ON THE JUMP BEHAVIOR OF DISTRIBUTIONS AND LOGARITHMIC AVERAGES

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ABSTRACT. The jump behavior and symmetric jump behavior of distributions are studied. We give several formulas for the jump of distributions in terms of logarithmic averages, this is done in terms of Cesàro-logarithmic means of decompositions of the Fourier transform and in terms of logarithmic radial and angular local asymptotic behaviors of harmonic conjugate functions. Application to Fourier series are analyzed. In particular, we give formulas for jumps of periodic distributions in terms of Cesàro-Riesz logarithmic means and Abel-Poisson logarithmic means of conjugate Fourier series.

1. INTRODUCTION

In this article we study several notions for jumps of distributions by using logarithmic averages. In the case that f is an ordinary function this is a classical subject, perhaps the place where this idea has been widely applied is in Fourier series. Let f be a function of period 2π having Fourier series,

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)\ . \tag{1.1}$$

Let

$$\sum_{n=1}^{\infty}\left(a_n\sin nx-b_n\cos nx\right) \tag{1.2}$$

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be its conjugate series. A classical theorem of F. Lukács [19], [35, Thm. 8.13] states that if f is $L^1[-\pi,\pi]$ and there is a number d such that

$$\lim_{h\to 0^+}\frac{1}{h}\int_0^h |f(x_0+t)-f(x_0-t)-d|\,\mathrm{d}t=0\ , \tag{1.3}$$

then

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}(a_n\sin nx_0-b_n\cos nx_0)=-\frac{d}{\pi}\ . \tag{1.4}$$

Relation (1.3) can be considered as a notion of jump at $x=x_0$ for the function f, we shall call it *symmetric Lebesgue jump behavior*, in analogy with the notion of Lebesgue point. The formula (1.4) for symmetric Lebesgue jump behaviors was extended later by A. Zygmund to the Abel-Poisson means of the conjugate Fourier series [35].

Recently many extensions of these results have been given [1, 7, 13, 14, 15, 16, 17, 20, 21, 34]. The study of the jump behavior and the determination of jumps by logarithmic or other types of means has become an important area because of its applications in edge detection [11, 12]. F. Móricz generalizes the Abel-Poisson version of F. Lukács result in [20, 21] by extending the notion of symmetric Lebesgue jump (1.3). He considered a more general notion for jump of integrable functions, namely, the existence of the limit

$$d=\lim_{h\to 0^+}\frac{1}{h}\int_0^h (f(x_0+t)-f(x_0-t))\,\mathrm{d}t\ , \tag{1.5}$$

and he showed that

$$\lim_{r\to 1^-}\frac{1}{\log(1-r)}\sum_{n=1}^{\infty}(a_n\sin nx_0-b_n\cos nx_0)r^n=\frac{1}{\pi}\,d\ . \tag{1.6}$$

It was noticed by one of the authors in [7] that the jump F. Móricz considered is a particular case of a symmetric jump behavior in the sense of distributions, that is, one can define it in terms of the very well known Łojasiewicz notion of limits of distributions at points [18]. Because of that reason, we should call (1.5) a *first order symmetric jump.* In the cited paper the author gave the corresponding generalization of F. Móricz result to distributions in terms of logarithmic Abel-Poisson means as well.

We will consider in this article two notions of jumps for distributions, the *distributional jump behavior* (Section 2) and the *distributional symmetric jump behavior* of distributions (Section 6). We will consider several logarithmic averages for both notions. In Section 2, we will give formulas for the jump occurring in the jump behavior case in terms of Cesàro-logarithmic means of a decomposition of the Fourier transform; it is remarkable that these results are applicable to general tempered distributions. Next, in Section 3 we study the boundary behavior of analytic representations of distributions at approaching angularly from the upper and lower semiplane to a point where the distribution possesses a jump behavior; it is shown they have an asymptotic logarithmic behavior related to the jump. Then, in the same section, we analyze harmonic conjugate functions in the upper semiplane having distributional boundary values on the real axis; it turns out that they have also a logarithmic angular asymptotic behavior related to the jump. In Section 4, we make some comments about the size of the set where a distribution can have a distributional jump behavior; in fact, it is seen that this set is at most countable. Section 5 is devoted to applications to Fourier series, we give formulas for the jump in terms of logarithmic averages by using Cesàro-Riesz means and Abel-Poisson means of the conjugate series; among our results, we recover (1.6) and a Cesàro version of (1.4). The last section of this article is dedicated to study some properties of the symmetric jump behavior of distributions, this notion is much more general than the jumps in the sense of (1.3) and (1.5); furthermore, we discuss the case of Fourier series of periodic distributions, generalizing the mentioned results from [19, 35, 20, 21, 7].

2. Jump behavior and logarithmic averages in Cesàro sense

The Schwartz spaces of test functions and distributions on the real line $\mathbb{R}$ are denoted by $\mathcal{D}$ and $\mathcal{D}'$, respectively; the spaces of rapidly decreasing functions and its dual, the space of tempered distributions, are denoted by $\mathcal{S}$ and $\mathcal{S}'$. We refer the reader to [23, 9, 32] for properties of these spaces.

In this section, we shall deal with tempered distributions having a jump at a point and study the logarithmic average in the Cesàro sense of the Fourier transform. We first consider the definition of *jump behavior* of distributions at points [29, 30].

Definition 2.1. A distribution $f \in \mathcal{D}'$ is said to have a *distributional jump behavior* (or *jump behavior*) at $x = x_0 \in \mathbb{R}$ if it satisfies the following distributional asymptotic relation

$$f\left(x_0 + \epsilon x\right) = \gamma_- H(-x) + \gamma_+ H(x) + o(1) \text{ as } \epsilon \to 0^+ \text{ in } \mathcal{D}', \tag{2.1}$$

where H is the Heaviside function, i.e., the characteristic function of $(0,\infty)$, and $\gamma_{\pm}$ are constants. The *jump* (or *saltus*) of f at $x = x_0$ is defined then as the number $[f]_{x=x_0} = \gamma_+ - \gamma_-$.

