of degree $-\sigma$ with respect to $L \equiv 1$, so (10.5.30) follows from Theorem 10.16. □

The next two theorems will be crucial in the next section. They generalize Theorems 10.19 and 10.16. We only give the proof at infinity, the proof at the origin is similar to that of Theorem 10.19.

Theorem 10.36. *Let b be a locally integrable associate asymptotically homogeneous function of degree zero at infinity with respect to the slowly varying function L defined on $[A, \infty)$. Then*

$$b(\lambda x)H(\lambda x - A) = b(\lambda)H(x) + L(\lambda)\beta H(x) \log x + o(L(\lambda)) \ , \tag{10.5.31}$$

as $\lambda \to \infty$ in the space $\mathcal{S}'(\mathbb{R})$.

Proof. Let λ_0 be any positive number. The function b can be decomposed as $b = b_1 + b_2$, where $b_1 \in L^1(\mathbb{R})$ has compact support and $b_2(x) = b(x)H(x - \lambda_0)$ is associate asymptotically homogeneous function of degree zero at infinity. Since b_1 satisfies the moment asymptotic expansion, it follows that $b_1(\lambda x) = O(\lambda^{-1}) = o(L(\lambda))$ as $\lambda \to \infty$ in $\mathcal{S}'(\mathbb{R})$. Therefore, we can always assume that $A = \lambda_0$, where λ_0 is selected at our convenience.

Our aim is to show that there is some $\lambda_0 > 1$ such that

$$J(x,\lambda) := \phi(x)\, \frac{b(\lambda x) - b(\lambda) - \beta L(\lambda) \log x}{L(\lambda)}\, H(\lambda x - \lambda_0)$$

is dominated by an integrable function, whenever $\phi \in \mathcal{S}(\mathbb{R})$, for the use of the Lebesgue dominated convergence theorem. For this goal, we can always assume that L is positive everywhere and satisfies the following estimate (see Section 2.10.1),

$$\frac{L(\lambda x)}{L(\lambda)} \leq M \max\left\{x^{-\frac{1}{4}}, x^{\frac{1}{4}}\right\}, \quad x, \lambda \in (0,\infty)\ , \tag{10.5.32}$$

for some positive constant M. Because of the uniformity of (10.5.29) on compact sets, there exists a $\lambda_0 > 1$ such that

$$|b(\lambda x) - b(\lambda) - \beta L(\lambda) \log x| < L(\lambda)\ , \quad x \in [1,2],\ \lambda_0 < \lambda\ .$$

Let n be a positive integer. We keep $\lambda_0 < \lambda$ and $x \in [2^n, 2^{n+1}]$. Then

$$\begin{aligned}
&|b(\lambda x) - b(\lambda) - \beta L(\lambda) \log x| \leq |b(\lambda x) - b(\lambda)| + |\beta|\, L(\lambda) \log x \\
&\leq |\beta|\, L(\lambda) \log x + |b(2(\lambda x/2)) - b(\lambda x/2) - \beta L(\lambda x/2) \log 2| \\
&\quad + |\beta|\, L(\lambda x/2) \log 2 + |b(\lambda x/2) - b(\lambda)| \\
&\leq |\beta|\, L(\lambda) \log x + (1 + |\beta| \log 2)\, L(\lambda x/2) + |b(\lambda x/2) - b(\lambda)| \\
&\leq (1 + |\beta| \log 2) \sum_{j=1}^{n} L\left(2^{-j} \lambda x\right) + |\beta|\, L(\lambda) \log 2x + L(\lambda) \\
&\leq \left(M x^{\frac{1}{4}} (1 + |\beta| \log 2) \sum_{j=1}^{n} (1/2)^{\frac{j}{4}} + |\beta| \log 2x + 1 \right) L(\lambda)\ ,
\end{aligned}$$

where the last inequality follows from (10.5.32). So if $\lambda_0 < \lambda$ and $1 < x$, then

$$\left| \frac{b(\lambda x) - b(\lambda) - \beta L(\lambda) \log x}{L(\lambda)} \right| \leq M_1 x^{\frac{1}{4}}\ ,$$

for some $M_1 > 0$. Now if $\lambda_0/\lambda < x < 1$, we have that

$$\begin{aligned}\left|\frac{b(\lambda x) - b(\lambda) - \beta L(\lambda)\log x}{L(\lambda)}\right| &\leq \left(1 + \frac{L(\lambda x)}{L(\lambda)}\right)|\beta \log x| \\ &\quad + \left|\frac{b(\lambda) - b(\lambda x) - \beta L(\lambda x)\log x^{-1}}{L(\lambda)}\right| \\ &\leq \left(1 + Mx^{-\frac{1}{4}}\right)|\beta \log x| \\ &\quad + \frac{L(\lambda x)}{L(\lambda)}\left|\frac{b(\lambda x(x^{-1})) - b(\lambda x) - \beta L(\lambda x)\log x^{-1}}{L(\lambda x)}\right| \\ &\leq \left(1 + Mx^{-\frac{1}{4}}\right)|\beta \log x| + MM_1 x^{-\frac{1}{2}}\ .\end{aligned}$$

Therefore $J(x, \lambda)$ is dominated by an integrable function for $\lambda > \lambda_0$, so we apply Lebesgue dominated convergence theorem to deduce that $\lim_{\lambda\to\infty}\int_0^\infty J(x,\lambda)\mathrm{d}x = 0$. Finally,

$$\begin{aligned}&\langle b(\lambda x)H(\lambda x - \lambda_0), \phi(x)\rangle - b(\lambda)\int_0^\infty \phi(x)\mathrm{d}x - \beta L(\lambda)\int_0^\infty \log x\, \phi(x)\mathrm{d}x \\ &= \int_{\lambda_0/\lambda}^\infty b(\lambda x)\phi(x)\mathrm{d}x - b(\lambda)\int_0^\infty \phi(x)\mathrm{d}x - \beta L(\lambda)\int_0^\infty \log x\, \phi(x)\mathrm{d}x \\ &= L(\lambda)\int_0^\infty J(x,\lambda)\mathrm{d}x + L(\lambda)O\left(\frac{\log\lambda}{\lambda}\right) + O\left(\frac{b(\lambda)}{\lambda}\right) \\ &= o(L(\lambda)) + L(\lambda)O\left(\frac{b(\lambda)}{\lambda L(\lambda)}\right) = o(L(\lambda))\ , \quad \lambda\to\infty\ ,\end{aligned}$$

where in the last equality we have used Lemma 10.35 and the fact that slowly varying functions are $o(\lambda^\sigma)$ for any $\sigma > 0$. This completes the proof of (10.5.31). □

Theorem 10.37. *Let b be a locally integrable associate asymptotically homogeneous function of degree zero at the origin with respect to the slowly varying function L defined on $(0, A]$. Then*

$$b(\varepsilon x)(H(x) - H(\varepsilon x - A)) = b(\varepsilon)H(x) + L(\varepsilon)\beta H(x)\log x + o(L(\varepsilon)) \qquad (10.5.33)$$

as $\varepsilon \to 0^+$ in $\mathcal{D}'(\mathbb{R})$.

Corollary 10.38. *Let b be an associate asymptotically homogeneous function of degree 0 with respect to the slowly varying function L. Then, there exists an associate asymptotically homogeneous function $c \in C^{\infty}(0,\infty)$ such that $b(x) = c(x) + o(L(x))$.*

Proof. By Lemma 10.4, we may assume that $L \in C^{\infty}(0,\infty)$. Find B such that b is locally bounded in $[B,\infty)$ (resp. $(0,B]$), this can be done because of Proposition 10.18. Take $\phi \in \mathcal{D}(\mathbb{R})$ such that $\int_0^\infty \phi(t)\mathrm{d}t = 1$ and set $c(x) = \int_{B/x}^\infty b(xt)\phi(t)\mathrm{d}t - \beta L(x)\int_0^\infty \phi(t)\log t\mathrm{d}t$ (resp. $\int_0^{B/x} b(xt)\phi(t)\mathrm{d}t - \beta L(x)\int_0^\infty \phi(t)\log t\mathrm{d}t$), the corollary now follows from Theorem 10.36 (resp. Theorem 10.37). □

We may also use Corollary 10.38 to obtain a representation formula for associate asymptotically homogeneous functions, this is the analog to [183, Theorem 1.2] for slowly varying functions.

Theorem 10.39. *The function b is associate asymptotically homogeneous of degree 0 at ∞ satisfying (10.5.29) if and only if there is a positive number A such that*

$$b(x) = \eta(x) + \int_A^x \frac{\tau(t)}{t}\,\mathrm{d}t\ , \quad x \geq A\ , \tag{10.5.34}$$

where η is a locally bounded measurable function on $[A,\infty)$ such that $\eta(x) = M + o(L(x))$ as $x \to \infty$, for some number M, and τ is a C^{∞}-function such that $\tau(x) \sim \beta L(x)$ as $x \to \infty$.

Proof. The converse follows easily from (10.5.34), so we show the other part. Assume first that b_1 is C^{∞}, defined on $[0,\infty)$ and satisfies the hypothesis of the theorem. We can find $L_1 \sim L$ which is C^{∞} and satisfies $xL_1'(x) = o(L(x))$ as $x \to \infty$ (Lemma 10.4). Let ϕ and c as in the proof of Corollary 10.38 corresponding to b_1 and L_1, additionally assume that $\operatorname{supp}\phi \subseteq (0,\infty)$. From Theorem 10.36, we have that

$$b_1'(\lambda x) = \frac{b_1(\lambda)}{\lambda}\delta(x) + \beta\frac{L(\lambda)}{\lambda}\,\mathrm{Pf}\left(\frac{H(x)}{x}\right) + o\left(\frac{L(\lambda)}{\lambda}\right) \quad \text{as } \lambda \to \infty$$

in $\mathcal{S}'(\mathbb{R})$, since distributional asymptotics can be differentiated. Then, for x positive

$$\begin{aligned} xc'(x) &= x\int_0^\infty b_1'(xt)t\phi(t)\mathrm{d}t - \beta x L_1'(x)\int_0^\infty \phi(t)\log t\,\mathrm{d}t \\ &= x\int_0^\infty b_1'(xt)t\phi(t)\mathrm{d}t + o(L(x)) \\ &= b_1(x)\cdot 0 + \beta L(x)\int_0^\infty \phi(t)\mathrm{d}t + o(L(x)) \\ &= \beta L(x) + o(L(x)) \quad \text{as} \quad x \to \infty\ . \end{aligned}$$

Set $\tau(x) = xc'(x)$. If $A > 0$, one has that $b_1(x) = c(A) + \int_A^x (\tau(t)/t)\mathrm{d}t + o(L(x))$.

In the general case, let A be a number such that b and L are locally bounded on $[A, \infty)$ and let b_1 be the function from Corollary 10.38 such that $b(x) = b_1(x) + o(L(x))$, then we can apply the previous argument to b_1 to find τ as before, so we obtain (10.5.34) with $\eta(x) = b(x) - \int_A^x (\tau(t)/t)\,\mathrm{d}t = c(A) + o(L(x))$. $\square$

A change of variables $x \leftrightarrow x^{-1}$ in Theorem 10.39 implies the analog result at 0.

Theorem 10.40. *The function b is associate asymptotically homogeneous of degree 0 at the origin satisfying (10.5.29) if and only if there is a positive number A such that*

$$b(x) = \eta(x) + \int_x^A \frac{\tau(t)}{t}\,\mathrm{d}t\ , \quad x \le A\ , \tag{10.5.35}$$

where η is a locally bounded measurable function on $(0, A]$ such that $\eta(x) = M + o(L(x))$ as $x \to 0^+$, for some number M, and τ is a C^∞-function such that $\tau(x) \sim \beta L(x)$ as $x \to 0^+$.

Remark 10.41. *A slightly different representation formula is given in [183, Theorem 2.13], but, except for the smoothness of τ, both are equivalent.*

Remark 10.42. *Note that the property (10.5.29) is exactly (10.5.12) with* $\alpha = 0$ *and* $\tau(a) = \beta \log a$*; indeed, when* $\alpha = 0$*, it can be shown [183, Theorem 2.9] that (10.5.12) forces* τ *to have this form. Associate asymptotically homogeneous functions of degree zero are called de Haan functions in [15], and have been very much studied. Some of the results of the present section overlap those from [15, Chap.3], however, the author independently rediscovered [212, 227] them because he was unaware of their existence.*

10.5.5 Structural Theorems for Negative Integral Degrees

This section is dedicated to the study of structural properties of quasiasymptotic behaviors with negative integral degree. The next lemma reduces the analysis of negative integral degrees to the case of degree -1.

Lemma 10.43. *Let* $f \in \mathcal{D}'(\mathbb{R})$ *and* $k \in \mathbb{Z}_+$*. Then* f *has the quasiasymptotic behavior*

$$f(\lambda x) = \gamma \lambda^{-k} L(\lambda)\, \delta^{(k-1)}(x) + \beta L(\lambda)(\lambda x)^{-k} + o\left(\lambda^{-k} L(\lambda)\right) \quad \text{in } \mathcal{D}'(\mathbb{R})$$

(at either 0 or ∞*) if and only if there exists a* k*- primitive* g *of* f *satisfying*

$$g(\lambda x) = \gamma \lambda^{-1} L(\lambda)\, \delta(x) + \frac{(-1)^{k-1}\beta}{(k-1)!} L(\lambda)(\lambda x)^{-1} + o\left(\lambda^{-1} L(\lambda)\right) \quad \text{in } \mathcal{D}'(\mathbb{R}) \ .$$

Proof. It follows directly from Proposition 10.22 and Theorem 10.16. □

We should introduce some notation that will be needed. In the following for all $n \in \mathbb{N}$ we denote by l_n the primitive of $\log|x|$ with the property that $l_n(0) = 0$ and $l_n' = l_{n-1}$. We have an explicit formula for them:

$$l_n(x) = \frac{x^n}{n!} \log|x| - \frac{x^n}{n!} \sum_{j=1}^{n} \frac{1}{j} \ , \quad x \in \mathbb{R} \ ,$$

which can be easily verified by direct differentiation. They satisfy

$$l_n(ax) = a^n l_n(x) + \frac{(ax)^n}{n!} \log a \ , \quad a > 0 \ . \tag{10.5.36}$$

We analyze the case at infinity, the treatment of quasiasymptotic behavior at the origin is similar.

Theorem 10.44. *Let $f \in \mathcal{D}'(\mathbb{R})$ have quasiasymptotic behavior*

$$f(\lambda x) = \gamma \frac{L(\lambda)\delta(x)}{\lambda} + \beta L(\lambda)\,(\lambda x)^{-1} + o\left(\frac{L(\lambda)}{\lambda}\right) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{D}'(\mathbb{R}) \ . \tag{10.5.37}$$

Then, there exist an associate asymptotically homogeneous function b satisfying

$$b(ax) = b(x) + \beta L(x) \log a + o(L(x)) \ , \quad x \to \infty \ , \tag{10.5.38}$$

an integer m, and an $(m+1)$-primitive $F \in L^1_{\text{loc}}(\mathbb{R})$ of f such that

$$F(x) = b\left(|x|\right) \frac{x^m}{m!} + \gamma \frac{x^m}{2m!} L\left(|x|\right) \operatorname{sgn} x - \beta L\left(|x|\right) \frac{x^m}{m!} \sum_{j=1}^{m} \frac{1}{j} + o\left(|x|^m L\left(|x|\right)\right) \tag{10.5.39}$$

as $x \to \pm\infty$, in the ordinary sense. Conversely, relation (10.5.39) implies the quasiasymptotic behavior (10.5.37). Furthermore, f is a tempered distribution and (10.5.37) holds in the space $\mathcal{S}'(\mathbb{R})$.

Proof. We shall study, as we have been doing, the coefficients of the integration of (10.5.37). For each $n \in \mathbb{N}$, choose an n primitive F_n of f satisfying $F_n' = F_{n-1}$. We now proceed to integrate (10.5.37) once, so we obtain

$$F_1(\lambda x) = b(\lambda) + \frac{\gamma}{2} L(\lambda) \operatorname{sgn} x + \beta L(\lambda) \log |x| + o(L(\lambda)) \quad \text{in } \mathcal{D}'(\mathbb{R}). \tag{10.5.40}$$

Now, using the standard trick of evaluating at $\phi \in \mathcal{D}(\mathbb{R})$ with the property $\int_{-\infty}^{\infty} \phi(x) dx = 1$, one obtains that

$$\begin{aligned}
& b(\lambda a) + \frac{\gamma}{2} L(\lambda a) \int_{-\infty}^{\infty} \operatorname{sgn} x \, \phi(x) \mathrm{d}x + \beta L(\lambda a) \int_{-\infty}^{\infty} \log |x| \, \phi(x) \mathrm{d}x + o(L(\lambda)) \\
&= \langle F_1(\lambda a x), \phi(x) \rangle = \frac{1}{a} \left\langle F_1(\lambda x), \phi\left(\frac{x}{a}\right) \right\rangle \\
&= b(\lambda) + \frac{\gamma}{2} L(\lambda) \int_{-\infty}^{\infty} \operatorname{sgn} x \phi(x) \mathrm{d}x + \beta L(\lambda) \int_{-\infty}^{\infty} \log |ax| \, \phi(x) \mathrm{d}x + o(L(\lambda)) \ ,
\end{aligned}$$

$\lambda \to \infty$, for each $a > 0$. So, we see that b satisfies (10.5.38) for each $a > 0$. Further integration of (10.5.40) gives,

$$F_{n+1}(\lambda x) = b(\lambda)\frac{(\lambda x)^n}{n!} + \sum_{j=1}^{n} \lambda^n b_j(\lambda)\frac{x^{n-j}}{(n-j)!} + \gamma L(\lambda)\operatorname{sgn} x \frac{(\lambda x)^n}{2n!}$$
$$+ \beta L(\lambda)\lambda^n l_n(x) + o\left(\lambda^n L(\lambda)\right) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{D}'(\mathbb{R}) \ .$$

As in the proof of Proposition 10.22, one shows that the b_j's are asymptotically homogeneous functions of degree $-j$ with respect to L. Hence, if we apply Theorem 10.16 to the b_j's, we obtain that

$$F_{m+1}(\lambda x) = b(\lambda)\frac{(\lambda x)^m}{m!} + \gamma L(\lambda)\frac{(\lambda x)^m}{2m!}\operatorname{sgn} x + \beta L(\lambda)\lambda^m l_m(x) + o\left(\lambda^m L(\lambda)\right) \tag{10.5.41}$$

in the sense of convergence in $\mathcal{D}'(\mathbb{R})$. Moreover, it follows from the definition of convergence in $\mathcal{D}'(\mathbb{R})$ there exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$ the distribution F_{m+1} is a continuous function and (10.5.41) holds uniformly for $x \in [-1, 1]$. Relation (10.5.39) is shown by making $x = \pm 1$ in (10.5.41) and then changing $\lambda \leftrightarrow x$.

Conversely, since only the behavior of b at infinity plays a roll in (10.5.39), we may assume that b is locally integrable, so the converse is obtained after application of Theorem 10.36 and then $(m+1)$ differentiations; Theorem 10.36 also shows that F is tempered, so is f, and that (10.5.37) holds in $\mathcal{S}'(\mathbb{R})$. □

Remark 10.45. *A similar statement holds for the the quasiasymptotic at the origin. We leave the formulation and proof to the reader.*

Remark 10.46. *The proof of Theorem 10.44 actually shows that m can be selected so that $F \in C(\mathbb{R})$ (resp. continuous near the origin in the case at the origin).*

Theorem 10.44 is a structural theorem, but we shall give a version free of b. We also state the assertion at the origin.

Theorem 10.47. *Let $f \in \mathcal{D}'(\mathbb{R})$. Then f has quasiasymptotic at infinity (resp. at the origin) of the form (10.5.37) if and only if there exists an $(m+1)$-primitive $F \in L^1_{\mathrm{loc}}(\mathbb{R})$ (resp. locally integrable near 0) of f, such that for each $a > 0$,*

$$\lim_{x\to\infty} \frac{m!\left(a^{-m}F(ax) - (-1)^m F(-x)\right)}{x^m L(x)} = \gamma + \beta \log a \quad (\textit{resp.} \lim_{x\to 0^+}). \tag{10.5.42}$$

Proof. The limit (10.5.42) follows from (10.5.39), (10.5.38) and (10.5.36) by direct computation. For the converse, rewrite (10.5.42) as

$$a^{-m}F(ax) - (-1)^m F(-x) = (\gamma + \beta \log a)\frac{x^m}{m!}L(x) + o\left(x^m L(x)\right),$$

for each $a > 0$. Set

$$b(x) = m! x^{-m} F(x) - \left(\frac{\gamma}{2} - \beta \sum_{j=1}^{m} \frac{1}{j}\right) L(x)\ , \quad x > 0.$$

By setting $a = 1$ in (10.5.42), one sees that for $x < 0$,

$$F(x) = b\left(|x|\right)\frac{x^m}{m!} + \gamma L\left(|x|\right)\frac{x^m}{2m!}\operatorname{sgn} x - \beta L\left(|x|\right)\frac{x^m}{m!}\sum_{j=1}^{m}\frac{1}{j} + o\left(|x|^m L\left(|x|\right)\right)\ .$$

Since

$$a^{-m}F(ax) - F(x) = \beta \frac{x^m}{m!} L(x) \log a + o\left(x^m L(x)\right),$$

it is clear that for each $a > 0$,

$$b(ax) = b(x) + \beta L(x) \log a + o(L(x)).$$

□

Remark 10.48. *It is remarkable that, initially, no uniform condition on a is assumed in (10.5.42). However, the proof of Theorem 10.47 forces this relation to hold uniformly for a in compact subsets, in view of the fact that such a property holds for associate asymptotically homogeneous functions (Lemma 10.34). Additionally, it is enough to know that (10.5.42) holds merely for a in a set of positive measure to conclude that it holds for each $a > 0$.*

We are now ready to state the general structural theorem for negative integral degrees which now follows directly from Lemma 10.43, Theorem 10.44 and Theorem 10.47.

Theorem 10.49. *Let $f \in \mathcal{D}'(\mathbb{R})$ and $k \in \mathbb{Z}_+$. Then f has the quasiasymptotic behavior*

$$f(\lambda x) = \gamma \frac{L(\lambda)}{\lambda^k} \, \delta^{(k-1)}(x) + \beta L(\lambda)(\lambda x)^{-k} + o\left(\frac{L(\lambda)}{\lambda^k}\right) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{D}'(\mathbb{R}) \tag{10.5.43}$$

if and only if there exist $m \in \mathbb{N}$, $m \geq k$, an associate asymptotically homogeneous function b of degree 0 at infinity with respect to L satisfying

$$b(ax) = b(x) + \frac{(-1)^{k-1}\beta}{(k-1)!} L(x) \log a + o(L(x)) \ , \quad x \to \infty \ ,$$

for each $a > 0$, and an m-primitive $F \in L^1_{\text{loc}}(\mathbb{R})$ of f which satisfies

$$\begin{aligned} F(x) =& b\left(|x|\right) \frac{x^{m-k}}{(m-k)!} + \gamma L\left(|x|\right) \frac{x^{m-k}}{2(m-k)!} \operatorname{sgn} x \\ & - \frac{(-1)^{k-1}\beta}{(k-1)!} L\left(|x|\right) \frac{x^{m-k}}{(m-k)!} \sum_{j=1}^{m-k} \frac{1}{j} + o\left(|x|^{m-k} L\left(|x|\right)\right) \end{aligned}$$

as $x \to \pm\infty$, in the ordinary sense. The last property is equivalent to

$$\lim_{x \to \infty} \frac{(m-k)! \left(a^{k-m} F(ax) - (-1)^{m-k} F(-x)\right)}{x^{m-k} L(x)} = \gamma + \frac{(-1)^{k-1}\beta}{(k-1)!} \log a \ , \tag{10.5.44}$$

for each $a > 0$. Furthermore, $f \in \mathcal{S}'(\mathbb{R})$ and (10.5.43) holds in the space $\mathcal{S}'(\mathbb{R})$.

Likewise, we have the structural theorem at the origin.

Theorem 10.50. *Let $f \in \mathcal{D}'(\mathbb{R})$ and $k \in \mathbb{Z}_+$. Then f has the quasiasymptotic behavior*

$$f(\varepsilon x) = \gamma \frac{L(\varepsilon)}{\varepsilon^k} \, \delta^{(k-1)}(x) + \beta L(\varepsilon)(\varepsilon x)^{-k} + o\left(\frac{L(\varepsilon)}{\varepsilon^k}\right) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{D}'(\mathbb{R})$$

if and only if there exist $m \in \mathbb{N}$, $m \geq k$, an associate asymptotically homogeneous function b of degree 0 at infinity with respect to L satisfying

$$b(ax) = b(x) + \frac{(-1)^{k-1}\beta}{(k-1)!} L(x) \log a + o(L(x)) \ , \quad x \to 0^+ \ ,$$

for each $a > 0$, and an m-primitive F of f which is locally integrable near the origin and satisfies

$$\begin{aligned}F(x) =& b\left(|x|\right)\frac{x^{m-k}}{(m-k)!} + \gamma L\left(|x|\right)\frac{x^{m-k}}{2(m-k)!}\operatorname{sgn} x \\ &- \frac{(-1)^{k-1}\beta}{(k-1)!}L\left(|x|\right)\frac{x^{m-k}}{(m-k)!}\sum_{j=1}^{m-k}\frac{1}{j} + o\left(|x|^{m-k}L\left(|x|\right)\right)\end{aligned}$$

as $x \to 0$, in the ordinary sense. The last property is equivalent to

$$\lim_{x\to 0^+}\frac{(m-k)!\left(a^{k-m}F(ax) - (-1)^{m-k}F(-x)\right)}{x^{m-k}L(x)} = \gamma + \frac{(-1)^{k-1}\beta}{(k-1)!}\log a\ , \quad (10.5.45)$$

for each $a > 0$.

It should be noticed that in (10.5.44) or (10.5.45) is not absolutely necessary to assume that the limit is of the form $\gamma + (-1)^{k-1}(\beta/(k-1)!)\log a$. Indeed, we have the following stronger result.

Theorem 10.51. *Let $f \in \mathcal{D}'(\mathbb{R})$. Then f has quasiasymptotic behavior at infinity (resp. at the origin) of degree $-k$, $k \in \mathbb{Z}_+$, if and only if there exists m-primitive $F \in L^1_{\text{loc}}(\mathbb{R})$ (resp. locally integrable near the origin) of f, $m \geq k$, such that the following limit exists*

$$\lim_{x\to\infty}\frac{\left(a^{k-m}F(ax) - (-1)^{m-k}F(-x)\right)}{x^{m-k}L(x)} = I(a) \quad (\textit{resp.} \lim_{x\to 0^+})\ , \qquad (10.5.46)$$

for each a merely in a subset $\mathfrak{B} \subset (0,\infty)$ having positive Lebesgue measure. In this case, there exist constants γ and β such that $I(a) = \gamma + (-1)^{k-1}(\beta/(k-1)!)\log a$, and (10.5.46) holds uniformly for a in any compact subset of $(0,\infty)$.

Proof. We may assume that $\mathfrak{B}$ is the maximal set of numbers a where (10.5.46) is valid. It is easy to see that $\mathfrak{B}$ is a multiplicative subgroup of $\mathbb{R}_+$ and has positive measure; consequently, Steinhaus theorem implies that $\mathfrak{B} = \mathbb{R}_+$. Next, we easily see that I is measurable and satisfies

$$I(ab) = I(a) + I(b) - I(1)\ ,$$

setting $h(x) = e^{I(x)-I(1)}$, one has that h is positive, measurable and satisfies the functional equation $h(ab) = h(a)h(b)$, from where it follows [183] that $h(x) = x^{\beta_1}$, for some β_1, and so I has the desired form upon setting $I(1) = \gamma$ and $\beta_1 = (1)^{k-1}\beta/(k-1)!$. The uniform convergence over compact subsets of $(0, \infty)$ follows from Remark 10.48. $\square$

10.6 Quasiasymptotic Boundedness

This section is intended to study the structure of the distributional relation

$$f(\lambda x) = O(\rho(\lambda)) \ , \tag{10.6.1}$$

where here $\lambda \to \infty$ or $\lambda \to 0^+$ and Our approach to the problem follows the exposition from [213]. In Section 1.8.1 we introduced quasiasymptotic boundedness with no restriction over the comparison function ρ. However, we will assume throughout this section that ρ is a regularly varying function, and we will obtain the structural properties of (10.6.1) under this assumption. In order to introduce some language, we state the following definition.

Definition 10.52. *Let L be a slowly varying function at infinity (resp. at the origin) and $\alpha \in \mathbb{R}$. We say f is quasiasymptotically bounded of degree α at infinity (at the origin) with respect to the slowly varying function L, if*

$$f(\lambda x) = O(\lambda^{\alpha} L(\lambda)) \qquad \text{as } \lambda \to \infty \ \text{ in } \mathcal{D}'(\mathbb{R}) \tag{10.6.2}$$

(resp. $\lambda \to 0^+$).

We may talk about (10.6.2) in other spaces of distributions. By translation, we can also formulate Definition 10.52 at any finite point.

In order to obtain the structure of quasiasymptotically bounded distributions For this aim, the program established in Section 10.5 will be followed. We will integrate the relation (10.6.2) and study the coefficients of integration.

10.6.1 Asymptotically Homogeneously Bounded Functions

The coefficients of integration of (10.6.2) will satisfy the properties of the next definition.

Definition 10.53. *Let b be a measurable function defined in some interval (A,∞) (resp. $(0,A)$), $A>0$, It is said to be asymptotically homogeneously bounded of degree α at infinity (resp. at the origin) with respect to the slowly varying function L if it is and for each $a>0$*

$$b(ax)=a^{\alpha}b(x)+O(L(x)),\quad x\to\infty\ \ (\textit{resp. } x\to 0^{+})\ . \tag{10.6.3}$$

If we set $c(x)=b(x)/x^{\alpha}$, then c satisfies

$$c(ax)=c(x)+O(x^{-\alpha}L(x))\ . \tag{10.6.4}$$

The class of functions satisfying the above relation has been very much studied by several authors, see for instance [183, Section 2.4] or [15, Chap.3]. In [15], more general classes, called $O\Pi$-classes, are defined and they contain functions satisfying (10.6.4). We now discuss some properties of asymptotically homogeneously bounded functions in connection with the structure of quasiasymptotically bounded distributions. Many of these properties of a asymptotically homogeneous function b can be deduced from those of the corresponding c by using the known results from [183, 15]. Alternatively, the reader may observe that most of the proofs of the following results are the analog to those for asymptotically homogeneous functions and can be obtained by replacing the o symbol by the O symbol and making obvious modifications to the estimates, therefore they will be omitted. We leave to the reader the details of such modifications.

Proceeding as in Lemma 10.13 and Proposition 10.15, or using the the results of [183, Section 2.4], one has the following result.

Proposition 10.54. *If (10.6.3) holds merely for a in a set of positive measure, then it remains valid for each $a > 0$. Moreover, (10.6.3) holds uniformly for a in compact subsets of $(0, \infty)$.*

One can also show the following series of results.

Proposition 10.55. *Let b be asymptotically homogeneously bounded at infinity (at the origin) with respect to the slowly varying function L. If the degree is negative (resp. positive), then $b(x) = O(L(x))$.*

Proposition 10.56. *Let b be asymptotically homogeneously bounded at infinity (at the origin) with respect to the slowly varying function L. If the degree α is positive (respectively negative), then there exits a constant γ such that $b(x) = \gamma x^{\alpha} + O(L(x))$.*

Corollary 10.57. *Let b be asymptotically homogeneously bounded function of degree 0 at infinity (at the origin) with respect to L. If $\sigma > 0$ (resp. $\sigma < 0$), then $b(x) = O(x^{\sigma})$. Consequently, it is locally integrable for large arguments (in a right neighborhood of the origin).*

The proof of the next proposition is totally analogous to those of Theorems 10.19 and 10.36, and therefore will be omitted again.

Proposition 10.58. *Let b be asymptotically homogeneously bounded of degree zero at infinity (at the origin) with respect to the slowly varying function L. Suppose that b is locally integrable on $[A, \infty)$ (respectively $(0, A]$). Then*

$$b(\lambda x)H(\lambda x - A) = b(\lambda)H(x) + O(L(\lambda)) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R}) \ , \tag{10.6.5}$$

(resp. $b(\varepsilon x)(H(x) - H(\varepsilon x - A) = b(\varepsilon)H(x) + O(L(\varepsilon))$ as $\varepsilon \to 0^{+}$ in $\mathcal{D}'(\mathbb{R})$).

Corollary 10.59. *Let b be an asymptotically homogeneously bounded function of degree 0 at infinity (at the origin) with respect to L. Then, there exists $c \in$*

$C^\infty(0,\infty)$, being also asymptotically homogeneously bounded of degree 0, such that $b(x)=c(x)+O(L(x))$.

Proof. We only show the assertion at infinity, the case at the origin is similar. Find A such that b is locally bounded in $[A,\infty)$. Take $\phi\in\mathcal{D}(\mathbb{R})$ supported in $(0,\infty)$ such that $\int_0^\infty \phi(t)\mathrm{d}t=1$ and set $c(x)=\int_{A/x}^\infty b(xt)\phi(t)\mathrm{d}t$, the corollary now follows from Proposition 10.58. $\square$

Using the ideas of Theorem 10.39, we can give a representation formula for asymptotically homogeneously bounded functions of degree 0. We start with the case at infinity.

Theorem 10.60. *A function b is associate asymptotically homogeneously bounded of degree 0 at ∞ with respect to the slowly varying function L if and only if there is a positive number A such that*

$$b(x)=\eta(x)+\int_A^x \frac{\tau(t)}{t}\,\mathrm{d}t\ , \quad x\geq A\ , \tag{10.6.6}$$

where η is a locally bounded measurable function on $[A,\infty)$ such that $\eta(x)=M+O(L(x))$, $x\to\infty$, for some number M, and τ is a C^∞-function such that $\tau(x)=O(L(x))$, $x\to\infty$.

Proof. The converse follows easily from (10.6.6), we concentrate on the other part. Assume first that b_1 is C^∞, defined on $[0,\infty)$, and satisfies the hypothesis of the theorem. Let ϕ be such that $\operatorname{supp}\phi\subseteq(0,\infty)$ and $\int_0^\infty\phi(t)\mathrm{d}t=1$. Set $c(x)=\int_0^\infty b_1(xt)\phi(t)\mathrm{d}t=b_1(x)+O(L(x))$. From Theorem 10.36, we have that

$$b_1'(\lambda x)=\frac{b_1(\lambda)}{\lambda}\delta(x)+O\left(\frac{L(\lambda)}{\lambda}\right)\quad \text{as } \lambda\to\infty \text{ in } \mathcal{S}'(\mathbb{R})\ ,$$

since distributional asymptotics can be differentiated. Then, for x positive

$$xc'(x)=x\int_0^\infty b_1'(xt)t\phi(t)\mathrm{d}t=b_1(x)\cdot 0+O(L(x))=O(L(x))\ .$$

Set $\tau(x) = xc'(x)$. If $A > 0$, one has that $b_1(x) = c(A) + \int_A^x (\tau(t)/t)\mathrm{d}t + O(L(x))$. In the general case, let A be a number such that b and L are locally bounded on $[A, \infty)$ and let b_1 be the function from Corollary 10.38 such that $b(x) = b_1(x) + O(L(x))$, then we can apply the previous argument to b_1 to find τ as before, so we obtain (10.6.6) with $\eta(x) = b(x) - \int_A^x (\tau(t)/t)\,\mathrm{d}t = c(A) + O(L(x))$. $\square$

A change of variables $x \leftrightarrow x^{-1}$ in Theorem 10.60 implies the analog result at 0.

Theorem 10.61. *A function b is associate asymptotically homogeneously bounded of degree 0 at the origin if and only if there is a positive number A such that*

$$b(x) = \eta(x) + \int_x^A \frac{\tau(t)}{t}\,\mathrm{d}t \ , \quad x \leq A \ , \tag{10.6.7}$$

where η is a locally bounded measurable function on $(0, A]$ such that $\eta(x) = M + O(L(x))$, $x \to 0^+$, for some number M, and τ is a C^∞-function such that $\tau(x) = O(L(x))$, $x \to 0^+$.

10.6.2 Structural Theorems

The main connection between quasiasymptotically bounded distributions and the class of asymptotically homogeneously bounded functions is given in the next proposition, again the proof will be omitted since it is analogous to that of Proposition 10.22.

Proposition 10.62. *Let $f \in \mathcal{D}'(\mathbb{R})$ be quasiasymptotically bounded of degree α at infinity (at the origin) with respect to the slowly varying function L. Let $m \in \mathbb{N}$. Then, for any given F_m, an m-primitive of f in $\mathcal{D}'(\mathbb{R})$, there exist functions $b_0, \dots, b_{m-1}$, continuous on $(0, \infty)$, such that*

$$F_m(\lambda x) = \sum_{j=0}^{m-1} \lambda^{\alpha+m} b_j(\lambda) \frac{x^{m-1-j}}{(m-1-j)!} + O\left(\lambda^{\alpha+m} L(\lambda)\right) \text{ in } \mathcal{D}'(\mathbb{R}) \ , \tag{10.6.8}$$

as $\lambda \to \infty$ (resp. $\lambda \to 0^+$), where each b_j is asymptotically homogeneously bounded of degree $-\alpha - j - 1$ with respect to L.

Thus we obtain from Propositions 10.55–10.62 our first structural theorem.

Theorem 10.63. *Let $f \in \mathcal{D}'(\mathbb{R})$ and $\alpha \notin \mathbb{Z}_-$. Then f is quasiasymptotically bounded of degree α at infinity (resp. at the origin) with respect to the slowly varying function L if and only if there exist $m \in \mathbb{N}$, $m+\alpha > -1$, and m-primitive $F \in L^1_{\text{loc}}(\mathbb{R})$ (resp. locally integrable in a neighborhood of the origin) of f such that*

$$F(x) = O\left(|x|^{m+\alpha} L\left(|x|\right)\right) \ , \tag{10.6.9}$$

$|x| \to \infty$ (resp. $x \to 0$), in the ordinary sense. Moreover, in the case at infinity, f belongs to $\mathcal{S}'(\mathbb{R})$ and is quasiasymptotically bounded of degree α with respect to L in $\mathcal{S}'(\mathbb{R})$.

Proof. We only discuss the case at infinity, the proof of the assertion at the origin is similar to this case. It follows from Proposition 10.62, Proposition 10.55 and Proposition 10.56 that given $m \in \mathbb{N}$ and an m-primitive F_m, there is a polynomial p_{m-1} of degree at most $m-1$ such that

$$F_m(\lambda x) = p_{m-1}(\lambda x) + O(\lambda^{\alpha+m} L(\lambda)) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{D}'(\mathbb{R}) \ , \tag{10.6.10}$$

from the definition of boundedness in $\mathcal{D}'(\mathbb{R})$ it follows that there is an $m > -\alpha$ such that (10.6.10) holds uniformly for $x \in [-1,1]$. We let $F = F_m - p_{m-1}$, so by taking $x = -1$, $x = 1$ and replacing λ by x in (10.6.10) we obtain (10.6.9). The converse follows by observing that (10.6.9) implies that $F(\lambda x) = O(\lambda^{\alpha+m} L(\lambda))$ in $\mathcal{S}'(\mathbb{R})$ which gives the result after differentiating m-times. □

We now analyze the case of negative integral degree.

Theorem 10.64. *Let $f \in \mathcal{D}'(\mathbb{R})$ and $k \in \mathbb{Z}_+$. Then f is quasiasymptotically bounded of degree $-k$ at infinity (at the origin) with respect to L if and only if there exist $k < m \in \mathbb{Z}_+$, an asymptotically homogeneously bounded function b of* *degree 0 at infinity (at the origin) with respect to L and an m-primitive $F \in L^1_{\mathrm{loc}}(\mathbb{R})$ (resp. locally integrable near the origin) of f such that*

$$F(x) = b\left(|x|\right) x^{m-k} + O\left(|x|^{m-k} L\left(|x|\right)\right) \ , \tag{10.6.11}$$

as $|x| \to \infty$ (resp. $x \to 0$). Moreover (10.6.11) is equivalent to have

$$a^{k-m} F(ax) - (-1)^{m-k} F(-x) = O\left(x^{m-k} L(x)\right) \ , \tag{10.6.12}$$

as $x \to \infty$ (resp. $x \to 0^+$), for each $a > 0$. In the case at infinity, it follows that f is tempered and quasiasymptotically bounded of degree $-k$ with respect to L in $\mathcal{S}'(\mathbb{R})$.

Proof. Again we only give the proof of the assertion at infinity, the case at the origin is similar. If $f(\lambda x) = O(\lambda^{-k} L(\lambda))$ in $\mathcal{D}'(\mathbb{R})$, then after $k-1$ integrations Proposition 10.62 and Proposition 10.56 provide us of a $(k-1)$-primitive of f which is quasiasymptotically bounded of degree -1 at infinity with respect to L, hence we may assume that $k = 1$. Next, Proposition 10.62, Proposition 10.55 and the definition of boundedness in $\mathcal{D}'(\mathbb{R})$ give to us the existence of an $m > 1$, an asymptotically homogeneously bounded function of degree -1 with respect to L and an m-primitive F of f such that $F(\lambda x)$ is continuous for $x \in [-1, 1]$ (hence F is continuous on $\mathbb{R}$ because of the dilation parameter) and $F(\lambda x) = \lambda^{m-1} b(\lambda) x^{m-1} + O(\lambda^{m-1} L(\lambda))$ as $\lambda \to \infty$ uniformly for $x \in [-1, 1]$, by taking $x = -1$, $x = 1$ and replacing λ by x one gets (10.6.11). Assume now (10.6.11), by using Corollary 10.59, we may assume that b is locally integrable on $[0, \infty)$, this allows the application of Proposition 10.58 to deduce that $F(\lambda x) = \lambda^{m-1} b(\lambda) x^{m-1} + O(\lambda^{m-1} L(\lambda))$ as $\lambda \to \infty$ in $\mathcal{S}'(\mathbb{R})$ and hence the converse follows by differentiating m-times. That (10.6.11) implies (10.6.12) is a simple calculation; conversely, setting $b(x) = x^{k-m} F(x)$ for $x > 0$, one obtains (10.6.11). □

It is not necessary to assume that (10.6.12) holds for all $a > 0$. Indeed, we have the following result.