The meaning of (2.1) is in the weak topology of $\mathcal{D}'$, in the sense that for each $\phi \in \mathcal{D}$,

$$\lim_{\epsilon\to 0^+} \langle f(x_0+\epsilon x), \phi(x)\rangle = \gamma_- \int_{-\infty}^{0} \phi(x)\,\mathrm{d}x + \gamma_+ \int_0^{\infty} \phi(x)\,\mathrm{d}x \ . \tag{2.2}$$

When $\gamma_- = \gamma_+$, we recover the usual notion of the value of a distribution at a point in the sense of Łojasiewicz [18]. Furthermore, the notion of jump behavior belongs to a more general type of asymptotic behaviors of distributions, the so-called *quasiasymptotic behaviors* at points [22, 31, 32]; therefore, from the results of [31], a structural characterization of the jump behavior of distributions can be given explicitly, that is, a distribution $f \in \mathcal{D}'$ has the jump behavior (2.1) if and only if there exist $m \in \mathbb{N}$ and a function F locally integrable on a neighborhood of x_0 such that $F^{(m)} = f$ near x_0 and

$$\lim_{x\to x_0^{\pm}} \frac{m!F(x)}{(x-x_0)^m} = \gamma_{\pm} \ . \tag{2.3}$$

The minimum m such that we can find an F satisfying (2.3) is called *the order of the jump behavior.* Obviously, if a locally integrable function has right and left limts at $x = x_0$, then it has a distributional jump behavior of order 0.

We now discuss two examples of particular types of jump behavior related to classical functions.

Example 2.2. *Lebesgue jumps.* Let f be a locally (Lebesgue) integrable function, then we say that f has a Lebesgue jump behavior if there are two numbers $\gamma_{\pm}$ such that

$$\lim_{h\to 0^{\pm}} \frac{1}{h} \int_{x_0}^{x_0+h} |f(x) - \gamma_{\pm}|\,\mathrm{d}x = 0 \ . \tag{2.4}$$

Example 2.3. *Jump behavior of the first order.* Let μ be a (regular) Borel measure. Then a jump behavior of the first order is nothing else than the existence of $\gamma_{\pm}$ such that

$$\lim_{h\to 0^{\pm}} \frac{1}{h} \int_{x_0}^{x_0+h} \mathrm{d}\mu(x) = \gamma_{\pm} \ . \tag{2.5}$$

A particular case is obtained if $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. Moreover, the first order jump behaviors can still be defined by an integral expression even if f is not locally (Lebesgue) integrable but just Denjoy locally integrable.

In such a case again the existence of the jump behavior of the first order is equivalent to the existence of the limits

$$\lim_{h\to 0^{\pm}} \frac{1}{h}\int_{x_0}^{x_0+h} f(x)\,\mathrm{d}x = \gamma_{\pm}\ , \tag{2.6}$$

where the last integral is taken in the Denjoy sense.

Example 2.4. It is worth to provide the reader with an example of jump behavior which is not included in last two cases. Consider the function

$$f(x) = \left(\gamma_- + A\,|\,x|^{\alpha}\, e^{i/x^{\beta}}\right) H(-x) + \left(\gamma_+ + Bx^{\alpha}e^{i/x^{\beta}}\right) H(x)\ . \tag{2.7}$$

For any choice of the constants, one can show that there is a tempered distribution having the distributional jump behavior (2.1) at $x = 0$ and coinciding with f on $\mathbb{R}\backslash\{0\}$ [18]. Observe that depending on the choice of the constants α and β the function is not a function of local bounded variation. In addition, the choice of the constants can be made so that f is not locally Denjoy integrable. One may also find values for α and β such that the order of the jump behavior is arbitrarily large [18].

Suppose now that $f \in \mathcal{S}'$. If we write $\mathcal{S}'$ instead of $\mathcal{D}'$ in Definition 2.1, we mean that (2.2) holds for all $\phi \in \mathcal{S}'$; in such a case we say that f has jump behavior in $\mathcal{S}'$ at $x = x_0$. It is well-known that if a tempered distribution has jump behavior (in $\mathcal{D}'$) at a point, then it will have the same jump behavior in $\mathcal{S}'$ at the given point [7]; actually, this property is true for any quasiasymptotic behavior at a point [31].

We shall deal in the rest of this section only with tempered distributions, since we want to make use of the Fourier transform. On test functions, we use the following Fourier transform

$$\hat{\phi}(x) = \int_{-\infty}^{\infty} \phi(t)e^{-ixt}\mathrm{d}t\ , \tag{2.8}$$

and as usual we extend it to $\mathcal{S}'$ by transposition.

Suppose then that $f \in \mathcal{S}'$ satisfies (2.1). Hence, since it holds in the weak sense, we are allowed to take Fourier transform in (2.1), so that it transforms into the equivalent asymptotic relation

$$e^{i\lambda x_0 x}\hat{f}(\lambda x) = 2\pi d_1\frac{\delta(x)}{\lambda} - id_2\mathrm{p.v.}\left(\frac{1}{\lambda x}\right) + o\left(\frac{1}{\lambda}\right)\ , \tag{2.9}$$

as $\lambda \to \infty$ in $\mathcal{S}'$, where $d_1 = (\gamma_- + \gamma_+)/2$, $d_2 = [f]_{x=x_0} = \gamma_+ - \gamma_-$, δ is the usual Dirac delta function and p.v.$(1/x)$ is the principal value distribution given by

$$\left\langle \mathrm{p.v.}\left(\frac{1}{x}\right), \phi(x) \right\rangle = \mathrm{p.v.}\int_{-\infty}^{\infty} \frac{\phi(x)}{x}\,\mathrm{d}x \ , \tag{2.10}$$

where p.v. stands for the Cauchy principal value of the integral. Notice that we have used here the formula $\widehat{H}(x) = \pi\delta(x) - i\,\mathrm{p.v.}\,(1/x)$. Needless to say that (2.9) is interpreted distributionally, i.e., the asymptotic formula holds after evaluation at test functions.