Theorem 10.65. *Let $f \in \mathcal{D}'(\mathbb{R})$ and $k \in \mathbb{Z}_+$. Then f is quasiasymptotically bounded at infinity (resp. at the origin) of degree $-k$ if and only if there exists m-primitive $F \in L^1_{\text{loc}}(\mathbb{R})$ (resp. locally integrable near the origin) of f, $m \geq k$, such that*

$$a^{k-m}F(ax) - (-1)^{m-k}F(-x) = O(x^{m-k}L(x)) \ , \tag{10.6.13}$$

$x \to \infty$ (resp. $x \to 0^+$), for each a merely in a subset $\mathfrak{B} \subset (0, \infty)$ having positive Lebesgue measure. In this case (10.6.13) holds uniformly for a in any compact subset of $(0, \infty)$.

Proof. Set $b(x) = F(x)/x^{m-k}$, for $x > 0$. Then, $b(ax) - b(x) = O(L(x))$, for $a \in \mathfrak{B}$. It follows from Proposition 10.54 that $b(ax) - b(x) = O(L(x))$ holds for each $a > 0$, and actually uniformly on compact subsets of $(0, \infty)$; but the latter is the same to say that (10.6.13) holds uniformly for a in any compact subset of $(0, \infty)$. □

10.7 Quasiasymptotic Extension Problems

We analyze some problems about which can be denominated as *quasiasymptotic extension problems*. Most of the results of the present section were obtained by the author in [212, 213, 227]. Let $\mathcal{U}$ and $\mathcal{A}$ be two suitable spaces of functions which are closed under dilation. Furthermore, assume that $\mathcal{U} \subset \mathcal{A}$ (not necessarily densely contained) with continuous inclusion. Suppose that $f \in \mathcal{U}'$ have quasiasymptotic behavior in $\mathcal{U}'$, that is,

$$\langle f(\lambda x), \phi(x) \rangle \sim \lambda^{\alpha} L(\lambda) \langle g(x), \phi(x) \rangle \ , \quad \forall \phi \in \mathcal{U}. \tag{10.7.1}$$

Suppose that either $f \in \mathcal{A}'$ or there is a suitable extension of f to $\mathcal{A}$. Sometimes, when corresponds, the existence of the extension is part of the problem. We are interested in the quasiasymptotic properties f (or its extensions) in $\mathcal{A}'$. We may classify the quasiasymptotic extension problems into two categories, each of them having subcategories.

1. $\mathcal{U}$ is dense in $\mathcal{A}$ (consequently, $\mathcal{A}' \subset \mathcal{U}'$). We may ask:

(QEP1.1) Suppose we know a priori $f \in \mathcal{A}'$. Does (10.7.1) hold for all $\phi \in \mathcal{A}$?

(QEP1.2) Would (10.7.1) be enough to conclude $f \in \mathcal{A}'$ and that (10.7.1) remains valid in $\mathcal{A}'$?

2. $\mathcal{U}$ is not dense in $\mathcal{A}$. We obtain a canonical map $\mathcal{A}' \to \mathcal{U}'$ via restriction of functionals (which is not necessarily onto nor one-to-one). The image of this map is precisely the set of elements of $\mathcal{U}'$ admitting extensions to $\mathcal{A}$. We may ask:

(QEP2.1) Suppose that f admits extensions to $\mathcal{A}$. What are the quasiasymptotic properties in $\mathcal{A}'$ of such extensions?

(QEP2.2) Would (10.7.1) be enough to conclude f has extensions to $\mathcal{A}$? In a positive case, what are the quasiasymptotic properties in $\mathcal{A}'$ of such extensions?

Observe that the problems just discussed also make sense for quasiasymptotic boundedness.

The positive answer for (QEP1.1) for $\mathcal{U}' = \mathcal{D}'(\mathbb{R})$, $\mathcal{A}' = \mathcal{S}'(\mathbb{R})$, and distributional point values has been widely used in the previous chapters in connection with Fourier inverse problems. In Section 10.7.2 we will treat the same question for the general quasiasymptotic behavior (and boundedness) at finite points.

The reader should have noticed that (QEP1.2) has been implicitly studied in the previous sections for quasiasymptotics at infinity in $\mathcal{D}'(\mathbb{R})$. Indeed, in Sections

10.5 and 10.6, we showed that if $f \in \mathcal{D}'(\mathbb{R})$ has quasiasymptotic behavior or is quasiasymptotic bounded at infinity with respect to a regularly varying function, then $f \in \mathcal{S}'(\mathbb{R})$ and the same quasiasymptotic properties are preserved in $f \in \mathcal{S}'(\mathbb{R})$. We will make a further study of this case in Section 10.7.3 for spaces of the form $\mathcal{A}' = \mathcal{K}'_{\beta}(\mathbb{R})$.

We will study (QEP2.1) and (QEP2.2) for $\mathcal{U}' = \mathcal{D}'(0,\infty)$ and $\mathcal{A}' = \mathcal{D}'(\mathbb{R})$ in Section 10.7.1.

10.7.1 Quasiasymptotic Extension from $(0,\infty)$ to $\mathbb{R}$

The purpose of this section is to study the extensions of distributions to $\mathbb{R}$ which are initially defined off the origin and have a prescribed asymptotic behavior, that is, $f \in \mathcal{D}'(\mathbb{R} \setminus \{0\})$ with a prescribed quasiasymptotic behavior at either the origin or infinity.

We want to make some comments about extension of distributions initially defined in $\mathbb{R}\setminus\{0\}$ to $\mathbb{R}$. Observe that this problem is of vital importance for renormalization procedures in Quantum Field Theory ([21, 125, 233, 234]). For simplicity, we discuss the problem of extending a distribution from $\mathbb{R}_+ = (0,\infty)$ to $\mathbb{R}$, the general case can be obviously reduced to this one.

Recall that the spaces $\mathcal{D}'(\overline{\mathbb{R}}_+)$ and $\mathcal{S}'(\overline{\mathbb{R}}_+)$, duals of $\mathcal{D}(\overline{\mathbb{R}}_+)$ and $\mathcal{S}(\overline{\mathbb{R}}_+)$, respectively, are identifiable [231, p.13] with the spaces of distributions and tempered distributions supported on $\overline{\mathbb{R}}_+$, respectively. Therefore, in discussing extensions of distributions defined on $\mathbb{R}_+$ to $\mathbb{R}$ is enough to considered the extension to the interval $\overline{\mathbb{R}}_+ = [0,\infty)$. In general, it is not true that a distribution $f_0 \in \mathcal{D}'(\mathbb{R}_+)$ should have an extension to $\mathcal{D}'(\overline{\mathbb{R}}_+)$. The necessary and sufficient condition [61] for a distribution $f_0 \in \mathcal{D}'(\mathbb{R}_+)$ to admit extensions to $\mathcal{D}'(\overline{\mathbb{R}}_+)$ is the existence of $\beta \in \mathbb{R}$ such that

$$f_0(\varepsilon x) = O(\varepsilon^{\beta}) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{D}'(\mathbb{R}_+). \tag{10.7.2}$$

We will recover this extension characterization below. We call $f_0 \in \mathcal{D}'(\mathbb{R}_+)$ extendable to $\overline{\mathbb{R}}_+$ if it satisfies this condition. In relation to the extendable distributions the notation $\mathcal{D}_{31}(\mathbb{R}_+)$ is used in [60, p.179] for those test functions from $\mathcal{D}(\mathbb{R})$ having support on $\overline{\mathbb{R}}_+$. Its dual is $\mathcal{D}'_{31}(\mathbb{R}_+)$. Notice that $\mathcal{D}(\mathbb{R}_+)$ is dense in $\mathcal{D}_{31}(\mathbb{R}_+)$; consequently, $\mathcal{D}'_{31}(\mathbb{R}_+) \subseteq \mathcal{D}'(\mathbb{R}_+)$. The space $\mathcal{D}_{31}(\mathbb{R}_+)$ is closed in $\mathcal{D}(\overline{\mathbb{R}}_+)$; hence every distribution of $\mathcal{D}'_{31}(\mathbb{R}_+)$, in view of Hanh-Banach theorem, admits an extension to $\mathcal{D}'(\overline{\mathbb{R}}_+)$. Moreover, $\mathcal{D}'_{31}(\mathbb{R}_+)$ coincides with the extendable distributions in $\mathcal{D}'(\mathbb{R}_+)$.

We now analyze our first extension problem where we suppose that f_0, defined on $(0, \infty)$, has quasiasymptotic behavior.

Theorem 10.66. *Let $f_0 \in \mathcal{D}'(\mathbb{R}_+)$ have the quasiasymptotic behavior*

$$f_0(\varepsilon x) \sim \varepsilon^\alpha L(\varepsilon) g(x) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{D}'(\mathbb{R}_+) \ . \tag{10.7.3}$$

Then f_0 is extendable to $\overline{\mathbb{R}}_+$. Moreover, if $f \in \mathcal{D}'(\overline{\mathbb{R}}_+)$ is an extension of f_0 to $\overline{\mathbb{R}}_+$, one has that:

(i) If $\alpha \notin \mathbb{Z}_-$, then there exist constants $a_0, a_1, \ldots, a_{m-1}$ such that

$$f(\varepsilon x) = \varepsilon^\alpha L(\varepsilon) g(x) + \sum_{j=0}^{m-1} a_j \frac{\delta^{(j)}(x)}{\varepsilon^{j+1}} + o(\varepsilon^\alpha L(\varepsilon)) \tag{10.7.4}$$

as $\varepsilon \to 0^+$ in $\mathcal{D}'(\mathbb{R})$.

(ii) If $\alpha = -k, k \in \mathbb{Z}_+$, then g is of the form $g(x) = C \operatorname{Pf}\left(H(x)/x^k\right)$ and there exist an associate asymptotically homogeneous function b satisfying

$$b(ax) = b(x) + \frac{(-1)^{k-1}}{(k-1)!} C L(x) \log a + o(L(x)), \quad x \to 0^+ \ , \tag{10.7.5}$$

for each $a > 0$, and constants $a_k, a_{k+1}, \ldots, a_{m-1}$ such that

$$f(\varepsilon x) = C \frac{L(\varepsilon)}{\varepsilon^k} \operatorname{Pf}\left(\frac{H(x)}{x^k}\right) + \frac{b(\varepsilon)}{\varepsilon^k} \delta^{(k-1)}(x) + \sum_{j=k}^{m-1} a_j \frac{\delta^{(j)}(x)}{\varepsilon^{j+1}} + o\left(\frac{L(\varepsilon)}{\varepsilon^k}\right) \tag{10.7.6}$$

as $\varepsilon \to 0^+$ in $\mathcal{D}'(\mathbb{R})$.

Proof. (i) Since α is not a negative integer and the quasiasymptotic behavior (10.7.3) holds on the positive part of the real line we have that

$$g(x) = \frac{Cx_+^\alpha}{\Gamma(\alpha+1)} \text{ for some constant } C \; .$$

In Proposition 10.22, we may replace the space $\mathcal{D}'(\mathbb{R})$ by $\mathcal{D}'(\mathbb{R}_+)$; as in the proof of Theorem 10.23, we have that there are a positive integer $m > -\alpha$, an m-primitive F_m of f_0 in $\mathcal{D}'(\mathbb{R}_+)$, which is continuous on the interval $(0,1)$, and a polynomial p such that

$$F_m(\varepsilon x) = C_+ L(\varepsilon)\frac{(\varepsilon x)_+^{\alpha+m}}{\Gamma(\alpha+m+1)} + o(\varepsilon^{\alpha+m}L(\varepsilon)) \qquad (10.7.7)$$

as $\varepsilon \to 0^+$, uniformly for $x \in [1/2, 1]$. Setting $x = 1$ and replacing x by ε, we obtain that

$$F_m(x) = C_+ L(x)\frac{x_+^{\alpha+m}}{\Gamma(\alpha+m+1)} + o(x^{\alpha+m}L(x)) \; ,$$

in the ordinary sense. Therefore, F is actually continuous on $[0,1)$ and the asymptotic formula (10.7.7) holds in $\mathcal{D}'(\mathbb{R})$. Let $f_1 = F_m^{(m)}$, differentiating (10.7.7) m-times, we see that f_1 has the quasiasymptotic behavior (10.7.3) in $\mathcal{D}'(\mathbb{R})$, and f_1 is an extension of f_0. The rest follows from the observation that $f - f_1$ is a distribution concentrated at the origin, and hence it is a sum of the Dirac delta distribution and its derivatives.

(ii) Let us observe that if we take the space $\mathcal{D}'(\mathbb{R}_+)$ instead of $\mathcal{D}'(\mathbb{R})$ in Proposition 10.22 and Lemma 10.43, they still hold. Hence, the arguments given in Theorem 10.44 are still applicable to conclude the existence of $m \in \mathbb{N}$, $m > k$, and F_m, an m-primitive of f_0 in $\mathcal{D}'(\mathbb{R}_+)$, which is continuous on the interval $(0,1)$, such that

$$F_m(x) = b_1(x)\frac{x^{m-k}}{(m-k)!} - \frac{(-1)^{k-1}C}{(k-1)!}L(x)\frac{x^{m-k}}{(m-k)!}\sum_{j=1}^{m-k}\frac{1}{j} + o(x^{m-k}L(x)), \quad x \to 0^+ \; ,$$

in the ordinary sense, where the function b_1 satisfies (10.7.5). Notice that F_m is then continuous on $[0,1)$. By Theorem 10.37, we have

$$F_m(\varepsilon x) = b_1(\varepsilon)\frac{(\varepsilon x)_+^{m-k}}{(m-k)!} + \frac{(-1)^{k-1}C}{(k-1)!}\varepsilon^{m-k}L(\varepsilon)l_{m-k}(x)H(x) + o(\varepsilon^{m-k}L(\varepsilon)) \ ,$$

as $\varepsilon \to 0^+$ in $\mathcal{D}'(\mathbb{R})$, where

$$l_{m-k}(x) = \frac{x^{m-k}}{(m-k)!}\log x - \frac{x^{m-k}}{(m-k)!}\sum_{j=1}^{m-k}\frac{1}{j} \ .$$

Differentiating the last expression $(m-k)$-times, we get

$$F_m^{(m-k)}(\varepsilon x) = b_1(\varepsilon)H(x) + \frac{(-1)^{k-1}C}{(k-1)!}L(\varepsilon)H(x)\log x + o(L(\varepsilon)) \ , \tag{10.7.8}$$

as $\varepsilon \to 0^+$ in $\mathcal{D}'(\mathbb{R})$. Set now $f_1 = F_m^{(m)} \in \mathcal{D}'(\overline{\mathbb{R}}_+)$, k more differentiations of (10.7.8) and the formula

$$\frac{d^{k-1}}{dx^{k-1}}\left(\mathrm{Pf}\left(\frac{H(x)}{x}\right)\right) = (-1)^{k-1}(k-1)!\mathrm{Pf}\left(\frac{H(x)}{x^k}\right) - \delta^{(k-1)}(x)\sum_{j=1}^{k}\frac{1}{j} \ ,$$

imply that

$$f_1(\varepsilon x) = C\frac{L(\varepsilon)}{\varepsilon^k}\mathrm{Pf}\left(\frac{H(x)}{x^k}\right) + \frac{b(\varepsilon)}{\varepsilon^k}\delta^{(k-1)}(x) + o\left(\frac{L(\varepsilon)}{\varepsilon^k}\right)$$

with $b(x) = b_1(x) - \dfrac{(-1)^{k-1}C}{(k-1)!}L(x)\displaystyle\sum_{j=1}^{k}\frac{1}{j}$. Since f_1 is an extension of f_0, then $f - f_1$ is concentrated at the origin, and hence we obtain (10.7.6). $\square$

Remark 10.67. *Theorem 10.66 extends the properties obtained by S. Łojasiewicz in [128] about the limit of a distribution at a point.*

We have a similar assertion for quasiasymptotic boundedness. The proof is almost the same as the case of quasiasymptotic behavior, we leave the details to the reader.

Theorem 10.68. *Let L be slowly varying at the origin. Let $f_0 \in \mathcal{D}'(\mathbb{R}_+)$ be such that*

$$f_0(\varepsilon x) = O(\varepsilon^\alpha L(\varepsilon)) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{D}'(\mathbb{R}_+) \ . \tag{10.7.9}$$

Then f_0 is extendable to $\overline{\mathbb{R}}_+$. Moreover, if $f \in \mathcal{D}'(\overline{\mathbb{R}}_+)$ is an extension of f_0 to $\overline{\mathbb{R}}_+$, one has that:

(i) If $\alpha \notin \mathbb{Z}_-$, then there exist constants $a_0, a_1, \dots, a_{m-1}$ such that

$$f(\varepsilon x) = \sum_{j=0}^{m-1} a_j \frac{\delta^{(j)}(x)}{\varepsilon^{j+1}} + O(\varepsilon^\alpha L(\varepsilon)) \tag{10.7.10}$$

as $\varepsilon \to 0^+$ in $\mathcal{D}'(\mathbb{R})$.

(ii) If $\alpha = -k, k \in \mathbb{Z}_+$, then there exist an asymptotically homogeneously bounded function b of degree 0 with respect to L and constants $a_k, a_{k+1}, \dots, a_{m-1}$ such that

$$f(\varepsilon x) = \frac{b(\varepsilon)}{\varepsilon^k}\delta^{(k-1)}(x) + \sum_{j=k}^{m-1} a_j \frac{\delta^{(j)}(x)}{\varepsilon^{j+1}} + O\left(\frac{L(\varepsilon)}{\varepsilon^k}\right) \tag{10.7.11}$$

as $\varepsilon \to 0^+$ in $\mathcal{D}'(\mathbb{R})$.

Therefore, we recover the characterization of extendable distributions.

Corollary 10.69. *A distribution $f_0 \in \mathcal{D}'(\mathbb{R}_+)$ is extendable to $\mathbb{R}$ if and only if (10.7.2) is satisfied.*

Proof. The first half of the statement follows from Theorem 10.68. On the other hand if f_0 is extendable to $\mathbb{R}$, find $m \in \mathbb{N}$ and F continuous in a neighborhood of the origin such that $F^m = f$; since F is bounded near the origin, then $F(\varepsilon x) = O(1)$, differentiating m-times, we obtain that $f(\varepsilon x) = O(\varepsilon^{-m})$ in $\mathcal{D}'(\mathbb{R}_+)$. $\square$

We now turn our attention to asymptotics at infinity. Suppose that a distribution $f \in \mathcal{D}'(\mathbb{R})$ with support in $[0, \infty)$ has quasiasymptotic behavior of degree α in the space $\mathcal{D}'(\mathbb{R}_+)$, that is, for each $\phi \in \mathcal{D}(\mathbb{R}_+)$

$$\langle f(\lambda x), \phi(x)\rangle \sim \lambda^\alpha L(\lambda) \langle g(x), \phi(x)\rangle \ . \tag{10.7.12}$$

What can we say about the quasiasymptotic properties of f in $\mathcal{D}'(\mathbb{R})$?

We can also apply the technique of Theorem 10.66 to give a complete answer to this question. The answer depends on α. We formulate the next theorem in more general terms. Recall that $\mathcal{S}(\mathbb{R}_+)$ is the closed subspace of $\mathcal{S}(\mathbb{R})$ consisting of functions supported in $[0,\infty)$. It is dual space $\mathcal{S}'(\mathbb{R}_+)$ coincides with the extendable distributions of $\mathcal{D}'(\mathbb{R}_+)$ which have tempered behavior at infinity.

Theorem 10.70. *Let $f_0 \in \mathcal{D}'(\mathbb{R}_+)$ be an extendable distribution to $\overline{\mathbb{R}}_+$. Let L be slowly varying at infinity and $\alpha \in \mathbb{R}$. Suppose that*

$$f_0(\lambda x) \sim \lambda^\alpha L(\lambda) g(x) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{D}'(\mathbb{R}_+). \tag{10.7.13}$$

Then $f_0 \in \mathcal{S}'(\mathbb{R}_+)$ and the quasiasymptotic behavior holds in $\mathcal{S}'(\mathbb{R}_+)$. Moreover, let $f \in \mathcal{S}'(\overline{\mathbb{R}}_+)$ be any extension of f_0.

(i) If $\alpha > -1$, then f has the quasiasymptotic behavior (10.7.13) in $\mathcal{S}'(\mathbb{R})$.

(ii) If $\alpha < -1$ and $\alpha \notin \mathbb{Z}_-$, then there exist constants $a_0, \dots, a_{n-1}$, $n < -\alpha$, such that

$$f(\lambda x) = \sum_{j=0}^{n-1} a_j \frac{\delta^{(j)}(x)}{\lambda^{j+1}} + \lambda^\alpha L(\lambda) g(x) + o(\lambda^\alpha L(\lambda)) \tag{10.7.14}$$

as $\lambda \to \infty$ in $\mathcal{S}'(\mathbb{R})$. The constants depend on the choice of the extension f.

(iii) If $\alpha = -k$, $k \in \mathbb{Z}_+$, then g is of the form $g(x) = C \operatorname{Pf}\left(H(x)/x^k\right)$ and there are $(k-1)$ constants $a_0, \dots, a_{k-2}$ and an associate asymptotically homogeneous function of degree 0 with respect to L satisfying

$$b(ax) = b(x) + \frac{(-1)^{k-1}}{(k-1)!} C L(x) \log a + o(L(x)), \quad x \to \infty\ , \tag{10.7.15}$$

such that

$$f(\lambda x) = C\frac{L(\lambda)}{\lambda^k} \operatorname{Pf}\left(\frac{H(x)}{x^k}\right) + \frac{b(\lambda)}{\lambda^k} \delta^{(k-1)}(x) + \sum_{j=0}^{k-2} a_j \frac{\delta^{(j)}(x)}{\lambda^{j+1}} + o\left(\frac{L(\lambda)}{\lambda^k}\right) \tag{10.7.16}$$

as $\lambda \to \infty$ in $\mathcal{S}'(\mathbb{R})$. The constants and the function b depend on the choice of the extension f.

Proof. Let $f \in \mathcal{S}'(\overline{\mathbb{R}}_+)$ be an extension of f_0.

(i) Let us start with the case $\alpha > -1$. Clearly g must be of the form $Cx_{+}^{\alpha}/\Gamma(\alpha+1)$, for some constant C. Next, Proposition 10.22 still holds replacing the space $\mathcal{D}'(\mathbb{R})$ by $\mathcal{D}'(\mathbb{R}_{+})$ (actually this holds without the restriction $\alpha > -1$). Hence, the same argument given in Theorem 10.23 applies here, but this time we only require the uniform convergence on $[1/2, 2]$, and hence we can still conclude the existence of the integer such that (10.5.21) holds with the limit taken only as $x \to \infty$. Actually, because $\alpha > -1$, relation (10.5.21) holds for any m-primitive of f. Let $f^{(-m)}$ be the m-primitive of f supported on the interval $[0, \infty)$, then we have that

$$f^{(-m)}(x) \sim \frac{Cx^{\alpha+m}L(x)}{\Gamma(\alpha+m+1)} \ , \quad x \to \infty \ ,$$

so we have that $f^{(-m)}(\lambda x) = CL(\lambda)(\lambda x)_{+}^{\alpha+m}/\Gamma(\alpha+m+1) + o(\lambda^{\alpha+m}L(\lambda))$ in the space $\mathcal{S}'(\mathbb{R})$, differentiating m-times, we obtain the result. (ii) Suppose now that $\alpha < -1$ and $\alpha \notin \mathbb{Z}_{-}$. This case differs from the last one essentially in one point, we cannot conclude (10.5.21) for every m-primitive of f but only for some of them. In any case, if $f^{(-m)}$ is the m-primitive (we keep $m > -\alpha - 1$) of f supported on $[0, \infty)$, we have that there exists a polynomial of degree at most $m - 1$ such that

$$f^{(-m)}(x) - p(x) \sim \frac{Cx^{\alpha+m}L(x)}{\Gamma(\alpha+m+1)} \ , \quad x \to \infty \ ;$$

therefore,

$$f^{(-m)}(\lambda x) = \frac{CL(\lambda)(\lambda x)_{+}^{\alpha+m}}{\Gamma(\alpha+m+1)} + \sum_{j=0}^{m-1} a_j (\lambda x)_{+}^{j} + o(\lambda^{\alpha+m}L(\lambda)) \quad \text{as } \lambda \to \infty \ ,$$

in the space $\mathcal{S}'(\mathbb{R})$, for some constants $a_0, \dots, a_{m-1}$. Thus, after m differentiations and a small rearrangement of constants, we obtain (10.7.14).

(iii) Reasoning as in the previous two cases, we obtain the existence of a positive integer $m > k$ such that $f^{(-m)}$ is continuous and

$$f^{(-m)}(x) = b_1(x)\frac{x^{m-k}}{(m-k)!} - \frac{(-1)^{k-1}C}{(k-1)!}L(x)\frac{x^{m-k}}{(m-k)!}\sum_{j=1}^{m-k}\frac{1}{j} + p_{m-1}(x) + o(x^{m-k}L(x)) \ ,$$

$x \to \infty$, where b_1 is a locally integrable associate asymptotically homogeneous function satisfying (10.7.15) and p_{m-1} is a polynomial of degree at most $m-1$. Throwing away the irrelevant terms of the polynomial p_{m-1} and using Theorem 10.36, we obtain the following asymptotic expansion as $\lambda \to \infty$ in the space $\mathcal{S}'(\mathbb{R})$,

$$\begin{aligned} f^{(-m)}(\lambda x) =& b_1(\lambda)\frac{(\lambda x)_+^{m-k}}{(m-k)!} + \frac{(-1)^{k-1}C}{(k-1)!}\lambda^{m-k}L(\lambda)l_{m-k}(x)H(x) \\ &+ \sum_{j=0}^{k-1} a_j \frac{(\lambda x)_+^{m-j-1}}{(m-j-1)!} + o(\lambda^{m-k}L(\lambda)) \ . \end{aligned}$$

Differentiating $(m-k)$-times this expansion, we have that

$$f^{(-m)}(\lambda x) = b_1(\lambda)H(x) + \frac{(-1)^{k-1}C}{(k-1)!}L(\lambda)H(x)\log x + \sum_{j=0}^{k-1} a_j \frac{(\lambda x)_+^{k-j-1}}{(k-j-1)!} + o(L(\lambda)) \ . \tag{10.7.17}$$

The well known formula

$$\frac{d^{k-1}}{dx^{k-1}}\left(\mathrm{Pf}\left(\frac{H(x)}{x}\right)\right) = (-1)^{k-1}(k-1)!\,\mathrm{Pf}\left(\frac{H(x)}{x^k}\right) - \delta^{(k-1)}(x)\sum_{j=1}^{k-1}\frac{1}{j} \ .$$

and k-times differentiations of (10.7.17) imply (10.7.16) with

$$b(x) = b_1(x) + \frac{(-1)^k C}{(k-1)!}\left(\sum_{j=1}^{k-1}\frac{1}{j}\right)L(x) \ .$$

□

Likewise, one shows.

Theorem 10.71. *Let $f_0 \in \mathcal{D}'(\mathbb{R}_+)$ be an extendable distribution to $\overline{\mathbb{R}}_+$. Let L be slowly varying at infinity and $\alpha \in \mathbb{R}$. Suppose that*

$$f_0(\lambda x) = O(\lambda^\alpha L(\lambda)) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{D}'(\mathbb{R}_+). \tag{10.7.18}$$

Then $f_0 \in \mathcal{S}'(\mathbb{R}_+)$ and (10.7.18) holds in $\mathcal{S}'(\mathbb{R}_+)$. Moreover, let $f \in \mathcal{S}'(\overline{\mathbb{R}}_+)$ be any extension of f_0.

(i) If $\alpha > -1$, then f is quasiasymptotically bounded of degree α at infinity with respect to L in $\mathcal{S}'(\mathbb{R})$.

(ii) If $\alpha < -1$ and $\alpha \notin \mathbb{Z}_-$, then there exist constants $a_0, \dots, a_{n-1}$, $n < -\alpha$, such that

$$f(\lambda x) = \sum_{j=0}^{n-1} a_j \frac{\delta^{(j)}(x)}{\lambda^{j+1}} + O(\lambda^\alpha L(\lambda)) \tag{10.7.19}$$

as $\lambda \to \infty$ in $\mathcal{S}'(\mathbb{R})$. The constants depend on the choice of the extension f.

(iii) If $\alpha = -k$, $k \in \mathbb{Z}_+$, then there are $(k-1)$ constants $a_0, \dots, a_{k-2}$ and an associate asymptotically homogeneously bounded function b of degree 0 with respect to L such that

$$f(\lambda x) = \frac{b(\lambda)}{\lambda^k}\delta^{(k-1)}(x) + \sum_{j=0}^{k-2} a_j \frac{\delta^{(j)}(x)}{\lambda^{j+1}} + O\left(\frac{L(\lambda)}{\lambda^k}\right) \tag{10.7.20}$$

as $\lambda \to \infty$ in $\mathcal{S}'(\mathbb{R})$. The constants and the function b depend on the choice of the extension f.

Example 10.72. *Theorems 10.66 and 10.70 show that if $\alpha \notin \mathbb{Z}_-$ we may select an extension having quasiasymptotic behavior. For the case $\alpha \in \mathbb{Z}_-$ this not longer true. Moreover, it is absolutely necessary to consider associate asymptotically homogeneous functions of degree 0 in Theorems 10.66 and 10.70, as shown by the following example. Consider*

$$f_0(x) = 2\frac{\log x}{x} \in \mathcal{D}'(\mathbb{R}_+) \ .$$

Then, $f(x) = g'(x)$, where $g(x) = H(x)\log^2 x$, is an extension of f_0 to $[0, \infty)$. Now

$$g(ax) = g(x) + 2\log a \log x + o(|\log x|) \ ,$$

as $x \to 0^+$ and $x \to \infty$. So g is associate asymptotically homogeneous of degree 0. Then we obtain the asymptotic expansions

$$g(\varepsilon x) = g(\varepsilon)H(x) - 2\log\varepsilon^{-1}H(x)\log x + o(\log\varepsilon^{-1}) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{S}'(\mathbb{R}) \ ,$$

and

$$g(\lambda x) = g(\lambda)H(x) + 2\log\lambda H(x)\log x + o(\log\lambda) \quad \textit{as } \lambda \to \infty \textit{ in } \mathcal{S}'(\mathbb{R}) \ ,$$

thus,

$$f(\varepsilon x) = \frac{g(\varepsilon)}{\varepsilon}\delta(x) - 2\frac{\log\varepsilon^{-1}}{\varepsilon}\mathrm{Pf}\left(\frac{H(x)}{x}\right) + o\left(\frac{\log\varepsilon^{-1}}{\varepsilon}\right) \quad \textit{as } \varepsilon \to 0^+ \textit{ in } \mathcal{S}'(\mathbb{R}) \ ,$$

and

$$f(\lambda x) = \frac{g(\lambda)}{\lambda}\delta(x) + 2\frac{\log\lambda}{\lambda}\mathrm{Pf}\left(\frac{H(x)}{x}\right) + o\left(\frac{\log\lambda}{\lambda}\right) \quad \textit{as } \lambda \to \infty \textit{ in } \mathcal{S}'(\mathbb{R}) \ .$$

The latter two expansions show that

$$f_0(\varepsilon x) = -2\frac{\log\varepsilon^{-1}}{\varepsilon x} + o\left(\frac{\log\varepsilon^{-1}}{\varepsilon}\right) \quad \textit{as } \varepsilon \to 0^+ \textit{ in } \mathcal{D}'(\mathbb{R}_+) \ ,$$

and

$$f_0(\lambda x) = 2\frac{\log\lambda}{\lambda x} + o\left(\frac{\log\lambda}{\lambda}\right) \quad \textit{as } \lambda \to \infty \textit{ in } \mathcal{D}'(\mathbb{R}_+) \ ,$$

but it is impossible to choose constants $a_0, ..., a_n$ which make disappear the function g in the expansion of an arbitrary extension $f + \sum_{j=0}^{n} a_j \delta^{(j)}$ of f_0. A counterexample for $\alpha = -k$ is constructed by considering $f_0^{(k-1)}$.

Example 10.73. *While for $\alpha \notin \mathbb{Z}_-$ Theorems 10.68 and 10.71 imply that we can select an extension which is also quasiasymptotically bounded, this is not longer true for $\alpha \in \mathbb{Z}_-$. In other words, for the negative integral degrees, it is absolutely necessary to consider asymptotically homogeneously bounded functions in Theorems 10.68 and 10.71. For instance, let g, f and f_0 be the function and the distributions from Example 10.72. Then, $g(x)$ is asymptotically homogeneously bounded of degree 0 with respect to $|\log x|$, both at infinity and the origin. Observe that $f_0^{(k-1)}$ is* *quasiasymptotically bounded in the space $\mathcal{D}'(\mathbb{R}_+)$ of degree $-k$ at both 0 and ∞ with respect to $|\log x|$. On the other hand,*

$$f^{(k-1)}(\varepsilon x) = \frac{g(\varepsilon)}{\varepsilon^k}\delta^{(k-1)}(x) + O\left(\frac{\log \varepsilon^{-1}}{\varepsilon^k}\right) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{S}'(\mathbb{R}) \ ,$$

and

$$f^{(k-1)}(\lambda x) = \frac{g(\lambda)}{\lambda^k}\delta^{(k-1)}(x) + O\left(\frac{\log \lambda}{\lambda^k}\right) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R}) \ ,$$

which show that $f_0^{(k-1)}$ has no extension to $[0,\infty)$ being quasiasymptotically bounded of degree $-k$ with respect to $|\log x|$.

10.7.2 Extensions of Quasiasymptotics at the Origin from $\mathcal{D}'(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$

We now study the following problem. Suppose that $f \in \mathcal{S}'(\mathbb{R})$ has quasiasymptotic behavior at the origin in the space $\mathcal{S}'(\mathbb{R})$, does f have the same quasiasymptotic behavior in $\mathcal{S}'(\mathbb{R})$? Such a question was posted as an open problem in [153, Remark 2], where a partial answer was given under the assumptions of boundedness for L and restrictions under the degree of the quasiasymptotic. We obtained a positive solution in [227] based in the structural theorems for quasiasymptotics at the origin; it will be the approach to be followed here. The solution is given by the following theorem, which we formulate for quasiasymptotics at finite points. The author was recently informed about a more general problem which was treated by Zavialov in [250] (though he has been unable to get a copy of the article).

Theorem 10.74. *Let $f \in \mathcal{S}'(\mathbb{R})$. If f has quasiasymptotic behavior at $x = x_0$ in $\mathcal{D}'(\mathbb{R})$, then f has the same quasiasymptotic behavior at $x = x_0$ in in the space $\mathcal{S}'(\mathbb{R})$.*

Proof. We may assume that $x_0 = 0$. Let α be the degree of the quasiasymptotic behavior. We shall divide the proof into three cases:

$\alpha \notin \{-1, -2, -3, \dots\}$,

$\alpha = -1,$

$\alpha = -2, -3, \dots .$

Suppose its degree is $\alpha \notin \mathbb{Z}_-$ and

$$f(\varepsilon x) = C_- L(\varepsilon) \frac{(\varepsilon x)_-^\alpha}{\Gamma(\alpha+1)} + C_+ L(\varepsilon) \frac{(\varepsilon x)_+^\alpha}{\Gamma(\alpha+1)} + o\left(\varepsilon^\alpha L(\varepsilon)\right), \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{D}'(\mathbb{R}).$$

Then, by using Theorem 10.28 and the fact $f \in \mathcal{S}'(\mathbb{R})$, we conclude the existence of an integer m, a real number β such that $m > -\alpha$, $\beta > m + \alpha$, and a continuous m-primitive F of f such that

$$F(x) = \frac{|x|^{m+\alpha}}{\Gamma(m+\alpha+1)} L(|x|) \left((-1)^m C_- H(-x) + C_+ H(x)\right) + o\left(|x|^{m+\alpha} L(|x|)\right) ,$$

$x \to 0^+$, and

$$F(x) = O\left(|x|^\beta\right), \quad |x| \to \infty . \tag{10.7.21}$$

We make the usual assumptions over L. Assume (Section 20.10.1) that L is positive, defined in $(0, \infty)$ and there exists $M_1 > 0$ such that

$$\frac{L(\varepsilon x)}{L(\varepsilon)} \leq M_1 \max\left\{x^{-\frac{1}{2}}, x^{\frac{1}{2}}\right\}, \quad \varepsilon, x \in (0, \infty) . \tag{10.7.22}$$

Let $\phi \in \mathcal{S}(\mathbb{R})$, then we can decompose $\phi = \phi_1 + \phi_2 + \phi_3$, where $\operatorname{supp} \phi_1 \subseteq (-\infty, 1]$, $\operatorname{supp} \phi_2$ is compact and $\operatorname{supp} \phi_3 \subseteq [1, \infty)$. Observe that since $\phi_2 \in \mathcal{D}(\mathbb{R})$ we have that

$$\langle f(\varepsilon x), \phi_2(x) \rangle \sim \varepsilon^\alpha L(\varepsilon) \left\langle \frac{C_- x_-^\alpha + C_+ x_+^\alpha}{\Gamma(\alpha+1)}, \phi_2(x) \right\rangle , \quad \varepsilon \to 0^+ . \tag{10.7.23}$$

So, if we want to show (10.7.23) for ϕ, it is enough to show it for ϕ_3 placed instead of ϕ_2 in the relation because by symmetry it would follow for ϕ_1 and hence for ϕ. Set

$$G(x) = \frac{F(x)}{x^{\alpha+m} L(x)}, \ x > 0.$$

Then

$$\lim_{x\to 0^+} G(x) = \frac{C_+}{\Gamma(\alpha+m+1)} \ , \tag{10.7.24}$$

On combining (10.7.21), (10.7.22) and (10.7.24), we find a constant $M_2 > 0$ such that

$$|G(x)| < M_2(1 + x^{\beta+\frac{1}{2}-m-\alpha}), \ x > 0 \ . \tag{10.7.25}$$

Relation (10.7.25) together with (10.7.22) show that for $\varepsilon \leq 1$,

$$\left| G(\varepsilon x)\frac{L(\varepsilon x)}{L(\varepsilon)} x^{\alpha+m}\phi_3^{(m)}(x) \right| \leq 2M_1M_2 x^{\beta+1} \left|\phi_3^{(m)}(x)\right| H(x-1) \ .$$

The right hand side of the last estimate belongs to $L^1(\mathbb{R})$ and thus we can use the Lebesgue dominated convergence theorem to obtain,

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^\alpha L(\varepsilon)} \langle f(\varepsilon x), \phi_3(x)\rangle &= \lim_{\varepsilon\to 0^+} (-1)^m \int_0^\infty G(\varepsilon x)\frac{L(\varepsilon x)}{L(\varepsilon)} x^{\alpha+m}\phi_3^{(m)}(x)\mathrm{d}x \\ &= (-1)^m \frac{C_+}{\Gamma(\alpha+m+1)} \int_0^\infty x^{\alpha+m}\phi_3^{(m)}(x)\mathrm{d}x \\ &= C_+ \left\langle \frac{x_+^\alpha}{\Gamma(\alpha+1)}, \phi_3(x) \right\rangle \ .\end{aligned}$$

This shows the result in the case $\alpha \notin \{-1,-2,-3,\dots\}$.

We now aboard the case $\alpha = -1$. Assume that

$$f(\varepsilon x) = \gamma\varepsilon^{-1}L(\varepsilon)\delta(x) + \beta\varepsilon^{-1}L(\varepsilon)x^{-1} + o\left(\varepsilon^{-1}L(\varepsilon)\right) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{D}'(\mathbb{R}) \ .$$

As in the last case, it suffices to assume that $\phi \in \mathcal{S}(\mathbb{R})$, $\operatorname{supp}\phi \subseteq [1,\infty)$ and show that

$$\lim_{\varepsilon\to 0^+} \frac{\varepsilon}{L(\varepsilon)} \langle f(\varepsilon x), \phi(x)\rangle = \beta \int_1^\infty \frac{\phi(x)}{x}\mathrm{d}x \ .$$

We may proceed as in the previous case to apply the structural theorem, but we rather reduce it to the previous situation. So, set $g(x) = xf(x)$, then

$$g(\varepsilon x) = \beta L(\varepsilon) + o(L(\varepsilon)) \ \text{ as } \varepsilon \to 0^+ \text{ in } \mathcal{D}'(\mathbb{R}) \ . \tag{10.7.26}$$

But $g \in \mathcal{S}'(\mathbb{R})$, then since the order of the quasiasymptotic is 0, first case implies that (10.7.26) is valid in $\mathcal{S}'(\mathbb{R})$. Therefore

$$\lim_{\varepsilon \to 0^+} \frac{\varepsilon}{L(\varepsilon)} \langle f(\epsilon x), \phi(x) \rangle = \lim_{\varepsilon \to 0^+} \frac{1}{L(\varepsilon)} \left\langle g(\varepsilon x), \frac{\phi(x)}{x} \right\rangle = \beta \int_1^\infty \frac{\phi(x)}{x} \mathrm{d}x \ .$$

This shows the case $\alpha = -1$.

It remains to show the theorem when $\alpha \in \{-2, -3, \dots\}$. Suppose the order is $-k$, $k \in \{2, 3, \dots\}$. It is easy to see that any primitive of order $(k-1)$ of f has quasiasymptotic behavior of order -1 at the origin with respect to L (in fact this is the content of Proposition 10.22 when combined with Theorem 10.16). The $(k-1)$-primitives of f are in $\mathcal{S}'(\mathbb{R})$, so we can apply the case $\alpha = -1$ to them, and then, by differentiation, it follows that f has quasiasymptotic at the origin in $\mathcal{S}'(\mathbb{R})$.

This completes the proof of Theorem 10.74. □

The analog to Theorem 10.74 is valid for quasiasymptotic boundedness with respect to regularly varying. Since the proof uses essentially the same arguments as those used in the proof of Theorem 10.74, we omit it and leave to the reader its verification.

Theorem 10.75. *Let $f \in \mathcal{S}'(\mathbb{R})$. If f is quasiasymptotically bounded of degree α at a point, with respect to a slowly varying function L, in $\mathcal{D}'(\mathbb{R})$, then f is quasiasymptotically bounded of degree α at the point with respect to L in in the space $\mathcal{S}'(\mathbb{R})$.*

If we now combine Theorems 10.64, 10.75, 10.66 and 10.68, we obtain the following corollary.

Corollary 10.76. *Let the hypotheses of Theorem 10.66 (resp. Theorem 10.68) be satisfied. If one assumes that $f_0 \in \mathcal{S}'(\mathbb{R}_+)$, then (10.7.3) (resp. (10.7.9)) holds in the space $\in \mathcal{S}'(\mathbb{R}_+)$. Furthermore, any extension f belongs to $\mathcal{S}'(\overline{\mathbb{R}}_+)$ and the* *asymptotic expansions (10.7.4) and (10.7.6) (resp. (10.7.10) and (10.7.11)) hold in $\mathcal{S}'(\mathbb{R})$.*

10.7.3 Extensions of Quasiasymptotics at Infinity from $\mathcal{S}'(\mathbb{R})$ to Spaces $\mathcal{K}'_\beta(\mathbb{R})$

Sometimes is very useful to have the right of evaluating a quasiasymptotic relation in more test functions than in $\mathcal{S}(\mathbb{R})$. For example, we confronted such a kind of problem in Chapter 7 when dealing with the $\phi-$transform and distributionally regulated functions. This section is dedicated to give some conditions under the test function which guarantee that quasiasymptotic behavior at infinity remains valid when evaluated at such a test function. We will consider test functions in the spaces $\mathcal{K}_\beta(\mathbb{R})$ (Section 1.2), $\beta \in \mathbb{R}$. Recall that $\mathcal{K}_\beta(\mathbb{R})$ consists of those test functions $\phi \in \mathcal{E}(\mathbb{R})$ such that

$$\phi(x) = O(|x|^\beta) \text{ strongly as } |x| \to \infty , \tag{10.7.27}$$

i.e., for each $m \in \{0, 1, 2, \dots\}$

$$\phi^{(m)}(x) = O(|x|^{\beta - m}) \text{ as } |x| \to \infty . \tag{10.7.28}$$

It is topologized in the obvious way [61]. These spaces and their dual spaces are very important in the theory of asymptotic expansions of distributions [61]. In fact, we have that $\mathcal{K}(\mathbb{R}) = \bigcup \mathcal{K}_\beta(\mathbb{R})$ (the union having a topological meaning), and $\mathcal{K}'(\mathbb{R}) = \bigcap \mathcal{K}'{}_\beta(\mathbb{R})$ (with projective limit topology) is the space of distributional small distributions at infinity [49, 61], they satisfy the moment asymptotic expansion at infinity [61].

The next theorem shows that if f is quasiasymptotically bounded with respect to a regularly varying functions at infinity, then the distributional evaluation of f at $\phi \in \mathcal{K}_\beta(\mathbb{R})$ makes sense under some conditions on β, specifically, we show that f has extensions to some of the spaces $\mathcal{K}'{}_\beta(\mathbb{R})$.

Theorem 10.77. *Let $f \in \mathcal{D}'(\mathbb{R})$ be quasiasymptotically bounded of degree α at infinity with respect to the slowly varying function L. If $\alpha+\beta < -1$, then f admits an extension to $\mathcal{K}_\beta(\mathbb{R})$.*

Proof. Let $\sigma > 0$ such that $\alpha + \beta + \sigma < -1$, then from Theorem 10.63, Theorem 10.64 and Corollary 10.57 we deduce that there exist $m \in \mathbb{N}$ and a continuous m-primitive of f, say F, such that

$$F(x) = O(|x|^{m+\alpha+\sigma}) \ , \quad |x| \to \infty \ . \tag{10.7.29}$$

Notice that here we have used that $L(x) = O(x^\sigma)$ as $x \to \infty$ (Section 1.7). So it is evident that an extension of f to $\mathcal{K}_\beta(\mathbb{R})$ is given by

$$\langle f_{\mathrm{e}}(x), \phi(x) \rangle = (-1)^m \int_{-\infty}^{\infty} F(x)\phi^{(m)}(x)\mathrm{d}x \ , \quad \phi \in \mathcal{K}_\beta(\mathbb{R}) \ , \tag{10.7.30}$$

which in view of (10.7.28) and (10.7.29) is well defined and defines an element of $\mathcal{K}'_\beta(\mathbb{R})$. □

We now show that the quasiasymptotic behavior remains valid in $\mathcal{K}'_\beta(\mathbb{R})$, with the assumption under β imposed in Theorem 10.77.

Theorem 10.78. *Let $f \in \mathcal{D}'(\mathbb{R})$ have quasiasymptotic behavior at ∞ of degree α with respect to a slowly varying function L, then f has an extention to $\mathcal{K}_\beta$ which has the same quasiasymptotic in $\mathcal{K}'_\beta(\mathbb{R})$, provided that $\alpha + \beta < -1$.*

Proof. The proof is similar to that of Theorem 10.74 with some modifications in the estimates. We use the extension from Theorem 10.77, which we keep calling $f = f_{\mathrm{e}}$. We shall divide the proof into two cases: $\alpha \notin \mathbb{Z}_-$ and $\alpha \in \mathbb{Z}_-$.

Suppose its degree is $\alpha \notin \mathbb{Z}_-$ and

$$f(\lambda x) = C_- L(\lambda)\frac{(\lambda x)_-^\alpha}{\Gamma(\alpha+1)} + C_+ L(\lambda)\frac{(\lambda x)_+^\alpha}{\Gamma(\alpha+1)} + o\left(\lambda^\alpha L(\lambda)\right) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{D}'(\mathbb{R}) \ .$$

Find $\sigma > 0$ such that $\alpha + \beta + \sigma < -1$. Then from Theorem 10.23, there are an m such that $m + \alpha > 0$ and a continuous m-primitive F of f such that

$$F(x) = \frac{x^m |x|^\alpha}{\Gamma(m+\alpha+1)} L(|x|)(C_- H(-x) + C_+ H(x)) + o\left(|x|^{m+\alpha} L(|x|)\right) ,$$

$x \to \infty$. We recall that H denotes the Heaviside function. We make the usual assumptions over L (Section 10.3.1), assume that L is positive, defined and continuous in $(0, \infty)$ and there exists $M_1 > 0$ such that

$$\frac{L(\lambda x)}{L(\lambda)} \leq M_1 \max\left\{x^\sigma, x^{-\sigma}\right\}, \ \lambda \geq 1, \ x \in (0, \infty) . \tag{10.7.31}$$

Let $\phi \in \mathcal{K}_\beta(\mathbb{R})$, then we can decompose $\phi = \phi_1 + \phi_2 + \phi_3$, where supp $\phi_1 \subseteq (-\infty, 1]$, supp ϕ_2 is compact and supp $\phi_3 \subseteq [1, \infty)$. Observe that since $\phi_2 \in \mathcal{D}(\mathbb{R})$ we have that

$$\langle f(\lambda x), \phi_2(x)\rangle \sim C_- \lambda^\alpha L(\lambda) \left\langle \frac{C_- x_-^\alpha + C_+ x_+^\alpha}{\Gamma(\alpha+1)}, \phi_2(x) \right\rangle \tag{10.7.32}$$

as $\lambda \to \infty$. If we want to show (10.7.32) for ϕ, it is enough to show it for ϕ_3 placed instead of ϕ_2 in the relation because by symmetry it would follow for ϕ_1 and hence for ϕ. Set

$$G(x) = \frac{F(x)}{x^{\alpha+m} L(x)} \text{ for } x \geq 1 , \tag{10.7.33}$$

then

$$\lim_{x\to\infty} G(x) = \frac{C_+}{\Gamma(\alpha+m+1)} . \tag{10.7.34}$$

So, we can find a constant $M_2 > 0$ such that

$$|G(x)| < M_2, \quad \text{globally.} \tag{10.7.35}$$

Relation (10.7.35) together with (10.7.31) show that for $\lambda \geq 1$,

$$\left| G(\lambda x) \frac{L(\lambda x)}{L(\lambda)} x^{\alpha+m} \phi_3^{(m)}(x) \right| \leq M_1 M_2 x^{\alpha+m+\sigma} \left|\phi_3^{(m)}(x)\right| H(x-1) .$$

Since $\phi_3 \in \mathcal{K}_\beta(\mathbb{R})$, the right hand side of the last estimate belongs to $L^1(\mathbb{R})$ and thus we can use the Lebesgue dominated convergence theorem to obtain,

$$\begin{aligned}\lim_{\lambda\to\infty} \frac{1}{\lambda^\alpha L(\lambda)} \left\langle f(\lambda x), \phi_3(x)\right\rangle &= \lim_{\lambda\to\infty} (-1)^m \int_0^\infty G(\lambda x)\frac{L(\lambda x)}{L(\lambda)} x^{\alpha+m}\phi_3^{(m)}(x)\mathrm{d}x \\ &= (-1)^m \frac{C_+}{\Gamma(\alpha+m+1)} \int_0^\infty x^{\alpha+m}\phi_3^{(m)}(x)\mathrm{d}x \\ &= C_+ \left\langle \frac{x_+^\alpha}{\Gamma(\alpha+1)}, \phi_3(x)\right\rangle \, .\end{aligned}$$

This shows the result in the case $\alpha \notin \{-1,-2,-3,\dots\}$.

We now aboard the case $\alpha = -k$, $k \in \mathbb{Z}_+$. Assume that

$$f(\lambda x) = \gamma\lambda^{-k}L(\lambda)\delta^{(k-1)}(x) + \beta\lambda^{-k}L(\lambda)x^{-k} + o\left(\lambda^{-k}L(\lambda)\right)$$

as $\lambda \to \infty$ in $\mathcal{D}'(\mathbb{R})$. As in the last case, it suffices to assume that $\phi \in \mathcal{K}_\beta(\mathbb{R})$, $\operatorname{supp}\phi \subseteq [1,\infty)$ and show that

$$\lim_{\lambda\to\infty} \frac{\lambda^k}{L(\lambda)} \left\langle f(\lambda x), \phi(x)\right\rangle = \beta \int_1^\infty \frac{\phi(x)}{x^k}\,\mathrm{d}x \, .$$

We may proceed as in the previous case to apply the structural theorem, but we rather reduce it to the previous situation. So, set $g(x) = x^k f(x)$, then

$$g(\lambda x) = \beta L(\lambda) + o(L(\lambda)) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{D}'(\mathbb{R}) \, . \tag{10.7.36}$$

But $\phi \in \mathcal{K}_\beta(\mathbb{R})$ implies $\phi(x)/x^k \in \mathcal{K}_{\beta-k}(\mathbb{R})$ then since the degree of the quasi-asymptotic behavior of g is 0, last case implies that (10.7.36) is valid in ${\mathcal{K}'}_{\beta-k}(\mathbb{R})$ because $\beta - k < -1$, therefore

$$\lim_{\lambda\to\infty} \frac{\lambda^k}{L(\lambda)} \left\langle f(\lambda x), \phi(x)\right\rangle = \lim_{\lambda\to\infty} \frac{1}{L(\lambda)} \left\langle g(\lambda x), \frac{\phi(x)}{x^k}\right\rangle = \beta \int_1^\infty \frac{\phi(x)}{x^k}\,\mathrm{d}x \, .$$

This completes the proof of Theorem 10.78 □

We have a similar result for quasiasymptotic boundedness. The same sort of arguments used in the proof of Theorem 10.78 lead to the next result; actually, the proof is even easier and we thus omit it.

Theorem 10.79. *Let $f \in \mathcal{D}'(\mathbb{R})$ satisfy $f(\lambda x) = O(\lambda^\alpha L(\lambda))$ as $\lambda \to \infty$ in the space $\mathcal{D}'(\mathbb{R})$. If $\alpha + \beta < -1$, then f has an extension to $\mathcal{K}_\beta(\mathbb{R})$, say f_{e}, satisfying*

$$f_{\mathrm{e}}(\lambda x) = O(\lambda^\alpha L(\lambda)) \quad \textit{as } \lambda \to \infty \textit{ in } \mathcal{K}'_\beta(\mathbb{R}) \ . \tag{10.7.37}$$

The importance of Theorems 10.78 and 10.79 lies in the fact that we can relax the growth restrictions on the test functions, this permits to apply quasiasymptotics to obtain ordinary asymptotics in many interesting situations, for example for certain integral transforms or for solutions to partial differential equations. We discuss a simple example.

Example 10.80. *Let $f \in \mathcal{D}'(\mathbb{R})$ have quasiasymptotic behavior at infinity of degree $\alpha < 1$,*

$$f(\lambda x) = \lambda^\alpha L(\lambda) g(x) + o\left(\lambda^\alpha L(\lambda)\right) \quad \textit{as } \lambda \to \infty \text{ in } \mathcal{D}'(\mathbb{R}) \ .$$

Consider the Poisson kernel,

$$P(t) = \frac{1}{\pi\left(t^2+1\right)} \ .$$

Clearly $P \in \mathcal{K}_{-2}(\mathbb{R})$. By Theorem 10.77, the evaluation of f at P is well defined. Thus

$$U(z) = U(x+yi) = \left\langle f(t), \frac{1}{y} P\left(\frac{x-t}{y}\right)\right\rangle$$

is a solution of the boundary value problem

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0, \quad U(x+i0^+) = f(x) \ .$$

Using Theorem 10.78, we can find the asymptotic behavior of U at infinity over cones. Indeed, let $0 < \sigma < \pi/2$, then Theorem 10.78 implies that as $r \to \infty$

$$U(re^{i\vartheta}) \sim \sin^\alpha(\vartheta) C_\vartheta r^\alpha L(r) \ , \quad \textit{uniformly for } \vartheta \in [\sigma, \pi-\sigma] \ ,$$

where $C_\theta = g * P(\cot\vartheta)$.

Chapter 11

Tauberian Theorems for the Wavelet Transform

11.1 Introduction

Local analysis of the wavelet transform at boundary points is the scope of this chapter. We make a complete wavelet analysis of asymptotic properties of distributions. The study is carried out via abelian and tauberian type results, connecting the boundary asymptotic behavior of the wavelet transform with local and non-local quasiasymptotic properties of elements in the Schwartz class of tempered distributions. The results to be discussed were obtained in collaboration with S. Pilipović and D. Rakać [228].

The wavelet transform is a powerful tool for studying local properties of functions. Usually, wavelet analysis presents two main important features [33, 95, 104, 137, 138]: the wavelet transform as a time-frequency analysis tool, and wavelet analysis as part of approximation theory (see also [29, 80] and references therein for another approach to time-frequency analysis). The existent applications of wavelet methods in local analysis are very rich. In [195], the wavelet transform is effectively applied to study differentiability properties of functions. A wavelet study of asymptotic and oscillatory behavior of functions can be found in [96, 104, 138]. Wavelet analysis can also be used to provide intrinsic characterizations of function and distribution spaces [137, 209]. Moreover, it is deeply involved in the analysis of regularity notions. One could mention the vital role it plays for Zygmund-Hölder type spaces (cf. [138] or [195]), and hence for the study of pseudodifferential operators within such classes (see [98, Sect. 8.5, 8.6]). Therefore, any result, as the ones of this chapter, connecting the wavelet transform with local properties of distributions might be used in this direction.

We are mainly concerned with the Schwartz class of tempered distributions. There are many classical ways to extend local regularity notions from functions to *certain* distributions which have been studied via the wavelet transform. Nevertheless such regularity notions are in somehow restrictive: they are not directly applicable to the nature of a distribution. In the past chapters we have made extensive use of the quasiasymptotic behavior of distributions. It should be now clear for the reader that the quasiasymptotics can be used to measure pointwise properties of very *general* distributions. Such a notion is more general and more suitable than others when one is only interested in the actual behavior of distributions around individual points.

Recently, wavelet methods have attracted the attention of many authors as a tool for the analysis of quasiasymptotic properties of distributions. Problems related to multiresolution expansions and orthogonal wavelets are studied in [163, 162, 169, 188, 205, 241, 242]. Abelian and tauberian results for the wavelet transform are obtained in [174, 175, 176].

The quasiasymptotic behavior is a very suitable concept for wavelet analysis. In fact, the wavelet transform can be thought as a sort of mathematical microscope analyzing a distribution on various length scales around any point of the real axis. On the other hand, the idea of the quasiasymptotic behavior itself is to study the asymptotic properties at small or large scale of the dilates of a distribution. In the case of small scales, the quasiasymptotic behavior uses only local information of the distribution at small scale around a point, and hence the natural connection between it and the boundary asymptotic behavior of the wavelet transform. Another reason that suggests the use of the quasiasymptotic behavior in wavelet analysis is that it is based on asymptotic comparison with regularly varying functions [15], which are actually a power function multiplied by an asymptotic invariant function under rescaling, and hence measures certain fractal behavior of distributions.

The chapter is organized as follows. We recall in Section 11.2 the basic facts from distribution wavelet analysis based on highly time-frequency localized function spaces, following Holschneider's book [95]. Sections 11.3 and 11.4 are devoted to connect the boundary asymptotic behavior of the wavelet transform through abelian theorems and tauberian characterizations of the quasiasymptotic behavior in the dual of the space of highly time-frequency localized functions. For finite points, Section 11.3 deals with global tauberian assumptions, while the results are later improved to a local version in Section 11.4. Since the results from Sections 11.3 and 11.4 lead to regard tempered distributions, and asymptotic relations, on a more restricted space of distributions, we study in Section 11.5 what this information tells us about the asymptotic properties in the space of tempered distributions. Sections 11.6 and 11.7 are the most important ones, there we obtain the tauberian theorems for quasiasymptotics of tempered distributions in terms of the wavelet transform; these are complete inverse theorems to the abelians from [175, 176]. It is shown that in some cases our tauberian theorems become full characterizations of asymptotic properties. They can also be considered as generalizations of the results from [96] to our distributional context. Finally, in Section 11.8, we indicate how to treat progressive and regressive distributions.

11.2 The Wavelet Transform of Distributions

We will follow the wavelet analysis for distributions from Holschneider's book [95]. For this, we will use the spaces of highly localized functions over the real line and the upper half-plane.

By a progressive function (or distribution), we mean a function whose Fourier transform, whenever the Fourier transform makes sense, is supported in $\overline{\mathbb{R}}_+$; similarly, the term regressive function (or distribution) refers to those whose Fourier transform is supported in $\overline{\mathbb{R}}_-$.

We define the space of highly time-frequency localized progressive functions over the real line as the set of those elements of $\mathcal{S}(\mathbb{R})$ which are progressive functions; this is a closed subspace of $\mathcal{S}(\mathbb{R})$ and it is denoted by $\mathcal{S}_+(\mathbb{R})$. The image of $\mathcal{S}_+(\mathbb{R})$ under the parity operator is denoted by $\mathcal{S}_-(\mathbb{R})$, that is, $\phi \in \mathcal{S}_-(\mathbb{R})$ if and only if $\check{\phi}(x) := \phi(-x) \in \mathcal{S}_+(\mathbb{R})$, equivalently, $\phi \in \mathcal{S}_-(\mathbb{R})$ if and only if $\overline{\phi}(x) := \overline{\phi(x)} \in \mathcal{S}_+(\mathbb{R})$. Observe that the elements of $\mathcal{S}_+(\mathbb{R})$ are precisely of those elements of $\mathcal{S}(\mathbb{R})$ which are in $H^2_+(\mathbb{R})$, the Hardy space of $L^2(\mathbb{R})$-boundary values of analytic functions on $\mathbb{H}$. The space $\mathcal{S}_0(\mathbb{R})$ is defined then as the direct sum of $\mathcal{S}_+(\mathbb{R})$ and $\mathcal{S}_-(\mathbb{R})$

$$\mathcal{S}_0(\mathbb{R}) = \mathcal{S}_-(\mathbb{R}) \oplus \mathcal{S}_+(\mathbb{R}) \ .$$

Alternatively, we may define $\mathcal{S}_0(\mathbb{R})$ as those elements of $\mathcal{S}(\mathbb{R})$ for which all the moments vanish, i.e., $\phi \in \mathcal{S}_0(\mathbb{R})$ if and only if

$$\int_{-\infty}^{\infty} x^n \phi(x) \mathrm{d}x = 0 \ , \tag{11.2.1}$$

for all $n \in \mathbb{N}$. We call $\mathcal{S}_0(\mathbb{R})$ the space of highly time-frequency localized functions over the real line. Note that $\mathcal{S}_0(\mathbb{R})$ is a closed subspace of $\mathcal{S}(\mathbb{R})$. The dual spaces of $\mathcal{S}_+(\mathbb{R})$, $\mathcal{S}_-(\mathbb{R})$ and $\mathcal{S}_0(\mathbb{R})$ (these spaces provided with the relative topology inhered from $\mathcal{S}(\mathbb{R})$) are $\mathcal{S}'_-(\mathbb{R}) = (\mathcal{S}_+(\mathbb{R}))'$, $\mathcal{S}'_+(\mathbb{R}) = (\mathcal{S}_-(\mathbb{R}))'$ and $\mathcal{S}'_0(\mathbb{R})$, respectively. It should be noticed that the space $\mathcal{S}'_+(\mathbb{R})$ defined above is different from the one used in [231], for example.

Observe that we have a well-defined continuous linear projector from $\mathcal{S}'(\mathbb{R})$ to $\mathcal{S}'_0(\mathbb{R})$ as the transpose of the trivial inclusion from the closed subspace $\mathcal{S}_0(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})$. Due to Hahn-Banach theorem, this map is subjective; however, there is no continuous right inverse for this projection [51]. Note also that the kernel of this projection is the space of polynomials; hence, the space $\mathcal{S}'_0(\mathbb{R})$ can be regarded as the quotient space of $\mathcal{S}'(\mathbb{R})$ by the space of polynomials. We do not want to introduce a notation for this map, so if $f \in \mathcal{S}'(\mathbb{R})$, we will keep calling by f the restriction of f to $\mathcal{S}_0(\mathbb{R})$. We will come back to this matter later, in Section 11.5, in connection with the quasiasymptotic behavior of distributions.

The corresponding space $\mathcal{S}(\mathbb{H})$ of highly localized function, over the upper half-plane $\mathbb{H} = \mathbb{R} \times \mathbb{R}_+$, is defined as those smooth functions on $\mathbb{H}$ such that

$$\sup_{(b,a)\in\mathbb{H}} \left(a + \frac{1}{a}\right)^m \left(1 + b^2\right)^{\frac{n}{2}} \left|\frac{\partial^{k+l}\Phi}{\partial a^k \partial b^l}(b,a)\right| < \infty \ ,$$

for all $m, n, k, l \in \mathbb{N}$. It is topologized in the obvious way. We will also consider its dual space, $\mathcal{S}'(\mathbb{H})$. Any locally integrable function F of "slow growth" on $\mathbb{H}$, that is,

$$|F(b,a)| \leq C \left(1 + b^2\right)^{\frac{l}{2}} \left(a + \frac{1}{a}\right)^m, \quad (b,a) \in \mathbb{H} \ ,$$

for some $C > 0$ and integers $m, l \in \mathbb{N}$, can be identified with an element of $\mathcal{S}'(\mathbb{H})$. Our convention for identifying it with an element of $\mathcal{S}'(\mathbb{H})$ is to keep using the notation $F \in \mathcal{S}'(\mathbb{H})$ and the evaluation of F at $\Phi \in \mathcal{S}(\mathbb{H})$ is given by

$$\langle F(b,a), \Phi(b,a)\rangle = \int_0^\infty \int_{-\infty}^\infty F(b,a)\Phi(b,a)\frac{\mathrm{d}b\mathrm{d}a}{a} \ .$$

By a wavelet (or analyzing wavelet) we simply mean an element $\psi \in \mathcal{S}_0(\mathbb{R})$. A wavelet η is called a *reconstruction wavelet* for the analyzing wavelet ψ if the two constants

$$c^{\pm}_{\psi,\eta} = \int_0^\infty \overline{\hat{\psi}}(\pm x)\hat{\eta}(\pm x)\frac{\mathrm{d}x}{x} < \infty$$

are non-zero and equal to each other; in such case we write

$$c_{\psi,\eta} = c^{+}_{\psi,\eta} = c^{-}_{\psi,\eta} = \frac{1}{2}\int_{-\infty}^\infty \overline{\hat{\psi}}(x)\hat{\eta}(x)\frac{\mathrm{d}x}{|x|} \ .$$

Note that any wavelet admits a reconstruction wavelet as long as $\operatorname{supp}\hat{\psi} \cap \mathbb{R}_+ \neq \emptyset$ and $\operatorname{supp}\hat{\psi} \cap \mathbb{R}_- \neq \emptyset$. In the case of a progressive wavelet ψ we require η to satisfy only the positive frequency part of the above condition. Analogously for regressive ones. We will mainly use wavelets admitting a reconstruction wavelet. If a wavelet is its own reconstruction wavelet, we say that it is *admissible.* An explicit example of an admissible wavelet is the wavelet ψ given in the Fourier side by $\hat{\psi}(x) = e^{-|x|+\frac{1}{|x|}},\ x \in \mathbb{R}$.

The *wavelet transform* of $f \in \mathcal{S}_0'(\mathbb{R})$ with respect to an analyzing wavelet ψ is given by the C^∞-function on $\mathbb{H}$

$$\mathcal{W}_\psi f(b,a) = \left\langle f(b+ax), \bar{\psi}(x) \right\rangle = \left\langle f(t), \frac{1}{a}\bar{\psi}\left(\frac{t-b}{a}\right) \right\rangle = f * \check{\bar{\psi}}_a(b), \qquad (11.2.2)$$

where $\psi_a(\cdot) = \frac{1}{a}\psi(\frac{\cdot}{a})$. For a $\Phi \in \mathcal{S}(\mathbb{H})$, we define the *wavelet synthesis operator* with respect to the wavelet ψ as

$$\mathcal{M}_\psi \Phi(x) = \int_0^\infty \int_{-\infty}^\infty \Phi(b,a) \frac{1}{a}\psi\left(\frac{x-b}{a}\right) \frac{\mathrm{d}b\mathrm{d}a}{a}\ , \quad x \in \mathbb{R}\ .$$

Observe that $\mathcal{W}_\psi : \mathcal{S}_0(\mathbb{R}) \to \mathcal{S}(\mathbb{H})$ and $\mathcal{M}_\psi : \mathcal{S}(\mathbb{H}) \to \mathcal{S}_0(\mathbb{R})$ are continuous linear maps [95, p.74]. Moreover, one has the reconstruction formula for the wavelet transform with respect an analyzing wavelet ψ

$$\mathrm{Id}_{\mathcal{S}_0(\mathbb{R})} = \frac{1}{c_{\psi,\eta}} \mathcal{M}_\eta \mathcal{W}_\psi\ ,$$

where η is a reconstruction wavelet for ψ. Because of the results of [95], we may have alternatively defined the wavelet transform of distributions by duality

$$\langle \mathcal{W}_\psi f(b,a), \Phi(b,a) \rangle = \left\langle f(x), \mathcal{M}_{\bar{\psi}} \Phi\ (x) \right\rangle, \quad \Phi \in \mathcal{S}(\mathbb{H})\ .$$

These two definitions coincide and we have, for $\Phi \in \mathcal{S}(\mathbb{H})$,

$$\langle \mathcal{W}_\psi f(b,a), \Phi(b,a) \rangle = \int_0^\infty \int_{-\infty}^\infty \mathcal{W}_\psi f(b,a) \Phi(b,a) \frac{\mathrm{d}b\mathrm{d}a}{a}\ .$$

Similarly, one defines the wavelet synthesis $\mathcal{M}_\psi : \mathcal{S}'(\mathbb{H}) \to \mathcal{S}_0'(\mathbb{R})$ by

$$\langle \mathcal{M}_\psi F(x), \phi(x) \rangle = \left\langle F(b,a), \mathcal{W}_{\bar{\psi}} \phi(b,a) \right\rangle\ .$$

Here we have again that if η is a reconstruction wavelet for ψ, then

$$\mathrm{Id}_{\mathcal{S}_0'(\mathbb{R})} = \frac{1}{c_{\psi,\eta}}\mathcal{M}_\eta \mathcal{W}_\psi \ .$$

Therefore, if $f \in \mathcal{S}_0'(\mathbb{R})$ and $\phi \in \mathcal{S}_0(\mathbb{R})$, we have the *desingularization* formula

$$\langle f(x), \phi(x)\rangle = \frac{1}{c_{\psi,\eta}} \int_0^\infty \int_{-\infty}^\infty \mathcal{W}_\psi f(b,a)\mathcal{W}_{\bar{\eta}}\phi(b,a)\frac{\mathrm{d}b\mathrm{d}a}{a} \ .$$

We will also make use of the projection operator of $\mathcal{S}'(\mathbb{H})$ onto the image of the wavelet transform [95], it is given by the projector

$$\frac{1}{c_{\psi,\eta}}\mathcal{W}_\psi \mathcal{M}_\eta \ .$$

Observe also that $\operatorname{Im}\mathcal{W}_\psi$ is a closed subspace of $\mathcal{S}'(\mathbb{H})$. Sometimes, for instance if the distribution is a locally integrable distribution of slow growth on $\mathbb{H}$, it is possible to write the projection by the integral transform

$$\frac{\mathcal{W}_\psi \mathcal{M}_\eta F(b,a)}{c_{\psi,\eta}} = \frac{1}{c_{\psi,\eta}} \int_0^\infty \int_{-\infty}^\infty \mathcal{W}_\psi \eta \left(\frac{b-b'}{a'}, \frac{a}{a'}\right) F(b',a') \frac{\mathrm{d}b'\mathrm{d}a'}{(a')^2} \ . \tag{11.2.3}$$

We can also define the wavelet transform of a tempered distribution $f \in \mathcal{S}'(\mathbb{R})$ by formula (11.2.2), but this integral transformation will be no longer invertible because the moment vanishing condition (11.2.1) gives that the wavelet transform of any polynomial vanishes.

Let us now turn our attention to quasiasymptotics. In this chapter we are mainly interested in tempered distributions. Besides quasiasymptotics in the space $\mathcal{S}'(\mathbb{R})$, we will consider quasiasymptotics in $\mathcal{S}_0'(\mathbb{R})$. Following our usual convention, we write

$$f(x_0 + \varepsilon x) \sim \varepsilon^\alpha L(\varepsilon) g(x) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{S}_0'(\mathbb{R}) \ , \tag{11.2.4}$$

or

$$f(x_0 + \varepsilon x) = \varepsilon^\alpha L(\varepsilon) g(x) + o(\varepsilon^\alpha L(\varepsilon)) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{S}_0'(\mathbb{R}) \ , \tag{11.2.5}$$

where L is slowly varying at the origin, if $f \in \mathcal{S}_0'(\mathbb{R})$ (or $\mathcal{S}'(\mathbb{R})$) and

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^\alpha L(\varepsilon)} \langle f(x_0 + \varepsilon x), \phi(x) \rangle = \langle g(x), \phi(x) \rangle, \forall \phi \in \mathcal{S}_0(\mathbb{R}) . \tag{11.2.6}$$

Observe that, by shifting to $x = x_0$, in most cases is enough to consider $x_0 = 0$.

Similarly, we consider the quasiasymptotics at infinity in the space $\mathcal{S}_0'(\mathbb{R})$,

$$f(\lambda x) \sim \lambda^\alpha L(\lambda) g(x) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}_0'(\mathbb{R}) , \tag{11.2.7}$$

where L is slowly varying at infinity.

Observe that slowly varying functions are very convenient objects to be employed in wavelet analysis since they are asymptotic invariant under rescaling at small scale (resp. large scale).

11.3 Wavelet Characterization of Quasiasymptotics in $\mathcal{S}_0'(\mathbb{R})$

Recently, Saneva and Bučkovska ([174, 175, 176]) investigated the asymptotic behavior of the wavelet transform of a distribution having quasiasymptotic behavior at a point. Indeed, it is fairly easy to show that if

$$f(x_0 + \varepsilon x) \sim \varepsilon^\alpha L(\varepsilon) g(x) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{S}'(\mathbb{R}) , \tag{11.3.1}$$

then

$$\mathcal{W}_\psi f(x_0, a) \sim L(a) \mathcal{W}_\psi g(0, a) = a^\alpha L(a) \mathcal{W}_\psi g(0, 1) \ , \quad a \to 0^+ . \tag{11.3.2}$$

The above result is of abelian nature. Let us mention that to conclude (11.3.2), it is enough to assume a weaker hypothesis. Indeed, if we only assume the quasiasymptotic behavior of the tempered distribution in the space $\mathcal{S}_0'(\mathbb{R})$, we are still able to deduce (11.3.2). Actually, the angular asymptotic behavior over cones with vertex at x_0 can also be obtained.

Theorem 11.1. *Let $f \in \mathcal{S}'(\mathbb{R})$ have quasiasymptotic behavior in $\mathcal{S}_0'(\mathbb{R})$,*

$$f\left(x_0 + \varepsilon x\right) \sim \varepsilon^{\alpha} L(\varepsilon) g(x) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{S}_0'(\mathbb{R}) \ . \tag{11.3.3}$$

Then, given any $0 < \sigma \leq \pi/2$ and $r > 0$, we have

$$\mathcal{W}_{\psi} f(x_0 + \varepsilon r \cos\vartheta, \varepsilon r \sin\vartheta) \sim \varepsilon^{\alpha} L\left(\varepsilon\right) \mathcal{W}_{\psi} g(r\cos\vartheta, r\sin\vartheta), \quad \varepsilon \to 0^+ \ , \tag{11.3.4}$$

uniformly for $\sigma \leq \vartheta \leq \pi - \sigma$.

Proof. In view of (11.3.3), Banach-Steinhaus theorem and the compactness of the set

$$\mathfrak{C}_{\sigma} = \left\{ \frac{1}{\sin\vartheta} \bar{\psi}\left(\frac{\ \cdot\ - \cos\vartheta}{\sin\vartheta} \right) \in \mathcal{S}_0(\mathbb{R}) : \sigma \leq \vartheta \leq \pi - \sigma \right\} \tag{11.3.5}$$

we have, as $\varepsilon \to 0^+$,

$$\begin{aligned}
\mathcal{W}_{\psi} f(x_0 + \varepsilon r \cos\vartheta, \varepsilon r \sin\vartheta) &= \left\langle f(x_0 + \varepsilon r \cos\vartheta + \varepsilon r \sin\vartheta x), \bar{\psi}(x) \right\rangle \\
&= \left\langle f(x_0 + \varepsilon r x), \frac{1}{\sin\vartheta} \bar{\psi}\left(\frac{x - \cos\vartheta}{\sin\vartheta} \right) \right\rangle \\
&\sim (r\varepsilon)^{\alpha} L(r\varepsilon) \left\langle g(x), \frac{1}{\sin\vartheta} \bar{\psi}\left(\frac{x - \cos\vartheta}{\sin\vartheta} \right) \right\rangle \\
&= \varepsilon^{\alpha} L(r\varepsilon) \left\langle g(rx), \frac{1}{\sin\vartheta} \bar{\psi}\left(\frac{x - \cos\vartheta}{\sin\vartheta} \right) \right\rangle \\
&= \varepsilon^{\alpha} L\left(r\varepsilon\right) \mathcal{W}_{\psi} g(r\cos\vartheta, r\sin\vartheta) \\
&\sim \varepsilon^{\alpha} L\left(\varepsilon\right) \mathcal{W}_{\psi} g(r\cos\vartheta, r\sin\vartheta) \ .
\end{aligned}$$

□

We have a similar assertion at ∞.

Theorem 11.2. *Let $f \in \mathcal{S}'(\mathbb{R})$ have quasiasymptotic behavior at infinity in $\mathcal{S}_0'(\mathbb{R})$*

$$f\left(\lambda x\right) \sim \lambda^{\alpha} L(\lambda) g(x) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}_0'(\mathbb{R}) \ . \tag{11.3.6}$$

Then, given any $0 < \sigma \leq \pi/2$ *and* $r > 0$*, we have*

$$\mathcal{W}_\psi f(\lambda r \cos\vartheta, \lambda r \sin\vartheta) \sim \lambda^\alpha L(\lambda)\, \mathcal{W}_\psi g(r\cos\vartheta, r\sin\vartheta), \quad \lambda \to \infty\ , \tag{11.3.7}$$

uniformly for $\sigma \leq \vartheta \leq \pi - \sigma$*.*

Proof. In view of (11.3.6), Banach-Steinhaus theorem and the compactness of the set $\mathfrak{C}_\sigma$ given by (11.3.5), we have, as $\lambda \to \infty$,

$$\begin{aligned}
\mathcal{W}_\psi f(\lambda r\cos\vartheta, \lambda r \sin\vartheta) &= \left\langle f(\lambda r\cos\vartheta + \lambda r \sin\vartheta x), \bar{\psi}(x)\right\rangle \\
&= \left\langle f(\lambda r x), \frac{1}{\sin\vartheta}\bar{\psi}\left(\frac{x-\cos\vartheta}{\sin\vartheta}\right)\right\rangle \\
&\sim (r\lambda)^\alpha L(r\lambda)\left\langle g(x), \frac{1}{\sin\vartheta}\bar{\psi}\left(\frac{x-\cos\vartheta}{\sin\vartheta}\right)\right\rangle \\
&\sim \lambda^\alpha L(\lambda)\, \mathcal{W}_\psi g(r\cos\vartheta, r\sin\vartheta)\ .
\end{aligned}$$

□

Our next goal is to provide an inverse theorem for these two abelian results, under some natural additional tauberian conditions. Actually, we characterize below the quasiasymptotics in $\mathcal{S}_0'(\mathbb{R})$ in terms of the wavelet transform. Later, we will use this characterization to study the quasiasymptotic behavior in the space $\mathcal{S}'(\mathbb{R})$ (Sections 11.6 and 11.7).

Theorem 11.3. *Let* $f \in \mathcal{S}_0'(\mathbb{R})$*. Let* $\psi \in \mathcal{S}_0(\mathbb{R})$ *be a wavelet admitting a reconstruction wavelet. The following two conditions:*

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^\alpha L(\varepsilon)} \mathcal{W}_\psi f\left(x_0 + \varepsilon b, \varepsilon a\right) = M_{b,a} < \infty, \quad (b,a) \in \mathbb{H}\,, \tag{11.3.8}$$

and the existence of constants $\gamma, \beta, M > 0$ *such that*

$$\frac{\left|\mathcal{W}_\psi f\left(x_0 + \varepsilon b, \varepsilon a\right)\right|}{\varepsilon^\alpha L(\varepsilon)} < M\left(a + \frac{1}{a}\right)^\gamma (1+|b|)^\beta, \quad (b,a)\in\mathbb{H}\,,\ \varepsilon < 1\,, \tag{11.3.9}$$

are necessary and sufficient for the existence of a homogeneous distribution g *of degree* α *such that*

$$f\left(x_0 + \varepsilon x\right) \sim \varepsilon^\alpha L(\varepsilon) g(x) \quad \textit{as } \varepsilon \to 0^+ \textit{ in } \mathcal{S}_0'(\mathbb{R})\ . \tag{11.3.10}$$

In this case we have $M_{b,a} = \mathcal{W}_\psi g(b,a), (b,a) \in \mathbb{H}$.

Proof. We may assume that $x_0 = 0$. Let η be a reconstruction wavelet for ψ. That (11.3.8) is necessary follows from the abelian theorem, Theorem 11.1. The necessity of (11.3.9) follows from the characterization of bounded sets in $\mathcal{S}_0'(\mathbb{R})$ (c.f. [95, Thm. 28.0.1]). For the converse, notice that (11.3.8) and (11.3.9) imply that the function given by $J(b,a) = M_{b,a}$, $(b,a) \in \mathbb{H}$, is measurable and satisfies the estimate

$$|J(b,a)| = |M_{b,a}| < M\left(a + \frac{1}{a}\right)^{\gamma}(1+|b|)^{\beta}, \quad (b,a) \in \mathbb{H}\ ,$$

hence it is in $\mathcal{S}'(\mathbb{H})$. Moreover, because of (11.3.8) and (11.3.9), we can use Lebesgue dominated convergence theorem and the wavelet desingularization formula to conclude that for each $\phi \in \mathcal{S}_0(\mathbb{R})$

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\left\langle \frac{f(\varepsilon x)}{\varepsilon^\alpha L(\varepsilon)}, \phi(x)\right\rangle &= \frac{1}{c_{\psi,\eta}}\lim_{\varepsilon\to 0^+}\int_0^\infty\int_{-\infty}^\infty \frac{\mathcal{W}_\psi f\,(\varepsilon b,\varepsilon a)}{\varepsilon^\alpha L(\varepsilon)}\mathcal{W}_{\bar\eta}\phi(b,a)\frac{\mathrm{d}b\mathrm{d}a}{a}\\ &= \frac{1}{c_{\psi,\eta}}\int_0^\infty\int_{-\infty}^\infty M_{b,a}\mathcal{W}_{\bar\eta}\phi(b,a)\frac{\mathrm{d}b\mathrm{d}a}{a}\ .\end{aligned}$$

Since the last limit exists for each $\phi \in \mathcal{S}_0(\mathbb{R})$, it follows that f has quasiasymptotic behavior in the space $\mathcal{S}_0'(\mathbb{R})$ and the existence of a homogeneous distribution g satisfying (11.3.10) and $M_{b,a} = \mathcal{W}_\psi g(b,a)$. □

Theorem 11.3 is of intermediate character, it will be improved in Section 11.4. It should be noticed that Theorem 11.3 uses global information of the wavelet transform; however, the quasiasymptotic behavior at a point is a local concept. Therefore, it is still somehow unsatisfactory. Nevertheless, this result will be used obtain a much better characterization only using local information on the transformed side (Theorem 11.5).

We now focus in the case of asymptotic behavior at ∞. Observe that the arguments given in the proof of Theorem 11.3 may lead us to a proof of the following theorem, but we choose to present an alternative version of the proof, where we use some basic results from functional analysis ([208]).