Therefore, if we want to study (2.1) is enough to study (2.9). The asymptotic relation (2.9) has been characterized by the authors in [29, 30]. The structural characterization given in [30] states that f satisfies (2.9) if and only if for any primitive of $e^{ix_0x}\hat{f}(x)$, say F, one has that there is a $k \in \mathbb{N}$ such that

$$\lim_{x\to\infty}\left(F(ax) - F(-x)\right) = 2\pi d_1 - i d_2 \log a \quad (\mathrm{C}, k) \ , \tag{2.11}$$

for each $a > 0$. Here (C, k) stands for limit in the Cesàro sense of distributions [6, 9, 30].

Example 2.5. When f is a Fourier series, i.e., $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$, where $\{c_n\}$ is a sequence of polynomial growth at infinity, one obtains that f has the jump behavior (2.1) at $x = x_0$ if and only if there is a $k \in \mathbb{N}$ such that for each $a > 0$

$$\lim_{x\to\infty} \sum_{-x<n\leq ax} c_n e^{inx_0} = d_1 + \frac{d_2}{2\pi i}\log a \quad (\mathrm{C}, k) \ . \tag{2.12}$$

In order to make further progress, we shall use in this section a different version of the characterization (2.11) of (2.9); however, this version is easily seen to be equivalent to (2.11) [27, 28]. In order to formulate the result, we first need some preliminaries.

Definition 2.6. A measurable function defined in some interval of the form $[A, \infty)$, $A \geq 0$, is said to be an asymptotically homogeneous function of degree 0 at infinity [4, 27] if for each $a > 0$,

$$\sigma(ax) = \sigma(x) + o(1) \quad \text{as } x \to \infty \ . \tag{2.13}$$

We remark that the real and imaginary parts of every asymptotically homogeneous function of degree 0 are the logarithm of slowly varying functions in the sense of Karamata; hence we are free to use the very well known properties of slowly varying functions [24], for instance (2.13) must hold uniformly for a on compact subsets of $(0, \infty)$ and $\sigma(x) = o(\log x)$ as $x \to \infty$.

Finally, let $l_k(x)$ be the k-primitive of $\log|x|$ satisfying the requirements $l_k^{(j)}(0)=0$ for $j<k$. Observe that it satisfies

$$l_k(ax)=a^k l_k(x)+\frac{(ax)^k}{k!}\log a\ , \quad a>0\ . \tag{2.14}$$

After all these preliminaries, let us state the mentioned structural theorem from [29, 30] (see also [27, 28]).

Theorem 2.7. *Let $g\in\mathcal{S}'$ have the following quasiasymptotic behavior in $\mathcal{S}'$*

$$g(\lambda x)=\gamma\frac{\delta(x)}{\lambda}+\beta\,\mathrm{p.v.}\left(\frac{1}{\lambda x}\right)+o\left(\frac{1}{\lambda}\right)\quad \textit{as } \lambda\to\infty\ . \tag{2.15}$$

Then, one can find a $k\in\mathbb{N}$, a continuous function G such that $G^{(k+1)}=g$, and an asymptotically homogeneous function σ of degree 0 such that,

$$G(x)=\sigma\left(|x|\right)\frac{x^k}{k!}+\frac{\gamma}{2}x^k\operatorname{sgn}x+\beta l_k(x)+o\left(|x|^k\right)\ \textit{as } |x|\to\infty\ , \tag{2.16}$$

in the ordinary sense. Conversely (2.16) implies (2.15).

We are now ready to state and show the main theorem of this section. It will enable us to study the logarithmic average behavior of $e^{ix_0}\hat{f}(x)$ separately for any decomposition as the sum of two tempered distributions having supports in $(-\infty,0]$ and $[0,\infty)$, respectively.

Theorem 2.8. *Let g have the quasiasymptotic behavior (2.15) in $\mathcal{S}'$. Then for any decomposition $g=g_+ + g_-$, where $\operatorname{supp} g_-\subseteq(-\infty,0]$ and $\operatorname{supp} g_+\subseteq[0,\infty)$, one has that*

$$g_\pm(\lambda x)=\pm\beta\,\frac{\log\lambda}{\lambda}\,\delta(x)+o\left(\frac{\log\lambda}{\lambda}\right)\quad \textit{as } \lambda\to\infty\quad \textit{in } \mathcal{S}'. \tag{2.17}$$

Proof. Let k , σ and G be as in Theorem 2.7. Then, to a decomposition $g=g_+ + g_-$, corresponds a decomposition $G=G_+ + G_-$, with $\operatorname{supp} G_-\subseteq(-\infty,0]$ and $\operatorname{supp} G_+\subseteq[0,\infty)$. Hence

$$\begin{aligned} G_\pm(x)&=\beta l_k(x)+o\left(|x|^k\log|x|\right),\ x\to\pm\infty\ , \\ &=\beta\frac{x^k}{k!}\log|x|+o\left(|x|^k\log|x|\right),\ x\to\pm\infty\ , \end{aligned} \tag{2.18}$$

since $\sigma(x) = o(\log x)$ as $x \to \infty$. This implies the distributional relations

$$\begin{aligned} G_{\pm}(\lambda x) &= \beta l_k(\lambda x) H(\pm x) + o\left(\lambda^k \log\lambda\right) \\ &= \beta\lambda^k l_k(x) H(\pm x) + \beta\lambda^k \log\lambda \frac{x^k}{k!}\, H(\pm x) + o\left(\lambda^k \log\lambda\right) \\ &= \beta\lambda^k \log\lambda \frac{x^k}{k!}\, H(\pm x) + o\left(\lambda^k \log\lambda\right) \text{ as } \lambda \to \infty\ , \end{aligned}$$

and the last relation holds in $\mathcal{S}'$. Therefore if we differentiate $(k+1)$-times, we obtain (2.17). □

Notice that (2.18) gives a logarithmic average in the Cesàro sense. We collect this in the next corollary for future reference.

Corollary 2.9. *Let g, g_+, g_-, k, G_- and G_+ be as in the last theorem, then*

$$G_{\pm}(x) \sim \beta \frac{x^k}{k!} \log|x|\,, \quad \text{as } x \to \pm\infty\ . \tag{2.19}$$

We now summarize our results.

Theorem 2.10. *Let $f \in \mathcal{S}'$ have the distributional jump behavior at $x = x_0$,*

$$f\left(x_0 + \epsilon x\right) = \gamma_- H(-x) + \gamma_+ H(x) + o(1) \quad \text{as } \epsilon \to 0^+ \text{ in } \mathcal{D}'. \tag{2.20}$$

Then for any decomposition $\hat{f} = \hat{f}_+ + \hat{f}_-$, where $\operatorname{supp}\hat{f}_- \subseteq (-\infty, 0]$ *and* $\operatorname{supp}\hat{f}_+ \subseteq [0,\infty)$*, we have that*

$$e^{i\lambda x_0 x} \hat{f}_{\pm}\left(\lambda x\right) = \pm[f]_{x=x_0} \frac{\log\lambda}{i\lambda} \delta(x) + o\left(\frac{\log\lambda}{\lambda}\right) \quad \text{as } \lambda \to \infty \tag{2.21}$$

in $\mathcal{S}'$. Furthermore, there exists $k \in \mathbb{N}$ such that

$$\left(e^{ix_0 t} \hat{f}_{\pm}(t) * t^k_{\pm}\right)(x) \sim \pm[f]_{x=x_0} \frac{|x|^k}{i} \log|x|\,, \quad \text{as } |x| \to \infty\ , \tag{2.22}$$

in the ordinary sense.

A special case is obtained in the next corollary which follows directly from Theorem 2.10.

Corollary 2.11. *Let $f \in \mathcal{S}'$ have the distributional jump behavior (2.20) at $x = x_0$. Suppose that its Fourier transform is given by a Borel measure μ, then there exists $k \in \mathbb{N}$ such that for any decomposition of*

$\mu = \mu_- + \mu_+$, as two Borel measures concentrated on $(-\infty, 0]$ and $[0, \infty)$, respectively,

$$\lim_{x\to\infty} \frac{i}{\log x} \int_0^x e^{\pm i x_0 t} \left(1 - \frac{t}{x}\right)^k \mathrm{d}\mu_\pm(\pm t) = \pm [f]_{x=x_0} \ . \tag{2.23}$$

3. Local asymptotic boundary behavior of analytic and harmonic functions

This section is devoted to the study of the local boundary behavior of analytic and harmonic representations of distributions having a jump behavior. Recall that given $f \in \mathcal{D}'$, we may see f as a hyperfunction, that is $f(x) = F(x+i0) - F(x-i0)$, where F is analytic for $\Im m\, z \neq 0$; moreover, this representation holds distributionally in the sense that

$$f(x) = \lim_{y\to 0^+} \left(F(x+iy) - F(x-iy)\right) \ , \tag{3.1}$$

where the last limit is taken in the weak topology of $\mathcal{D}'$ [3]. In such a case, we say that F is an analytic representation of f. Note that, initially, we are not assuming that $F(x \pm i0)$ belong to $\mathcal{D}'$ separately, but that their difference does; however, it is shown in [5, Section 5] that the existence of the distributional jump of F across the real axis implies the existence of $F(x \pm i0)$, separately, in $\mathcal{D}'$.

In the next theorem we obtain the angular behavior of $F(z)$ when z approaches a point where f has a jump behavior. We remark this is done separately when z approaches angularly the point from the upper and lower semiplanes.

Given $0 < \eta \leq \pi/2$ and $x_0 \in \mathbb{R}$, we define the subset of the upper semiplane $\Delta_\eta^+(x_0)$ as the set of those z such that $\eta \leq \arg(z - x_0) \leq \pi - \eta$, similarly, we define the subset of the lower semiplane $\Delta_\eta^-(x_0)$ as the set of those z such that $\eta - \pi \leq \arg(z - x_0) \leq -\eta$.

Theorem 3.1. *Let $f \in \mathcal{D}'$ have the distributional jump behavior*

$$f\left(x_0 + \epsilon x\right) = \gamma_- H(-x) + \gamma_+ H(x) + o(1) \quad \text{as } \epsilon \to 0^+ \ \text{ in } \mathcal{D}'. \tag{3.2}$$

Suppose that F is an analytic representation of f. Then for any $0 < \eta \leq \pi/2$,

$$\lim_{z\to z_0,\ z\in\Delta_\eta^\pm(x_0)} \frac{F(z)}{\log|z - x_0|} = -\frac{[f]_{x=x_0}}{2\pi i} \ . \tag{3.3}$$

Proof. Note first that if (3.3) holds for one analytic representation, then it holds for any analytic representation of f. In fact by the very well known edge of wedge theorem, any two such analytic representations differ by an entire function, and for entire functions (3.3) gives 0. Next, we see that we may assume that $f \in \mathcal{S}'$. Indeed we can decompose $f = f_1 + f_2$ where f_2 is zero in a neighborhood of x_0 and $f_1 \in \mathcal{S}'$. Let F_1 and F_2 be analytic representations of f_1 and f_2, respectively; then F_2 can be continued across a neighborhood of x_0 (edge of wedge theorem once again), hence $F_2(z) = F_2(x_0) + O\left(|z - x_0|\right) = o\left(|\log|z - x_0||\right)$ as $z \to x_0$. Additionally, f_1 has the same jump behavior as f. Thus, we assume that $f \in \mathcal{S}'$. Let $\hat{f} = \hat{f}_+ + \hat{f}_-$ be a decomposition such that $\operatorname{supp} \hat{f}_- \subseteq (-\infty, 0]$ and $\operatorname{supp} \hat{f}_+ \subseteq [0, \infty)$. Then,

$$F(z) = \begin{cases} \dfrac{1}{2\pi}\left\langle \hat{f}_+(t), e^{izt} \right\rangle, & \Im m\ z > 0\ , \\ -\dfrac{1}{2\pi}\left\langle \hat{f}_-(t), e^{izt} \right\rangle, & \Im m\ z < 0\ , \end{cases}$$

is an analytic representation of f [3, p.83]. Keep the number m on a compact set and $\lambda > 0$, then

$$\begin{aligned} F\left(x_0 + \frac{m}{\lambda}, \frac{\pm 1}{\lambda}\right) &= \pm\frac{1}{2\pi}\left\langle \lambda e^{i\lambda x_0 x} \hat{f}_\pm(\lambda x), e^{i(m+i)x} \right\rangle \\ &= \frac{[f]_{x=x_0}}{2\pi i} \log\lambda + o\left(\log\lambda\right) \end{aligned}$$