Theorem 11.4. *Let $f \in \mathcal{S}_0'(\mathbb{R})$. Let $\psi \in \mathcal{S}_0(\mathbb{R})$ be a wavelet admitting a reconstruction wavelet. The following two conditions:*

$$\lim_{\lambda\to\infty} \frac{1}{\lambda^\alpha L(\lambda)} \mathcal{W}_\psi f\,(\lambda b, \lambda a) = M_{b,a} < \infty,\ (b,a) \in \mathbb{H}, \tag{11.3.11}$$

and the existence of constants $\gamma, \beta, M > 0$ such that

$$\frac{|\mathcal{W}_\psi f\,(\lambda b, \lambda a)|}{\lambda^\alpha L(\lambda)} < M\left(a + \frac{1}{a}\right)^\gamma (1+|b|)^\beta, \quad (b,a) \in \mathbb{H}\,,\ \lambda > 1\,, \tag{11.3.12}$$

are necessary and sufficient for the existence of a homogeneous distribution g of degree α such that

$$f\,(\lambda x) \sim \lambda^\alpha L(\lambda) g(x) \quad \textit{as } \lambda \to \infty \textit{ in } \mathcal{S}_0'(\mathbb{R})\ . \tag{11.3.13}$$

In this case we have $M_{b,a} = \mathcal{W}_\psi g(b,a), (b,a) \in \mathbb{H}$.

Proof. The necessity is clear, we concentrate in the sufficiency. Let

$$\mathfrak{B} = \left\{\psi_{b,a} := a^{-1}\bar{\psi}\left(a^{-1}(\ \cdot - b)\right), (b,a) \in \mathbb{H}\right\}\ .$$

We claim that the linear span of $\mathfrak{B}$ is dense in $\mathcal{S}_0(\mathbb{R})$. Let $h \in \mathcal{S}_0'(\mathbb{R})$. If we suppose that

$$\left\langle h(x), \frac{1}{a}\bar{\psi}\left(\frac{x-b}{a}\right)\right\rangle = \mathcal{W}_\psi h\,(b,a) = 0, \quad \text{for all } (b,a) \in \mathbb{H}\ ,$$

then, by wavelet desingularization, we have that for every $\phi \in \mathcal{S}_0(\mathbb{R})$,

$$\langle h(x), \phi(x)\rangle = \frac{1}{c_{\psi,\eta}} \langle \mathcal{W}_\psi h\,(b,a)\,, \mathcal{W}_{\bar{\eta}}\phi\,(b,a)\rangle = 0\ ;$$

and hence $h = 0$. Thus, by the Hahn-Banach theorem, we conclude that the linear span of $\mathfrak{B}$ is dense in $\mathcal{S}_0(\mathbb{R})$. Furthermore, let $\mathfrak{F} = \{f_\lambda; \lambda \geq 1\}$ where $f_\lambda = f(\lambda\,\cdot)/(\lambda^\alpha L(\lambda))$. The estimate (11.3.12) and the characterization of bounded sets in $\mathcal{S}'_0(\mathbb{R})$ (c.f. [95, Thm. 28.0.1]) imply that $\mathfrak{F}$ is a bounded family in $\mathcal{S}'_0(\mathbb{R})$, which in turn implies, by Banach-Steinhaus theorem, that $\mathfrak{F}$ is an equicontinuous set. It is known that, for equicontinuous sets, the pointwise convergence over a complete test space and over some dense subset coincide. But observe that (11.3.11) exactly gives us the convergence over the linear span of $\mathfrak{B}$; so, for some $g \in \mathcal{S}'_0(\mathbb{R})$, we have $f_\lambda \to g$, $\lambda \to \infty$, in the weak sense. $\square$

In conclusion, we have characterized the quasiasymptotic behavior of distributions in the space $\mathcal{S}'_0(\mathbb{R})$ in terms of the asymptotic behavior of the wavelet transform at approaching points of the boundary. Our main aim is now to extend these results to $\mathcal{S}'(\mathbb{R})$, that is, we want to give tauberian theorems for quasiasymptotics at points and infinity of tempered distributions in terms of the behavior of the wavelet transform. We have reduced this question to the following one: if $f \in \mathcal{S}'(\mathbb{R})$ has quasiasymptotic at $x = x_0$ or $x = \infty$ in $\mathcal{S}'_0(\mathbb{R})$, what can we say about the existence of the quasiasymptotic of f at $x = x_0$ or $x = \infty$ in $\mathcal{S}'(\mathbb{R})$? We postpone the study of this question for Section 5, where we will give a complete answer.

11.4 Tauberian Characterization with Local Conditions

As we remarked before, conditions (11.3.8) and (11.3.9) are of global character, we want to replace them by local conditions. This is done in the next theorem. We may refer to relation (11.4.2) as a tauberian condition of Vladimirov-Drozhzhinov-Zavialov type, because they have made extensive use of these types of conditions in the study of tauberian theorems for local behavior of generalized functions in terms of several integral transforms such as the Laplace transform and Mellin convolution type transforms, among others [39, 40, 231, 232].

Theorem 11.5. *Let $f \in \mathcal{S}_0'(\mathbb{R})$. Let $\psi \in \mathcal{S}_0(\mathbb{R})$ be a wavelet admitting a reconstruction wavelet. The following two conditions:*

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^\alpha L(\varepsilon)} \mathcal{W}_\psi f\,(x_0+\varepsilon b, \varepsilon a) = M_{b,a} < \infty \tag{11.4.1}$$

exists for each $(b,a) \in \mathbb{H}$ satisfying $a^2+b^2=1$ and $a>0$, and there exists $m \in \mathbb{N}$ such that

$$\limsup_{\varepsilon\to 0^+} \sup_{a^2+b^2=1,\, a>0} \frac{a^m}{\varepsilon^\alpha L(\varepsilon)} \left|\mathcal{W}_\psi f\,(x_0+\varepsilon b, \varepsilon a)\right| < \infty\ , \tag{11.4.2}$$

are necessary and sufficient for the existence of a homogeneous distribution g of degree α such that

$$f\,(x_0+\varepsilon x) \sim \varepsilon^\alpha L(\varepsilon) g(x) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{S}_0'(\mathbb{R})\ . \tag{11.4.3}$$

In this case we have $M_{b,a} = \mathcal{W}_\psi g(b,a)$.

Proof. The necessity follows from Theorem 11.3. Therefore, we concentrate in showing the sufficiency.

We assume that $x_0 = 0$ for simplicity. Let η be a reconstruction wavelet for ψ.

Let $F = \chi_I \mathcal{W}_\psi f$ where χ_I is the characteristic function of the set $I = |b| \le 1,\ 0 < a \le 1$. Let $G = \mathcal{W}_\psi f - F$. Consider $f_0 = c_{\psi,\eta}^{-1}\ \mathcal{M}_\eta F$ and $h_0 = c_{\psi,\eta}^{-1} \mathcal{M}_\eta G$. Notice that $\mathcal{W}_\psi f = \mathcal{W}_\psi h_0 + \mathcal{W}_\psi f_0$, and hence $f = h_0 + f_0$.

The plan is to show that an extension h of $h_0 \in \mathcal{S}_0'(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$ is C^∞ in a neighborhood of the origin and that f_0 has quasiasymptotic behavior at the origin. This would imply that $h_0(\varepsilon x) = o\left(\varepsilon^\infty\right)$ in $\mathcal{S}_0'(\mathbb{R})$, that is, $h_0(\varepsilon x) = o\left(\varepsilon^\sigma\right)$ for every $\sigma > 0$. Hence, we would have that $f\,(\varepsilon x) - f_0\,(\varepsilon x) = o\left(\varepsilon^\infty\right)$ in $\mathcal{S}_0'(\mathbb{R})$, showing that f has the quasiasymptotic behavior (11.4.3) in $\mathcal{S}_0'(\mathbb{R})$ if and only if f_0 does.

Let h be an extension of h_0 to $\mathcal{S}'(\mathbb{R})$. We will show that $\mathcal{W}_\psi h(b,a) = o\left(a^\infty\right)$ uniformly for b in a neighborhood of the origin as $a \to 0^+$. Let σ be a positive real number. Find $n \in \mathbb{N}$ and $B > 0$ such that

$$|G(b,a)| \leq B\left(a+\frac{1}{a}\right)^n (1+|b|)^n$$

and

$$|\mathcal{W}_\psi \eta(b,a)| \leq B\left(a+\frac{1}{a}\right)^{-1-2n-\sigma} (1+|b|)^{-2-n}.$$

If $|b| \leq \frac{1}{2}$ and $a < 1$, then, by (11.2.3),

$$\begin{aligned}
|c_{\psi,\eta}\mathcal{W}_\psi h(b,a)| &= \left|\int_1^\infty \int_{|b'|\geq 1} \mathcal{W}_\psi \eta\left(\frac{b-b'}{a'},\frac{a}{a'}\right) G(b',a') \frac{\mathrm{d}b'\mathrm{d}a'}{(a')^2}\right| \\
&\leq 4^n B^2 \int_1^\infty \int_{|b'|\geq 1} |b'|^n (a')^n \left(\frac{a'}{|b-b'|}\right)^{2+n} \left(\frac{a}{a'}\right)^{1+2n+\sigma} \frac{\mathrm{d}b'\mathrm{d}a'}{(a')^2} \\
&\leq a^{1+2n+\sigma} 4^n B^2 \left(\int_{|b'|\geq 1} \frac{|b'|^n \,\mathrm{d}b'}{\left(|b'|-\frac{1}{2}\right)^{n+2}}\right)\left(\int_1^\infty \frac{\mathrm{d}a'}{(a')^{\sigma+1}}\right) \\
&= o\left(a^\sigma\right) .
\end{aligned}$$

We use the characterization of the singular support of distributions given in [95, Thm. 27.0.2], and conclude that h is C^∞ in $(-1/2, 1/2)$.

Next, we show that $\mathcal{W}_\psi f_0$ has quasiasymptotic in $\mathcal{S}_0'(\mathbb{R})$. For this, in view of Theorem 11.3, it is enough to show that f_0 satisfies (11.3.8) and an estimate of the form (11.3.9).

We first show that $\mathcal{W}_\psi f_0$ satisfies

$$\frac{1}{\varepsilon^\alpha L(\varepsilon)} |\mathcal{W}_\psi f_0(\varepsilon b, \varepsilon a)| \leq M\left(a+\frac{1}{a}\right)^\gamma (1+|b|)^\beta , \tag{11.4.4}$$

for some constants $\gamma, \beta, M > 0$ and all $(b,a) \in \mathbb{H},\ 0 < \varepsilon \leq 1$. Observe that (11.4.2) implies

$$\frac{a^m}{\varepsilon^\alpha L(\varepsilon)} |\mathcal{W}_\psi f\left(\varepsilon b, \varepsilon a\right)| < M_1, \text{ for all } a^2+b^2 = 1,\ a > 0,\ 0 < \varepsilon \leq \varepsilon_0 ,$$

for some $M_1 > 0$ and ε_0. After rescaling we can put $\varepsilon_0 = \sqrt{2}$. Let $a' \in (0, \varepsilon^{-1})$ and $b' \in (-\varepsilon^{-1}, \varepsilon^{-1})$. Then we have for $\varepsilon < 1$ that $\varepsilon\sqrt{(a')^2+(b')^2} \leq \sqrt{2}$. So, if we replace a, b and ε by $a'/\sqrt{(a')^2+(b')^2}, b'/\sqrt{(a')^2+(b')^2}$ and $\varepsilon\sqrt{(a')^2+(b')^2}$, we obtain that for $a' \in (0, \varepsilon^{-1}), b' \in (-\varepsilon^{-1}, \varepsilon^{-1})$

$$\frac{(a')^m \left|\mathcal{W}_\psi f\left(\varepsilon b', \varepsilon a'\right)\right|}{\varepsilon^\alpha \left(\sqrt{(a')^2+(b')^2}\right)^{m+\alpha} L\left(\varepsilon\sqrt{(a')^2+(b')^2}\right)} < M_1 \ , \quad 0 < \varepsilon \leq 1 \ . \tag{11.4.5}$$

In addition, we can assume that $\alpha + m \geq 1$. We also need to make a technical assumption over L which can be always made since only the values of L near 0 matter for the quasiasymptotic at zero; indeed, as seen in Section 10.3.1 of Chapter 10, we can assume (see also [227, Section 3],[15, p.25]) that there exists a constant $M_2 > 0$ such that

$$\frac{L(\varepsilon x)}{L(\varepsilon)} \leq M_2 \max\left\{x, x^{-1}\right\} \leq M_2 \, \frac{1+x^2}{x} \ , \quad \text{for all } \varepsilon, x > 0 \ . \tag{11.4.6}$$

Let

$$\beta = \alpha + m + 3, \quad \gamma = \max\left\{m+2, \alpha+\beta+1\right\} \ . \tag{11.4.7}$$

Find now a constant $M_3 > 0$ such that

$$\left|\mathcal{W}_\psi \eta(b, a)\right| \leq M_3 \left(a + \frac{1}{a}\right)^{-\gamma} (1+|b|)^{-\beta} \ . \tag{11.4.8}$$

In the following, we will also make repeated use of the elementary inequality

$$1 + |x+y| \leq (1+|x|)\,(1+|y|) \ . \tag{11.4.9}$$

Then for $0 < \varepsilon \leq 1$, we have from (11.4.5), (11.4.6) and (11.4.9) that

$$
\begin{aligned}
&|c_{\psi,\eta}\mathcal{W}_\psi f_0(\varepsilon b, \varepsilon a)| = \left| \int_0^\infty \int_{-\infty}^\infty \mathcal{W}_\psi \eta \left(\frac{\varepsilon b - b'}{a'}, \frac{\varepsilon a}{a'} \right) F(b', a') \frac{\mathrm{d}b'\mathrm{d}a'}{(a')^2} \right| \\
&= \left| \int_0^{\varepsilon^{-1}} \int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} \mathcal{W}_\psi \eta \left(\frac{b - b'}{a'}, \frac{a}{a'} \right) \mathcal{W}_\psi f(\varepsilon b', \varepsilon a') \frac{\mathrm{d}b'\mathrm{d}a'}{(a')^2} \right| \\
&\leq M_1 \varepsilon^\alpha \int_0^{\varepsilon^{-1}} \int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} \left(\sqrt{(a')^2 + (b')^2} \right)^{\alpha+m} \\
&\quad L\left(\varepsilon \sqrt{(a')^2 + (b')^2} \right) \left| \mathcal{W}_\psi \eta \left(\frac{b - b'}{a'}, \frac{a}{a'} \right) \right| \frac{\mathrm{d}b'\mathrm{d}a'}{(a')^{m+2}} \\
&\leq M_1 M_2 \varepsilon^\alpha L(\varepsilon) \int_0^{\varepsilon^{-1}} \int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} \left(\sqrt{(a')^2 + (b')^2} \right)^{\alpha+m-1} \\
&\quad \left(1 + (a')^2 + (b')^2 \right) \left| \mathcal{W}_\psi \eta \left(\frac{b - b'}{a'}, \frac{a}{a'} \right) \right| \frac{\mathrm{d}b'\mathrm{d}a'}{(a')^{m+2}} \\
&\leq M_1 M_2 \varepsilon^\alpha L(\varepsilon) \int_0^{\varepsilon^{-1}} \int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} (a' + |b'|)^{\alpha+m-1} (1 + a')^2 (1 + |b'|)^2 \\
&\quad \left| \mathcal{W}_\psi \eta \left(\frac{b - b'}{a'}, \frac{a}{a'} \right) \right| \frac{\mathrm{d}b'\mathrm{d}a'}{(a')^{m+2}} \\
&\leq M_1 M_2 \varepsilon^\alpha L(\varepsilon) \left(4I_1 + 4I_2 + 2^{\alpha+m+1} I_3 \right) \ ,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \int_0^1 \int_{|b-b'|\leq 1} (1 + |b'|)^{\alpha+m+1} \left| \mathcal{W}_\psi \eta \left(\frac{b - b'}{a'}, \frac{a}{a'} \right) \right| \frac{\mathrm{d}b'\mathrm{d}a'}{(a')^{m+2}} \ , \\
I_2 &= \int_0^1 \int_{1\leq|b-b'|} (1 + |b'|)^{\alpha+m+1} \left| \mathcal{W}_\psi \eta \left(\frac{b - b'}{a'}, \frac{a}{a'} \right) \right| \frac{\mathrm{d}b'\mathrm{d}a'}{(a')^{m+2}} \ , \\
I_3 &= \int_1^\infty \int_{-\infty}^\infty (a')^{\alpha-1} (1 + |b'|)^{\alpha+m+1} \left| \mathcal{W}_\psi \eta \left(\frac{b - b'}{a'}, \frac{a}{a'} \right) \right| \mathrm{d}b'\mathrm{d}a' \ .
\end{aligned}
$$

To estimate the last three integrals, we make use of (11.4.7), (11.4.8) and the elementary inequality (11.4.9). We have

$$
\begin{aligned}
I_1 &\leq M_3 \int_0^1 \int_{|b'|\leq 1+|b|} (1 + |b'|)^{\alpha+m+1} \left(\frac{a'}{a} \right)^\gamma \frac{\mathrm{d}b'\mathrm{d}a'}{(a')^{m+2}} \\
&\leq 2M_3 \left(\frac{1}{a} \right)^\gamma (1 + |b|)(2 + |b|)^{\alpha+m+1} \int_0^1 (a')^{\gamma-m-2} \,\mathrm{d}a' \\
&< 2^{\alpha+m+2} M_3 \left(a + \frac{1}{a} \right)^\gamma (1 + |b|)^\beta \ ;
\end{aligned}
$$

for I_2,

$$\begin{aligned}
I_2 &\leq M_3 \int_0^1 \int_{1<|b-b'|} (1+|b'|)^{\alpha+m+1} \left(\frac{a'}{a}\right)^{\gamma} \left(\frac{a'}{a'+|b-b'|}\right)^{\beta} \frac{\mathrm{d}b'\mathrm{d}a'}{(a')^{m+2}} \\
&\leq M_3 \left(a+\frac{1}{a}\right)^{\gamma} \left(\int_0^1 (a')^{\gamma+\beta-m-2}\,\mathrm{d}a'\right) \left(\int_{1<|b-b'|} \frac{(1+|b'|)^{\alpha+m+1}}{|b-b'|^{\beta}}\,\mathrm{d}b'\right) \\
&\leq M_3 \left(a+\frac{1}{a}\right)^{\gamma} \int_{1<|b'|} \frac{(1+|b'|+|b|)^{\alpha+m+1}}{|b'|^{\beta}}\,\mathrm{d}b' \\
&\leq 2M_3 \left(a+\frac{1}{a}\right)^{\gamma} (1+|b|)^{\alpha+m+1} \int_1^{\infty} \frac{(1+b')^{\alpha+m+1}}{(b')^{\beta}}\,\mathrm{d}b' \\
&\leq 2^{\alpha+m+2} M_3 \left(a+\frac{1}{a}\right)^{\gamma} (1+|b|)^{\beta}\ ;
\end{aligned}$$

and finally I_3,

$$\begin{aligned}
I_3 &\leq M_3 a^{\gamma} \int_1^{\infty} \int_{-\infty}^{\infty} \frac{(a')^{\alpha+\beta-\gamma-1}(1+|b'|)^{\alpha+m+1}}{(a'+|b-b'|)^{\beta}}\,\mathrm{d}b'\mathrm{d}a' \\
&\leq M_3 \left(a+\frac{1}{a}\right)^{\gamma} \left(\int_{-\infty}^{\infty} \frac{(1+|b'|)^{\alpha+m+1}}{(1+|b-b'|)^{\beta}}\mathrm{d}b'\right) \left(\int_1^{\infty} \frac{\mathrm{d}a'}{(a')^{\gamma+1-\beta-\alpha}}\right) \\
&\leq M_3 \left(a+\frac{1}{a}\right)^{\gamma} (1+|b|)^{\beta} \int_{-\infty}^{\infty} \frac{\mathrm{d}b'}{(1+|b'|)^2} \\
&\leq 2M_3 \left(a+\frac{1}{a}\right)^{\gamma} (1+|b|)^{\beta}\ .
\end{aligned}$$

Hence (11.4.4) is satisfied with $M = 2^{\alpha+m+6} M_1 M_2 M_3$.

In order to apply Theorem 11.3 to f_0, it remains to show that f_0 satisfies (11.3.8). Let us show that (11.4.1) is valid for all $(b,a) \in \mathbb{H}$. Indeed, for $(b,a) \in \mathbb{H}$ fixed, write $b = r\cos\vartheta$ and $a = r\sin\vartheta$, with $r>0$ and $0<\vartheta<\pi$. Then we have that

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \frac{\mathcal{W}_{\psi} f\,(\varepsilon b, \varepsilon a)}{\varepsilon^{\alpha} L(\varepsilon)} = \lim_{\varepsilon\to 0^+} \frac{\mathcal{W}_{\psi} f\,(\varepsilon r\cos\vartheta, \varepsilon r\sin\vartheta)}{\varepsilon^{\alpha} L(\varepsilon)} \\
&= r^{\alpha} \lim_{\varepsilon\to 0^+} \frac{L(\varepsilon)}{L(\varepsilon/r)}\, \frac{\mathcal{W}_{\psi} f\,(\varepsilon\cos\vartheta, \varepsilon\sin\vartheta)}{\varepsilon^{\alpha} L(\varepsilon)} = r^{\alpha} M_{\cos\vartheta,\,\sin\vartheta} := M_{b,a}\ .
\end{aligned}$$

Define $J(b,a) = M_{b,a}$, for $(b,a)\in\mathbb{H}$, we can use the above relation in combination with (11.4.5) to conclude that $J \in \mathcal{S}'(\mathbb{H})$ is a function of slow growth. In fact, for $(b,a)\in\mathbb{H}$

$$|J(b,a)| = r^{\alpha}\,|M_{\cos\vartheta,\,\sin\vartheta}| \leq M_1 \frac{\left(\sqrt{a^2+b^2}\right)^{\alpha}}{(\sin\vartheta)^m} \leq M_1 \frac{(a+|b|)^{\alpha+m}}{a^m}\ .$$

Moreover, relation (11.4.5) and (11.4.6) also imply the estimate (already used above!),

$$\left|\frac{\mathcal{W}_{\psi}f\left(\varepsilon b,\varepsilon a\right)}{\varepsilon^{\alpha}L(\varepsilon)}\right| \leq M_1 M_2 \frac{\left(\sqrt{a^2+b^2}\right)^{\alpha+m}}{a^m} \ \max\left\{\sqrt{a^2+b^2}, \frac{1}{\sqrt{a^2+b^2}}\right\} \ ,$$

for $0<\varepsilon\sqrt{a^2+b^2}\leq\sqrt{2}$.

Finally, using the last two facts and Lebesgue dominated convergence theorem, we conclude that

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+}\frac{\mathcal{W}_{\psi}f_0\left(\varepsilon b,\varepsilon a\right)}{\varepsilon^{\alpha}L(\varepsilon)} \\
&=\lim_{\varepsilon\to 0^+}\frac{1}{c_{\psi,\eta}}\int_0^{\varepsilon^{-1}}\int_{-\varepsilon^{-1}}^{\varepsilon^{-1}}\mathcal{W}_{\psi}\eta\left(\frac{b-b'}{a'},\frac{a}{a'}\right)\frac{\mathcal{W}_{\psi}f(\varepsilon b',\varepsilon a')}{\varepsilon^{\alpha}L(\varepsilon)}\frac{\mathrm{d}b'\mathrm{d}a'}{(a')^2} \\
&=\frac{1}{c_{\psi,\eta}}\int_0^{\infty}\int_{-\infty}^{\infty}\mathcal{W}_{\psi}\eta\left(\frac{b-b'}{a'},\frac{a}{a'}\right)J(b',a')\frac{\mathrm{d}b'\mathrm{d}a'}{(a')^2},
\end{aligned}$$

and this completes the argument. □

Remark 11.6. *In Theorem 11.5, we have used the half-circle $a^2+b^2=1$, $a>0$, to formulate (11.4.1) and (11.4.2), but it is clear from the proof that if we use any half-circle $a^2+b^2=r^2$, $a>0$, with r being any positive number, the conclusions of the theorem would still hold.*

11.5 Quasiasymptotic Extension from $\mathcal{S}_0'(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$.

In this section we investigate what the quasiasymptotic in $\mathcal{S}_0'(\mathbb{R})$ tells us about the quasiasymptotic in $\mathcal{S}'(\mathbb{R})$. Since $\mathcal{S}_0'(\mathbb{R})$ is the quotient space of $\mathcal{S}'(\mathbb{R})$ module the space of polynomials, we expect naturally all of our results hold module polynomials. We reformulate the problem with the aid of the Fourier transform.

Let $\mathcal{S}(\mathbb{R}_+)$, respectively $\mathcal{S}(\mathbb{R}_-)$, be the closed subspace of $\mathcal{S}(\mathbb{R})$ consisting of functions having support in $\overline{\mathbb{R}}_+$, respectively $\overline{\mathbb{R}}_-$. Note that $\mathcal{F}(\mathcal{S}_+(\mathbb{R}))=\mathcal{S}(\mathbb{R}_+)$ and $\mathcal{F}(\mathcal{S}_-(\mathbb{R}))=\mathcal{S}(\mathbb{R}_-)$. The space $\mathcal{D}(\mathbb{R}_+)$ has different nature than $\mathcal{S}(\mathbb{R}_+)$, it is defined as the set of those elements of $\phi\in\mathcal{D}(\mathbb{R})$ such that $\operatorname{supp}\phi\subset\mathbb{R}_+$ (not $\overline{\mathbb{R}}_+$).

Similarly for $\mathcal{D}(\mathbb{R}_-)$. Their dual spaces are then $\mathcal{S}'(\mathbb{R}_-) = (\mathcal{S}(\mathbb{R}_-))'$, $\mathcal{S}'(\mathbb{R}_+) = (\mathcal{S}(\mathbb{R}_+))'$, $\mathcal{D}'(\mathbb{R}_-) = (\mathcal{D}(\mathbb{R}_-))'$ and $\mathcal{D}'(\mathbb{R}_+) = (\mathcal{D}(\mathbb{R}_+))'$. Let $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$. We also consider spaces $\mathcal{D}(\mathbb{R}_0)$, $\mathcal{S}(\mathbb{R}_0) := \mathcal{S}(\mathbb{R}_-) \oplus \mathcal{S}(\mathbb{R}_+)$ and their dual spaces $\mathcal{D}'(\mathbb{R}_0)$ and $\mathcal{S}'(\mathbb{R}_0)$, respectively.

The problem of extending distributions from $\mathcal{S}'_0(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$, together with its asymptotic properties, can be reduced to that of extending distributions from $\mathcal{S}'(\mathbb{R}_0)$ to $\mathcal{S}'(\mathbb{R})$. For $\mathcal{S}'_0(\mathbb{R})$ different extensions to $\mathcal{S}'(\mathbb{R})$ differ by polynomials, and on $\mathcal{S}'(\mathbb{R}_0)$ extensions to $\mathcal{S}'(\mathbb{R})$ differ by distributions concentrate at the origin, i.e., finite sums of δ, the Dirac delta distribution, and its derivatives. Indeed, the images under Fourier transform of $\mathcal{S}'_+(\mathbb{R})$ and $\mathcal{S}'_-(\mathbb{R})$ are $\mathcal{F}(\mathcal{S}'_+(\mathbb{R})) = \mathcal{S}'(\mathbb{R}_+)$ and $\mathcal{F}(\mathcal{S}'_-(\mathbb{R})) = \mathcal{S}'(\mathbb{R}_-)$, respectively; finally the image of $\mathcal{S}'_0(\mathbb{R})$ under Fourier transform is $\mathcal{S}'(\mathbb{R}_0)$.

Suppose now that $f \in \mathcal{S}'(\mathbb{R})$ and

$$f\left(x_0 + \varepsilon x\right) \sim \varepsilon^{\alpha} L(\varepsilon) g(x) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{S}'_0(\mathbb{R}), \tag{11.5.1}$$

then if we take Fourier transform and replace $\varepsilon = \lambda^{-1}$, we obtain the equivalent expression

$$e^{i\lambda x_0 x} \hat{f}(\lambda x) \sim \lambda^{-1-\alpha} L(\lambda^{-1}) \hat{g}(x) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R}_0) \ . \tag{11.5.2}$$

Therefore the problem we are addressing is equivalent to the problem of determining the asymptotic behavior of a tempered distribution at infinity upon knowledge of the quasiasymptotic at infinity in $\mathcal{S}'(\mathbb{R}_0)$. Since $\mathcal{S}'(\mathbb{R}_0) = \mathcal{S}'(\mathbb{R}_-) \oplus \mathcal{S}'(\mathbb{R}_+)$ is enough to work in the space $\mathcal{S}'(\mathbb{R}_+)$. Observed that, as pointed out in Chapter 10, such problem has much relevance in renormalization procedures in Quantum Field Theory [21, 125, 233, 234] and in the study of singular integral equations on spaces of distributions [60]. Furthermore, the solution to the *quasiasymptotic extension problem* from $\mathcal{S}'(\mathbb{R}_+)$ to $\mathcal{S}'(\overline{\mathbb{R}}_+)$ was completely solved in Section 10.7.1 of Chapter

10, after adding the information of Corollary 10.76. Thus, Theorem 10.70 implies the following result. Notice that new terms appear in the extension, polynomial terms, as expected, and asymptotically homogeneous functions of degree zero.

Theorem 11.7. *Let $f \in \mathcal{S}'(\mathbb{R})$. Let L be slowly varying at the origin and $x_0, \alpha \in \mathbb{R}$. Suppose that*

$$f(x_0 + \varepsilon x) \sim \varepsilon^{\alpha} L(\varepsilon) g(x) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{S}_0'(\mathbb{R}) \ . \tag{11.5.3}$$

(i) If $\alpha < 0$, then f has the quasiasymptotic behavior (11.5.3) in $\mathcal{S}'(\mathbb{R})$.

(ii) If $\alpha > 0$ and $\alpha \notin \mathbb{Z}_+$, then there exists a polynomial p, of degree less than α, such that

$$f(x_0 + \varepsilon x) = p(\varepsilon x) + \varepsilon^{\alpha} L(\varepsilon) g(x) + o(\varepsilon^{\alpha} L(\varepsilon)) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{S}'(\mathbb{R}) \ . \tag{11.5.4}$$

(iii) If $\alpha = k$, $k \in \mathbb{N}$, then g is of the form $g(x) = C_- x_-^k + C_+ x_+^k + \beta x^k \log|x|$, and there are a polynomial p of degree at most $(k-1)$ and an associate asymptotically homogeneous function of degree 0 at the origin with respect to L, satisfying $c(ax) = c(x) + \beta L(x) \log a + o(L(x))$, such that

$$f(x_0 + \varepsilon x) = p(\varepsilon x) + c(\varepsilon)\varepsilon^k x^k + \varepsilon^k L(\varepsilon) g(x) + o(\varepsilon^k L(\varepsilon)) \ . \tag{11.5.5}$$

as $\varepsilon \to 0^+$ in $\mathcal{S}'(\mathbb{R})$.

Proof. As in (11.5.2), take Fourier transform to $f(x_0 + \cdot)$. In $\mathcal{S}_0'(\mathbb{R})$, we have unique decompositions $e^{ix_0 x}\hat{f} = f_- + f_+$ and $\hat{g} = g_- + g_+$, where $f_\pm, g_\pm \in \mathcal{S}'(\mathbb{R}_\pm)$. A direct application of Theorem 10.70 to $f_\pm, g_\pm$ and $L(1/\lambda)$ yields the result on the Fourier side, after a routine calculation which is left to the reader. The fact that c in (11.5.5) is associate asymptotically homogeneous follows from a computation, but we can also verify it directly; indeed, take $\phi \in \mathcal{D}(\mathbb{R})$ such that $\int_{-\infty}^{\infty} x^k \phi(x) \mathrm{d}x = 1$, and $\int_{-\infty}^{\infty} x^j \phi(x)\mathrm{d}x = 0$, for $j < k$, if we evaluate it in (11.5.5), we have that

$$\begin{aligned}
&(a\varepsilon)^k c(a\varepsilon) \int_{-\infty}^{\infty} x^k \phi(x)\mathrm{d}x + (a\varepsilon)^k L(a\varepsilon) \langle g(x), \phi(x)\rangle + o(\varepsilon^k L(\varepsilon)) \\
&= \langle f(a\varepsilon x), \phi(x)\rangle = \frac{1}{a}\left\langle f(\varepsilon x), \phi\left(\frac{x}{a}\right)\right\rangle \\
&= (a\varepsilon)^k c(\varepsilon) \int_{-\infty}^{\infty} x^k \phi(x)\mathrm{d}x + \varepsilon^k L(\varepsilon) \langle g(ax), \phi(x)\rangle + o(\varepsilon^k L(\varepsilon)) \ ,
\end{aligned}$$

and so $c(a\varepsilon) = c(\varepsilon) + \beta L(\varepsilon)\log a + o(L(\varepsilon))$, $\varepsilon \to 0^+$. $\square$

The same arguments given in the proof of Theorem 11.7, but now using Theorem 10.66 and Corollary 10.76, lead to the following theorem.

Theorem 11.8. *Let $f \in \mathcal{S}'(\mathbb{R})$. Let L be slowly varying at infinity and $\alpha \in \mathbb{R}$. Suppose that*

$$f(\lambda x) \sim \lambda^\alpha L(\lambda) g(x) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}_0'(\mathbb{R}) \ . \tag{11.5.6}$$

(i) If $\alpha \notin \mathbb{N}$, then there exists a polynomial p, which may be chosen to be divisible by $x^{\max\{0,\,[\alpha]+1\}}$, such that

$$f(\lambda x) = p(\lambda x) + \lambda^\alpha L(\lambda) g(x) + o(\lambda^\alpha L(\lambda)) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R}) \ . \tag{11.5.7}$$

(ii) If $\alpha = k$, $k \in \mathbb{N}$, then g is of the form $g(x) = C_- x_-^k + C_+ x_+^k + \beta x^k \log|x|$, and there are a polynomial p, which may be chosen divisible by x^{k+1}, and an associate asymptotically homogeneous function of degree 0 at infinity with respect to L, satisfying $c(ax) = c(x) + \beta L(x)\log a + o(L(x))$, such that,

$$f(x_0 + \lambda x) = p(\lambda x) + c(\lambda)\lambda^k x^k + \lambda^k L(\lambda) g(x) + o(\lambda^k L(\lambda)) \ , \tag{11.5.8}$$

as $\lambda \to \infty$ in $\mathcal{S}'(\mathbb{R})$.

11.6 Wavelet Tauberian Theorems for Quasiasymptotics at Points

As a consequence of our analysis from Sections 11.4 and 11.5, we obtain the tauberian theorems for quasiasymptotics at points of tempered distributions. The proofs of the next three theorems follow at once by applying Theorem 11.5 and Theorem 11.7.

Theorem 11.9. *Let $f \in \mathcal{S}'(\mathbb{R})$ and $\alpha < 0$. Suppose the wavelet $\psi \in \mathcal{S}_0(\mathbb{R})$ admits a reconstruction wavelet. Necessary and sufficient conditions in terms of the wavelet transform for f to have quasiasymptotic behavior at $x = x_0$ of degree α with respect to a slowly varying function L are the existence of the limits*

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^\alpha L(\varepsilon)} \mathcal{W}_\psi f\,(x_0 + \varepsilon b, \varepsilon a) = M_{b,a} < \infty \ , \quad a^2 + b^2 = 1, \ a > 0 \ , \tag{11.6.1}$$

and the existence of m such that

$$\limsup_{\varepsilon\to 0^+} \sup_{a^2+b^2=1,\, a>0} \frac{a^m}{\varepsilon^\alpha L(\varepsilon)} \left|\mathcal{W}_\psi f\,(x_0 + \varepsilon b, \varepsilon a)\right| < \infty \ . \tag{11.6.2}$$

In such a case there is a homogeneous distribution g of degree α such that $M_{b,a} = \mathcal{W}_\psi g(b,a)$.

Theorem 11.10. *Let $f \in \mathcal{S}'(\mathbb{R})$ and $\alpha > 0$, $\alpha \notin \mathbb{N}$. Suppose the wavelet $\psi \in \mathcal{S}_0(\mathbb{R})$ admits a reconstruction wavelet. Conditions (11.6.1) and (11.6.2) are necessary and sufficient for the existence of a polynomial p of degree less than α such that $f - p$ has quasiasymptotic behavior of degree α with respect to L at the point $x = x_0$. In such a case there is a homogeneous distribution g of degree α such that $M_{b,a} = \mathcal{W}_\psi g(b,a)$.*

Theorem 11.11. *Let $f \in \mathcal{S}'(\mathbb{R})$ and $k \in \mathbb{N}$. Suppose the wavelet $\psi \in \mathcal{S}_0(\mathbb{R})$ admits a reconstruction wavelet. Conditions (11.6.1) and (11.6.2) with $\alpha = k$ are necessary and sufficient for the existence of a polynomial of degree at most $k-1$, an associate asymptotically homogeneous function c of degree 0 with respect to L, satisfying $c(ax) = c(x) + \beta L(x)\log a + o(L(x))$, and two constants C_- and C_+ such that*

$$f\left(x_0+\varepsilon x\right)=p(\varepsilon x)+c(\varepsilon)\varepsilon^k x^k+\varepsilon^k L(\varepsilon)\left(C_- x_-^k+C_+ x_+^k+\beta x^k\log|x|\right)+o(\varepsilon^k L(\varepsilon)) \ , \tag{11.6.3}$$

as $\varepsilon\to 0^+$ *in* $\mathcal{S}'(\mathbb{R})$.

We may formulate tauberian conditions in order to guarantee quasiasymptotic behavior in the case $\alpha\geq 0$. We also point out that test functions can always be found satisfying the hypothesis of the next two theorems [44].

Theorem 11.12. *Let the hypotheses of Theorem 11.10 be satisfied. Let* $n=[\alpha]$. *Let* $\varphi\in\mathcal{S}(\mathbb{R})$ *be such that its moments* $\mu_j:=\int_{-\infty}^{\infty}x^j\varphi(x)\mathrm{d}x\neq 0$, *for* $0\leq j\leq n$. *The condition*

$$\langle f(x_0+\varepsilon x),\varphi(x)\rangle=O\left(\varepsilon^\alpha L(\varepsilon)\right), \quad \varepsilon\to 0^+ \ , \tag{11.6.4}$$

implies that f *has quasiasymptotic behavior of degree* α *with respect to* L *at the point* $x=x_0$.

Proof. By Theorem 11.11, there exist $(n+1)$ constants $c_0,c_1,\dots,c_n$ and a homogeneous distribution g of degree α such that

$$f(\varepsilon x)=\sum_{j=0}^{n}c_j\varepsilon^j x^j+\varepsilon^\alpha L(\varepsilon)g(x)+o\left(\varepsilon^\alpha L(\varepsilon)\right) \quad \text{as } \varepsilon\to 0^+ \text{ in } \mathcal{S}'(\mathbb{R}) \ .$$

Evaluating the last asymptotic expansion at φ and comparing with (11.6.4), one has that

$$\sum_{j=0}^{n}\varepsilon^j c_j\mu_j=O\left(\varepsilon^\alpha L(\varepsilon)\right)$$

which readily implies that $c_j=0$, for each $0\leq j\leq n$. □

Theorem 11.13. *Let the hypotheses of Theorem 11.11 be satisfied and* $\alpha=k$. *Let* $\varphi\in\mathcal{S}(\mathbb{R})$ *be such that its moments* $\mu_j:=\int_{-\infty}^{\infty}x^j\varphi(x)\mathrm{d}x\neq 0$, *for* $0\leq j\leq k$. *The condition*

$$\langle f(x_0+\varepsilon x),\varphi(x)\rangle\sim C\varepsilon^k L(\varepsilon) \ , \quad \varepsilon\to 0^+ \ , \tag{11.6.5}$$

for some constant C, implies that f has quasiasymptotic behavior of degree k with respect to L at the point $x = x_0$.

Proof. Comparison between (11.6.5) and (11.6.3), evaluated at φ, gives that the polynomial vanishes and the asymptotic relation

$$c(\varepsilon) \sim \frac{L(\varepsilon)}{\mu_k}\left(C - C_- \int_0^\infty x^k \varphi(-x)\mathrm{d}x - C_+ \int_0^\infty x^k \varphi(x)\mathrm{d}x\right) ,$$

from where we obtain the result. ☐

Our next tauberian theorems makes use of *quasiasymptotic boundedness* [213] as the tauberian condition. Recall the distribution f is quasiasymptotic bounded of degree α at $x = x_0$ with respect to a function L, slowly varying at the origin, if $f(x_0 + \varepsilon\,\cdot)/(\varepsilon^\alpha L(\varepsilon))$ is a weak bounded set in $\mathcal{S}'(\mathbb{R})$, for ε small enough, and we denote the order relation $f(x_0 + \varepsilon x) = O(\varepsilon^\alpha L(\varepsilon))$ as $\varepsilon \to 0^+$ in $\mathcal{S}'(\mathbb{R})$.

Theorem 11.14. *Let $f \in \mathcal{S}'(\mathbb{R})$, $x_0 \in \mathbb{R}$, $\alpha \notin \mathbb{N}$, and L be a slowly varying function at the origin. Let $\psi \in \mathcal{S}_0(\mathbb{R})$ be a wavelet admitting a reconstruction wavelet. Suppose that the following limits exist:*

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^\alpha L(\varepsilon)} \mathcal{W}_\psi f\,(x_0 + \varepsilon b, \varepsilon a) = M_{b,a} < \infty \ , \quad a^2 + b^2 = 1,\ a > 0 \ . \tag{11.6.6}$$

Then, the tauberian condition

$$f(x_0 + \varepsilon x) = O(\varepsilon^\alpha L(\varepsilon)) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{S}'(\mathbb{R}) \ , \tag{11.6.7}$$

implies the existence of a homogeneous distribution g of degree α such that $M_{b,a} = \mathcal{W}_\psi g(b,a)$ and

$$f(x_0 + \varepsilon x) \sim \varepsilon^\alpha L(\varepsilon) g(x) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{S}'(\mathbb{R}) \ . \tag{11.6.8}$$

Conversely, the quasiasymptotic behavior (11.6.8) implies (11.6.6) and (11.6.7).

Proof. The converse is clear; indeed the abelian theorem, Theorem 11.1, implies (11.6.6), and, obviously, quasiasymptotic behavior implies quasiasymptotic boundedness. On the other hand, relation (11.6.7) holds in particular in $\mathcal{S}'_0(\mathbb{R})$, hence, the characterization of bounded sets of $\mathcal{S}'_0(\mathbb{R})$ [95, Thm.28.0.1] implies that (11.6.2) is satisfied. If $\alpha < 0$, then Theorem 11.9 implies (11.6.8). Now, if $\alpha > 0$, we can always select a test function φ such that its moments $\mu_j := \int_{-\infty}^{\infty} x^j \varphi(x)\mathrm{d}x \neq 0$, for $0 \leq j \leq [\alpha]$. But if we evaluate (11.6.7) at φ, we obtain (11.6.4), and thus, Theorem 11.12 yields the result in this case. □

Theorem 11.15. *Let $f \in \mathcal{S}'(\mathbb{R})$, $x_0 \in \mathbb{R}$, $k \in \mathbb{N}$, and L be a slowly varying function at the origin. Let $\psi \in \mathcal{S}_0(\mathbb{R})$ be a wavelet admitting a reconstruction wavelet. Suppose that the following limits exist:*

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^k L(\varepsilon)} \mathcal{W}_\psi f\,(x_0 + \varepsilon b, \varepsilon a) = M_{b,a} < \infty\ , \quad a^2 + b^2 = 1,\ a > 0\ . \tag{11.6.9}$$

Then, the tauberian condition

$$f(x_0 + \varepsilon x) = O(\varepsilon^k L(\varepsilon)) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{S}'(\mathbb{R})\ , \tag{11.6.10}$$

implies the existence of a distribution of the form $g(x) = C_- x_-^k + C_+ x_+^k + \beta x^k \log|x|$, and an associate asymptotically homogeneous function c of degree zero with respect to L such that $M_{b,a} = \mathcal{W}_\psi g(b, a)$ and

$$f(x_0 + \varepsilon x) = c(\varepsilon)\varepsilon^k x^k + \varepsilon^k L(\varepsilon) g(x) + o(\varepsilon^k L(\varepsilon)) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{S}'(\mathbb{R})\ . \tag{11.6.11}$$

Moreover, $c(\varepsilon) = O(L(\varepsilon))$. Additionally, if there exists a test function $\varphi \in \mathcal{S}'(\mathbb{R})$ satisfying (11.6.5) and having non-zero k^{th}-moment, i.e. $\mu_k = \int_{-\infty}^{\infty} x^k \varphi(x)\mathrm{d}x \neq 0$, then f has quasiasymptotic behavior of degree k with respect to L.

Proof. As in the proof of Theorem 11.14, we obtain that (11.6.9) and (11.6.10) imply f satisfies an asymptotic expansion of the form (11.6.3); furthermore, evaluating the asymptotic relation (11.6.3) at a ϕ with non-zero first k moments and using the quasiasymptotic boundedness (11.6.10), we obtain (11.6.11) and $c(\varepsilon) = O(L(\varepsilon))$. Evaluating (11.6.11) at φ, we obtain that $c(\varepsilon) \sim BL(\varepsilon)$, for some constant B. This completes the proof. $\square$

11.7 Wavelet Tauberian Theorems for Quasiasymptotics at Infinity

We now state the tauberian theorems for asymptotic behavior at infinity, the proofs of Theorem 11.16 and Theorem 11.17 follow immediately from Theorem 11.4 and Theorem 11.8. The proofs of Theorems 11.18–11.21 are analogous to those of Theorems 11.12–11.15, and then we choose to omit them.

Theorem 11.16. *Let $f \in \mathcal{S}'(\mathbb{R})$ and $\alpha \notin \mathbb{N}$. Suppose the wavelet $\psi \in \mathcal{S}_0(\mathbb{R})$ admits a reconstruction wavelet. Necessary and sufficient conditions in terms of the wavelet transform for the existence of a polynomial p such that $f - p$ has quasiasymptotic behavior at infinity of degree α with respect to a slowly varying function L are the existence of the limits*

$$\lim_{\lambda \to \infty} \frac{1}{\lambda^{\alpha} L(\lambda)} \mathcal{W}_{\psi} f\,(\lambda b, \lambda a) = M_{b,a}, \quad \textit{for each } (b,a) \in \mathbb{H}\ , \tag{11.7.1}$$

and the existence constants of $\gamma, \beta, M > 0$ such that

$$\frac{1}{\lambda^{\alpha} L(\lambda)} \left|\mathcal{W}_{\psi} f\,(\lambda b, \lambda a)\right| < M \left(a + \frac{1}{a}\right)^{\gamma} (1 + |b|)^{\beta}, \tag{11.7.2}$$

for all $(b,a) \in \mathbb{H}$ and $\lambda \geq 1$. In such a case there is a homogeneous distribution g of degree α such that $M_{b,a} = \mathcal{W}_{\psi} g(b,a), (b,a) \in \mathbb{H}$.

Theorem 11.17. *Let $f \in \mathcal{S}'(\mathbb{R})$ and $k \in \mathbb{N}$. Suppose the wavelet $\psi \in \mathcal{S}_0(\mathbb{R})$ admits a reconstruction wavelet. The conditions (11.7.1) and (11.7.2) with $\alpha = k$ are necessary and sufficient for the existence of a polynomial p, which is divisible by x^{k+1}, an associate asymptotically homogeneous function c of degree 0 with respect to L, satisfying $c(ax) = c(x) + \beta L(x) \log a + o(L(x))$, and two constants C_{-} and*

C_+ such that

$$f(\lambda x) = p(\lambda x) + c(\lambda)x^k + \lambda^k L(\lambda)\left(C_- x_-^k + C_+ x_+^k + \beta x^k \log|x|\right) + o(\lambda^k L(\lambda)), \tag{11.7.3}$$

as $\lambda \to \infty$ in $\mathcal{S}'(\mathbb{R})$.

Theorem 11.18. *Let the hypotheses of Theorem 11.16 be satisfied. Set $n = [\alpha]$. Let $\varphi \in \mathcal{S}(\mathbb{R})$ be such that its moments $\mu_j := \int_{-\infty}^{\infty} x^j \varphi(x)\mathrm{d}x \neq 0$, for $n < j$. The condition*

$$\langle f(\lambda x), \varphi(x)\rangle = O(\lambda^k L(\lambda)) , \quad \lambda \to \infty , \tag{11.7.4}$$

implies that f has quasiasymptotic behavior of degree α with respect to L at infinity.

Theorem 11.19. *Let the hypotheses of Theorem 11.17 be satisfied. Let $\varphi \in \mathcal{S}(\mathbb{R})$ be such that its moments $\mu_j := \int_{-\infty}^{\infty} x^j \varphi(x)\mathrm{d}x \neq 0$, for $k \leq j$. The condition*

$$\langle f(\lambda x), \varphi(x)\rangle \sim C\lambda^k L(\lambda) , \quad \lambda \to \infty , \tag{11.7.5}$$

for some constant C, implies that f has quasiasymptotic behavior of degree k with respect to L at infinity.

Theorem 11.20. *Let $f \in \mathcal{S}'(\mathbb{R})$, $\alpha \notin \mathbb{N}$, and L be a slowly varying function at infinity. Let $\psi \in \mathcal{S}_0(\mathbb{R})$ be a wavelet admitting a reconstruction wavelet. Suppose that the following limits exist:*

$$\lim_{\lambda\to\infty} \frac{1}{\lambda^\alpha L(\lambda)} \mathcal{W}_\psi f\,(\lambda b, \lambda a) = M_{b,a}, \quad \textit{for each } (b,a) \in \mathbb{H} . \tag{11.7.6}$$

Then, the tauberian condition

$$f(\lambda x) = O(\lambda^\alpha L(\lambda)) \quad \textit{as } \lambda \to \infty \textit{ in } \mathcal{S}'(\mathbb{R}) , \tag{11.7.7}$$

implies the existence of a homogeneous distribution g of degree α such that $M_{b,a} = \mathcal{W}_\psi g(b,a)$ and

$$f(\lambda x) \sim \lambda^\alpha L(\lambda) g(x) \quad \textit{as } \lambda \to \infty \textit{ in } \mathcal{S}'(\mathbb{R}) . \tag{11.7.8}$$

Conversely, the quasiasymptotic behavior (11.7.8) implies (11.7.6) and (11.7.7).

Theorem 11.21. *Let $f \in \mathcal{S}'(\mathbb{R})$, $k \in \mathbb{N}$, and L be a slowly varying function at infinity. Let $\psi \in \mathcal{S}_0(\mathbb{R})$ be a wavelet admitting a reconstruction wavelet. Suppose that the following limits exist:*

$$\lim_{\lambda\to\infty} \frac{1}{\lambda^k L(\lambda)} \mathcal{W}_\psi f\,(\lambda b, \lambda a) = M_{b,a}, \quad \textit{for each } (b,a) \in \mathbb{H}\ . \tag{11.7.9}$$

Then, the tauberian condition

$$f(\lambda x) = O(\lambda^\alpha L(\lambda)) \quad \textit{as } \lambda \to \infty \textit{ in } \mathcal{S}'(\mathbb{R})\ , \tag{11.7.10}$$

implies the existence of a distribution of the form $g(x) = C_- x_-^k + C_+ x_+^k + \beta x^k \log|x|$, and an associate asymptotically homogeneous function c of degree zero with respect to L such that $M_{b,a} = \mathcal{W}_\psi g(b,a)$ and

$$f(\lambda x) = c(\lambda)\lambda^k x^k + \lambda^k L(\lambda) g(x) + o(\lambda^k L(\lambda)) \quad \textit{as } \lambda \to \infty \textit{ in } \mathcal{S}'(\mathbb{R})\ . \tag{11.7.11}$$

Moreover, $c(\lambda) = O(L(\lambda))$. Additionally, if there exists $\varphi \in \mathcal{S}'(\mathbb{R})$ satisfying (11.7.5) and having non-zero k^{th}-moment, i.e. $\mu_k = \int_{-\infty}^{\infty} x^k \varphi(x)\mathrm{d}x \neq 0$, then f has quasiasymptotic behavior of degree k with respect to L.

11.8 Remarks on Progressive and Regressive Distributions

We end this chapter with some comments about progressive and regressive distributions.

Suppose first that $f \in \mathcal{S}'_+(\mathbb{R})$. Since only the positive frequency part of a wavelet is relevant for the wavelet transform of f, and any non-vanishing $\psi \in \mathcal{S}_+(\mathbb{R})$ is a progressive admissible wavelet (hence it is its own reconstruction wavelet), it is enough to assume in Theorems 11.3–11.5 that ψ is an arbitrary non-zero element of $\mathcal{S}_+(\mathbb{R})$. Likewise, if $f \in \mathcal{S}'_-(\mathbb{R})$, Theorems 11.3–11.5 hold for an arbitrary non-zero regressive $\psi \in \mathcal{S}_-(\mathbb{R})$.

Assume now that $f \in \mathcal{S}'(\mathbb{R})$ is a progressive distribution, that is, $\operatorname{supp} \hat{f} \subseteq [0, \infty)$. Then, Theorems 11.9–11.21 hold if ψ is an arbitrary non-zero element of $\mathcal{S}_+(\mathbb{R})$. Similarly, for a regressive distribution, they hold for a arbitrary non-zero regressive $\psi \in \mathcal{S}_-(\mathbb{R})$.

Chapter 12

Measures and the Multidimensional $\phi-$transform

12.1 Introduction

The aim of this chapter is to use the distributional $\phi-$transform, introduced in Chapter 7 (see also [215]) in the one variable case, and here in the multidimensional case, in order to characterize the positive measures that belong to the distribution space $\mathcal{D}'(\mathbb{R}^n)$. As seen in Section 7.6, the $\phi-$transform is a very powerful tool to study global properties of distributions from local information. It is also our objective to provide extensions of the results from Section 7.4 to the multidimensional setting.

We use the notation $\mathbb{H} = \{(\mathbf{x}, t) : \mathbf{x} \in \mathbb{R}^n \text{ and } t > 0\}$. Let $F(\mathbf{x}, t)$, $(\mathbf{x}, t) \in \mathbb{H}$, be the $\phi-$transform of a distribution $f \in \mathcal{D}'(\mathbb{R}^n)$, namely

$$F(\mathbf{x}, t) = \langle f(\mathbf{x} + t\mathbf{y}), \phi(\mathbf{y}) \rangle \ ,$$

where ϕ is a fixed *positive* test function of the space $\mathcal{D}(\mathbb{R}^n)$. We prove that f is a positive measure if and only if the inferior limit of $F(\mathbf{x}, t)$, as $(\mathbf{x}, t)$ approaches *any* point in the boundary $\partial\mathbb{H} = \mathbb{R}^n \times \{0\}$, in an angular fashion, is positive. Since any positive measure is equal to a function almost everywhere, this result provides a technique to show the existence of the almost everywhere angular limits of the $\phi-$transform of a distribution.

The plan of this chapter is as follows. We start by giving some necessary background in Section 12.2, we are particularly interested in the concepts of distributional point values and Cesàro order symbols for distributions of several variables. We then continue by proving some useful properties of the multidimensional $\phi-$transform in Section 12.3, an important result to be established is the distributional convergence of the transform to the analyzed distribution. Finally, we consider the characterization of positive measures in Section 12.4. The results to be discussed in this chapter have already been published in [219].

12.2 Preliminaries

We will discuss in this section how to extend the notions of distributional point values and Cesàro order symbols to distributions of several variables. The definitions are essentially the same as in the one-dimensional case.

12.2.1 Distributional Point Values in Several Variables

We shall use the notion of the distributional point value of distributions introduced by Łojasiewicz for the multidimensional case in [129]. Actually, the definition does not differ from the one variable case. Let $f \in \mathcal{D}'(\mathbb{R}^n)$, and let $\mathbf{x}_0 \in \mathbb{R}^n$. We say that f has the distributional point value γ at $\mathbf{x} = \mathbf{x}_0$, and write

$$f(\mathbf{x}_0) = \gamma, \quad \text{distributionally}, \tag{12.2.1}$$

if $\lim_{\varepsilon \to 0} f(\mathbf{x}_0 + \varepsilon \mathbf{x}) = \gamma$ in the space $\mathcal{D}'(\mathbb{R}^n)$, that is, if

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^n} \left\langle f(\mathbf{x}), \phi\left(\frac{\mathbf{x} - \mathbf{x_0}}{\varepsilon}\right)\right\rangle = \gamma \int_{\mathbb{R}^n} \phi(\mathbf{x})\, \mathrm{d}\mathbf{x}, \tag{12.2.2}$$

for all test functions $\phi \in \mathcal{D}(\mathbb{R}^n)$. It can be shown that $f(\mathbf{x_0}) = \gamma$, distributionally, if and only if there exists a multi-index $\mathbf{k}_0 \in \mathbb{N}^n$ such that for all multi-indices $\mathbf{k} \geq \mathbf{k}_0$ there exists a $\mathbf{k}$ primitive of f, G with $\mathbf{D}^{\mathbf{k}} G = f$, that is a continuous function in a neighborhood of $\mathbf{x} = \mathbf{x}_0$ and satisfies

$$G(\mathbf{x}) = \frac{\gamma (\mathbf{x} - \mathbf{x}_0)^{\mathbf{k}}}{\mathbf{k}!} + o\left(|\mathbf{x} - \mathbf{x}_0|^{|\mathbf{k}|}\right), \quad \text{as } \mathbf{x} \to \mathbf{x}_0. \tag{12.2.3}$$

It is important to observe that the distributional point values determine a distribution if they exist *everywhere,* that is, if $f \in \mathcal{D}'(\mathbb{R}^n)$ is such that $f(\mathbf{x_0}) = 0$ distributionally $\forall \mathbf{x_0} \in \Omega$, where Ω is an open set, then $f = 0$ in Ω [128, 129].

There is a related but different notion of distributional point value in several variables, that of a *radial symmetric* value. It extends the corresponding one-dimensional notion studied in Section 3.10. We say that f has the (radial) symmetric distributional value γ at $\mathbf{x} = \mathbf{x}_0$, and write

$$f_{\mathrm{sym}}\left(\mathbf{x}_0\right) = \gamma, \quad \text{distributionally}\,, \tag{12.2.4}$$

if (12.2.2) holds for *radial* test functions, that is, test functions $\phi \in \mathcal{D}'(\mathbb{R}^n)$ such that $\phi(x) = \phi(Tx)$, for all $T \in \mathrm{O}(n)$, the group of orthogonal linear transformations. In the one variable case this means that $(f\left(x_0 + x\right) + f\left(x_0 - x\right))/2$ has the distributional value γ at $x = 0$. In several variables it means that $R\left(r\right) = \int_{\mathbb{S}} f\left(\mathbf{x}_0 + r\omega\right)\, \mathrm{d}\sigma\left(\omega\right)$, when suitable extended to $\mathcal{D}'\left(\mathbb{R}\right)$, has the value γ at $r = 0$, where $\mathbb{S}$ is the unit sphere. A distribution like $\delta'\left(x\right)$ has the symmetric value 0 at all points, so, in general, the symmetric distributional point values *do not* determine a distribution uniquely.

12.2.2 Multidimensional Cesàro Order Symbols

We shall follow [49, 61] for the notions related to Cesàro behavior of distributions. If $f \in \mathcal{D}'\left(\mathbb{R}^n\right)$ and $\alpha \in \mathbb{R}$ is not a negative integer, we say that f is bounded by $\left|\mathbf{x}\right|^{\alpha}$ in the Cesàro sense for $\left|\mathbf{x}\right|$ large, and write

$$f\left(\mathbf{x}\right) = O\left(\left|\mathbf{x}\right|^{\alpha}\right) \quad (\mathrm{C})\ , \quad \text{as } \left|\mathbf{x}\right| \to \infty\,, \tag{12.2.5}$$

if there exists a multi-index $\mathbf{k} \in \mathbb{N}^n$ and a $\mathbf{k}$- primitive, $\mathbf{D}^{\mathbf{k}}G = f$, which is a (locally integrable) function for $\left|\mathbf{x}\right|$ large and satisfies the *ordinary* order relation

$$G\left(\mathbf{x}\right) = O\left(\left|\mathbf{x}\right|^{\alpha + \left|\mathbf{k}\right|}\right)\ , \quad \text{as } \left|\mathbf{x}\right| \to \infty\,. \tag{12.2.6}$$

Naturally (12.2.6) will not hold for all primitives of f, and if it holds for $\mathbf{k}$ it will also hold for bigger multi-indices. Naturally, we may also include $\mathbf{k}$ in the notation by writing $(\mathrm{C}, \mathbf{k})$ in (12.2.5) instead of (C), but the nature of the problems to be consider will not require us to do so.

12.3 The Multidimensional $\phi-$transform

In this section we explain how we can extend to several variables the $\phi-$transform introduced in Section 7.4 of Chapter 7 (see also [215, 40, 41]). Let $\phi \in \mathcal{D}(\mathbb{R}^n)$ be a fixed *normalized* test function, that is, one that satisfies

$$\int_{\mathbb{R}^n} \phi(\mathbf{x})\, \mathrm{d}\mathbf{x} = 1\,. \tag{12.3.1}$$

If $f \in \mathcal{D}'(\mathbb{R}^n)$ we introduce the function of $n+1$ variables $F = F_\phi\{f\}$ by the formula

$$F(\mathbf{x},t) = \langle f(\mathbf{x}+t\mathbf{y}), \phi(\mathbf{y})\rangle\ , \tag{12.3.2}$$

where $(\mathbf{x},t) \in \mathbb{H}$, the half-space $\mathbb{R}^n \times (0,\infty)$. Naturally the evaluation in (12.3.2) is with respect to the variable $\mathbf{y}$. We call F the $\phi-$transform of f. This transform also receives other names, such as the standard average with kernel ϕ [40, 41]. Whenever we consider $\phi-$transforms we assume that ϕ satisfies (12.3.1).

The definition of the $\phi-$transform tells us that if $f(\mathbf{x}_0) = \gamma$, distributionally, then $F(\mathbf{x}_0,t) \to \gamma$ as $t \to 0^+$, but actually $F(\mathbf{x},t) \to \gamma$ as $(\mathbf{x},t) \to (\mathbf{x}_0,0)$ in an *angular* or *non-tangential* fashion, that is if $|\mathbf{x}-\mathbf{x}_0| \le Mt$ for some $M > 0$ (just replace $\phi(\mathbf{x})$ in (12.2.2) by $\phi(\mathbf{x}-r\omega)$ where $|\omega| = 1$ and $0 \le r \le M$). On the other hand, if $f_{\mathrm{sym}}(\mathbf{x}_0) = \gamma$, distributionally, then $F(\mathbf{x}_0,t) \to \gamma$ as $t \to 0^+$ whenever ϕ is radial. However, in general $F(\mathbf{x},t)$ does not approach γ radially for general test functions and in general $F(\mathbf{x},t)$ does not approach γ in an angular fashion even if ϕ is radial.

We can also consider the $\phi-$transform if $\phi \in \mathcal{A}(\mathbb{R}^n)$ satisfies (12.3.1) and $f \in \mathcal{A}'(\mathbb{R}^n)$, where $\mathcal{A}(\mathbb{R}^n)$ is a suitable space of test functions, such as $\mathcal{S}(\mathbb{R}^n)$ or $\mathcal{K}(\mathbb{R}^n)$.

We start with the distributional convergence of the $\phi-$transform.

Theorem 12.1. *If $\phi \in \mathcal{D}(\mathbb{R}^n)$ and $f \in \mathcal{D}'(\mathbb{R}^n)$, then*

$$\lim_{t\to 0^+} F(\mathbf{x},t) = f(\mathbf{x}) \,, \tag{12.3.3}$$

distributionally in the space $\mathcal{D}'(\mathbb{R}^n)$, that is, if $\rho \in \mathcal{D}(\mathbb{R}^n)$ then

$$\lim_{t\to 0^+} \langle F(\mathbf{x},t), \rho(\mathbf{x})\rangle = \langle f(\mathbf{x}), \rho(\mathbf{x})\rangle \,. \tag{12.3.4}$$

Proof. We have that

$$\langle F(\mathbf{x},t), \rho(\mathbf{x})\rangle = \langle \varrho(t\mathbf{y}), \phi(\mathbf{y})\rangle \,, \tag{12.3.5}$$

where

$$\varrho(\mathbf{z}) = \langle f(\mathbf{x}), \rho(\mathbf{x}-\mathbf{z})\rangle \,, \tag{12.3.6}$$

is a smooth function of $\mathbf{z}$. The Łojasewicz point value $\varrho(\mathbf{0})$ exists and equals the ordinary value and thus

$$\lim_{t\to 0^+} \langle \varrho(t\mathbf{y}), \phi(\mathbf{y})\rangle = \varrho(\mathbf{0}) = \langle f(\mathbf{x}), \rho(\mathbf{x})\rangle \,, \tag{12.3.7}$$

as required. □

The result of the Theorem 12.1 also hold in other cases. In order to obtain those results we need some lemmas. Recall that an asymptotic order relation is *strong* if it remains valid after differentiation of any order. So, we write

$$\phi(\mathbf{x}) = O\left(|\mathbf{x}|^{\beta}\right), \quad \text{strongly as } |\mathbf{x}| \to \infty \,,$$

if $\phi \in \mathcal{E}(\mathbb{R}^n)$ and it satisfies

$$\mathbf{D}^{\mathbf{j}}\phi(\mathbf{x}) = O\left(|\mathbf{x}|^{\beta-|\mathbf{j}|}\right) \,,$$

for each multi-index $\mathbf{j} \in \mathbb{N}^n$.

Lemma 12.2. *Let $f \in \mathcal{E}'(\mathbb{R}^n)$ be a distribution with compact support K. Let $\phi \in \mathcal{E}(\mathbb{R}^n)$ be a test function that satisfies (12.3.1) and*

$$\phi(\mathbf{x}) = O\left(|\mathbf{x}|^{\beta}\right), \quad \textit{strongly as } |\mathbf{x}| \to \infty\,, \tag{12.3.8}$$

where $\beta < -n$. Then

$$\lim_{t\to 0^+} F(\mathbf{x}, t) = 0\,, \tag{12.3.9}$$

uniformly on compacts of $\mathbb{R}^n \setminus K$.

Proof. There exits a constants $M > 0$ and $q \in \mathbb{N}$ such that

$$|\langle f(\mathbf{y}), \rho(\mathbf{y})\rangle| \le M \sum_{|\mathbf{j}|=0}^{q} \left\|\mathbf{D}^{\mathbf{j}}\rho\right\|_{K,\infty} \qquad \forall \rho \in \mathcal{E}(\mathbb{R}^n)\,, \tag{12.3.10}$$

where $\|\rho\|_{K,\infty} = \sup\{|\rho(\mathbf{x})| : \mathbf{x} \in K\}$. There exist $r_0 > 0$ and constants $M_{\mathbf{j}} > 0$ such that $\left|\mathbf{D}^{\mathbf{j}}\phi(\mathbf{x})\right| \le M_{\mathbf{j}} |\mathbf{x}|^{\beta - |\mathbf{j}|}$ for $|\mathbf{x}| \ge r_0$ and $|\mathbf{j}| \le q$. Let L be a compact subset of $\mathbb{R}^n \setminus K$, and let $t_0 > 0$ be such that if $0 < t \le t_0$ then $t^{-1}|\mathbf{x} - \mathbf{y}| \ge r_0$ for all $\mathbf{x} \in L$, $\mathbf{y} \in K$. Then, since

$$F(\mathbf{x}, t) = t^{-n}\left\langle f(\mathbf{y}), \phi\left(t^{-1}(\mathbf{y} - \mathbf{x})\right)\right\rangle\,, \tag{12.3.11}$$

it follows that for $0 < t \le t_0$,

$$|F(\mathbf{x}, t)| \le M_2 t^{-n-\beta}, \qquad \forall \mathbf{x} \in L, \tag{12.3.12}$$

where $M_2 = M \sum_{|\mathbf{j}|=0}^{q} M_{\mathbf{j}}$ is a constant. Since $-\beta - n > 0$, we obtain that (12.3.9) holds uniformly on $\mathbf{x} \in L$. $\square$

The definition of the Łojasiewicz point value is that if $f \in \mathcal{D}'(\mathbb{R}^n)$ then $f(\mathbf{x}_0) = \gamma$, distributionally, if

$$\lim_{\varepsilon\to 0} \langle f(\mathbf{x}_0 + \varepsilon\mathbf{x}), \phi(\mathbf{x})\rangle = \gamma \int_{\mathbb{R}^n} \phi(\mathbf{x})\, \mathrm{d}\mathbf{x}\,, \tag{12.3.13}$$

whenever $\phi \in \mathcal{D}(\mathbb{R}^n)$. If f belongs to a smaller class of distributions, then the evaluation $\langle f(\mathbf{x}_0 + \varepsilon\mathbf{x}), \phi(\mathbf{x})\rangle$ will be defined for test functions of a larger class, not only for those of $\mathcal{D}(\mathbb{R}^n)$, and one may ask whether (12.3.13) remains true in that case. There are cases where (12.3.13) is not true, for instance if $f \in \mathcal{E}'(\mathbb{R})$ sometimes there are $\phi \in \mathcal{E}(\mathbb{R})$ that do not satisfy (12.3.13) (see Remark 7.5). However, it was shown in [54], and already used in Chapter 7, that in the one variable case, (12.3.13) holds if $f(x_0) = \gamma$, distributionally, and the following conditions are satisfied:

$$f(x) = O(|x|^{\alpha}) \quad (\mathrm{C}), \quad \text{as } |x| \to \infty, \tag{12.3.14}$$

$$\phi(x) = O(|x|^{\beta}), \quad \text{strongly as } |x| \to \infty, \tag{12.3.15}$$

$$\alpha + \beta < -1, \quad \beta < -1. \tag{12.3.16}$$

In particular, (12.3.13) is valid when $f \in \mathcal{S}'(\mathbb{R})$ and $\phi \in \mathcal{S}(\mathbb{R})$ [54, 153, 227]. Actually a corresponding result is valid in several variables, and the proof is basically the same.

Theorem 12.3. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$ with $f(\mathbf{x}_0) = \gamma$, distributionally. Let $\phi \in \mathcal{E}(\mathbb{R}^n)$. Suppose that*

$$f(\mathbf{x}) = O(|\mathbf{x}|^{\alpha}) \quad (\mathrm{C}), \quad \textit{as } |\mathbf{x}| \to \infty, \tag{12.3.17}$$

$$\phi(\mathbf{x}) = O\left(|\mathbf{x}|^{\beta}\right), \quad \textit{strongly as } |\mathbf{x}| \to \infty, \tag{12.3.18}$$

$$\alpha + \beta < -n, \quad \textit{and } \beta < -n. \tag{12.3.19}$$

Then

$$\lim_{\varepsilon \to 0} \langle f(\mathbf{x}_0 + \varepsilon \mathbf{x}), \phi(\mathbf{x}) \rangle = \gamma \int_{\mathbb{R}^n} \phi(\mathbf{x})\, \mathrm{d}\mathbf{x}. \tag{12.3.20}$$

Proof. Suppose that $\mathbf{x}_0 = \mathbf{0}$. There exists a multi-index $\mathbf{k}$ and two primitives of f, $\mathbf{D}^{\mathbf{k}} G_1 = \mathbf{D}^{\mathbf{k}} G_2 = f$ such that they are continuous and

$$G_1(\mathbf{x}) = O\left(|\mathbf{x}|^{\alpha + |\mathbf{k}|}\right), \quad \text{as } |\mathbf{x}| \to \infty, \tag{12.3.21}$$

$$G_2(\mathbf{x}) = \frac{\gamma \mathbf{x}^{\mathbf{k}}}{\mathbf{k}!} + o\left(|\mathbf{x}|^{|\mathbf{k}|}\right), \quad \text{as } |\mathbf{x}| \to 0. \tag{12.3.22}$$

Hence we can combine them into a single function G that satisfies

$$G(\mathbf{x}) = \frac{\gamma \mathbf{x}^{\mathbf{k}}}{\mathbf{k}!} + o\left(|\mathbf{x}|^{|\mathbf{k}|}\right) , \quad \text{as } |\mathbf{x}| \to 0 ,$$
$$|G(\mathbf{x})| \le M |\mathbf{x}|^{|\mathbf{k}|} , \quad \text{for } |\mathbf{x}| \le 1 ,$$
$$|G(\mathbf{x})| \le M |\mathbf{x}|^{\alpha+|\mathbf{k}|} , \quad \text{for } |\mathbf{x}| \ge 1 ,$$

and

$$f = g + \mathbf{D}^{\mathbf{k}} G , \tag{12.3.23}$$

where g has compact support and g vanishes near the origin. Then (12.3.20) holds for g (with $\gamma = 0$), because of the Lemma 12.2. Therefore it is enough to prove (12.3.20) if $f = \mathbf{D}^{\mathbf{k}} G$; but in this case we may use the Lebesgue dominated convergence theorem to obtain

$$\begin{aligned}
\lim_{\varepsilon \to 0} \langle f(\varepsilon \mathbf{x}), \phi(\mathbf{x}) \rangle &= \lim_{\varepsilon \to 0} (-1)^{|\mathbf{k}|} \varepsilon^{-|\mathbf{k}|} \int_{\mathbb{R}^n} G(\varepsilon \mathbf{x}) \mathbf{D}^{\mathbf{k}} \phi(\mathbf{x}) \, \mathrm{d}\mathbf{x} \\
&= \frac{(-1)^{|\mathbf{k}|} \gamma}{\mathbf{k}!} \int_{\mathbb{R}^n} \mathbf{x}^{\mathbf{k}} \mathbf{D}^{\mathbf{k}} \phi(\mathbf{x}) \, \mathrm{d}\mathbf{x} \\
&= \gamma \int_{\mathbb{R}^n} \phi(\mathbf{x}) \, \mathrm{d}\mathbf{x} ,
\end{aligned}$$

as required. $\square$

In particular, (12.3.20) holds if $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$.

Corollary 12.4. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$ with $f(\mathbf{x}_0) = \gamma$, distributionally. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$\lim_{\varepsilon \to 0} \langle f(\mathbf{x}_0 + \varepsilon \mathbf{x}), \phi(\mathbf{x}) \rangle = \gamma \int_{\mathbb{R}^n} \phi(\mathbf{x}) \, \mathrm{d}\mathbf{x} . \tag{12.3.24}$$

Using the same argument as in the last proof we can prove that if $f(\mathbf{x}) = 0$ for $\mathbf{x} \in \Omega$, an open set, and the conditions (12.3.17), (12.3.18), and (12.3.19) are satisfied, then the convergence in (12.3.20) is uniform on compacts of Ω.

Corollary 12.5. *If $\phi \in \mathcal{E}(\mathbb{R}^n)$ and $f \in \mathcal{D}'(\mathbb{R}^n)$ satisfy the conditions (12.3.17), (12.3.18), and (12.3.19). Then*

$$\lim_{t \to 0^+} F(\mathbf{x}, t) = 0\,, \tag{12.3.25}$$

uniformly on compact subsets of $\mathbb{R}^n \setminus \operatorname{supp} f$. *In particular, (12.3.25) holds if $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$.*

We can now extend the distributional convergence of the $\phi-$transform, Theorem 12.1, to other cases.

Theorem 12.6. *If $\phi \in \mathcal{E}(\mathbb{R}^n)$ and $f \in \mathcal{D}'(\mathbb{R}^n)$ satisfy the conditions (12.3.17), (12.3.18), and (12.3.19), then*

$$\lim_{t \to 0^+} F(\mathbf{x}, t) = f(\mathbf{x})\ , \tag{12.3.26}$$

distributionally in the space $\mathcal{D}'(\mathbb{R}^n)$, that is, if $\rho \in \mathcal{D}(\mathbb{R}^n)$, then

$$\lim_{t \to 0^+} \langle F(\mathbf{x}, t), \rho(\mathbf{x})\rangle = \langle f(\mathbf{x}), \rho(\mathbf{x})\rangle\ . \tag{12.3.27}$$

In particular, distributional convergence, (12.3.26), holds if $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$ actually in the space $\mathcal{S}'(\mathbb{R}^n)$.

Proof. We proceed as in the proof of the Theorem 12.1 by observing that

$$\langle F(\mathbf{x}, t), \rho(\mathbf{x})\rangle = \langle \varrho(t\mathbf{y}), \phi(\mathbf{y})\rangle\ ,$$

where $\varrho(\mathbf{z}) = \langle f(\mathbf{x}), \rho(\mathbf{x}-\mathbf{z})\rangle$. Next we observe that ϱ is a smooth function, and that it satisfies $\varrho(\mathbf{x}) = O(|\mathbf{x}|^\alpha)\,(\mathrm{C})$, as $|\mathbf{x}| \to \infty$. Indeed, there exists a multi-index $\mathbf{k}$ and a primitive of f of that order, $\mathbf{D}^{\mathbf{k}} G = f$, which is an ordinary function for large arguments and satisfies $|G(\mathbf{x})| = O\left(|\mathbf{x}|^{|\mathbf{k}|+\alpha}\right)$ as $|\mathbf{x}| \to \infty$. We have then that

$$\begin{aligned} \varrho(\mathbf{z}) &= \left\langle \mathbf{D}^{\mathbf{k}}_{\mathbf{x}} G(\mathbf{x}), \rho(\mathbf{x}-\mathbf{z})\right\rangle \\ &= \mathbf{D}^{\mathbf{k}}_{\mathbf{z}} \langle G(\mathbf{x}), \rho(\mathbf{x}-\mathbf{z})\rangle\ , \end{aligned}$$

and $\langle G(\mathbf{x}), \rho(\mathbf{x}-\mathbf{z})\rangle = \int_{\operatorname{supp}\rho} G(\mathbf{x}+\mathbf{z})\,\rho(\mathbf{x})\;\mathrm{d}\mathbf{x} = O\left(|\mathbf{z}|^{|\mathbf{k}|+\alpha}\right)$ as $|\mathbf{z}| \to \infty$, since $\operatorname{supp}\rho$ is compact. Hence, Theorem 12.3 allows us to obtain that

$$\lim_{t\to 0^+} \langle \varrho(t\mathbf{y}), \phi(\mathbf{y})\rangle = \varrho(\mathbf{0}) = \langle f(\mathbf{x}), \rho(\mathbf{x})\rangle \ .$$

□

Remark 12.7. *Observe also if $\phi \in \mathcal{E}(\mathbb{R}^n)$ and $f \in \mathcal{D}'(\mathbb{R}^n)$ satisfy the conditions (12.3.17), (12.3.18), and (12.3.19), then when the distributional point value $f(\mathbf{x}_0)$ exists, then $F(\mathbf{x}, t) \to f(\mathbf{x}_0)$ as $(\mathbf{x}, t) \to (\mathbf{x}_0, 0)$ in an angular fashion, while if the distributional symmetric value $f_{\text{sym}}(\mathbf{x}_0)$ exists and ϕ is radial then $F(\mathbf{x}_0, t) \to f_{\text{sym}}(\mathbf{x}_0)$ as $t \to 0^+$.*

12.4 Measures and the ϕ−transform

We shall use the following nomenclature. As usual, a positive (Radon) measure μ is a *positive* functional in the space of compactly supported continuous functions, which would be denoted by integral notation, or by distributional notation, $f = f_\mu$, so that

$$\langle f, \phi\rangle = \int_{\mathbb{R}^n} \phi(\mathbf{x})\;\mathrm{d}\mu(\mathbf{x})\,. \tag{12.4.1}$$

Recall [180] that a distribution f is a positive measure if and only if $\langle f, \phi\rangle \geq 0$ whenever $\phi \geq 0$. A signed measure is a real continuous functional in the space of compactly supported continuous functions, denoted as, say ν, or as $g = g_\nu$. Observe that any signed measure can be written as $\nu = \nu_+ - \nu_-$, where $\nu_\pm$ are positive measures concentrated on disjoint sets. We shall also use the Lebesgue decomposition, according to which any signed measure ν can be written as $\nu = \nu_{\text{abs}} + \nu_{\text{sig}}$, where ν_{abs} is absolutely continuous with respect to the Lebesgue measure, so that it corresponds to a regular distribution, while ν_{sig} is a signed measure concentrated on a set of Lebesgue measure zero. We shall also need to consider the positive measures $(\nu_{\text{sig}})_\pm = (\nu_\pm)_{\text{sig}}$, the positive and negative singular parts of ν.

Our first results are very simple, but useful.

Theorem 12.8. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$. Let U be an open set of $\mathbb{R}^n$. Then f is a positive measure in U if and only if its $\phi-$transform $F = F_\phi\{f\}$ with respect to a given normalized, positive test function $\phi \in \mathcal{D}(\mathbb{R}^n)$ satisfies $F(\mathbf{x}, t) \geq 0$ for all $(\mathbf{x}, t) \in \mathfrak{U}$, where $\mathfrak{U}$ is some open subset of $\mathbb{H}$ with $U \subset \overline{\mathfrak{U}} \cap \partial\mathbb{H}$.*

Proof. If f is a positive measure in U, and $\phi(\mathbf{x}) = 0$ for $|\mathbf{x}| \geq R$, then $F(\mathbf{x}, t) \geq 0$ if the ball of center $\mathbf{x}$ and radius Rt is contained in U, and the set of such points $(\mathbf{x}, t) \in \mathbb{H}$ could be taken as $\mathfrak{U}$. Conversely, if such $\mathfrak{U}$ exists then $\langle f, \psi\rangle = \lim_{t\to 0}\langle F(\mathbf{x}, t), \psi(\mathbf{x})\rangle \geq 0$ whenever $\psi \in \mathcal{D}(\mathbb{R}^n)$, $\psi \geq 0$, and $\operatorname{supp}\psi \subset U$. □

Theorem 12.9. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$. Then f is a positive measure if and only if its $\phi-$transform $F = F_\phi\{f\}$ with respect to a given normalized, positive test function $\phi \in \mathcal{D}(\mathbb{R}^n)$ satisfies $F(\mathbf{x}, t) \geq 0$ for all $(\mathbf{x}, t) \in \mathbb{H}$.*

Proof. The proof is clear. □

If $\mathbf{x}_0 \in \mathbb{R}^n$ we shall denote by $C_{\mathbf{x}_0,\vartheta}$ the cone in $\mathbb{H}$ starting at $\mathbf{x}_0$ of angle $\vartheta \geq 0$,

$$C_{\mathbf{x}_0,\vartheta} = \{(\mathbf{x}, t) \in \mathbb{H} : |\mathbf{x} - \mathbf{x}_0| \leq (\tan\vartheta)t\} . \tag{12.4.2}$$

If $f \in \mathcal{D}'(\mathbb{R}^n)$ is real valued and $\mathbf{x}_0 \in \mathbb{R}^n$ then we consider the upper and lower angular values of its $\phi-$transform,

$$f^+_{\phi,\vartheta}(\mathbf{x}_0) = \limsup_{\substack{(\mathbf{x},t)\to(\mathbf{x}_0,0)\\(\mathbf{x},t)\in C_{\mathbf{x}_0,\vartheta}}} F(\mathbf{x}, t) , \tag{12.4.3}$$

$$f^-_{\phi,\vartheta}(\mathbf{x}_0) = \liminf_{\substack{(\mathbf{x},t)\to(\mathbf{x}_0,0)\\(\mathbf{x},t)\in C_{\mathbf{x}_0,\vartheta}}} F(\mathbf{x}, t) . \tag{12.4.4}$$

The quantities $f^\pm_{\phi,\vartheta}(\mathbf{x}_0)$ are well defined at all points $\mathbf{x}_0$, but, of course, they could be infinite.