as $\lambda \to \infty$, where here we have used (2.21). □

We say that $U(z)$, harmonic on $\Im m\ z > 0$, is a harmonic representation of $f \in \mathcal{D}'$ [3] if

$$\lim_{y \to 0^+} U(x, y) = f(x) \qquad \text{in } \mathcal{D}'. \tag{3.4}$$

The problem of finding the angular behavior of U when approaching a point where f has a jump behavior has been discussed in [7] by studying the Poisson kernel and in [28] by using Fourier transform methods and the structural theorem for quasiasymptotic behaviors at ∞. Indeed, if $f \in \mathcal{D}'$ satisfies (3.2), then

$$\lim_{z \to x_0,\ z \in l_\theta} U(z) = d_1 + \frac{\theta}{\pi}[f]_{x=x_0}\ , \tag{3.5}$$

where l_θ is a ray in the upper semiplane starting at x_0 and making an angle θ with the ray $x = x_0$. As usual $d_1 = (\gamma_+ + \gamma_-)/2$ and $[f]_{x=x_0} = \gamma_+ - \gamma_-$. Actually (3.5) holds uniformly for $|\theta| < \eta < \pi/2$. The details are available in the literature.

Our next goal is to study the angular behavior of harmonic conjugate functions. This is the content of the next theorem.

Theorem 3.2. *Let $f \in \mathcal{D}'$ have the jump behavior (3.2) and U be a harmonic representation of f in the upper semiplane. Then if V is a* *harmonic conjugate to* U*, one has that*

$$\lim_{z\to x_0,\ z\in\Delta^{+}_{\eta}(x_0)} \frac{V(z)}{\log|z-x_0|} = \frac{1}{\pi}[f]_{x=x_0}\ , \tag{3.6}$$

for each $0<\eta\leq \pi/2$.

Proof. Since harmonic conjugates to U differ by a constant, it is enough to show (3.6) for any particular harmonic conjugate to U.

We now show that we may work with any harmonic representation U of f we want. Suppose that U_1 and U_2 are two harmonic representations of f, then $U=U_1-U_2$ represents the zero distribution. Then by applying the reflection principle to the real and imaginary parts of U [2, Section 4.5], [26, Section 3.4], we have that U admits a harmonic extension to a (complex) neighborhood of x_0. Consequently, if V_1 and V_2 are harmonic conjugates to U_1 and U_2, we have that $V=V_1-V_2$ is harmonic conjugate to U, and thus it admits a harmonic extension to a (complex) neighborhood of x_0 as well. Therefore $V(z)=O(1)=o(-\log|z-x_0|)$, which shows that V_1 satisfies (3.6) if and only if V_2 does.

Let F be an analytic representation of f on $\Im m\ z\neq 0$. We can assume then that $U(z)=F(z)-F(\bar{z})$, $\Im m\ z>0$. We have that $V(z)=-i\left(F(z)+F(\bar{z})\right)$, $\Im m\ z>0$, is a harmonic conjugate to U. Therefore, an application of Theorem 3.1 yields to (3.6). □

Example 3.3. As an example, we discuss our results in the context of the spaces $L^p(\mathbb{R})$ with $1<p<\infty$. Let $f\in L^p(\mathbb{R})$ and assume that it has the distributional jump behavior (3.2). A particular case is obtained when f has a Lebesgue jump (Example 2.2), but we remark that our assumption is much weaker. A harmonic representation of f is given by the Poisson representation, i.e., by integration against the Poisson kernel. Among all the harmonic conjugates to the Poisson representation, the natural choice is

$$V(z)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Re e\ z-t}{|z-t|^2}\,f(t)\,\mathrm{d}t\ . \tag{3.7}$$

As a corollary of Theorem 3.2, we obtain the angular asymptotic behavior of this integral: it is indeed given by (3.6). Note that the harmonic function $V(z)$ has as boundary value a function $\tilde{f}\in L^p(\mathbb{R})$, which in fact is the Hilbert transform of f [8, 25]. The asymptotic behavior of V suggests that $\tilde{f}$ has the following quasiasymptotic behavior at $x=x_0$ in $\mathcal{D}'$,

$$\tilde{f}(x_0+\epsilon x)=\frac{1}{\pi}[f]_{x=x_0}\log\epsilon+o\left(\log\left(\frac{1}{\epsilon}\right)\right)\quad \text{as } \epsilon\to 0^{+}, \tag{3.8}$$

which is actually the case. A proof of the last relation can be given by using the fact that the Fourier transform of $\tilde{f}$ is $-i\left(\hat{f}_+ - \hat{f}_-\right)$, for a suitable decomposition of $\hat{f}$, by using the Theorem 2.10, and then taking Fourier inverse transform. If we work on the circle, i.e., on $L^p(\mathbb{T})$, we obtain similar conclusions for the conjugate function; we will do this in Section 5 but in a more general distributional setting obtaining several logarithmic asymptotic behaviors of the conjugate Fourier series.

4. The set where jump behavior occurs

In [29] the authors have defined the class of distributionally regulated functions, these are functions which correspond to distributions having a jump behavior at every point. We have shown that if $f \in \mathcal{D}'$ admits a jump behavior at every point, then $[f]_{x=x_0} = 0$, except for at most a countable set [29, Thm.8], so one obtains that f has a Łojasiewicz point value except perhaps in a countable set. In the proof of the cited result from [29], we did not use the fact that the distribution had a jump behavior everywhere, so exactly the same arguments can be used to give a proof of the following theorem.

Theorem 4.1. *Let $f \in \mathcal{D}'$. Then the set*

$$\mathfrak{S} = \{x_0 \in \mathbb{R} : f \text{ has jump behavior at } x_0 \text{ and } [f]_{x=x_0} \neq 0\} \tag{4.1}$$

is at most countable.

It is also possible to construct a direct proof based on (3.5) by slightly modifying the arguments given in [29]; however, we choose to omit it since the details are straightforward.