Theorem 12.10. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$. Let U be an open set. Then f is a positive measure in U if and only if its $\phi-$transform $F = F_\phi\{f\}$ with respect to a given normalized, positive test function $\phi \in \mathcal{D}(\mathbb{R}^n)$ satisfies*

$$f_{\phi,\vartheta}^{-}(\mathbf{x}) \geq 0 \qquad \forall \mathbf{x} \in U\,,\ \forall \vartheta \in [0, \pi/2)\,. \tag{12.4.5}$$

Proof. If f is a positive measure in U, then $F \geq 0$ in some open set of $\mathbb{H}$, $\mathfrak{U}$ with $U \subset \overline{\mathfrak{U}} \cap \partial\mathbb{H}$, and thus (12.4.5) is satisfied. Conversely, let us show that if f is not a positive measure in U then (12.4.5) is not satisfied. First, if f is not a positive measure then there exists $\sigma > 0$ such that $g = f + \sigma$ is not a positive measure; let G be the $\phi-$transform of g. There exists an open ball B, with $\overline{B} \subset U$, such that g is not a positive measure in B. Using Theorem 12.8, if $0 < \varepsilon < 1$ we can find $(\mathbf{x}_1, t_1) \in \mathbb{H}$ with $\mathbf{x}_1 \in \overline{B}$ and $t_1 < \varepsilon$, such that $G(\mathbf{x}_1, t_1) < 0$.

The test function ϕ has compact support, so suppose that $\phi(\mathbf{x}) = 0$ for $|\mathbf{x}| \geq R$. Since $G(\mathbf{x}_1, t_1)$ depends only on the values of g on the closed ball $|\xi - \mathbf{x}_1| \leq Rt_1$, it follows that g is not a positive measure in that ball and consequently given $S > R$ and σ small enough, there exist t_σ and ξ_σ with $|\xi_\sigma - \mathbf{x}_1| \leq St_1$ such that $G(\xi_\sigma, t_\sigma) < 0$. Let $0 < \alpha < 1$, and choose ε such that the distance from $\overline{B}$ to the complement of U is bigger than $S\varepsilon(1-\alpha)^{-1}$. Hence we can define recursively two sequences $\{\mathbf{x}_n\}$ and $\{t_n\}$ such that

$$|\mathbf{x}_n - \mathbf{x}_{n-1}| \leq St_{n-1}\,, \qquad 0 < t_n < \alpha t_{n-1}\,, \qquad G(\mathbf{x}_n, t_n) < 0\,. \tag{12.4.6}$$

The sequence $\{\mathbf{x}_n\}$ converges to some $\mathbf{x}^*$, because $\sum_{n=1}^{\infty} |\mathbf{x}_{n+1} - \mathbf{x}_n|$ converges, due to the inequality $|\mathbf{x}_{n+1} - \mathbf{x}_n| \leq S\alpha^{n-1}t_1$. Then $\mathbf{x}^* \in U$, since $|\mathbf{x}^* - \mathbf{x}_1| \leq S\varepsilon(1-\alpha)^{-1}$. Actually,

$$|\mathbf{x}^* - \mathbf{x}_n| \leq \sum_{k=n}^{\infty} |\mathbf{x}_{k+1} - \mathbf{x}_k| \leq \frac{St_n}{1-\alpha}\,, \tag{12.4.7}$$

and it also follows that $(\mathbf{x}_n, t_n) \in C_{\mathbf{x}^*,\vartheta}$ if $\tan\vartheta = S(1-\alpha)^{-1}$, and thus

$$g_{\phi,\vartheta}^{-}(\mathbf{x}^*) \leq 0\,. \tag{12.4.8}$$

But (12.4.8) in turn yields that $f^{-}_{\phi,\vartheta}(\mathbf{x}^{*}) < -\sigma < 0$, in contradiction with the hypothesis. $\square$

If f is a signed measure then it has distributional point values almost everywhere and thus the angular limit of its $\phi-$transform exists almost everywhere and equals the absolutely continuous part of the distribution. Therefore we immediately obtain the following result.

Theorem 12.11. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$. Suppose its $\phi-$transform $F = F_{\phi}\{f\}$ with respect to a given normalized, positive test function $\phi \in \mathcal{D}(\mathbb{R}^n)$ satisfies*

$$f^{-}_{\phi,\vartheta}(\mathbf{x}) \geq -M\,, \qquad \forall \mathbf{x} \in U\,,\ \forall \vartheta \in [0, \pi/2)\,, \tag{12.4.9}$$

where U is an open set and where M is a constant. Then the angular boundary limit

$$f_{\text{ang}}(\mathbf{x}) = \lim_{\substack{(\mathbf{x},t)\to(\mathbf{x}_0,0) \\ \text{angular}}} F(\mathbf{x}, t)\ , \tag{12.4.10}$$

exists almost everywhere in U and defines a locally integrable function. Also there exists a singular positive measure μ_{+} such that in U

$$f = f_{\text{ang}} + \mu_{+}\,. \tag{12.4.11}$$

Proof. Indeed, Theorem 12.10 yields that $f + M$ is a positive measure in U, whose Lebesgue decomposition yields (12.4.11), after a small rearrangement of terms. $\square$

We also obtain the following result on the existence of almost everywhere angular limits of the $\phi-$transform.

Theorem 12.12. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$. Suppose its $\phi-$transform $F = F_{\phi}\{f\}$ with respect to a given normalized, positive test function $\phi \in \mathcal{D}(\mathbb{R}^n)$ satisfies*

$$M_{+} \geq f^{+}_{\phi,\vartheta}(\mathbf{x}) \geq f^{-}_{\phi,\vartheta}(\mathbf{x}) \geq -M_{-}\,, \qquad \forall \mathbf{x} \in U\,,\ \forall \vartheta \in [0, \pi/2)\,. \tag{12.4.12}$$

where U is an open set and where $M_{\pm}$ are constants. Then the angular boundary limit

$$f_{\text{ang}}(\mathbf{x}) = \lim_{\substack{(\mathbf{x},t)\to(\mathbf{x}_0,0) \\ \text{angular}}} F(\mathbf{x},t)\ , \tag{12.4.13}$$

exists almost everywhere in U and defines a function in $L^{\infty}(U)$, and the distribution f is a regular distribution equal to f_{ang} in U :

$$\langle f(\mathbf{x}), \psi(\mathbf{x})\rangle = \int_{\mathbb{R}^n} f_{\text{ang}}(\mathbf{x})\, \psi(\mathbf{x})\ \mathrm{d}\mathbf{x}\,, \tag{12.4.14}$$

for all $\psi \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp}\psi \subset U$.

We end this chapter with an useful remark.

Remark 12.13. *It is clear from the proof of Theorem 12.10 that is enough to assume (12.4.5), (12.4.9), or (12.4.12) for just some $\vartheta > \arctan R_{\phi}$ in order to obtain the same conclusions of Theorems 12.10, 12.11, or 12.12, respectively, where the number R_{ϕ} is given by $R_{\phi} = \inf\{R > 0 : \operatorname{supp}\phi \subseteq [-R, R]\}$.*

Chapter 13

Characterizations of the Support of Distributions

13.1 Introduction

In a recent study, González Vieli and Graham [75] characterized the support of certain tempered distributions in several variables in terms of the uniform convergence over compacts of the *symmetric* Cesàro means of its Fourier inversion formula. Indeed, they proved that for a large class of tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$, if for some $k \in \mathbb{N}$

$$\lim_{r\to\infty} \int_{|\mathbf{u}|\leq r} \hat{f}(\mathbf{u})\, e^{i\mathbf{u}\bullet\mathbf{x}} \mathrm{d}\mathbf{u} = 0 \quad (\mathrm{C},k) \ , \tag{13.1.1}$$

uniformly on compacts of an open set $\Omega \subset \mathbb{R}^n$, then $\Omega \subset \mathbb{R}^n \backslash \operatorname{supp} f$. See also [72, 74, 78, 79]. Results on this subject have a rich tradition that goes back to the work of Kahane and Salem [105] and that of Walter [236]. Here we use the constants in the Fourier transform such that $\hat{f}(\mathbf{u}) = \int_{\mathbb{R}^n} f(\mathbf{x})\, e^{-i\mathbf{u}\bullet\mathbf{x}} \mathrm{d}\mathbf{x}$ if the integral exists. Hence, the inversion formula becomes $f(\mathbf{x}) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{f}(\mathbf{u})\, e^{i\mathbf{u}\bullet\mathbf{x}} \mathrm{d}\mathbf{u}$ when the integral makes sense. If instead of *uniform* convergence one has only *pointwise* convergence, then it is easy to see that maybe $\Omega \cap \operatorname{supp} f \neq \varnothing$.

The aims of this chapter are the following:

1. To obtain the characterization of the support of *any* tempered distribution.

2. To prove the result under weaker conditions than uniform convergence of the means, in particular, when the means are locally L^1 bounded.

3. To obtain the corresponding result for other summability methods such as Abel summability and Gauss-Weierstrass summability.

It should be pointed out that in the one-variable case one can completely characterize the support of a tempered distribution in term of the *pointwise* Cesàro behavior if one uses slightly asymmetric means. It was proved in Chapter 3 (see also [47]) that a periodic distribution of period 2π, $f \in \mathcal{S}'(\mathbb{R})$, with Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$, has the distributional point value $f(x_0) = \gamma$ in the Łojasiewicz sense if and only if there exists k such that $\forall a > 0$,

$$\lim_{y\to\infty} \sum_{-ay<n\leq y} c_n e^{inx_0} = \gamma \qquad (\mathrm{C},k) \ . \tag{13.1.2}$$

This result was recently generalized to arbitrary tempered distributions [215, 216], we presented a complete discussion of such a generalization in Chapter 3 and obtained in Theorem 3.21 that if $f \in \mathcal{S}'(\mathbb{R})$ then

$$f(x_0) = \gamma \quad \text{distributionally}\,, \tag{13.1.3}$$

if and only if

$$\text{e.v.} \left\langle \hat{f}(u)\,, e^{iux_0} \right\rangle = 2\pi\gamma \qquad (\mathrm{C},k) \ , \tag{13.1.4}$$

where the e.v. involves slightly Cesàro asymmetric means of the distributional evaluation.

Therefore, since the Łojasiewicz point values determine a distribution completely if they exist at all points [128], we obtain the following characterization of the support of a distribution.

Theorem 13.1. *Let $f \in \mathcal{S}'(\mathbb{R})$. Let Ω be an open set of $\mathbb{R}$. If there exists k such that*

$$\text{e.v.} \left\langle \hat{f}(u)\,, e^{iux_0} \right\rangle = 0 \qquad (\mathrm{C},k) \ , \qquad \forall x_0 \in \Omega\,, \tag{13.1.5}$$

then $\Omega \subset \mathbb{R} \backslash \operatorname{supp} f$.

We introduced in Section 3.11 the principal value distributional evaluations in the Cesàro sense. Naturally the Theorem 13.1 is not true for principal value evaluations, as the example $f(x) = \delta'(x)$ shows, since here the means converge to zero in the p.v. sense for all $x \in \mathbb{R}$.

The plan of the chapter is as follows. The basic summability procedures for the Fourier inversion formula, and their relation with the distributional $\phi-$transform are presented in Section 13.2; we observe that summability results for the Fourier transform and its inverse can be considered as particular cases of the results for the distributional $\phi-$transform that were obtained in Section 12.3. In Section 13.3 we show the uniform convergence on compacts of the distributional $\phi-$transform of a function continuous in an open set and its converse, and consequently for summability in the Fourier inversion formula. Finally in Section 13.4 we characterize the complement of the support of a distribution in the case when the means are locally L^1 bounded.

It should be mentioned that the results of this chapter will be published soon in [221].

13.2 Summability Methods in Several Variables

In this section we explain several methods of summability that one can use in connection with the multidimensional Fourier inversion formula. We start with the $\psi-$summability.

13.2.1 The $\psi-$summability

Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be any function with $\psi(\mathbf{0}) = 1$. If $g \in \mathcal{S}'(\mathbb{R}^n)$ and ρ is a smooth function in $\mathbb{R}^n$ with $\rho g \in \mathcal{S}'(\mathbb{R}^n)$, then the evaluation

$$\langle g(\mathbf{x}), \rho(\mathbf{x})\rangle , \tag{13.2.1}$$

is not defined, in general, because ρ may not longer belong to $\mathcal{S}(\mathbb{R}^n)$. However, if $\varepsilon > 0$, the evaluation

$$G(\varepsilon) = \langle g(\mathbf{x}), \rho(\mathbf{x})\psi(\varepsilon\mathbf{x})\rangle , \tag{13.2.2}$$

is well-defined. If

$$\lim_{\varepsilon\to 0} G(\varepsilon) = S , \tag{13.2.3}$$

exists, then we say that the evaluation $\langle g(\mathbf{x}), \rho(\mathbf{x})\rangle$ is $\psi-$summable to S, and write

$$\langle g(\mathbf{x}), \rho(\mathbf{x})\rangle = S \qquad (\psi) \ . \tag{13.2.4}$$

When g is locally integrable, then (13.2.4) can be written as

$$\int_{\mathbb{R}^n} g(\mathbf{x})\,\rho(\mathbf{x})\ \mathrm{d}\mathbf{x} = S \qquad (\psi) \ , \tag{13.2.5}$$

while if $g(\mathbf{x}) = \sum_{n=1}^{\infty} c_n \delta(\mathbf{x} - \mathbf{b}_n)$, then (13.2.4) becomes

$$\sum_{n=1}^{\infty} c_n \rho(\mathbf{b}_n) = S \qquad (\psi) \ . \tag{13.2.6}$$

In particular, if $\psi(\mathbf{x}) = e^{-|\mathbf{x}|^2}$ then the (ψ) summability becomes the Gauss-Weierestrass summability; we may write $\langle g(\mathbf{x}), \rho(\mathbf{x})\rangle$ (G-W) in this case.

Proposition 13.2. *Let* $\psi \in \mathcal{S}(\mathbb{R}^n)$ *with* $\psi(\mathbf{0}) = 1$. *Let* $f \in \mathcal{S}'(\mathbb{R}^n)$. *Then*

$$f(\mathbf{x}) = \frac{1}{(2\pi)^n}\left\langle \hat{f}(\mathbf{u}), e^{i\mathbf{u}\bullet\mathbf{x}}\right\rangle \qquad (\psi) \ , \tag{13.2.7}$$

distributionally in the space $\mathcal{S}'(\mathbb{R}^n)$, *that is,* $\forall\rho \in \mathcal{S}(\mathbb{R}^n)$,

$$\lim_{\varepsilon\to 0^+}\left\langle \frac{1}{(2\pi)^n}\left\langle \hat{f}(\mathbf{u}), e^{i\mathbf{u}\bullet\mathbf{x}}\psi(\varepsilon\mathbf{u})\right\rangle \ , \ \rho(\mathbf{x})\right\rangle = \langle f(\mathbf{x}), \rho(\mathbf{x})\rangle \ . \tag{13.2.8}$$

Moreover, relation (13.2.7) holds pointwise at any point $\mathbf{x} = \mathbf{x}_0$ *where the distributional point value* $f(\mathbf{x}_0)$ *exists.*

Proof. The result follows immediately from Theorem 12.6 and Corollary 12.4 because

$$\frac{1}{(2\pi)^n}\left\langle \hat{f}(\mathbf{u}), e^{i\mathbf{u}\bullet\mathbf{x}}\psi(\varepsilon\mathbf{u})\right\rangle = F(\mathbf{x}, \varepsilon) \ , \tag{13.2.9}$$

where F is the $\phi-$transform of f for $\phi(\mathbf{x}) = (2\pi)^{-n}\hat{\psi}(\mathbf{x})$. □

Observe, in particular, that the Fourier inversion formula is always valid distributionally, in the space $\mathcal{D}'(\mathbb{R}^n)$, in the Gauss-Weierestrass summability sense for *any* tempered distribution.

We also have pointwise convergence at all points where the symmetric point value exists, provided that ψ is radial.

Proposition 13.3. *Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be a radial test function with $\psi(\mathbf{0}) = 1$. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Let $\mathbf{x}_0 \in \mathbb{R}^n$ be a point where the distributional symmetric value $f_{\mathrm{sym}}(\mathbf{x}_0)$ exists. Then*

$$f_{\mathrm{sym}}(\mathbf{x}_0) = \frac{1}{(2\pi)^n} \left\langle \hat{f}(\mathbf{u}), e^{i\mathbf{u}\bullet\mathbf{x}_0} \right\rangle \qquad (\psi) \, . \tag{13.2.10}$$

13.2.2 Abel Summability

The Abel method of summability follows by taking $\psi(\mathbf{x}) = e^{-|\mathbf{x}|}$ in the (ψ) summability procedure:

$$\langle g(\mathbf{x}), \rho(\mathbf{x}) \rangle = S \qquad (\mathrm{A}) \, . \tag{13.2.11}$$

if

$$\lim_{\varepsilon \to 0^+} \left\langle g(\mathbf{x}), \rho(\mathbf{x}) e^{-\varepsilon|\mathbf{x}|} \right\rangle = S \, . \tag{13.2.12}$$

There is an obvious problem in the application of this method, namely, the function $e^{-|\mathbf{x}|}$ does not belong to $\mathcal{S}(\mathbb{R}^n)$ since it is not differentiable at $\mathbf{x} = \mathbf{0}$. It is fair to say, however, that $e^{-|\mathbf{x}|}$ does have the behavior of the space $\mathcal{S}(\mathbb{R}^n)$ as $|\mathbf{x}| \to \infty$. If g satisfies certain conditions near $\mathbf{x} = \mathbf{0}$, then $\left\langle g(\mathbf{x}), \rho(\mathbf{x}) e^{-\varepsilon|\mathbf{x}|} \right\rangle$ can be computed, for instance, if g is a locally integrable function in a neighborhood of $\mathbf{x} = \mathbf{0}$, or more generally if it is a Radon measure in such a neighborhood.

We can consider Abel means for general g if we accept that in some cases these means are not unique. Indeed, let $e(g)$ be an extension of $g \in \mathcal{S}'(\mathbb{R}^n)$ to the dual space $\left(\mathcal{X}\widehat{\otimes}\mathcal{D}(\mathbb{S})\right)'$, where we use polar coordinates $\mathbf{x} = r\omega$, $r \geq 0$, $\omega \in \mathbb{S}$, and where $\mathcal{X}$ is the space of restrictions of functions $\rho(r)$ for $\rho \in \mathcal{S}(\mathbb{R})$ to $[0, \infty)$, i.e., $\mathcal{X} = \mathcal{S}[0, \infty)$. Then $\rho(\mathbf{x}) e^{-\varepsilon|\mathbf{x}|}$ belongs to $\mathcal{X}\hat{\otimes}\mathcal{D}(\mathbb{S})$ and thus we can consider the Abel means $G(\varepsilon) = \left\langle e(g)(\mathbf{x}), \rho(\mathbf{x}) e^{-\varepsilon|\mathbf{x}|} \right\rangle$, and its limit as $\varepsilon \searrow 0$ instead of (13.2.12). Some g have canonical extensions $e(g)$, but in general $e(g)$ is not uniquely defined.

If we use Abel summability in the Fourier inversion formula, we obtain the means

$$U(\mathbf{x},t)=\frac{1}{(2\pi)^n}\left\langle e\left(\hat{f}\right)(\mathbf{u}),e^{i\mathbf{u}\bullet\mathbf{x}-t|\mathbf{u}|}\right\rangle, \tag{13.2.13}$$

which is harmonic in $\mathbb{H}$: $U_{tt}+\sum_{j=1}^{n}U_{x_jx_j}=0$. A similar analysis to that of Proposition 13.2 yields

$$\lim_{t\to0^+}U(\mathbf{x},t)=f(\mathbf{x}). \tag{13.2.14}$$

We also observe that for a fixed $t>0$ the function $U(\mathbf{x},t)$ belongs to $\mathcal{S}'(\mathbb{R}^n)$.

We can thus say that the Abel means in the Fourier inversion formula of a tempered distribution $f\in\mathcal{S}'(\mathbb{R}^n)$ are those harmonic functions in $\mathbb{H}$ with these properties. Functions like $U(\mathbf{x},t)=t$ or $U(\mathbf{x},t)=3x_j^2t-t^3$ are Abel means of $f=0$, and thus the source of non-uniqueness.

If $f\in\mathcal{E}'(\mathbb{R}^n)$, or more generally if $f(\mathbf{x})=O(1)$ (C) as $|\mathbf{x}|\to\infty$, then one can define a canonical Abel mean for the Fourier inversion formula as

$$U(\mathbf{x},t)=c_n\left\langle f(\mathbf{y}),\frac{t}{\left(t^2+\|\mathbf{x}-\mathbf{y}\|^2\right)^{\frac{n+1}{2}}}\right\rangle, \tag{13.2.15}$$

where

$$c_n=\frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}=\left(\int_{\mathbb{R}^n}\frac{\mathrm{d}\mathbf{y}}{\left(1+\|\mathbf{y}\|^2\right)^{\frac{n+1}{2}}}\right)^{-1}, \tag{13.2.16}$$

and where the kernel in (13.2.15) is the Poisson kernel for $\mathbb{H}$. In this case $U(\mathbf{x},t)$ is the $\phi-$transform of f for $\phi(\mathbf{y})=c_n\left(1+\|\mathbf{y}\|^2\right)^{-\frac{n+1}{2}}$.

Observe that if the distributional symmetric value $f_{\mathrm{sym}}(\mathbf{x}_0)$ exists then for any Abel mean $U(\mathbf{x},t)$ we have that $U(\mathbf{x}_0,t)\to f_{\mathrm{sym}}(\mathbf{x}_0)$, that is,

$$f_{\mathrm{sym}}(\mathbf{x}_0)=\frac{1}{(2\pi)^n}\left\langle\hat{f}(\mathbf{u}),e^{i\mathbf{u}\bullet\mathbf{x}_0}\right\rangle\quad(\mathrm{A}). \tag{13.2.17}$$

13.2.3 Cesàro Summability

We can also consider Cesàro summability by spherical means [61, Section 6.8]. Summability by spherical means can actually be reduced to summability in one variable since using polar coordinates, $\mathbf{x} = r\omega$, $r \geq 0$, $\omega \in \mathbb{S}$, we obtain

$$\langle f(\mathbf{x}), 1_{\mathbf{x}} \rangle = \left\langle F(r), r^{n-1} \right\rangle \qquad (\mathrm{C}), \tag{13.2.18}$$

where

$$F(r) = \langle f(r\omega), 1_{\omega} \rangle_{\mathcal{D}'(\mathbb{S}) \times \mathcal{D}(\mathbb{S})} . \tag{13.2.19}$$

The distribution F is not uniquely defined at $r = 0$, however we can always write $f = f_1 + f_2$, where f_1 has compact support and where $\mathbf{0} \notin \operatorname{supp} f_2$. The evaluation $\langle f_1(\mathbf{x}), \phi(\mathbf{x}) \rangle$ is well-defined for any $\phi \in \mathcal{E}(\mathbb{R}^n)$, so we need to consider only the case when $f = f_2$ satisfies that $\operatorname{supp} f \subset \{\mathbf{x} : |\mathbf{x}| \geq a\}$ for some $a > 0$. Then F will be uniquely defined if we require that $\operatorname{supp} F \subset [a, \infty)$.

We now explain when $\langle f(\mathbf{x}), \phi(\mathbf{x}) \rangle$ is Cesàro summable by spherical means of order N,

$$\langle f(\mathbf{x}), \phi(\mathbf{x}) \rangle = \ell \qquad (\mathrm{C},N)_{\mathrm{r}} . \tag{13.2.20}$$

If $\phi \equiv 1$ the $(\mathrm{C},N)_{\mathrm{r}}$ summability means that the one-variable evaluation

$$\left\langle F(r), r^{n-1} \right\rangle = \ell \qquad (\mathrm{C},N) , \tag{13.2.21}$$

exists in the (C,N) sense. For a general ϕ it means that $\langle \phi(\mathbf{x}) f(\mathbf{x}), 1_{\mathbf{x}} \rangle = \ell$ $(\mathrm{C},N)_{\mathrm{r}}$. The notation $(\mathrm{C})_{\mathrm{r}}$ is used for Cesàro summability by spherical means, namely when there exists some N such that the evaluation is $(\mathrm{C},N)_{\mathrm{r}}$.

Observe that the $(\mathrm{C},N)_{\mathrm{r}}$ summability corresponds to the case where

$$\psi_N(\mathbf{x}) = H(1 - |\mathbf{x}|) \frac{(1 - |\mathbf{x}|)^N}{N!} , \tag{13.2.22}$$

in the $\psi-$summability. Here H is the Heaviside function.

If $f \in \mathcal{K}'(\mathbb{R}^n)$ and $\phi \in \mathcal{K}(\mathbb{R}^n)$, then the evaluation $\langle f, \phi \rangle$ exists in the $(\mathrm{C})_{\mathrm{r}}$ sense, that is, it exists $(\mathrm{C},N)_{\mathrm{r}}$ for some N. The value of N depends on ϕ in this case: Consider the example where $f(x) = e^{ix}$ and $\phi(x) = x^n$. On the other hand, if $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ then the evaluation $\langle f, \phi \rangle$ also exists $(\mathrm{C})_{\mathrm{r}}$ since $\langle f, \phi \rangle = \langle \phi f, 1 \rangle$, and $\phi f \in \mathcal{K}'(\mathbb{R}^n)$, but now if $f \in \mathcal{S}'(\mathbb{R}^n)$ is fixed then there exists N such that $\langle f, \phi \rangle$ exists $(\mathrm{C},N)_{\mathrm{r}}$ for *all* test functions $\phi \in \mathcal{S}(\mathbb{R}^n)$.

The Cesàro means of the Fourier inversion formula will converge distributionally, as in the case of the Abel means and the (ψ) means, but this happens if N is large.

Proposition 13.4. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then there exists N such that*

$$f(\mathbf{x}) = \frac{1}{(2\pi)^n} \left\langle \hat{f}(\mathbf{u}), e^{i\mathbf{u}\bullet\mathbf{x}} \right\rangle \qquad (C,N)_r\,, \tag{13.2.23}$$

distributionally in the space $\mathcal{S}'(\mathbb{R}^n)$, in the sense that for each $\rho \in \mathcal{S}(\mathbb{R}^n)$

$$\lim_{\varepsilon\to 0^+} \left\langle \frac{1}{(2\pi)^n} \left\langle \hat{f}(\mathbf{u}), e^{i\mathbf{u}\bullet\mathbf{x}} \psi_N(\varepsilon\mathbf{u}) \right\rangle, \rho(\mathbf{x}) \right\rangle = \langle f(\mathbf{x}), \rho(\mathbf{x}) \rangle \quad (C,N)_r\,. \tag{13.2.24}$$

Proof. Indeed,

$$\left\langle \frac{1}{(2\pi)^n} \left\langle \hat{f}(\mathbf{u}), e^{i\mathbf{u}\bullet\mathbf{x}} \psi_N(\varepsilon\mathbf{u}) \right\rangle\,,\ \rho(\mathbf{x}) \right\rangle = \frac{1}{(2\pi)^n} \left\langle \hat{f}(\mathbf{u}), \hat{\rho}(-\mathbf{u})\, \psi_N(\varepsilon\mathbf{u}) \right\rangle\,, \tag{13.2.25}$$

and there exists N such that the evaluation $\left\langle \hat{f}, \phi \right\rangle$ exists $(\mathrm{C},N)_{\mathrm{r}}$ for all test functions $\phi \in \mathcal{S}(\mathbb{R}^n)$, in particular for $\phi(\mathbf{u}) = \hat{\rho}(-\mathbf{u})$. But since

$$(2\pi)^{-n} \left\langle \hat{f}(\mathbf{u}), \hat{\rho}(-\mathbf{u}) \right\rangle = \langle f(\mathbf{x}), \rho(\mathbf{x}) \rangle\ ,$$

then (13.2.24) is obtained. □

It is interesting to observe if $f \in \mathcal{E}'(\mathbb{R}^n)$ then there is no need to use Cesàro summability in (13.2.23), that is, we actually get convergence of the spherical means. Similarly, if f is periodic of periods in $\prod_{j=1}^n \tau_j \mathbb{Z}$, so that its Fourier transform is concentrated on a discrete set, and the Fourier inversion formula is the

Fourier series, then we also get convergence. However, for a general $f \in \mathcal{S}'(\mathbb{R}^n)$ there is a value N for which (13.2.23) holds, but the spherical means are not (C,M) summable if $M < N$.

When the distributional symmetric value $f_{\text{sym}}(\mathbf{x}_0)$ exists then (13.1.4) implies that we have pointwise Cesàro summability,

$$f_{\text{sym}}(\mathbf{x}_0) = \frac{1}{(2\pi)^n} \left\langle \hat{f}(\mathbf{u}), e^{i\mathbf{u}\bullet\mathbf{x}_0} \right\rangle \qquad (\mathrm{C},N)_{\mathrm{r}} ,$$

if N is large.

13.3 Continuity

If $U(\mathbf{x},t)$ is harmonic in $\mathbb{H}$, with distributional boundary value $f(\mathbf{x}) = U(\mathbf{x},0^+) \in \mathcal{S}'(\mathbb{R}^n)$, and f is continuous in an open set $\Omega \subset \mathbb{R}^n$, then it is well-known that actually $U(\mathbf{x},t)$ can be extended as a continuous function to $\mathbb{H} \cup (\Omega \times \{0\})$, and consequently, $U(\mathbf{x},t) \to f(\mathbf{x})$ uniformly on compacts of Ω. In fact, this is a general result for the $\phi-$transform.

Proposition 13.5. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$ and let $F(\mathbf{x},t)$ be its $\phi-$transform. Suppose that $\phi \in \mathcal{D}(\mathbb{R}^n)$ or that (12.3.17), (12.3.18), and (12.3.19) are satisfied. If f is an ordinary bounded function in a neighborhood of a point $\mathbf{x}_0$ and that function is continuous at $\mathbf{x} = \mathbf{x}_0$ then*

$$\lim_{(\mathbf{x},t)\to(\mathbf{x}_0,0)} F(\mathbf{x},t) = f(\mathbf{x}_0) , \tag{13.3.1}$$

so that F can be extended as a continuous function to $\mathbb{H} \cup (\{\mathbf{x}_0\} \times \{0\})$.

Proof. The results of Section 12.3 show that (13.3.1) holds if $\mathbf{x}_0 \in \mathbb{R}^n \setminus \operatorname{supp} f$. Hence, it is enough to prove (13.3.1) when f is an ordinary bounded function with compact support. Let $\varepsilon > 0$, and let B be an open neighborhood of $\mathbf{x}_0$, with compact closure, such that $|f(\mathbf{y}) - f(\mathbf{x}_0)| < \varepsilon$ for $\mathbf{y} \in B$. Write $F(\mathbf{x},t) - f(\mathbf{x}_0) =$

$G_1(\mathbf{x},t) + G_2(\mathbf{x},t)$, where

$$G_1(\mathbf{x},t) = t^{-n} \int_B (f(\mathbf{y}) - f(\mathbf{x}_0))\,\phi\left(t^{-1}(\mathbf{y}-\mathbf{x})\right) \mathrm{d}\mathbf{y}\,, \tag{13.3.2}$$

$$G_2(\mathbf{x},t) = t^{-n} \int_{\mathbb{R}^n\setminus B} (f(\mathbf{y}) - f(\mathbf{x}_0))\,\phi\left(t^{-1}(\mathbf{y}-\mathbf{x})\right) \mathrm{d}\mathbf{y}\,. \tag{13.3.3}$$

Then $G_2(\mathbf{x},t) \to 0$ as $t \to 0$ uniformly on compacts of B, while

$$|G_1(\mathbf{x},t)| \le \varepsilon \int_{\mathbb{R}^n} |\phi(\mathbf{y})|\, \mathrm{d}\mathbf{y}\,, \tag{13.3.4}$$

and (13.3.1) follows. $\square$

Observe that if the conditions of the Proposition 13.5 are satisfied and $f(\mathbf{x}_0) = \gamma$ distributionally then $F(\mathbf{x},t) \to \gamma$ as $(\mathbf{x},t) \to (\mathbf{x}_0,0)$ in a non-tangential fashion, while if the distributional symmetric value exists, $f_{\mathrm{sym}}(\mathbf{x}_0) = \gamma$, and ϕ is radial then $F(\mathbf{x}_0,t) \to \gamma$ as $t \to 0^+$. According to Proposition 13.5 if f is continuous at $\mathbf{x} = \mathbf{x}_0$ then $F(\mathbf{x},t) \to \gamma$ as $(\mathbf{x},t) \to (\mathbf{x}_0,0)$ in an unrestricted fashion.

Proposition 13.6. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$ and let $F(\mathbf{x},t)$ be its $\phi-$transform. Suppose that $\phi \in \mathcal{D}(\mathbb{R}^n)$ or that (12.3.17), (12.3.18), and (12.3.19) are satisfied. If f is a continuous function in an open set $\Omega \subset \mathbb{R}^n$ then F can be extended as a continuous function to $\mathbb{H}\cup(\Omega\times\{0\})$, and $F(\mathbf{x},t) \to f(\mathbf{x})$ uniformly on compacts of Ω. Conversely, if $F(\mathbf{x},t) \to f(\mathbf{x})$ uniformly on compacts of Ω, then f is a continuous function in Ω.*

Proof. The direct part follows immediately from the previous proposition, while the converse result follows because uniform convergence on compacts implies distributional convergence. $\square$

In particular, we have the following result for summability of the Fourier inversion formula.

Corollary 13.7. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$. If f is a continuous function in an open set $\Omega \subset \mathbb{R}^n$ then the ψ means, for any $\psi \in \mathcal{S}(\mathbb{R}^n)$, any Abel means, or the Cesàro means of large order converge to f uniformly on compacts of Ω :*

$$f(\mathbf{x}) = \frac{1}{(2\pi)^n} \left\langle \hat{f}(\mathbf{u}), e^{i\mathbf{u}\bullet\mathbf{x}} \right\rangle \qquad (\mathrm{T}) , \tag{13.3.5}$$

uniformly on $\mathbf{x} \in K$, K a compact subset of Ω, for $(\mathrm{T}) = (\psi)$, (A), or $(C,N)_{\mathrm{r}}$ for N large. Conversely, if (13.3.5) holds uniformly on compacts of Ω then f is a continuous function on Ω.

13.4 The Support of a Distribution

We now show how we can obtain a characterization of the complement of the support of a distribution if we add some extra conditions to the pointwise convergence to zero of the symmetric means. Naturally, the uniform convergence to zero of the means on compacts of an open set Ω gives that $\Omega \subset \mathbb{R}^n \setminus \operatorname{supp} f$, because of the Corollary 13.7; this is the result of González Vieli and Graham [75] when $(\mathrm{T}) = (\mathrm{C},N)_{\mathrm{r}}$ for N large.

Let us start with the $\phi-$transform.

Theorem 13.8. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$ and let $F(\mathbf{x}, t)$ be its $\phi-$transform. Assume that $\phi(\mathbf{x}) \geq 0$ $\forall \mathbf{x} \in \mathbb{R}^n$, while $\phi(\mathbf{0}) > 0$. Suppose that $\phi \in \mathcal{D}(\mathbb{R}^n)$ or that (12.3.17), (12.3.18), and (12.3.19) are satisfied. Suppose that pointwise*

$$\lim_{t \to 0^+} F(\mathbf{x}, t) = 0 , \quad \forall \mathbf{x} \in \Omega , \tag{13.4.1}$$

where Ω is an open set. Let $p \in [1, \infty]$ and suppose that for $0 < t \leq t_0$ the function $F(\mathbf{x}, t)$ is locally bounded in $L^p(\Omega)$, i.e., if K is compact in Ω, there exists a constant $M = M(K, p)$ such that

$$\left(\int_K |F(\mathbf{x}, t)|^p \, \mathrm{d}\mathbf{x} \right)^{1/p} \leq M , \tag{13.4.2}$$

for $p<\infty,$ *or if* $p=\infty,$

$$\sup\left\{ \left| F\left(\mathbf{x},t\right)\right| :\mathbf{x}\in K\right\} \leq M\,. \tag{13.4.3}$$

Then $\Omega\subset\mathbb{R}^{n}\setminus\operatorname{supp}f$.

Proof. It is enough to do it when $p=1$, since local boundedness in $L^{q}\left(\Omega\right)$ for $q\geq1$ implies local boundedness in $L^{1}\left(\Omega\right).$ Now, local boundedness in $L^{1}\left(\Omega\right)$ plus distributional convergence yield that f is a Radon measure in Ω : if $\{t_{n}\}$ is any sequence of positive numbers that converges to zero then local boundedness in $L^{1}\left(\Omega\right)$ implies that there exists a subsequence $\{t_{n_{k}}\}$ such that $F\left(\mathbf{x},t_{n_{k}}\right)$ converges $\ast-$weakly in the dual space of $C_{c}\left(\Omega\right),$ the continuous functions with compact support in Ω, that is, $F\left(\mathbf{x},t_{n_{k}}\right)\rightarrow\nu\left(\mathbf{x}\right)$ where ν is a measure in Ω; but clearly $f=\nu$ in Ω.

We can then write, in Ω, $f=f_{\text{ac}}+f_{\text{dis}}+f_{\text{sin}}$, where f_{ac}, the absolutely continuous part, is a locally integrable function in Ω, $f_{\text{dis}}\left(\mathbf{x}\right)=\sum_{\mathbf{a}\in A}c_{\mathbf{a}}\delta\left(\mathbf{x}-\mathbf{a}\right)$ where A is countable at the most and $\sum_{\mathbf{a}\in A\cap K}\left|c_{\mathbf{a}}\right|$ converges for all K compact with $K\subset\Omega$, and where f_{sin} is a continuous measure concentrated on a set of Lebesgue measure zero. But the distributional point value $f_{\text{ac}}\left(\mathbf{x}\right)$ exists almost everywhere because f_{ac} is locally integrable and equals the distributional point value $f\left(\mathbf{x}\right)$ almost everywhere since $f_{\text{dis}}\left(\mathbf{x}\right)=f_{\text{sin}}\left(\mathbf{x}\right)=0$ almost everywhere, and from (13.4.1) those values are 0, so that the *function* f_{ac} is null a.e. in Ω, and so the *distribution* $f_{\text{ac}}=0$ in Ω. On the other hand, if $c_{\mathbf{a}_{0}}\neq0$ then the contributions form $\sum_{\mathbf{a}\in A\setminus\{\mathbf{a}_{0}\}}c_{\mathbf{a}}\delta\left(\mathbf{x}-\mathbf{a}\right)$ and from $f_{\text{sin}}\left(\mathbf{x}\right)$ give parts of $F\left(\mathbf{a}_{0},t\right)$ that are of order $o\left(t^{-n}\right)$ as $t\rightarrow0^{+}$, so that the main contribution comes from $c_{\mathbf{a}_{0}}\delta\left(\mathbf{x}-\mathbf{a}_{0}\right),$ which yields $F\left(\mathbf{a}_{0},t\right)\sim c_{\mathbf{a}_{0}}t^{-n}\phi\left(\mathbf{0}\right)$ as $t\rightarrow0^{+}$. However, this is not possible because of (13.4.1); hence the discrete part f_{dis} also vanishes. Thus $f=f_{\text{sin}}=\mu$, a singular measure. We can write $\mu=\mu_{+}-\mu_{-}$, where $\mu_{\pm}$ are positive continuous measures, concentrated on disjoint sets, $Z_{\pm}$. But using the results of [173, Chap.4], the set of points $\mathbf{x}_0$ with infinite upper symmetric derivative

$$\limsup_{\varepsilon\to 0^+} \varepsilon^{-n} \int_{|\mathbf{x}-\mathbf{x}_0|<\varepsilon} \mathrm{d}\mu_{\pm}(\mathbf{x}) = \infty\,, \tag{13.4.4}$$

is of full measure with respect to $|\mu|$, and at those points, because $\phi(\mathbf{x}) \geq 0$ $\forall \mathbf{x} \in \mathbb{R}^n$ and $\phi(\mathbf{0}) > 0$,

$$\limsup_{\varepsilon\to 0^+} |F(\mathbf{x}_0, \varepsilon)| \geq \limsup_{\varepsilon\to 0^+} \varepsilon^{-n} \int_{|\mathbf{x}-\mathbf{x}_0|<\varepsilon} \phi(\mathbf{0})\ \mathrm{d}\mu_{\pm}(\mathbf{x})\ , \tag{13.4.5}$$

contradicting (13.4.1); therefore $f_{\sin} = 0$. $\square$

We immediately obtain a corresponding result for the characterization of the complement of the support in the Fourier inversion formula.

Corollary 13.9. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Suppose that pointwise*

$$\frac{1}{(2\pi)^n} \left\langle \hat{f}(\mathbf{u}), e^{i\mathbf{u}\bullet\mathbf{x}} \right\rangle = 0 \qquad (\mathrm{T})\ , \tag{13.4.6}$$

for all $\mathbf{x} \in \Omega$, where Ω is an open set, and where $(\mathrm{T}) = (\psi)$, (A), or $(C,N)_r$ for N large. If the means are locally bounded in $L^p(\Omega)$ for some $p \in [1,\infty]$ then $\Omega \subset \mathbb{R}^n \setminus \operatorname{supp} f$.

Chapter 14
Global Behavior of Integral Transforms

14.1 Introduction

In this chapter we investigate global estimates for various integral transforms of a certain class of functions.

In a recent article, R. Berndt [13] obtained the following global estimate for the Fourier sine transform of the function f,

$$\frac{A}{x}f\left(\frac{1}{x}\right) \leq \int_0^\infty f(u)\sin(ux)\,\mathrm{d}u \leq \frac{B}{x}f\left(\frac{1}{x}\right)\,, \ \forall x>0\,, \tag{14.1.1}$$

where A and B are positive constants, provided that f is a differentiable function defined on $(0,\infty)$ that satisfies

$$c_1\frac{f(x)}{x} \leq -f'(x) \leq c_2\frac{f(x)}{x}\,, \tag{14.1.2}$$

where c_1 and c_2 are constants with

$$0 < c_1 \leq c_2 < 2\,. \tag{14.1.3}$$

It should be remarked that asymptotic estimates of the behavior of the sine and of other integral transforms of regularly varying functions [183] in terms of the function $f\,(1/x)$ had been obtained before [189, 190, 191], both as $x \to 0^+$ and as $x \to \infty$. However, (14.1.1) is a global estimate, that considers not only the endpoint behavior but that holds for all $x > 0$.