5. Logarithmic averages of Fourier Series

We now apply our results to the Fourier series of 2π-periodic distributions. Suppose that $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$, where the series converges in $\mathcal{S}'$. Assume also that f has the jump behavior (2.1). Then Theorem 2.10 implies at once that there exists $k \in \mathbb{N}$ such that

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{0\le n\le x} c_n e^{inx_0} \left(1 - \frac{n}{x}\right)^k = \frac{[f]_{x=x_0}}{2\pi i}\ , \tag{5.1}$$

and

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{1\le n\le x} c_{-n} e^{-inx_0} \left(1 - \frac{n}{x}\right)^k = -\frac{[f]_{x=x_0}}{2\pi i}\ , \tag{5.2}$$

which gives us a logarithmic average for the Cesàro-Riesz means of these two series.

The conjugate Fourier series is $\tilde{f}(x) = \sum_{n=-\infty}^{\infty} \tilde{c}_n e^{inx}$, where $\tilde{c_0} = 0$ and $\tilde{c}_n = -i \operatorname{sgn} n\, c_n$. It follows from the above relations that it has the quasiasymptotic behavior at x_0,

$$\tilde{f}\,(x_0 + \epsilon x) = \frac{1}{\pi}[f]_{x=x_0} \log \epsilon + o\left(\log\left(\frac{1}{\epsilon}\right)\right) \text{ as } \epsilon \to 0^+ \text{ in } \mathcal{D}'. \tag{5.3}$$

Moreover, since $V(z)$, $\Im m\ z > 0$, given by

$$V(z) = \sum_{n=-\infty}^{-1} \tilde{c}_n e^{i\bar{z}n} + \sum_{n=1}^{\infty} \tilde{c}_n e^{izn}\ , \tag{5.4}$$

is a harmonic conjugate to a harmonic representation of f, one deduces from Theorem 3.2 that for $0 < \eta \leq \frac{\pi}{2}$

$$\lim_{z \to x_0,\ z \in \Delta_\eta^+(x_0)} \frac{1}{\log|z - x_0|}\left(\sum_{n=-\infty}^{-1} \tilde{c}_n e^{i\bar{z}n} + \sum_{n=1}^{\infty} \tilde{c}_n e^{izn}\right) = \frac{1}{\pi}[f]_{x=x_0}\ . \tag{5.5}$$

Hence we obtain the jump as the logarithmic angular average of the harmonic representation of the conjugate series. In particular, if we take $\eta = \pi/2$,

$$\lim_{y \to 0^+} \frac{1}{\log y} \sum_{n=1}^{\infty} \left(\tilde{c}_n e^{ix_0 n} + \tilde{c}_{-n} e^{-ix_0 n}\right) e^{-yn} = \frac{1}{\pi}[f]_{x=x_0}\ . \tag{5.6}$$

If we now use the sines and cosines series for f, i.e.,

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)\ , \tag{5.7}$$

where $a_n = c_n + c_{-n}$, $b_n = i(c_n - c_{-n})$, then $\tilde{c}_n = \frac{1}{2}\left(-b_{|n|} - i \operatorname{sgn} n\, a_{|n|}\right)$ and

$$\tilde{f}(x) = \sum_{n=1}^{\infty} \left(a_n \sin nx - b_n \cos nx\right)\ . \tag{5.8}$$

Therefore (5.6) is equivalent to

$$\lim_{r \to 1^{-1}} \frac{1}{\log(1 - r)} \sum_{n=1}^{\infty} \left(a_n \sin nx_0 - b_n \cos nx_0\right) r^n = \frac{1}{\pi}[f]_{x=x_0}\ , \tag{5.9}$$

which exhibits the jump now as the Abel-Poisson logarithmic means of the conjugate Fourier series. In fact, also using the sines and cosines series expression for the conjugate series and (5.1)-(5.2), one obtains the jump as the logarithmic average of the symmetric partial sums of the conjugate series in the Cesàro-Riesz means

$$\lim_{x\to\infty}\frac{1}{\log x}\sum_{0<n\leq x}(a_n\sin nx_0-b_n\cos nx_0)\left(1-\frac{x}{n}\right)^k=-\frac{1}{\pi}[f]_{x=x_0}\ . \tag{5.10}$$

In the next section, we will obtain (5.9) and (5.10) under weaker assumptions, namely, under a symmetric jump behavior.

6. Symmetric jumps

We conclude this article by analyzing the case when the distribution f has a *distributional symmetric jump behavior* (or *symmetric jump behavior*) at $x=x_0$.

The symmetric jump behavior means that the jump distribution

$$\psi_{x_0}(x)=f(x_0+x)-f(x_0-x)\ , \tag{6.1}$$

has jump behavior at $x=0$. Since ψ_{x_0} is odd one easily concludes that this jump behavior must be of the form

$$\psi_{x_0}(\epsilon x)=d\operatorname{sgn}x+o(1)\quad\text{as }\epsilon\to0^+\text{ in }\mathcal{D}'\ , \tag{6.2}$$

where here $\operatorname{sgn}x$ is the signum function, i.e., $H(x)-H(-x)$. One says that $d=[f]_{x=x_0}$ is the *symmetric jump* of f at $x=x_0$. Note that jump behavior implies symmetric jump behavior, but the converse is not true as shown by $\delta(x)$, which has a symmetric jump 0 at $x=0$ but does not have jump behavior at the origin. Also note that the two notions for ordinary functions mentioned at the introduction, symmetric Lebesgue jump and first order symmetric jump, are special cases of our distributional symmetric jump behavior.

We use our results from Section 2 and Section 3, applied to the jump distribution, to deduce some logarithmic averages in the case of symmetric jump behavior.

Theorem 6.1. *Suppose that $f\in\mathcal{S}'$ has a symmetric jump at $x=x_0$. Then for any decomposition $\hat{f}=\hat{f}_-+\hat{f}_+$, where* $\operatorname{supp}\hat{f}_-\subseteq(-\infty,0]$ *and* $\operatorname{supp}\hat{f}_+\subseteq[0,\infty)$, *we have that*

$$e^{i\lambda x_0x}\hat{f}_+(\lambda x)-e^{-i\lambda x_0x}\hat{f}_-(-\lambda x)=2\,[f]_{x=x_0}\frac{\log\lambda}{i\lambda}\delta(x)+o\left(\frac{\log\lambda}{\lambda}\right) \tag{6.3}$$

as $\lambda\to\infty$ in $\mathcal{S}'$. Consequently, there exists k such that

$$\left(\left(e^{ix_0t}\hat{f}_+(t)-e^{-ix_0t}\hat{f}_-(-t)\right)*t_+^k\right)(x)\sim\frac{2}{i}\,[f]_{x=x_0}\,x^k\log x \tag{6.4}$$

as $x\to\infty$, in the ordinary sense.