Our aim is to generalize (14.1.1) in two directions. On the one hand, we want to consider other kernels than sine, so we shall give conditions on the kernel $k\,(x)$ such that an estimate of the form

$$\frac{A'}{x}f\left(\frac{1}{x}\right) \leq \int_0^\infty f(u)k(ux)\,\mathrm{d}u \leq \frac{B'}{x}f\left(\frac{1}{x}\right)\,, \ \forall x>0\,, \tag{14.1.4}$$

holds if f satisfies (14.1.2).

On the other hand, we shall remove the condition $c_2 < 2$ for the sine transform. Actually, this condition was imposed by R. Berndt to guarantee the integrability of $\sin(ux)f(x)$ at $x = 0$; if $c_2 \geq 2$, it may not be longer integrable near from 0. In such a case, the ordinary sine transform of f will not exist, but one may consider regularizations of f which are tempered distribution of the space $\mathcal{S}'(\mathbb{R})$, and whose Fourier sine transforms satisfy a global estimate as in (14.1.1), modulo a polynomial. In this way, we remove the problem of nonintegrability at $x = 0$. We are also able to remove the integrability condition (in general, if $c_2 \geq 1$, f may not be integrable at 0) and obtain global estimates modulo a polynomial for the Laplace transform of f.

Our analysis is based on a characterization of the class of function $\mathfrak{V}$, which consists of those differentiable functions that satisfy (14.1.2). This characterization is given in Section 14.3. Using this characterization we are able to give several global estimates for integral transforms of elements of $\mathfrak{V}$ both for general oscillatory kernels, particularly for the sine transform, and for the Laplace transform; it is done in Sections 14.4 and 14.5.

The results of the chapter are already published in [214].

14.2 Preliminaries

We shall briefly discuss the concept of regularization [48, 61, 108]. Let f be a real-valued function, which we assume to be locally integrable in $\mathbb{R} \setminus \{0\}$; we say that a distribution $\tilde{f} \in \mathcal{S}'(\mathbb{R})$ is a *regularization* of f at 0, if for all $\phi \in \mathcal{S}(\mathbb{R})$ with $\operatorname{supp} \phi \subseteq (-\infty, 0) \cup (0, \infty)$, we have

$$\left\langle \tilde{f}(x), \phi(x) \right\rangle = \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}x\,.$$

In other words, $\tilde{f}$ is an extension of $f \in \mathcal{S}'(\mathbb{R} \setminus \{0\})$ to the whole real line. The function f has a regularization at $x = 0$ if and only if it has algebraic growth near the origin, in the Cesàro sense [48]. If a function f has a regularization at 0, then it has infinitely many regularizations at 0, and all of them are obtained by adding a linear combination of the Dirac delta function and its derivatives concentrated at 0 [61, 208, 252]. Thus, given $\tilde{f}$ and $\tilde{f}_1$, two regularizations of f at 0, they satisfy

$$\tilde{f}_1(x) = \tilde{f}(x) + \sum_{i=0}^{n} a_i \delta^{(i)}(x) , \tag{14.2.1}$$

for some constants $a_0, \ldots, a_n$.

We shall define the *sine* transform of an *odd* tempered distribution by duality. Note that if $\phi \in \mathcal{S}(\mathbb{R})$ is an odd function, then its sine transform, defined as

$$\int_0^\infty \phi(u) \sin(xu)\, \mathrm{d}u = \frac{i}{2} \hat{\phi}(x) , \tag{14.2.2}$$

is also an odd element of $\mathcal{S}(\mathbb{R})$. So, the sine transform is an isomorphism on the subspace of odd elements of $\mathcal{S}(\mathbb{R})$. We define the sine transform for odd distributions in $\mathcal{S}'(\mathbb{R})$ as the transpose of the sine transform on the subspace of $\mathcal{S}(\mathbb{R})$ consisting of odd functions. Alternatively, the sine transform of an odd distribution $f \in \mathcal{S}'(\mathbb{R})$ is the odd tempered distribution $F \in \mathcal{S}'(\mathbb{R})$ given by

$$F = \frac{i}{2} \hat{f} . \tag{14.2.3}$$

14.3 Characterization of the Class $\mathfrak{V}$

In this section we shall define and characterize the class of functions $\mathfrak{V}$. The study of integral transforms of elements in this class will be the central subject of this chapter.

Definition 14.1. *A positive, differentiable function f defined on $(0, \infty)$ is said to be an element of $\mathfrak{V}$ if it satisfies*

$$c_1 \frac{f(x)}{x} \leq -f'(x) \leq c_2 \frac{f(x)}{x} , \tag{14.3.1}$$

where c_1 and c_2 are positive numbers.

We shall prove that the functions in $\mathfrak{V}$ satisfy a variational property. Let us start by setting

$$\epsilon(t) = \frac{-tf'(t)}{f(t)} \, . \tag{14.3.2}$$

It follows that ϵ satisfies

$$c_1 \leq \epsilon(t) \leq c_2 \, , \quad \forall t > 0 \, . \tag{14.3.3}$$

By integrating $-\epsilon(t)/t$, we obtain

$$\log f(x) = -\int_1^x \frac{\epsilon(t)}{t} \, \mathrm{d}t + \log f(1) \, , \tag{14.3.4}$$

and hence

$$f(x) = f(1) \exp\left\{ -\int_1^x \frac{\epsilon(t)}{t} \, \mathrm{d}t \right\} \, , \tag{14.3.5}$$

which gives us a representation formula for f. Conversely, if (14.3.5) and (14.3.3) hold, then f satisfies (14.3.1). This fact is stated in the following lemma.

Lemma 14.2. *A function f defined on $(0, \infty)$ belongs to the class $\mathfrak{V}$ if and only if it satisfies (14.3.5), where ϵ satisfies (14.3.3).*

We now give another characterization of the elements of $\mathfrak{V}$.

Theorem 14.3. *A function f, defined on $(0, \infty)$, belongs to $\mathfrak{V}$ if and only if it is a positive differentiable function and satisfies*

$$\frac{1}{u^{c_1}} \leq \frac{f(ux)}{f(x)} \leq \frac{1}{u^{c_2}} \, , \quad \forall x \in (0, \infty) \, , \quad \forall u \in (0, 1] \, , \tag{14.3.6}$$

and

$$\frac{1}{u^{c_2}} \leq \frac{f(ux)}{f(x)} \leq \frac{1}{u^{c_1}} \, , \quad \forall x \in (0, \infty) \, , \quad \forall u \in [1, \infty) \, . \tag{14.3.7}$$

Proof. We assume that $f \in \mathfrak{V}$. By the Lemma 14.2,

$$f(x) = f(1) \exp\left\{ -\int_1^x \frac{\epsilon(t)}{t} \, \mathrm{d}t \right\} \, ,$$

where $c_1 \leq \epsilon(t) \leq c_2$. Therefore,

$$\frac{f(ux)}{f(x)} = \exp\left\{\int_1^x \frac{\epsilon(t)}{t}\,\mathrm{d}t - \int_1^{xu} \frac{\epsilon(t)}{t}\,\mathrm{d}t\right\}. \tag{14.3.8}$$

Let us take $u \in (0, 1]$. Then we have

$$\int_1^x \frac{\epsilon(t)}{t}\,\mathrm{d}t - \int_1^{xu} \frac{\epsilon(t)}{t}\,\mathrm{d}t = \int_{xu}^x \frac{\epsilon(t)}{t}\,\mathrm{d}t\,. \tag{14.3.9}$$

Moreover,

$$\log\left(\frac{1}{u^{c_1}}\right) = c_1 \int_{xu}^x \frac{\mathrm{d}t}{t} \leq \int_{xu}^x \frac{\epsilon(t)}{t}\,\mathrm{d}t \leq c_2 \int_{xu}^x \frac{\mathrm{d}t}{t} = \log\left(\frac{1}{u^{c_2}}\right).$$

Therefore, (14.3.6) holds. By using a similar argument, we can see that (14.3.7) follows.

Let us now assume the converse. First of all, we shall show that f is a decreasing function. Let us take $y \geq x$; by setting $u = x/y$ in (14.3.6), we obtain

$$\frac{f(x)}{f(y)} = \frac{f(y(x/y))}{f(y)} \geq \left(\frac{x}{y}\right)^{-c_1} \geq 1\,,$$

and so f is a decreasing function. Set now $g(y) = \log f(e^y)$; by (14.3.6), we have

$$-c_1 u \leq g(y+u) - g(y) \leq -c_2 u, \ \forall\ u < 0\,,$$

or

$$-c_2 \leq \frac{g(y+u) - g(x)}{u} \leq -c_1, \ \forall\ u < 0\,.$$

Taking $u \to 0^-$, we obtain

$$-c_2 \leq g'(y) \leq -c_1\,,$$

and hence

$$c_1 \leq \frac{-f'(e^y)}{f(e^y)} e^y \leq c_2\,.$$

Therefore,

$$\frac{c_1 f(x)}{x} \leq -f'(x) \leq \frac{c_2 f(x)}{x}\,.$$

and thus $f \in \mathfrak{V}$. $\square$

Corollary 14.4. *If f belongs to $\mathfrak{V}$, with constants c_1 and c_2, then*

$$f(t) = O\left(\frac{1}{t^{c_2}}\right), \quad t \to 0^+ . \tag{14.3.10}$$

Proof. According to the Theorem 14.3,

$$t^{-c_1} \le \frac{f(t)}{f(1)} \le t^{-c_2}, \text{ for all } t \in (0,1] .$$

Thus,

$$0 < t^{c_2} f(t) \le f(1), \text{ for all } t \in (0,1] ,$$

as required. □

Note that the last corollary implies the integrability of $f(u)\sin(ux)$ (with respect to u), in any interval $(0,a)$, $a < \infty$, only for $c_2 < 2$. Moreover, if k is continuous on $(0,\infty)$ and

$$k(t) = O\left(t^{\alpha}\right), \text{ as } t \to 0 ,$$

then for the integrability of $f(u)k(ux)$ at 0 it is sufficient to have $c_2 < \alpha + 1$. We observe also that the corollary implies that any $f \in \mathfrak{V}$ admits regularizations in the space $\mathcal{S}'(\mathbb{R})$ since $f(t)$ is bounded by a power of t as $t \to 0^+$.

It is interesting that one may obtain inequalities similar to (14.3.6) and (14.3.7) for functions that do not belong to $\mathfrak{V}$. Indeed, the following result applies to oscillatory functions like $f(x) = x^{-c}(2 + \sin \ln x)$.

Theorem 14.5. *Let f be a positive function defined in $(0,\infty)$. Suppose that for each compact set $J \subset (0,\infty)$ there are constants $m = m(J)$ and $M = M(J)$ with $0 < m < M$ such that*

$$m \le \frac{f(ux)}{f(x)} \le M , \quad \forall x \in (0,\infty) , \quad \forall u \in J . \tag{14.3.11}$$

Then there exist constants K_q, $1 \le q \le 4$, and c_1, c_2 such that

$$\frac{K_1}{u^{c_1}} \le \frac{f(ux)}{f(x)} \le \frac{K_2}{u^{c_2}} , \quad \forall x \in (0,\infty) , \quad \forall u \in (0,1] , \tag{14.3.12}$$

and

$$\frac{K_3}{u^{c_2}} \leq \frac{f(ux)}{f(x)} \leq \frac{K_4}{u^{c_1}}\,, \quad \forall x \in (0,\infty)\ , \quad \forall u \in [1,\infty)\ . \tag{14.3.13}$$

Proof. Let

$$M_+\left(u\right) = \sup\left\{\frac{f\left(ux\right)}{f\left(x\right)} : x \in (0,\infty)\right\} . \tag{14.3.14}$$

Then M_+ is locally bounded in $(0,\infty)$ and satisfies

$$M_+\left(uv\right) \leq M_+\left(u\right) M_+\left(v\right)\ . \tag{14.3.15}$$

If we now write $\log u = n + \theta$, where $n \in \mathbb{N}$ and where $0 \leq \theta < 1$, for $u \geq 1$, we obtain

$$M_+\left(u\right) \leq \sup\left\{M_+\left(e^{\theta}\right) : 0 \leq \theta \leq 1\right\}\ M_+\left(e\right)^{\log u}, \tag{14.3.16}$$

whenever $u \geq 1$, and thus the right inequality in (14.3.12) follows with $K_2 = \sup\left\{M_+\left(e^{\theta}\right) : 0 \leq \theta \leq 1\right\}$ and $c_2 = -\log\max\left\{M_+\left(e\right), 1\right\}$. This also gives us the left inequality in (14.3.13) with $K_3 = 1/K_2$. The proof of the other two inequalities is similar (or can be obtained by applying what we already proved to the function $1/f$). □

14.4 Oscillatory Kernels

Let $f \in \mathfrak{V}$. Suppose that $c_2 < 2$ in Definition 14.1. It was proved by R. Berndt [13, 14] that its sine transform satisfies

$$\frac{A}{x} f\left(\frac{1}{x}\right) \leq \int_0^{\infty} f(u)\sin(ux)\,\mathrm{d}u \leq \frac{B}{x} f\left(\frac{1}{x}\right)\ ,\ \forall x > 0\,. \tag{14.4.1}$$

The previous inequality provides us with an estimate of the global behavior for the sine transform of f in terms of $f\left(1/x\right)$.

Our aim is to generalize (14.4.1) in two directions. First , we want to consider other kernels than sine, so we shall give conditions on the kernel such that an estimate similar to (14.4.1) holds. Second, we shall remove the condition $c_2 < 2$ for the sine transform; in such a case, the sine transform of f will exist as a tempered distribution satisfying a global estimate as in (14.4.1), modulo a polynomial.

For our first goal, we define the k transform of f as the function F given by

$$F(x) = \int_0^\infty k(xu) f(u)\,\mathrm{d}u\,. \tag{14.4.2}$$

We shall assume that k satisfies:

1. k is continuous on $[0, \infty)$.

2. k has only simple zeros, located at $t = \lambda_n$, where $\{\lambda_n\}_{n=0}^\infty$ satisfies that $\lambda_0 = 0$, and $\lambda_0 < \lambda_1 < \ldots < \lambda_n < \ldots$, where $\lambda_n \to \infty$ as $n \to \infty$; k changes sign at every λ_n, being positive on (λ_0, λ_1), and

$$\left| \int_{\lambda_n}^{\lambda_{n+1}} k(t)\,\mathrm{d}t \right| \geq \left| \int_{\lambda_{n+1}}^{\lambda_{n+2}} k(t)\,\mathrm{d}t \right|\,. \tag{14.4.3}$$

3. $k(t) = O(t^\alpha)$, $\alpha \geq 0$, $t \to 0$.

We can now state our first theorem.

Theorem 14.6. *Let f be an element of the class $\mathfrak{V}$. If k satisfies (1), (2) and (3), and $c_2 < \alpha + 1$, then*

$$F(x) = \frac{1}{x} f\left(\frac{1}{x}\right) h(x)\,,\ \forall\ x > 0\,, \tag{14.4.4}$$

where h is continuous and bounded above and below by positive constants. Hence there exist positive constants A and B such that

$$\frac{A}{x} f\left(\frac{1}{x}\right) \leq F(x) \leq \frac{B}{x} f\left(\frac{1}{x}\right)\,,\ \forall\ x > 0\,. \tag{14.4.5}$$

Note that Theorem 14.6 is applicable to a wide variety of kernels. For example, it applies to the Hankel kernel defined by

$$k(t) = t^{1/2} J_\nu(t),\ \nu > -\frac{1}{2}\,, \tag{14.4.6}$$

under the assumption $c_2 < \nu + \frac{3}{2}$. Let us consider the proof of the Theorem 14.6.

Proof. If we perform a change of variables we obtain

$$F(x) = x^{-1} \int_0^\infty f\left(\frac{u}{x}\right) k(u)\,\mathrm{d}u\,. \tag{14.4.7}$$

Let

$$d_n(x) = \int_{\lambda_n}^{\lambda_{n+1}} f\left(\frac{u}{x}\right) k(u)\,\mathrm{d}u\ . \tag{14.4.8}$$

It follows that

$$F(x) = x^{-1} \sum_{n=0}^{\infty} d_n(x)\,. \tag{14.4.9}$$

Since $\sum_{n=0}^{\infty} d_n(x)$ is an alternating series and $|d_n(x)|$ decreases to zero as $n \to \infty$, we have

$$x^{-1} \sum_{j=0}^{2n+1} d_j(x) \le F(x) \le x^{-1} \sum_{j=0}^{2n} d_j(x)\,, \quad n \ge 0\,, \tag{14.4.10}$$

which is equivalent to

$$\int_0^{\lambda_{2n+2}} \frac{f\left(\frac{u}{x}\right)}{f\left(\frac{1}{x}\right)} k(u)\,\mathrm{d}u \le \frac{F(x)}{x^{-1} f\left(\frac{1}{x}\right)} \le \int_0^{\lambda_{2n+1}} \frac{f\left(\frac{u}{x}\right)}{f\left(\frac{1}{x}\right)} k(u)\,\mathrm{d}u\,. \tag{14.4.11}$$

In particular, for $n = 0$,

$$\int_0^{\lambda_2} \frac{f\left(\frac{u}{x}\right)}{f\left(\frac{1}{x}\right)} k(u)\,\mathrm{d}u \le \frac{F(x)}{x^{-1} f\left(\frac{1}{x}\right)} \le \int_0^{\lambda_1} \frac{f\left(\frac{u}{x}\right)}{f\left(\frac{1}{x}\right)} k(u)\,\mathrm{d}u\,. \tag{14.4.12}$$

Next, we shall find positive constants A, $B < \infty$ such that

$$\int_0^{\lambda_1} \frac{f\left(\frac{u}{x}\right)}{f\left(\frac{1}{x}\right)} k(u)\,\mathrm{d}u \le B\,,\ \forall\ x > 0\,, \tag{14.4.13}$$

and

$$\int_0^{\lambda_2} \frac{f\left(\frac{u}{x}\right)}{f\left(\frac{1}{x}\right)}\,\mathrm{d}u \ge A\,,\ \forall\ x > 0\,, \tag{14.4.14}$$

and then (14.4.5) will follow. By Theorem 14.3,

$$\frac{f\left(\frac{u}{x}\right)}{f\left(\frac{1}{x}\right)} \le \max\left\{\frac{1}{u^{c_1}}, \frac{1}{u^{c_2}}\right\}\,, \tag{14.4.15}$$

and hence

$$\int_0^{\lambda_1} \frac{f\left(\frac{u}{x}\right)}{f\left(\frac{1}{x}\right)} k(u)\,\mathrm{d}u \le \int_0^{\lambda_1} \max\left\{\frac{1}{u^{c_1}}, \frac{1}{u^{c_2}}\right\} k(u)\,\mathrm{d}u\,. \tag{14.4.16}$$

If we set

$$B = \int_0^{\lambda_1} \max\left\{\frac{1}{u^{c_1}}, \frac{1}{u^{c_2}}\right\} k(u)\,\mathrm{d}u\,, \tag{14.4.17}$$

then (14.4.13) follows. Since f is a decreasing function and k is negative on (λ_1, λ_2),

$$\begin{aligned}
&\int_0^{\lambda_1} \frac{f\left(\frac{u}{x}\right)}{f\left(\frac{1}{x}\right)} k(u)\,\mathrm{d}u + \int_{\lambda_1}^{\lambda_2} \frac{f\left(\frac{u}{x}\right)}{f\left(\frac{1}{x}\right)} k(u)\,\mathrm{d}u \\
&\qquad \geq \int_0^{\lambda_1} \frac{f\left(\frac{u}{x}\right)}{f\left(\frac{1}{x}\right)} k(u)\,\mathrm{d}u + \int_{\lambda_1}^{\lambda_2} \frac{f\left(\frac{\lambda_1}{x}\right)}{f\left(\frac{1}{x}\right)} k(u)\,\mathrm{d}u \\
&\qquad = \int_0^{\lambda_1} \frac{\left(f\left(\frac{u}{x}\right) - f\left(\frac{\lambda_1}{x}\right)\right)}{f\left(\frac{1}{x}\right)} k(u)\,\mathrm{d}u + \frac{f\left(\frac{\lambda_1}{x}\right)}{f\left(\frac{1}{x}\right)} \int_0^{\lambda_2} k(u)\,\mathrm{d}u\,,
\end{aligned}$$

so that

$$\int_0^{\lambda_2} \frac{f\left(\frac{u}{x}\right)}{f\left(\frac{1}{x}\right)} k(u)\,\mathrm{d}u \geq \int_0^{\lambda_1} \frac{\left(f\left(\frac{u}{x}\right) - f\left(\frac{\lambda_1}{x}\right)\right)}{f\left(\frac{1}{x}\right)} k(u)\,\mathrm{d}u\,. \tag{14.4.18}$$

Therefore, applying the mean value theorem, we obtain

$$f\left(\frac{u}{x}\right) - f\left(\frac{\lambda_1}{x}\right) = -f'\left(\frac{\eta}{x}\right)\left(\frac{\lambda_1 - u}{x}\right),$$

for some point $\eta \in (u, \lambda_1)$. Then, by the left inequality in the Definition 14.1,

$$f\left(\frac{u}{x}\right) - f\left(\frac{\lambda_1}{x}\right) \geq c_1 f\left(\frac{\eta}{x}\right) \frac{\lambda_1 - u}{\eta}\,.$$

Since $\frac{1}{\eta} f\left(\frac{\eta}{x}\right) \geq \frac{1}{\lambda_1} f\left(\frac{\lambda_1}{x}\right)$, we have

$$f\left(\frac{u}{x}\right) - f\left(\frac{\lambda_1}{x}\right) \geq f\left(\frac{\lambda_1}{x}\right) \frac{c_1\left(\lambda_1 - u\right)}{\lambda_1} \geq c_1 f\left(\frac{\lambda_1}{x}\right).$$

Combining (14.4.18) and the last inequality, it follows that

$$\int_0^{\lambda_2} \frac{f\left(\frac{u}{x}\right)}{f\left(\frac{1}{x}\right)} k(u)\,\mathrm{d}u \geq \frac{f\left(\frac{\lambda_1}{x}\right)}{f\left(\frac{1}{x}\right)} \int_0^{\lambda_1} c_1 k(u)\,\mathrm{d}u\,.$$

By Theorem 14.3, this implies that

$$\int_0^{\lambda_2} \frac{f\left(\frac{u}{x}\right)}{f\left(\frac{1}{x}\right)} k(u)\,\mathrm{d}u \geq c_1 \min\left\{\frac{1}{\lambda_1^{c_1}}, \frac{1}{\lambda_1^{c_2}}\right\} \int_0^{\lambda_1} k(u)\,\mathrm{d}u\,. \tag{14.4.19}$$

Setting A equal to the right side of the last inequality, the relation (14.4.14) has been proved.

Set now

$$h(x) = \frac{F(x)}{x^{-1} f\left(\frac{1}{x}\right)}\,, \quad x > 0\,, \tag{14.4.20}$$

so that

$$h(x) = \lim_{n\to\infty} \sum_{j=0}^{2n} \frac{d_j(x)}{f\left(\frac{1}{x}\right)}\,. \tag{14.4.21}$$

We shall show that each d_j is continuous. Pick $x_0 \in (0, \infty)$ and choose a such that $a > \max\{x_0, 1\}$. By Theorem 14.3

$$\left| f\left(\frac{u}{x}\right) k(u) \right| \le \max\{x^{c_1}, x^{c_2}\} f(u) k(u)\,,$$

so that, for any $x \in (0, a]$, it follows that

$$\left| f\left(\frac{u}{x}\right) k(u) \right| \le a^{c_2} f(u)\,|k(u)|\,.$$

We have found an integrable function that dominates $f\left(\frac{u}{x}\right) k(u)$ for $x \in (0, a]$, this implies that

$$\lim_{x\to x_0} d_j(x) = d_j(x_0)\,.$$

Finally, we show that h is continuous. We claim that the convergence in (14.4.21) is uniform on each interval $[a, b]$, $0 < a < b < \infty$. By (14.4.10),

$$\left| h(x) - \sum_{j=0}^{2n} \frac{d_j(x)}{f\left(\frac{1}{x}\right)} \right| \le \frac{|d_{2n+1}(x)|}{f\left(\frac{1}{x}\right)}\,.$$

We also have

$$\begin{aligned}
\frac{|d_{2n+1}(x)|}{f\left(\frac{1}{x}\right)} &= \int_{\lambda_{2n+1}}^{\lambda_{2n+2}} \frac{f\left(\frac{u}{x}\right)}{f\left(\frac{1}{x}\right)} |k(u)|\, \mathrm{d}u \\
&\le \frac{1}{f\left(\frac{1}{a}\right)} \int_{\lambda_{2n+1}}^{\lambda_{2n+2}} f\left(\frac{u}{x}\right) |k(u)|\, \mathrm{d}u \\
&\le \frac{f\left(\frac{\lambda_{2n+1}}{b}\right)}{f\left(\frac{1}{a}\right)} \int_{\lambda_{2n+1}}^{\lambda_{2n+2}} |k(u)|\, \mathrm{d}u \\
&\le \frac{f\left(\frac{\lambda_{2n+1}}{b}\right)}{f\left(\frac{1}{a}\right)} \int_{0}^{\lambda_1} k(u)\, \mathrm{d}u\,.
\end{aligned}$$

Since the last term approaches to 0 as $n \to \infty$, the convergence in (14.4.21) is uniform on any interval $[a, b]$, $0 < a < b < \infty$. Therefore, h is continuous. $\square$

We now consider the second generalization of the estimate (14.4.1). We want to emphasize that the sine transform in this analysis shall be considered as a tempered distribution, so that we shall take a regularization of f, instead of f. If we let $c_2 > 2$ with no restriction, the sine transform of f may not exist, as we remarked at the end of Section 14.3. In order to define a regularization of f, we need to extend f to the whole real line; we do this by setting $f(x) = -f(-x)$ for $x < 0$, so that it becomes an odd function; for the sake of simplicity, we shall keep denoting this extension by f.

We state our second result.

Theorem 14.7. *Let f be an odd function such that its restriction to $(0, \infty)$ belongs to $\in \mathfrak{V}$. Suppose that $\tilde{f}$ is any regularization of f in $\mathcal{S}'(\mathbb{R})$ which defines an odd distributions. Denote the sine transform of $\tilde{f}$ by F. Then, for $x > 0$, either*

$$F(x) = \frac{h(x)}{x} f\left(\frac{1}{x}\right) + P(x)\,, \tag{14.4.22}$$

or

$$F(x) = -\frac{h(x)}{x} f\left(\frac{1}{x}\right) + P(x)\,, \tag{14.4.23}$$

where h is continuous and bounded above and below by positive constants and P is a polynomial.

Proof. It is known that any two odd regularization of f, say $\tilde{f}$ and $\tilde{f}_1$, satisfy

$$\tilde{f}\,(x) = \tilde{f}_1\,(x) + \sum_{i=0}^{m} a_i \delta^{(2i+1)}\,(x)\ , \tag{14.4.24}$$

where $a_0, a_1, \ldots, a_m$ are constants. Observe that the sine transform of the sum of delta functions and its derivatives on the right side is a polynomial. To see this fact, let ϕ be a test function of the space $\mathcal{S}(\mathbb{R})$, $k \in \mathbb{N}$; then,

$$\left\langle \delta^{(k)}(x),\ \int_0^\infty \phi(u)\sin(ux)\,\mathrm{d}u \right\rangle = 0,\ \text{if } k \text{ is even}\,;$$

$$\left\langle \delta^{(k)}(x),\ \int_0^\infty \phi(u)\sin(ux)\,\mathrm{d}u \right\rangle = \int_0^\infty (-x)^k \phi(x)\,\mathrm{d}x\,,\ \text{if } k = 4j+1\,;$$

$$\left\langle \delta^{(k)}(x),\ \int_0^\infty \phi(x)\sin(ux)\,\mathrm{d}u \right\rangle = \int_0^\infty x^k \phi(x)\,\mathrm{d}x\,,\ \text{if } k = 4j+3\,.$$

Therefore, it suffices to work with any particular odd regularization of f. So we shall find a regularization of f for which the conclusion of the theorem holds. We shall suppose that $c_2 \geq 2$; otherwise, the conclusion of this theorem would be a consequence of the Theorem 4.1. Let n be the unique natural number such that

$$2n+1 \leq c_2 < 2n+3\,. \tag{14.4.25}$$

We shall divide the proof into two cases. We consider the cases when n is odd and then when n is even.

Assume first that n is odd. Define now $\widetilde{f}$ as

$$\begin{aligned}\left\langle \widetilde{f}(x),\ \phi(x) \right\rangle = \text{p.v.} \int_{-2\pi}^{2\pi} f(x) \left(\phi(x) - \sum_{i=0}^{2n+1} \frac{\phi^{(i)}(0)}{i!} x^i \right) \mathrm{d}x \\ + \int_{2\pi \leq |x|} f(x)\phi(x)\,\mathrm{d}x\,,\end{aligned} \tag{14.4.26}$$

for $\phi \in \mathcal{S}(\mathbb{R})$. Here p.v. stands for the Cauchy principal value of the integral at the origin, that is, p.v. $\int = \lim_{\varepsilon \to 0^+} \int_{\varepsilon \leq |x|}$. We shall prove that $\widetilde{f}$ is well-defined. Let $\phi \in \mathcal{S}(\mathbb{R})$, then by Corollary 14.4

$$f(x) \left(\phi(x) - \sum_{i=0}^{2n+1} \frac{\phi^{(i)}(0)}{i!} x^i \right) = O(x^{2n+2-c_2})\,,\ x \to 0\,,$$

and so, by (14.4.25), it is integrable on $(0, 2\pi)$. The integrability on $(2\pi, \infty)$ is clear since $\phi \in \mathcal{S}(\mathbb{R})$. By an standard argument, $\widetilde{f} \in \mathcal{S}'(\mathbb{R})$. Observe that $\widetilde{f}$ is odd, in fact the principal value integral in the definition of the distribution ensures that $\left\langle \widetilde{f}(x), \phi(x) \right\rangle = 0$, if ϕ is an even test functions. On the other hand, if the test function ϕ is odd, then

$$\left\langle \widetilde{f}(x),\ \phi(x) \right\rangle = 2\int_0^{2\pi} f(x)\left(\phi(x) - \sum_{i=0}^{n} \frac{\phi^{(2i+1)}(0)}{(2i+1)!}x^{2i+1}\right) \mathrm{d}x \qquad (14.4.27)$$
$$+ 2\int_{2\pi}^{\infty} f(x)\phi(x)\,\mathrm{d}x\,,$$

We shall prove the formula for the sine transform of $\widetilde{f}$. Denote by $\widetilde{F}$ the sine transform of $\widetilde{f}$. Let us now set

$$K(x) = \sin x - \sum_{i=0}^{n} \frac{(-1)^i}{(2i+1)!}x^{2i+1}. \qquad (14.4.28)$$

Since n is odd,

$$K(x) \geq 0, \text{ for } x \geq 0\,.$$

Using the definition of $\widetilde{F}$, we have for an odd test function ϕ,

$$\begin{aligned}
\left\langle \widetilde{F}(x), \phi(x) \right\rangle &= \left\langle \widetilde{f}(x), \int_0^{\infty} \phi(u)\sin(xu)\,\mathrm{d}u \right\rangle \\
&= 2\int_0^{2\pi} f(x)\left(\int_0^{\infty} \phi(u)K(xu)\,\mathrm{d}u\right)\mathrm{d}x \\
&\quad + 2\int_{2\pi}^{\infty} f(x)\left(\int_0^{\infty} \phi(u)\sin(xu)\,\mathrm{d}u\right)\mathrm{d}x \\
&= 2\int_0^{\infty} \frac{\phi(x)}{x}\left(\int_0^{2\pi} f\left(\frac{u}{x}\right)K(u)\,\mathrm{d}u + \int_{2\pi}^{\infty} f\left(\frac{u}{x}\right)\sin u\,\mathrm{d}u\right)\mathrm{d}x\,,
\end{aligned}$$

For a general $\phi \in \mathcal{S}(\mathbb{R})$, we then obtain that

$$\left\langle \widetilde{F}(x), \phi(x) \right\rangle = \text{p.v.} \int_{-\infty}^{\infty} \phi(x)\widetilde{F}(x)\mathrm{d}x\,, \qquad (14.4.29)$$

where

$$\widetilde{F}(x) = \frac{1}{|x|}\left[\int_0^{2\pi} f\left(\frac{u}{x}\right)K(u)\,\mathrm{d}u + \int_{2\pi}^{\infty} f\left(\frac{u}{x}\right)\sin u\,\mathrm{d}u\right]. \qquad (14.4.30)$$

Hence $\widetilde{F}$ can be identified with a classical function, in the sense that $\widetilde{F}$ is the distribution generated by the function given by (14.4.30) .

Next we set

$$h(x) = \frac{\widetilde{F}(x)}{x^{-1}f\left(\frac{1}{x}\right)}, \text{ for } x > 0\,. \qquad (14.4.31)$$

We shall find two constants, A and B, so that

$$A \le h(x) \le B\,, \quad x > 0\,. \tag{14.4.32}$$

Notice that

$$h(x) - \int_0^{2\pi} \frac{f\left(\frac{u}{x}\right)}{f\left(\frac{1}{x}\right)} K(u)\,\mathrm{d}u = \int_{2\pi}^{\infty} \frac{f\left(\frac{u}{x}\right)}{f\left(\frac{1}{x}\right)} \sin u\,\mathrm{d}u\,. \tag{14.4.33}$$

We also have that

$$\int_{2\pi}^{4\pi} \frac{f\left(\frac{u}{x}\right)}{f\left(\frac{1}{x}\right)} \sin u\,\mathrm{d}u \le \int_{2\pi}^{\infty} \frac{f\left(\frac{u}{x}\right)}{f\left(\frac{1}{x}\right)} \sin u\,\mathrm{d}u \le \int_{2\pi}^{3\pi} \frac{f\left(\frac{u}{x}\right)}{f\left(\frac{1}{x}\right)} \sin u\,\mathrm{d}u\,. \tag{14.4.34}$$

We can apply the argument that we used in Theorem 14.6 to find positive constants A' and B' such that

$$\int_{2\pi}^{3\pi} \frac{f\left(\frac{u}{x}\right)}{f\left(\frac{1}{x}\right)} \sin u\,\mathrm{d}u \le B'\,, \tag{14.4.35}$$

and

$$A' \le \int_{2\pi}^{4\pi} \frac{f\left(\frac{u}{x}\right)}{f\left(\frac{1}{x}\right)} \sin u\,\mathrm{d}u\,, \tag{14.4.36}$$

for all $x \in (0, \infty)$. Using the last inequalities, we obtain that

$$A' \le \int_{2\pi}^{\infty} \frac{f\left(\frac{u}{x}\right)}{f\left(\frac{u}{x}\right)} \sin u\,\mathrm{d}u \le B'\,. \tag{14.4.37}$$

It follows that

$$\int_0^{2\pi} \min\left\{\frac{1}{u^{c_1}}, \frac{1}{u^{c_2}}\right\} K(u)\,\mathrm{d}u + A' \le h(x)\,, \tag{14.4.38}$$

and

$$h(x) \le \int_0^{2\pi} \max\left\{\frac{1}{u^{c_1}}, \frac{1}{u^{c_2}}\right\} K(u)\,\mathrm{d}u + B'\,, \tag{14.4.39}$$

which shows that h is bounded above and below by positive constants.

We now prove the continuity of h. The continuity of

$$\int_{2\pi}^{\infty} f\left(\frac{u}{x}\right) \sin u\,\mathrm{d}u$$

follows from the proof of Theorem 14.6. Moreover, since

$$\frac{f\left(\frac{u}{x}\right)}{f\left(\frac{1}{x}\right)} K(u) \le \max\left\{\frac{1}{u^{c_1}}, \frac{1}{u^{c_2}}\right\} K(u)\,, \tag{14.4.40}$$

it follows by the Lebesgue Dominated Convergence Theorem that

$$h(x) - \int_{2\pi}^{\infty} \frac{f\left(\frac{u}{x}\right)}{f\left(\frac{1}{x}\right)} \sin u \, \mathrm{d}u \,,$$

is continuous, and so is $h(x)$. This completes the proof for the odd case.

We now assume that n is an even number. Define $\tilde{f}$ as

$$\left\langle \tilde{f}(x), \phi(x) \right\rangle = \text{p.v.} \int_{-3\pi}^{3\pi} f(x) \left(\phi(x) - \sum_{i=0}^{2n+1} \frac{\phi^{(i)}(0)}{i!} x^i \right) \mathrm{d}x \qquad (14.4.41)$$
$$+ \int_{3\pi \le |x|} f(x)\phi(x) \, \mathrm{d}x \,,$$

for $\phi \in \mathcal{S}(\mathbb{R})$. It follows that $\tilde{f} \in \mathcal{S}'(\mathbb{R})$. Set

$$J(x) = \sum_{i=0}^{n} \frac{(-1)^i}{(2i+1)!} x^{2i+1} - \sin x \,, \qquad (14.4.42)$$

which is a positive function, since n is an even number. Let $\tilde{F}$ be the sine transform of $\tilde{f}$. We have that if $x > 0$

$$\tilde{F}(x) = \frac{1}{x} \left[- \int_0^{3\pi} f\left(\frac{u}{x}\right) J(u) \, \mathrm{d}u + \int_{3\pi}^{\infty} f\left(\frac{u}{x}\right) \sin u \, \mathrm{d}u \right] . \qquad (14.4.43)$$

Set

$$h(x) = - \frac{\tilde{F}(x)}{x^{-1} f\left(\frac{1}{x}\right)} \,, \ x > 0 \,. \qquad (14.4.44)$$

It follows that

$$h(x) = \int_0^{3\pi} \frac{f\left(\frac{u}{x}\right)}{f\left(\frac{1}{x}\right)} J(u) \, \mathrm{d}u - \int_{3\pi}^{\infty} \frac{f\left(\frac{u}{x}\right)}{f\left(\frac{1}{x}\right)} \sin u \, \mathrm{d}u \,, \qquad (14.4.45)$$

for $x > 0$. We can find two positive constants, A'' and B'', such that

$$- \int_{3\pi}^{4\pi} \frac{f\left(\frac{u}{x}\right)}{f\left(\frac{1}{x}\right)} \sin u \, \mathrm{d}u \le B'' \,,$$

and

$$- \int_{3\pi}^{5\pi} \frac{f\left(\frac{u}{x}\right)}{f\left(\frac{1}{x}\right)} \sin u \, \mathrm{d}u \ge A'' \,.$$

From these inequalities, it follows that

$$\int_0^{3\pi} \min\left\{ \frac{1}{u^{c_1}}, \frac{1}{u^{c_2}} \right\} J(u) \, \mathrm{d}u + A'' \le h(x) \,,$$

and,

$$h(x) \leq \int_0^{3\pi} \max\left\{\frac{1}{u^{c_1}}, \frac{1}{u^{c_2}}\right\} J(u)\, \mathrm{d}u + B'' ,$$

which proves the required inequalities. The continuity of h can be established as in the odd case. $\square$

14.5 Laplace Transform

In this section, we shall give a result analogous to Theorem 14.7 for the Laplace transform. The estimate is as follows.

Theorem 14.8. *Let $f \in \mathfrak{V}$. Suppose that $\tilde{f}$ is any regularization of f in $\mathcal{S}'(\mathbb{R})$ such that* $\operatorname{supp} f \subseteq [0,\infty)$*. Then, for $x > 0$, the Laplace transform satisfies either*

$$\mathcal{L}\left\{\tilde{f}; x\right\} = \frac{h(x)}{x} f\left(\frac{1}{x}\right) + P(x)\,, \tag{14.5.1}$$

or

$$\mathcal{L}\left\{\tilde{f}; x\right\} = -\frac{h(x)}{x} f\left(\frac{1}{x}\right) + P(x)\,, \tag{14.5.2}$$

where h is continuous and bounded above and below by positive constants, and P is a polynomial.

Proof. We proceed as in Theorem 14.7. It suffices to consider a particular regularization of f. Let n be the integer part of c_2. We shall consider two cases. First, we assume that n is odd, and then we consider the even case.

Assume that n is odd. Define $\tilde{f}$ as

$$\begin{aligned}\left\langle \tilde{f}(x), \phi(x) \right\rangle = \int_0^1 f(x) \left(\phi(x) - \sum_{i=0}^{n} \frac{\phi^i(0)}{i!}\right) \mathrm{d}x \\ + \int_1^\infty f(x)\phi(x)\, \mathrm{d}x\,,\end{aligned} \tag{14.5.3}$$

for $\phi \in \mathcal{S}(\mathbb{R})$. Then, $\widetilde{f}$ is a regularization of f in $\mathcal{S}'(\mathbb{R})$. Since $\operatorname{supp} \widetilde{f} = [0,\infty)$, its Laplace transform is well-defined. Let us denote its Laplace transform by $\widetilde{L}$, so that

$$\widetilde{L}(x) = \int_0^1 f(u)\left(e^{-ux} - \sum_{i=0}^{n} \frac{(-ux)^i}{i!}\right) \mathrm{d}u + \int_1^\infty f(u)e^{-ux}\,\mathrm{d}u$$
$$= \frac{1}{x}\left[\int_0^1 f\left(\frac{u}{x}\right)\left(e^{-u} - \sum_{i=0}^{n} \frac{(-u)^i}{i!}\right) \mathrm{d}u + \int_1^\infty f\left(\frac{u}{x}\right) e^{-u}\,\mathrm{d}u\right].$$

We now consider the following inequality,

$$e^{-x} - \sum_{i=0}^{n} \frac{(-x)^i}{i!} > 0, \text{ for } x > 0\,. \tag{14.5.4}$$

Set

$$h(x) = \frac{\tilde{L}(x)}{x^{-1} f\left(\frac{1}{x}\right)}\,, \tag{14.5.5}$$

and

$$K(x) = e^{-x} - \sum_{i=0}^{n} \frac{(-x)^i}{i!}\,. \tag{14.5.6}$$

Then, we have

$$\int_0^1 \frac{K(u)}{u^{c_1}}\,\mathrm{d}u + \int_1^\infty \frac{e^{-u}}{u^{c_2}}\,\mathrm{d}u \le h(x) \le \int_0^1 \frac{K(u)}{u^{c_2}}\,\mathrm{d}u + \int_0^\infty \frac{e^{-u}}{u^{c_1}}\,\mathrm{d}u\,.$$

This completes the proof for the odd case.