Proof. We can apply Theorem 2.10 directly, since $\hat{\psi}_{x_0}(x) = e^{ix_0x}\hat{f}(x) - e^{-ix_0x}\hat{f}(-x)$, and a decomposition $\hat{f} = \hat{f}_- + \hat{f}_+$ leads to the decomposition

$$\hat{\psi}_{x_0}(x) = \left(e^{ix_0x}\hat{f}_+(x) - e^{-ix_0x}\hat{f}_-(-x)\right) + \left(e^{ix_ox}\hat{f}_-(x) - e^{-ix_0x}\hat{f}_+(-x)\right).$$

□

We now obtain the announced Cesàro-Riesz logarithmic version of F. Lukács Theorem.

Corollary 6.2. *Let $f \in \mathcal{S}'$ be a 2π-periodic distribution having the following Fourier series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \ . \tag{6.5}$$

If f has a symmetric jump behavior at $x = x_0$, then there is a $k \in \mathbb{N}$ such that

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{n=1}^{\infty} (a_n \sin nx_0 - b_n \cos nx_0) \left(1 - \frac{n}{x}\right)^k = -\frac{1}{\pi} \left[f\right]_{x=x_0} . \tag{6.6}$$

Proof. Notice that the jump distribution has Fourier series,

$$\psi_{x_0}(x) = -2 \sum_{n=1}^{\infty} (a_n \sin nx_0 - b_n \cos nx_0) \sin nx \ ,$$

then

$$\hat{\psi}_{x_0}(x) = 2\pi i \sum_{n=1}^{\infty} (a_n \sin nx_0 - b_n \cos nx_0) \left(\delta\left(x - n\right) - \delta\left(x + n\right)\right) \ .$$

Therefore one has that

$$\sum_{n=1}^{\infty} (a_n \sin nx_0 - b_n \cos nx_0)\, \delta(\lambda x - n) = -\frac{1}{\pi} \left[f\right]_{x=x_0} \log \lambda \frac{\delta(x)}{\lambda} + o\left(\frac{\log \lambda}{\lambda}\right)$$

as $\lambda \to \infty$ in $\mathcal{S}'$, from where we deduce (6.6). □

We now give the radial version of Theorem 3.2 in the case of symmetric jump behavior.

Theorem 6.3. *Let $f \in \mathcal{D}'$ have a symmetric jump behavior at $x = x_0$. Then if V is any harmonic conjugate to a harmonic representation of f on $\Im m\ z > 0$, one has that*

$$\lim_{y\to 0^+} \frac{V(x_0, y)}{\log y} = \frac{1}{\pi} \left[f\right]_{x=x_0} \ . \tag{6.7}$$

Proof. As is the proof of Theorem 3.1 and Theorem 3.2 we may assume that f is tempered distribution and

$$V(z) = -\frac{i}{2\pi}\left(\left\langle \hat{f}_{+}(t), e^{izt}\right\rangle - \left\langle \hat{f}_{-}(t), e^{i\bar{z}t}\right\rangle\right) ,$$

where $\hat{f} = \hat{f}_{-} + \hat{f}_{+}$ is any decomposition with $\operatorname{supp}\hat{f}_{-} \subseteq (-\infty, 0]$ and $\operatorname{supp}\hat{f}_{+} \subseteq [0, \infty)$. Hence by Theorem 6.1, we obtain that

$$\begin{aligned} V(x_0, y) &= -\frac{i}{2\pi}\left\langle \hat{f}_{+}(t)e^{ix_0 t} - \hat{f}_{-}(-t)e^{-ix_0 t}, e^{-yt}\right\rangle \\ &= -\frac{i}{2\pi}\left\langle \frac{2}{i}\left[f\right]_{x=x_0}\log\left(\frac{1}{y}\right)\delta(t), e^{-yt}\right\rangle + o\left(\log\frac{1}{y}\right) \\ &= \frac{1}{\pi}\left[f\right]_{x=x_0}\log y + o\left(\log\frac{1}{y}\right) \text{ as } y \to 0^{+} , \end{aligned}$$

as required. □

In the case when f is the boundary value of an analytic function, one can get a much better result. As was obtained in [7, Thm.5], one has the angular asymptotic logarithmic behavior. We give a new proof of this fact.

Theorem 6.4. *Let F be analytic in the upper semiplane, with distributional boundary values $f(x) = F(x+i0)$. Suppose f has a distributional symmetric jump behavior at $x = x_0$. Then, for any $0 < \eta \leq \pi/2$*

$$F(z) \sim \frac{i}{\pi}[f]_{x=x_0}\log(z - x_0) \quad \textit{as } z \in \Delta_{\eta}^{+}(x_0) \to x_0 \ . \tag{6.8}$$

Proof. Let ψ_{x_0} be the jump distribution at $x = x_0$. Then ψ_{x_0} has a jump behavior at $x = 0$ and $[\psi_{x_0}]_{x=0} = 2[f]_{x=x_0}$. Observe that $U(z) = F(x_0 + z) - F(x_0 - \bar{z})$ is a harmonic representation of ψ_{x_0} and $V(z) = -i\left(F(x_0 + z) + F(x_0 - \bar{z})\right)$ is a harmonic conjugate. Hence, we can apply (3.5) and Theorem 3.2 to U and V and obtain that $F(x_0 - \bar{z}) = F(x_0+z)+O(1)$ and $F(x_0+z)+F(x_0-\bar{z}) = \frac{2i}{\pi}[f]_{x=x_0}\log|z|+o\left(|\log|z||\right)$ as $z \in \Delta_{\eta}^{+}(0) \to 0$; and therefore (6.8) follows. □

We end this article with an immediate corollary of Theorem 6.3, this is the result from [7] which generalizes F. Móricz result, namely, we express the symmetric jump as a logarithmic average of the Abel-Poisson means of the conjugate series.