Assume now that n is even. Set

$$J(x) = \sum_{i=0}^{n} \frac{(-x)^i}{i!} - e^{-x}\,; \tag{14.5.7}$$

it follows that

$$J(x) > 0\,, \quad \text{for } x > 0\,.$$

Take $A > 1$ such that

$$\int_0^1 \frac{J(u)}{u^{c_1}}\,\mathrm{d}u - \int_A^\infty \frac{e^{-u}}{u^{c_1}}\,\mathrm{d}u > 0\,, \tag{14.5.8}$$

and

$$\int_0^1 \frac{J(u)}{u^{c_2}}\,\mathrm{d}u - \int_A^\infty \frac{e^{-u}}{u^{c_2}}\,\mathrm{d}u\, < 0\,. \tag{14.5.9}$$

We define $\tilde{f}$, a regularization of f, as

$$\left\langle \tilde{f}(x), \phi(x) \right\rangle = \int_0^A f(x) \left(\phi(x) - \sum_{i=0}^{n} \frac{\phi^{(i)}(0)}{i!} \right) \mathrm{d}x \qquad (14.5.10)$$
$$+ \int_A^\infty f(x)\phi(x)\,\mathrm{d}x\,.$$

It follows that $\tilde{L}$, the Laplace transform of $\tilde{f}$, is given by

$$\tilde{L}(x) = \frac{1}{x}\left(-\int_0^A f\left(\frac{u}{x}\right) J(u)\,\mathrm{d}u + \int_A^\infty f\left(\frac{u}{x}\right) e^{-u}\,\mathrm{d}u \right)\,. \qquad (14.5.11)$$

Define now h by

$$h(x) = \frac{-\tilde{L}(x)}{x^{-1} f\left(\frac{1}{x}\right)}\,. \qquad (14.5.12)$$

We have that

$$\int_0^1 \frac{J(u)}{u^{c_1}}\,\mathrm{d}u + \int_1^A \frac{J(u)}{u^{c_2}}\,\mathrm{d}u - \int_A^\infty \frac{e^{-u}}{u^{c_1}}\,\mathrm{d}u \le h(x)\,,$$

and

$$h(x) \le \int_0^1 \frac{J(u)}{u^{c_2}}\,\mathrm{d}u + \int_1^A \frac{J(u)}{u^{c_1}}\,\mathrm{d}u - \int_A^\infty \frac{e^{-u}}{u^{c_2}}\,\mathrm{d}u\,,$$

so h is bounded above and below by positive constants. $\square$

References

[1] D. D. Adamović, *Sur quelques propriétés des functions á croissance lente de Karamata I*, Matematički Vesnik **3** (1966), 123–136.

[2] D. D. Adamović, *Sur quelques propriétés des functions á croissance lente de Karamata II*, Matematički Vesnik **3** (1966), 161–172.

[3] S. Aljančić, R. Bojanić, M. Tomić, *Sur la valuer asymptotique d'une classe des integrales defines,* Publ. Inst. Math. Acad. Serbe Sci. **7** (1954), 81–94.

[4] L. Alvarez, J.M. Morel, *Formalization and computational aspects of image analysis,* Acta Numer. (1994), 1–59.

[5] K. Ananda Rau, *On the converse of Abel's theorem,* J. London Math. Soc. **3** (1928), 349–355.

[6] P. Antosik, J. Mikusiński, R. Sikorski, *Theory of distributions. The sequential approach,* Elsevier Scientific Publishing Co., Amsterdam; PWN—Polish Scientific Publishers, Warsaw, 1973.

[7] W. Arendt, C. Batty, M Hieber, F. Neubrander, *Vector-valued Laplace transforms and Cauchy problems,* Monographs in Mathematics, 96, Birkhäuser Verlag, Basel, 2001.

[8] J. M. Ash, P. Erdős, L. A. Rubel, *Very slowly varying functions*, Aequationes Math. **10** (1974), 1–9.

[9] M. Avdispahić, *On the determination of the jump of a function by its Fourier series,* Acta Math. Hungar. **48** (1986), 267–271.

[10] R. G. Bartle, *A Modern Theory of Integration,* A.M.S., Providence, 2001.

[11] E. J. Beltrami, M. R. Wohlers, *Distributions and the Boundary Values of Analytic Functions,* Academic Press, New York, 1966.

[12] B. C. Berndt, R. J. Evans, *Extensions of asymptotic expansions from Chapter 15 of Ramanujan's second notebook,* J. Reine Angew. Math. **361** (1985), 118–134.

[13] R. Berndt, *A formula for the Fourier transform of certain odd differentiable functions,* J. Math. Anal. Appl. **85** (2003), 349–355.

[14] R. Berndt, *Singular Integrals with New Singularities,* Dissertation, University of Minnesota, 2003.

[15] N. H. Bingham, C. M. Goldie, J. L. Teugels, *Regular variation,* Encyclopedia of Mathematics and its Applications 27, Cambridge University Press, Cambridge, 1989.

[16] S. Bochner, *Summation of multiple Fourier series by spherical means*, Trans. Amer. Math. Soc. **40** (1936), 175–207.

[17] S. Bochner, *Lectures on Fourier Integrals,* Annals of Mathematics Studies, No. 42, Princeton University Press, Princeton, N.J., 1959.

[18] S. Bochner, K. Chandrasekharan, *Summations over lattice points in κ-space*, Quart. J. Math. Oxford Ser. **19** (1948), 238–248.

[19] S. Bochner, K. Chadrasekharan, *Fourier transforms,* Annals of Mathematics Studies, No. 19, Princeton University Press, Princeton, N.J., 1949.

[20] J. Boersma, *A nonharmonic trigonometric series,* SIAM Review **37** (1995), 443–445.

[21] N. N. Bogoliubov, D.V. Shirkov, *Introduction to the Theory of Quantized Fields,* Wiley, New York, 1959.

[22] R. Bojanić, J. Karamata, *On a class of functions of regular asymptotic behavior*, Math. Res. Centre, U.S. Army, Madison, Wis., Tech. Summary Rep. No. 436 (1963).

[23] V. A. Borovikov, F. J. Mendoza, *The problem of the pointwise Fourier inversion for piecewise smooth functions of several variables*, J. Fourier Anal. Appl. **8** (2002), 399–406.

[24] H. Bremermann, *Distributions, Complex Variables and Fourier Transforms,* Addison-Wesley, Reading, Massachusetts, 1965.

[25] Yu. A. Brychkov, A. P. Prudnikov, *Integral transforms of generalized functions*, Gordon and Breach Science Publishers, New York, 1989.

[26] J. Campos Ferreira, *Introduction to the Theory of Distributions,* Longman, London, 1997.

[27] L. Carleson, *On convergence and growth of partial sumas of Fourier series*, Acta Math. **116** (1966), 135–157.

[28] K. Chandrasekharan, S. Minakshisundaram, *Typical means,* Oxford University Press, Oxford, 1952.

[29] O. Christensen, *An introduction to frames and Riesz bases,* Applied and Numerical Harmonic Analysis, Birkhäuser, Boston, 2003.

[30] J. F. Colombeau, *Elementary introduction to new generalized functions,* North-Holland Mathematics Studies, 113, Notes on Pure Mathematics, 103, North-Holland Publishing Co., Amsterdam, 1985.

[31] F. Constantinescu, *Boundary values of analytic functions,* Comm. Math. Phys. **7** (1968), 225–233.

[32] J. B. Conway, *Functions of One Complex Variable II,* Springer-Verlag, New York, 1995.

[33] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Phyladelphia, 1992.

[34] A. Denjoy, *Sur l'intégration des coefficients différentiels d'ordre supérieur* Fund. Math. **25** (1935), 237–320.

[35] R. DeVore, A. Ronald, B. Lucier, *Wavelets,* Acta numerica **92** (1992), 1–56.

[36] J. Dieudonné, *Foundations of Modern Analysis,* Academic Press, New York, 1969.

[37] Ju. N. Drožžinov, B. I. Zav'jalov, *Quasiasymptotic behavior of generalized functions, and Tauberian theorems in the complex domain,* Mat. Sb. (N.S.) **102**(144) (1977), 372–390 (in Russian).

[38] Yu. N. Drozhzhinov, B. I. Zav'yalov, *Theorems of Hardy-Littlewood type for measures of variable sign in a cone,* Sb. Math. **186** (1995), 675–693.

[39] Y. N. Drozhzhinov, B. I. Zav'yalov, *Tauberian-type theorems for a generalized multiplicative convolution*, Izv. Math. **64** (2000), 35–92.

[40] Y. N. Drozhzhinov, B. I. Zav'yalov, *Tauberian theorems for generalized functions with values in Banach spaces*, Izv. Math. **66** (2002), 701–769.

[41] Yu. N. Drozhzhinov, B. I. Zav'yalov, *Multidimensional Tauberian theorems for generalized functions with values in Banach spaces,* Sb. Math. **194** (2003), 1599–1646.

[42] Yu. N. Drozhzhinov, B. I. Zav'yalov, *Asymptotically homogeneous generalized functions in a spherical representation and some applications*, Dokl. Akad. Nauk **405** (2005), 18–21 (in Russian).

[43] Yu. N. Drozhzhinov, B. I. Zav'yalov, *Asymptotically homogeneous generalized functions and boundary properties of functions holomorphic in tubular cones*, Izv. Math. **70** (2006), 1117–1164.

[44] A. L. Durán, R. Estrada, *Strong moment problems for rapidly decreasing smooth functions,* Proc. Amer. Math. Soc. **120** (1994), 529–534.

[45] A. L Durán, R. Estrada, R. P. Kanwal, *Extensions of the Poisson summation formula,* J. Math. Anal. Appl. **218** (1998), 581–606.

[46] R. Estrada, *The asymptotic expansion of certain series considered by Ramanujan,* Appl. Anal. **43** (1992), 191–228.

[47] R. Estrada, *Characterization of the Fourier series of distributions having a value at a point*, Proc. Amer. Math. Soc. **124** (1996), 1205–1212.

[48] R. Estrada, *Regularization of distributions,* Internat. J. Math. & Math. Sci. **21** (1998), 625–636.

[49] R. Estrada, *The Cesàro behaviour of distributions,* Proc. Roy. Soc. London Ser. A **454** (1998), 2425–2443.

[50] R. Estrada, *Abel summability and angular convergence,* Scientia **9** (2003), 45–51.

[51] R. Estrada, *The nonexistence of regularization operators,* J. Math. Anal. Appl. **286** (2003), 1–10.

[52] R. Estrada, *Boundary values of analytic functions without distributional point values,* Tamkang J. Math. **35** (2004), 53–60.

[53] R. Estrada, *Distributional radius of curvature,* Math. Meth. Appl. Scs. **29** (2005), 427–444.

[54] R. Estrada, *A distributional version of the Ferenc Lukács theorem*, Sarajevo J. Math. **1** (2005), 75–92.

[55] R. Estrada, *One-sided cluster sets of distributional boundary values of analytic functions,* Complex Var. and Elliptic Eqns. **51** (2006), 661–673.

[56] R. Estrada, R. P. Kanwal, *Distributional boundary values of harmonic and analytic functions,* J. Math. Anal. Appl. **89** (1982), 262–289.

[57] R. Estrada, R. P. Kanwal, *A distributional theory of asymptotic expansions,* Proc. Roy. Soc. London Ser. A. **428** (1990), 399–430.

[58] R. Estrada, R. P. Kanwal, *Asymptotic separation of variables*, J. Math. Anal. Appl. **178** (1993), 130–142.

[59] R. Estrada, R. P. Kanwal, *Moment expansion analysis for time-domain asymptotics,* Math. Meth. Appl. Scs. **21** (1998), 489–499.

[60] R. Estrada, R. P. Kanwal, *Singular Integral Equations,* Birkhäuser, Boston, 2000.

[61] R. Estrada, R. P. Kanwal, *A distributional approach to Asymptotics. Theory and Applications,* second edition, Birkhäuser, Boston, 2002.

[62] L. Fejér, *Untersuchungen über Fouriersche Reihen*, Math. Ann. **58** (1904), 501–569 (in German).

[63] L. Fejér, *Über die Bestimmung des Sprunges der Funktion aus ihrer Fourierreihe,* J. Reine Angew. Math. **142** (1913), 165–188 (in German). See also the Collected Works of L. Fejér: Gesammelte Arbeiten, Akadémiai Kiadó, Budapest, 1970, pp. 718-744.

[64] W. Feller, *An introduction to probability theory and its applications,* vol.II , second edition, John Wiley & Sons, Inc., New York-London-Sydney, 1971.

[65] I. Fraga Alves, L. de Haan, T. Lin, *Third order extended regular variation,* Publ. Inst. Math. (Beograd) (N.S.) **80**(94) (2006), 109–120.

[66] A. Gelb, E. Tadmor, *Detection of edges in spectral data,* Appl. Comput. Harmon. Anal. **7** (1999), 101–135.

[67] A. Gelb, E. Tadmor, *Detection of edges in spectral data. II. Nonlinear enhancement,* SIAM J. Numer. Anal. **38** (2000), 1389–1408.

[68] I. M. Gel'fand, G. E. Shilov, *Generalized functions. Vol. 1: Properties and operations,* Academic Press, New York-London, 1964.

[69] M. L. Glasser, *A nonharmonic trigonometric series,* SIAM Review **37** (1995), 443.

[70] B. I. Golubov, *Determination of the jump of a function of bounded p-variation from its Fourier series,* Mat. Zametki **12** (1972), 19–28 (in Russian).

[71] F. J. González Vieli, *Intégrales trigonométriques et pseudofonctions,* Ann. Inst. Fourier (Grenoble) **44** (1994), 197–211.

[72] F. J. González Vieli, *Inversion de Fourier ponctuelle des distributions à support compact,* Arch. Math. (Basel) **75** (2000), 290-298.

[73] F. J. González Vieli, *Rate of convergence in the Fourier inversion of indicator functions,* Asymptot. Anal. **41** (2005), 179–187.

[74] F. J. González Vieli, *Pointwise Fourier inversion of distributions,* Anal. Theory Appl. **24** (2008), 87–92.

[75] F. J. González Vieli, C. C. Graham, *On the support of tempered distributions,* Arch. Math. (Basel) **88** (2007), 133–142.

[76] R. A. Gordon, *The integrals of Lebesgue, Denjoy, Perron, and Henstock,* A.M.S., Providence, 1994.

[77] D. Gottlieb, C. Shu, *On the Gibbs phenomenon and its resolution,* SIAM Rev. **39** (1997), 644–668.

[78] C. C. Graham, *The support of pseudomeasures in* $\mathbb{R}$, Math. Proc. Camb. Phil. Soc. **142** (2007), 149–152.

[79] C. C. Graham, *The support of tempered distributions,* Math. Proc. Camb. Phil. Soc. **144** (2008), 495–498.

[80] K. Gröchenig, *Foundation of Time-Frequency Analysis,* Birkhäuser, Boston, 2001.

[81] T. H. Gronwall, *Über eine Summationsmethode und ihre Anwendung auf die Fouriersche Reihe,* J. reine angew. Math. **147** (1916), 16–35 (in German).

[82] A. Grossman, G. Loupias, E. M. Stein, *An algebra of pseudodifferential operators and Quantum Mechanics in phase space,* Ann. Inst. Fourier **18** (1968), 343–368.

[83] O. von Grudzinski, *Quasihomogeneous distributions,* North-Holland Mathematics Studies 165, North-Holland Publishing Co., Amsterdam, 1991.

[84] L. de Haan, *On regular variation and its application to the weak convergence of sample extremes,* Mathematical Centre Tracts, 32, Mathematisch Centrum, Amsterdam, 1970.

[85] G. H. Hardy, *Divergent Series,* Clarendon Press, Oxford, 1949.

[86] G. H. Hardy, J. E. Littlewood, *Contributions to the arithmetic theory of series,* Proc. London Math. Soc. **11** (1913), 411-478.

[87] G. H. Hardy, J. E. Littlewood, *Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive,* Proc. London Math. Soc. **13** (1914), 174–191.

[88] G. H. Hardy, J. E. Littlewood, *Some theorems concerning Dirichlet's series* Messenger of Mathematics **43** (1914), 134–147.

[89] G. H. Hardy, J. E. Littlewood, *Solution of the Cesàro summability problem for power series and Fourier series,* Math. Z. **19** (1923), 67–96.

[90] G. H. Hardy, J. E. Littlewood, *The Fourier series of a positive function,* J. London Math. Soc. **1** (1926), 134–138.

[91] G. H. Hardy, M. Riesz, *The general theory of Dirichlet's series,* Cambridge Tracts in Mathematics and Mathematical Physics, No. 18, Cambridge University Press, Cambridge, 1952.

[92] G. H. Hardy, W. W. Rogosinski, *Notes on Fourier series. III. Asymptotic formulae for the sums of certain trigonometrical series,* Quart. J. Math. Oxford Ser. **16** (1945), 49–58.

[93] G. H. Hardy, W. W. Rogosinski, *Fourier series,* second edition, Cambridge Tracts in Mathematics and Mathematical Physics, no. 38, Cambridge University Press, Cambridge, 1950.

[94] E. W. Hobson, *The theory of functions of a real variable and the theory of Fourier's series,* vol. II, Dover Publications, New York, 1958.

[95] M. Holschneider, *Wavelets. An analysis tool,* The Clarendon Press, Oxford University Press, New York, 1995.

[96] M. Holschneider, Ph. Tchamitchian, *Pointwise analysis of Riemann's "non-differentiable" function,* Invent. Math. **105** (1991), 157–175.

[97] L. Hörmander, *The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis,* second edition. Grundlehren der Mathematischen 256, Springer-Verlag, Berlin, 1990.

[98] L. Hörmander, *Lectures on Nonlinear Hyperbolic Differential Equations*, Mathematiques & Applications 26, Springer-Verlag, Berlin, 1997.

[99] J. Horváth, *Topological Vector Spaces and Distributions*, vol.I., Addison-Wesley, Reading, Massachusetts, 1966.

[100] A. E. Ingham, *The equivalence theorem for Cesàro and Riesz summability,* Publ. Ramanujan Inst. **1** (1968), 107–113.

[101] A. E. Ingham, *The distribution of prime numbers*, Cambridge University Press, Cambridge, 1990.

[102] A. Ivić, *On summatory functions of additive functions and regular variation,* (Dédié à Jovan Karamata) Publ. Inst. Math. (Beograd) **71** (2002), 27–39.

[103] A. Ivić, *The Riemann zeta-function. Theory and applications,* Dover Publications, New York, 2003.

[104] S. Jaffard, Y. Meyer, *Wavelet Methods for Pointwise Regularity and Local Oscillations of Functions*, Memoirs of the American Mathematical Society, vol.123, No 587, 1996.

[105] J.P. Kahane, R. Salem, *Ensembles Parfaits et Séries Trigonométriques,* Hermann, Paris, 1963.

[106] A. Kamiński, *Convolution, product and Fourier transform of distributions,* Studia Math. **74** (1982), 83–96.

[107] A. Kaneko, *Introduction to hyperfunctions,* Mathematics and its Applications (Japanese Series), Kluwer Academic Publishers Group, Dordrecht; SCI-PRESS, Tokyo, 1988.

[108] R. P. Kanwal, *Generalized Functions: Theory and Technique*, second edition, Birkhäuser, Boston, 1998.

[109] J. Karamata, *Über die Hardy-Littlewoodschen Umkehrungen des Abelschen Stetigkeitssatzes,* Math. Z. **32** (1930), 319–320 (in German).

[110] J. Karamata, *Neuer Beweis und Verallgemeinerung der Tauberschen Sätze, welche die Laplacesche and Stieltjessche Transformation betreffen,* J. Reine Angew. Math. **164** (1931), 27–39 (in German).

[111] J. Karamata, *Sur un mode de croissance régulière des fonctions,* Mathematica (Cluj) **4** (1933), 38–53.

[112] J. Karamata, *Sur un mode de croissance régulière. Théorèmes fondamentaux,* Bull. Soc. Math. France **61** (1933), 55–62.

[113] P. Koosis, *Introduction to H_p spaces,* London Mathematical Society Lecture Note Series, 40, Cambridge University Press, Cambridge-New York, 1980.

[114] H. Korevaar, T. van Aardenne-Ehrenfest, N. G. de Bruijn, *A note on slowly oscillating functions*, Nieuw. Arch. Wisk. **23** (1949), 77–88.

[115] J. Korevaar, *Tauberian theory. A century of developments,* Grundlehren der Mathematischen Wissenschaften, 329., Springer-Verlag, Berlin, 2004.

[116] J. Korevaar, *Distributional Wiener-Ikehara theorem and twin primes,* Indag. Math. (N.S.) **16** (2005), 37–49.

[117] J. Korevaar, *The Wiener-Ikehara theorem by complex analysis,* Proc. Amer. Math. Soc. **134** (2006), 1107–1116.

[118] G. Kvernadze, *Determination of the jumps of a bounded function by its Fourier series,* J. Approx. Theory **92** (1998), 167–190.

[119] G. Kvernadze, *Approximation of the singularities of a bounded function by the partial sums of its differentiated Fourier series,* Appl. Comput. Harmon. Anal. **11** (2001), 439–454.

[120] G. Kvernadze, T. Hagstrom and H. Shapiro, *Locating discontinuities of a bounded function by the partial sums of its Fourier series,* J. Sci. Comput. **14** (1999), 301–327.

[121] G. Kvernadze, T. Hagstrom and H. Shapiro, *Detecting the singularities of a function of V_p class by its integrated Fourier series,* Comput. Math. Appl. **39** (2000), 25–43.

[122] E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen,* vol.1, second edition, Chelsea Publishing Co., New York, 1953 (in German).

[123] J. Lavoine, O. P. Misra, *Théoremes abeliens pour la transformation de Stieltjes des distributions*, C. R Acad. Sci. Paris Série A **279** (1974), 99–102.

[124] J. Lavoine, O. P. Misra, *Abelian theorems for the distributional Stieltjes transformation,* Math. Proc. Camb. Phil. Soc. **86** (1979), 287–293.

[125] S. Lazzarini, J. M. Gracia-Bondía, *Improved Epstein-Glaser renormalization. II. Lorentz invariant framework,* J. Math. Phys. **44** (2003), 3863–3875.

[126] M. J. Lighthill, *Introduction to Fourier analysis and generalized functions*, Cambridge Univ. Press, London, 1958.

[127] J. E. Littlewood, *The converse of Abel's theorem on power series,* Proc. London Math. Soc. **9** (1911), 434–448.

[128] S. Łojasiewicz, *Sur la valeur et la limite d'une distribution en un point*, Studia Math. **16** (1957), 1–36.

[129] S. Łojasiewicz, *Sur la fixation des variables dans une distribution,* Studia Math. **17** (1958), 1–64.

[130] L. H. Loomis, *The converse of the Fatou theorem for positive harmonic functions,* Trans. Amer. Math. Soc. **53** (1943), 239–250.

[131] F. Lukács, *Über die Bestimmung des Sprunges einer Funktion aus ihrer Fourierreihe,* J. Reine Angew. Math. **150** (1920), 107–112 (in German).

[132] J. Lützen, *The Prehistory of the Theory of Distributions,* Springer-Verlag, New York, 1982.

[133] Y. Maday, S. M. Ould Kaber, E. Tadmor, *Legendre pseudospectral viscosity method for nonlinear conservation laws,* SIAM J. Numer. Anal. **30** (1993), 321–342.

[134] A. Majda, J. McDonough, S. Osher, *The Fourier method for nonsmooth initial data,* Math. Comp. **32** (1978), 1041–1081.

[135] J. Marcinkiewicz, *Sur les séries de Fourier,* Fundamenta Mathematicae **27** (1936), 38–69.

[136] V. Marić, *Regular variation and differential equations,* Lecture Notes in Mathematics 1726, Springer-Verlag, Berlin, 2000.

[137] Y. Meyer, *Wavelets and Operators*, Cambridge Univ. Press, Cambridge, 1992.

[138] Y. Meyer, *Wavelets, vibrations and scalings*, CRM Monograph series 9, American Mathematical Society, Providence, 1998.

[139] O. P. Misra, J. Lavoine, *Transform Analysis of Generalized Functions,* North-Holland, Amsterdam, 1986.

[140] F. Móricz, *Determination of jumps in terms of Abel-Poisson means,* Acta. Math. Hungar. **98** (2003), 259–262.

[141] F. Móricz, *Ferenc Lukács type theorems in terms of the Abel-Poisson means of conjugate series,* Proc. Amer. Math. Soc. **131** (2003), 1243–1250.

[142] F. Móricz, *Determination of jumps in terms of the Abel-Poisson mean of double conjugate series,* Acta Sci. Math. (Szeged) **69** (2003), 677–686.

[143] F. Móricz, *Fejér type theorems for Fourier-Stieltjes series,* Anal. Math. **30** (2004), 123–136.

[144] M. Morimoto, *An introduction to Sato's hyperfunctions,* Translations of Mathematical Monographs 129, American Mathematical Society, Providence, 1993.

[145] S. Moritoh, *Wavelet transforms in Euclidean spaces—their relation with wave front sets and Besov, Triebel-Lizorkin spaces,* Tohoku Math. J. **47** (1995), 555–565.

[146] M. Oberguggenberger, *Multiplication of distributions and applications to partial differential equations,* Pitman Research Notes in Mathematics Series, 259, Longman Sci. Tech., Essex, 1992.

[147] M. Orton, *Hilbert transforms, Plemelj relations, and Fourier transforms,* SIAM J. Math. Anal. **4** (1973), 656–670.

[148] J. C. Oxtoby, *Measure and Category,* second edition, Graduate Texts in Mathematics, vol.2 , Springer-Verlag, NY, 1980.

[149] J. Peetre, *On the value of a distribution at a point,* Portugal Math. **27** (1968), 149–159.

[150] S. Pilipović, *On the quasiasymptotic of Schwartz distributions*, Math. Nachr. **137** (1988), 19–25.

[151] S. Pilipović, *Some properties of the quasiasymptotic of Schwartz distributions. I. Quasiasymptotic at* $\pm\infty$*,* Publ. Inst. Math. (Beograd) **43** (1988), 125–130.

[152] S. Pilipović, *Some properties of the quasiasymptotic of Schwartz distributions, Part II-Quasiasymptotic at 0,* Publ. Inst. Math. (Beograd) **43** (1988), 131–135.

[153] S. Pilipović, *On the behavior of a distribution at the origin,* Math. Nachr. **141** (1989), 27–32.

[154] S. Pilipović, *Quasiasymptotics and* S*-asymptotics in* $\mathcal{S}'$ *and* $\mathcal{D}'$, Publ. Inst. Math. (Beograd) **72** (1995), 13–20.

[155] S. Pilipović, B. Stanković, S*-asymptotic of a distribution,* Pliska Stud. Math. Bulgar. **10** (1989), 147–156.

[156] S. Pilipović, B. Stanković, *Structural theorems for the* S*-asymptotic and quasiasymptotic of distributions,* Math. Pannon. **4** (1993), 23–35.

[157] S. Pilipović, B. Stanković, *Wiener Tauberian theorems for distributions*, J. London Math. Soc. **47** (1993), 507–515.

[158] S. Pilipović, B. Stanković, *Quasi-asymptotics and* S*-asymptotics of ultradistributions,* Bull. Acad. Serbe Sci. Arts Cl. Sci. Math. Natur. **20** (1995), 61–74.

[159] S. Pilipović, B. Stanković, *Tauberian Theorems for Integral Transforms of Distributions,* Acta Math. Hungar. **74** (1997), 135–153.

[160] S. Pilipović, B. Stanković, A. Takači, *Asymptotic Behaviour and Stieltjes Transformation of Distributions*, Teubner-Texte zuer Mathematik, Leipzig, 1990.

[161] S. Pilipović, M. Stojanović, *Quasiasymptotic analysis in Colombeau algebra,* J. Math. Anal. Appl. **302** (2005), 97–128.

[162] S. Pilipović, A. Takači, N. Teofanov, *Wavelets and quasiasymptotics at a point,* J. Approx. Theory **97** (1999), 40–52.

[163] S. Pilipović, N. Teofanov, *Multiresolution expansion, approximation order and quasiasymptotic behavior of tempered distributions*, J. Math. Anal. Appl. **331** (2007), 455–471.

[164] M. A. Pinsky, *Pointwise Fourier inversion in several variables,* Notices Amer. Math. Soc. **42** (1995), 330–334.

[165] M. A. Pinsky, *Inverse problems in Fourier analysis and summability theory,* Methods Appl. Anal. **11** (2004), 317–330.

[166] M. Plancherel, *Sur la convergence et sur la sommation par les moyennes de Cesàro de* $lim_{z\to\infty}\int_a^z f(x)cosxy\mathrm{d}x$, Math. Annalen **76** (1915), 315–326.

[167] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps,* Springer-Verlag, Berlin, 1992.

[168] C.T. Rajagopal, *Note on some Tauberian theorems of O. Szász,* Pacific J. Math. **2** (1952), 377–384.

[169] D. Rakić, *Quasiasymptotic in* $\mathcal{D}'_{L_q}(\mathbb{R}^n)$, Novi Sad J. Math. **37** (2007), 259–267.

[170] S. Ramanujan, *Notebooks*, vol.2, Tata Institute of Fundamental Research, Bombay, 1957.

[171] A. Rényi, *On a Tauberian theorem of O. Szász,* Acta Univ. Szeged. Sect. Sci. Math. **11** (1946), 119–123.

[172] M. Riesz, *Une méthode de sommation équivalente à la méthode des moyennes arithmétiques,* C. R. Acad. Sci. Paris (1911), 1651–1654.

[173] S. Saks, *Theory of the Integral,* second edition, Dover, New York, 1964.

[174] K. Saneva, A. Bučkovska, *Asymptotic behaviour of the distributional wavelet transform at 0,* Math. Balkanica (N.S.) **18** (2004), 437–441.

[175] K. Saneva, A. Bučkovska, *Asymptotic expansion of distributional wavelet transform,* Integral Transforms Spec. Funct. **17** (2006), 85–91.

[176] K. Saneva, A. Bučkovska, *Tauberian theorems for distributional wavelet transform,* Integral Transforms Spec. Funct. **18** (2007), 359–368.

[177] K. Saneva, *Asymptotic behaviour of wavelet coefficients,* Integral Transforms Spec. Funct. **20** (2008), 333–339.

[178] G. Scharf, *Finite Quantum Electromechanics: the Causal Approach,* second edition, Springer, Berlin, 1995.

[179] L. Schwartz, *Sur l'impossibilit de la multiplication des distributions,* C. R. Acad. Sci. Paris **239** (1954), 847–848.

[180] L. Schwartz, *Théorie des Distributions*, Hermann, Paris, 1966.

[181] J. Sebastião e Silva, *Integrals and order of distributions,* Proc. Internat. Summer Inst., Lisbon, pp. 327–390, Inst. Gubelkijan de Ciencia, Lisboa 35, 1964.

[182] J. Sebastião e Silva, *Les séries de multipoles des physiciens et la théorie des ultradistributions*, Math. Ann. **174** (1967), 109–142.

[183] E. Seneta, *Regularly Varying Functions,* Lecture Notes in Mathematics, 598, Springer Verlag, Berlin, 1976.

[184] V. L. Shapiro, *Fourier series in several variables,* Bull. Amer. Math. Soc. **70** (1964), 48–93.

[185] V. M. Shelkovich, *Associated and quasi associated homogeneous distributions (generalized functions),* J. Math. Anal. Appl. **338** (2008), 48–70.

[186] Q. Shi, X. Shi, *Determination of jumps in terms of spectral data,* Acta Math. Hungar. **110** (2006), 193–206.

[187] P. Sjölin, *Convergence of generalized conjugate partial Fourier integrals,* Math. Z. **256** (2007), 265–278.

[188] B. K. Sohn, D. H. Pahk, *Pointwise convergence of wavelet expansion of* $\mathcal{K}_r^{M'}(\mathbb{R})$*,* Bull. Korean Math. Soc. **38** (2001), 81–91.

[189] K. Soni, R. Soni, *Slowly Varying Functions and Asymptotic Behavior of a Class of Integral Transforms I,* J. Math. Appl. Anal. **9** (1975), 166–174.

[190] K. Soni, R. Soni, *Slowly Varying Functions and Asymptotic Behavior of a Class of Integral Transforms II,* J. Math. Appl. Anal. **9** (1975), 477–495.

[191] K. Soni, R. Soni,*Slowly Varying Functions and Asymptotic Behavior of a Class of Integral Transforms III,* J. Math. Appl. Anal. **9** (1975), 612–628.

[192] B. Stanković, *Structural problems of the asymptotic behaviour of generalized functions,* in: Generalized functions and their applications (Varanasi, 1991), 251–263, Plenum, New York, 1993.

[193] B. Stanković, *Asymptotics of regular convolution quotiens,* Int. J. Math. Math. Sci. **1** (1992), 631-634.

[194] B. Stanković, *Fourier hyperfunctions having the S-asymptotics,* Bull. Cl. Sci. Math. Nat. Sci. Math. **24** (1999), 67–75.

[195] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, New Jersey, 1970.

[196] H. Steinhaus, *Sur les distances des points de measure positive,* Fund. Math. **1**, 93–104.

[197] R. S. Strichartz, *A guide to distribution theory and Fourier transforms,* World Scientific Publ., Singapore, 2003.

[198] O. Szász, *Verallgemeinerung eines Littlewoodschen Satzes über Potenzreihen,* J. London Math. Soc. **3** (1928), 254–262 (in German).

[199] O. Szász, *Generalization of two theorems of Hardy and Littlewood on power series,* Duke Math. J. **1** (1935), 105–111.

[200] O. Szász, *Converse theorems of summability for Dirichlet's series,* Trans. Amer. Math. Soc. **39** (1936), 117–130.

[201] O. Szàsz, *On a Tauberian theorem for Abel summability,* Pacific J. Math. **1** (1951), 117–125.

[202] A. Takači, *On the equivalence of distributions at infinity,* Univ. u Novom Sadu. Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. **15** (1) (1985), 175–187.

[203] A. Takači, *On Sebastiao e Silva's order of growth of distributions,* Univ. u Novom Sadu. Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. **20** (1) (1990), 155–166.

[204] A. Tauber, *Ein Satz aus der Theorie der unendlichen Reihen,* Monatsh. Math. Phys. **8** (1897), 273–277 (in German).

[205] N. Teofanov, *Convergence of multiresolution expansions in the Schwartz class,* Math. Balkanica (N.S.) **20** (2006), 101–111.

[206] E. C. Titchmarsh, *Introduction to the theory of Fourier Integrals,* second edition, Clarendon Press, Oxford, 1948.

[207] E. C. Titchmarsh, *The Theory of Functions,* second edition, Oxford University Press, Oxford, 1979.

[208] F. Trèves, *Topological Vector Spaces, Distributions and Kernel*, Academic Press, New York, 1967.

[209] H. Triebel, *Wavelet frames for distributions; local and pointwise regularity,* Studia Math. **154** (2003), 59–88.

[210] Ch. J. de la Vallée Poussin, *Sur l'approximation des fonctions d'une variable réelle et leurs dérivées par les polynômes et les suites limitées de Fourier*, Bull. de l'Acd. Royale de Belgique (1908), 193–254.

[211] H. Vandeven, *Family of spectral filters for discontinuous problems,* J. Sci. Comput. **6** (1991), 159–192.

[212] J. Vindas, *Structural theorems for quasiasymptotics of distributions at infinity*, Pub. Inst. Math. (Beograd) (N.S.) **84**(98) (2008), 159–174.

[213] J. Vindas, *The structure of quasiasymptotics of Schwartz distributions*, in: Proceedings of the International Conference on Generalized Function GF 07 (Bedlewo, Poland), in press. To be published by Banach Center Publications.

[214] J. Vindas, R. Estrada, *Global behavior of integral transforms*, Journal of Inequalities and Applications, vol. 2006, Article ID 61980, 2006.

[215] J. Vindas, R. Estrada, *Distributionally regulated functions*, Studia Math. **181** (2007), 211–236.

[216] J. Vindas, R. Estrada, *Distributional Point Values and Convergence of Fourier Series and Integrals*, J. Fourier Anal. Appl. **13** (2007), 551–576.

[217] J. Vindas, R. Estrada, *A tauberian theorem for distributional point values*, Arch. Math. (Basel) **91** (2008), 247–253.

[218] J. Vindas, R. Estrada, *On the jump behavior of distributions and logarithmic averages*, J. Math. Anal. Appl. **347** (2008), 597–606.

[219] J. Vindas, R. Estrada, *Measures and the distributional ϕ-transform*, Integral Transforms Spec. Funct. **20** (2009), 325–332.

[220] J. Vindas, R. Estrada, *A quick distributional way to the prime number theorem*, Indag. Math. (N.S.), in press.

[221] J. Vindas, R. Estrada, *On the support of tempered distributions*, Proc. Edinb. Math. Soc., in press.

[222] J. Vindas, R. Estrada, *Determination of jumps of distributions by differentiated means*, Acta Math. Hungar., in press

[223] J. Vindas, R. Estrada, *On the order of summability of the Fourier inversion formula* , submitted.

[224] J. Vindas, R. Estrada, *On the Cesàro behavior and some asymptotic notions for Schwartz distributions*, preprint.

[225] J. Vindas, R. Estrada, *Local boundary behavior of harmonic and analytic functions I: Abelian theorems*, preprint.

[226] J. Vindas, R. Estrada, *On Tauber's second tauberian theorem*, preprint.

[227] J. Vindas, S. Pilipović, *Structural theorems for quasiasymptotics of distributions at the origin*, Math. Nachr., in press.

[228] J. Vindas, S. Pilipović, D. Rakić, *Tauberian theorems for the wavelet transform*, submitted.

[229] V. S. Vladimirov, *A multidimensional generalization of the Tauberian theorem of Hardy and Littlewood,* Math. USSR-Izv. **10** (1976), 1031–1048.

[230] V. S. Vladimirov, *Methods of the Theory of Generalized Functions,* Taylor & Francis, London, 2002.

[231] V. S. Vladimirov, Y. N. Drozhzhinov, B. I. Zavialov, *Tauberian theorems for generalized functions,* Kluwer Academic Publishers Group, Dordrecht, 1988.

[232] V. S. Vladimirov, Y. N. Drozhzhinov, B. I. Zavialov, *Tauberian theorems for generalized functions in a scale of regularly varying functions and functionals,* (Dédié à Jovan Karamata) Publ. Inst. Math. (Beograd) (N.S.) **71** (2002), 123–132 (in Russian).

[233] V. S. Vladimirov, B. I. Zavialov, *Tauberian Theorems in Quantum Field Theory*, Theoret. Mat. Fiz. **40** (1979), 155–178.

[234] V. S. Vladimirov, B. I. Zavialov, *Tauberian Theorems in Quantum Field Theory*, in: Current Problems in Mathematics, vol.15 (in Russian), pp.95–130, 228, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1980.

[235] P. Wagner, *On the quasi-asymptotic expansion of the casual fundamental solution of hyperbolic operators and systems*, Z. Anal. Anwendungen **10** (1991), 159–167.

[236] G. Walter, *Pointwise convergence of distribution expansions,* Studia Math. **26** (1966), 143-154.

[237] G. Walter, *Fourier series and analytic representation of distributions*, SIAM Review **12** (1970), 272–276.

[238] G. Walter, *Local boundary behavior of harmonic functions*, Illinois J. Math. **16** (1972), 491–501.

[239] G. Walter, *Sampling bandlimited functions of polynomial growth,* SIAM J. Math. Anal. **19** (1988), 1198–1203.

[240] G. Walter, *Abel summability for a distribution sampling theorem,* in: Generalized Functions, Convergence Structures, and Their Applications, edited by Stanković, B., Pap, E., Pilipović, S., and Vladimirov, V.S., Plenum Press, New York, 1988, 349–357.

[241] G. Walter, *Pointwise convergence of wavelet expansions,* J. Approx. Theory **80** (1995), 108–118.

[242] G. Walter, X. Shen, *Wavelets and other orthogonal systems,* second edition, Studies in Advanced Mathematics, Chapman & Hall/CRC, Boca Raton, 2001.

[243] R. Wawak, *On the value of a distribution at a point,* in: Generalized Functions, Convergence Structures, and Their Applications, edited by Stanković, B., Pap, E., Pilipović, S., and Vladimirov, V.S., Plenum Press, New York, 1988, 359–364.

[244] M. Wei, A. G. Martínez, A. R. De Pierro, *Detection of edges from spectral data: new results,* Appl. Comput. Harmon. Anal. **22** (2007), 386–393.

[245] D. V. Widder, *The Laplace transform,* Princeton Univ. Press, Princeton, 1941.

[246] N. Wiener, *Tauberian theorems,* Ann. of Math. **33** (1932), 1–100.

[247] T. Xin, H. Wong, *A delta function model of facets,* Surface Science Letters, **487** (2002), 529–533.

[248] D. Yu, P. Zhou and S. P. Zhou, *On determination of jumps in terms of Abel-Poisson mean of Fourier series,* J. Math. Anal. Appl. **341** (2008), 12–23.

[249] B. I. Zavialov, *Automodel asymptotic of electromagnetic form factors and the behavior of their Fourier transforms*, Teoret. Mat. Fiz. **17** (1973), 178–188.

[250] B. I. Zav'yalov, *Asymptotic properties of functions that are holomorphic in tubular cones*, Math. USSR-Sb. **64** (1989), 97–113.

[251] A. H. Zemanian, *Generalized Integral Transforms*, Interscience, New York, 1965.

[252] A. H. Zemanian, *Distributions Theory and Transform Analysis,* Dover Publications Inc., New York, 1987.

[253] P. Zhou and S. P. Zhou, *More on determination of jumps,* Acta Math. Hungar. **118** (2008), 41–52.

[254] Z. Zieleźny, *Über die Mengen der regulären und singulären Punkte einer Distribution,* Studia Math. **19** (1960), 27–52 (in German).

[255] A. Zygmund, *Sur un théorème de M. Gronwall,* Bull. Acad. Polon. (1925), 207–217.

[256] A. Zygmund, *Trigonometric Series*, vols. I, II, second edition, Cambridge University Press, Cambridge, 1959.

Vita

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