Corollary 6.5. *Let $f \in \mathcal{S}'$ be a 2π-periodic distribution with Fourier series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right) \ . \tag{6.9}$$

If f has a symmetric jump behavior at $x = x_0$, then its conjugate series has the following logarithmic Abel-Poisson average value

$$\lim_{r\to 1^-} \frac{1}{\log(1-r)} \sum_{n=1}^{\infty} \left(a_n \sin nx_0 - b_n \cos nx_0\right) r^n = \frac{1}{\pi} [f]_{x=x_0} \ . \tag{6.10}$$

References

[1] M. Avdispahić, *On the determination of the jump of a function by its Fourier series,* Acta Math. Hungar. 48 (1986), 267–271.

[2] E. J. Beltrami and M. R. Wohlers, *Distributions and the Boundary Values of Analytic Functions,* Academic Press, N.Y., 1966.

[3] H. Bremermann, *Distributions, Complex Variables and Fourier Transforms,* Addison-Wesley, Reading, Massachusetts, 1965.

[4] R. Estrada, *Characteriztion of the Fourier series of distributions having a value at a point,* Proc. Amer. Math. Soc. 124 (1996), 1205–1212.

[5] R. Estrada, *Regularization of distributions,* Internat. J. Math. & Math. Sci. 21 (1998), 625–636.

[6] R. Estrada, *The Cesàro behavior of distributions,* Proc. R. Soc. Lond. A 454 (1998), 2425–2443.

[7] R. Estrada, *A Distributional Version of the Ferenc Lukács Theorem,* Sarajevo J. Math. 1 (2005), 75–92.

[8] R. Estrada and R. P. Kanwal, *Singular integral equations,* Birkhäuser, Boston, 2000.

[9] R. Estrada and R. P. Kanwal, *A distributional approach to Asymptotics: Theory and Applications, Second Edition,* Birkhäuser, Boston, 2002.

[10] L. Fejér, *Über die Bestimmung des Sprunges der Funktion aus ihrer Fourierreihe* J. Reine Angew. Math., 142 (1913), 165–188.

[11] A. Gelb and E. Tadmor, *Detection of edges in spectral data,* Appl. Comput. Harmon. Anal. 7 (1999), 101–135.

[12] A. Gelb and E. Tadmor, *Detection of edges in spectral data. II. Nonlinear enhancement,* SIAM J. Numer. Anal. 38 (2000), 1389–1408.

[13] B. I. Golubov, *Determination of the jump of a function of bounded p-variation from its Fourier series,* (Russian) Mat. Zametki 12 (1972), 19–28.

[14] G. Kvernadze, *Determination of the jumps of a bounded function by its Fourier series,* J. Approx. Theory 92 (1998), 167–190.

[15] G. Kvernadze, *Approximation of the singularities of a bounded function by the partial sums of its differentiated Fourier series,* Appl. Comput. Harmon. Anal. 11 (2001), 439–454.

[16] G. Kvernadze, T. Hagstrom and H. Shapiro, *Locating discontinuities of a bounded function by the partial sums of its Fourier series,* J. Sci. Comput. 14 (1999), 301–327.

[17] G. Kvernadze, T. Hagstrom and H. Shapiro, *Detecting the singularities of a function of V_p class by its integrated Fourier series,* Comput. Math. Appl. 39 (2000), 25–43.

[18] S. Łojasiewicz, *Sur la valuer et la limite d'une distribution en un point*, Studia Math. 16 (1957), 1–36.

[19] F. Lukács, *Über die Bestimmung des Sprunges einer Funktion aus ihrer Fourierreihe,* J. Reine Angew. Math. 150 (1920), 107–112.
[20] F. Móricz, *Determination of jumps in terms of Abel-Poisson means,* Acta. Math. Hungar. 98 (2003), 259–262.
[21] F. Móricz, *Ferenc Lukács type theorems in terms of the Abel-Poisson means of conjugate series,* Proc. Amer. Math. Soc., 131 (2003), 1243–1250.
[22] S. Pilipović, B. Stanković and A. Takači, *Asymptotic behaviour and Stieltjes Transformation of Distributions*, Teubner-Texte zur Mathematik, Leipzig, 1990.
[23] L. Schwartz, *Théorie des Distributions,* Hermann, Paris, 1966.
[24] E. Seneta, *Regularly Varying Functions,* Springer, Berlin, 1976.
[25] E. C. Titchmarsh, *Introduction to the theory of Fourier Integrals,* Second edition, Clarendon Press, Oxford, 1948.
[26] E. C. Titchmarsh, *The Theory of Functions,* Second edition, Oxford University Press, Oxford, 1979.
[27] J. Vindas, *Structural Theorems for Quasiasymptotics of Distributions at Infinity,* Pub. Inst. Math. (Beograd), in press.
[28] J. Vindas, *The structure of quasiasymptotics of Schwartz distributions*, preprint.
[29] J. Vindas and R. Estrada, *Distributionally regulated functions*, Studia Math. 181 (2007), 211–236.
[30] J. Vindas and R. Estrada, *Distributional Point Values and Convergence of Fourier Series and Integrals*, J. Fourier. Anal. Appl. 13 (2007), 551–576.
[31] J. Vindas and S. Pilipović, *Structural Theorems for Quasiasymptotics of Distributions at the Origin,* Math. Nachr., in press.
[32] V. S. Vladimirov, Yu. N. Drozzinov and B. I. Zavialov, *Tauberian Theorems for Generalized Functions,* Kluver Academic Publishers, Maine, 1988.
[33] G. Walter, *Local boundary behavior of harmonic functions*, Illinois J. Math. 16 (1972), 491–501.
[34] D. Yu, P. Zhou and S. P. Zhou, *On determination of jumps in terms of Abel-Poisson mean of Fourier series,* J. Math. Anal. Appl. 341 (2008), 12–23.
[35] A. Zygmund, *Trigonometric Series*, Third edition, Cambridge University Press, Cambridge, 2002.

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