REGULARIZATIONS AT THE ORIGIN OF DISTRIBUTIONS HAVING PRESCRIBED ASYMPTOTIC PROPERTIES

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Abstract. By a regularization at the origin is meant an extension to $\mathbb{R}^n$ of a suitable distribution initially defined off the origin. We study the regularizations of distributions when the generalized functions to be regularized have prescribed asymptotic properties. A complete description of the asymptotic properties of the regularizations is obtained.

1. Introduction

Regularization of distributions refers to the problem of extending distributions which are a priori defined on a smaller set. Typically, this situation arises when one constructs distributions out of functions (or generalized functions) which have mild singularities at a point [10, 9]. This is a very important subject for both theoretical mathematics and mathematical physics.

In quantum field theory [3], regularization is also known as renormalization. The fundamental problem is often to find a suitable regularization in such a way it be consistent with the experimental considerations. Scaling asymptotic properties of distributions have shown to have a valuable role in this respect, they bring new insights into the problem [1, 23, 24]. On the other hand, the relationship between regularizations and asymptotic properties of distributions is also of importance from the point of view of pure mathematics, for instance, in areas such as singular integral equations [8], the study of boundary properties of holomorphic functions [5], or in Tauberian theory for integral transforms [4, 12, 13, 22, 23]. In fact, as shown in recent studies [4, 11, 22], the asymptotic analysis of various integral transforms may be completely reduced to the study of asymptotic properties of regularizations of distributions; this is the case for the Laplace and wavelet transforms.

In this article we study the regularizations at the origin when the distribution to be regularized possesses prescribed quasiasymptotic properties

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[9, 12, 23] at either the origin itself or infinity. Our aim is to provide a full description of the asymptotic properties of the regularizations. We emphasize that such a problem is essentially a Tauberian one and may be restated in terms of Mellin convolution type integral transforms: quasiasymptotic behavior is nothing but knowledge of asymptotic information over (Mellin) convolution transforms for all kernels in a Schwartz space of test functions. Recently, this problem has been investigated in [5, 18, 19, 20, 22]; we shall give extensions of those results, and in particular we provide more detailed asymptotic information for critical degrees than that from [5]. We shall consider distributions with values in a Banach space. The main results of this paper are presented in Section 3.

2. Notation and Preliminaries

The space E always denotes a fixed, but arbitrary, Banach space with norm $\|\cdot\|$. If $\mathbf{h}:\mathbb{R}_+\mapsto E$ and $T:\mathbb{R}_+\mapsto\mathbb{R}_+$, we write $\mathbf{h}(\lambda)=o(T(\lambda))$ if $\|\mathbf{h}(\lambda)\|=o(T(\lambda))$, and similarly for the big O landau symbol; let $\mathbf{v}\in E$, we write $\mathbf{h}(\lambda)\sim T(\lambda)\mathbf{v}$ if $\mathbf{h}(\lambda)=T(\lambda)\mathbf{v}+o(T(\lambda))$. Let $m\in\mathbb{N}^n$, we use the notation $\varphi^{(m)}=(\partial^{|m|}/\partial x^m)\varphi$.

2.1. **Spaces of Distributions.** The Schwartz spaces [14] of smooth compactly supported and rapidly decreasing test functions are denoted by $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$. We denote by $\mathcal{D}^0(\mathbb{R}^n)\subset\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{S}^0(\mathbb{R}^n)\subset\mathcal{S}(\mathbb{R}^n)$ the closed subspaces consisting of those functions vanishing at the origin together with all their partial derivatives of any order; we provide them with the relative topologies inhered from $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$, respectively.

Let $\mathcal{A}(\mathbb{R}^n)$ be a topological vector space of test function over $\mathbb{R}^n$. We denote by $\mathcal{A}'(\mathbb{R}^n,E)=L_b(\mathcal{A}(\mathbb{R}^n),E)$, the space of continuous linear mappings from $\mathcal{A}(\mathbb{R}^n)$ to E with the topology of uniform convergence over bounded subsets of $\mathcal{A}(\mathbb{R}^n)$. We will mostly have $\mathcal{A}=\mathcal{D},\mathcal{S},\mathcal{D}^0$, or $\mathcal{S}^0$. For vector-valued distributions, we refer to [17]. If $f\in\mathcal{A}(\mathbb{R}^n)$ is a scalar generalized function and $\mathbf{v}\in E$, we denote by $f\mathbf{v}\in\mathcal{A}'(\mathbb{R}^n,E)$ the E-valued generalized function given by $\langle f(x)\mathbf{v},\varphi(x)\rangle=\langle f(x),\varphi(x)\rangle\,\mathbf{v}$.

Observe that [6, 9] (see also [7]) the elements of ${\mathcal{D}^0}'(\mathbb{R}^n,E)$ are precisely those distributions defined on $\mathbb{R}^n\setminus\{0\}$ which admit extensions to $\mathbb{R}^n$, while the elements of ${\mathcal{S}^0}'(\mathbb{R}^n,E)$ are those distributions of slow growth at infinity defined on $\mathbb{R}^n\setminus\{0\}$ and having extensions to $\mathbb{R}^n$ as tempered E-valued distributions. So, if $\mathbf{f}_0\in{\mathcal{S}^0}'(\mathbb{R}^n,E)$ (resp. ${\mathcal{D}^0}'(\mathbb{R}^n,E)$), there exists $\mathbf{f}\in\mathcal{S}'(\mathbb{R}^n,E)$ (resp. $\mathcal{D}'(\mathbb{R}^n,E)$) such that $\langle\mathbf{f}_0,\varphi\rangle=\langle\mathbf{f},\varphi\rangle$, for each $\varphi\in\mathcal{S}^0(\mathbb{R}^n)$ (resp. $\mathcal{D}^0(\mathbb{R}^n)$). Any $\mathbf{f}$ satisfying such a property is called a *regularization* at the origin of $\mathbf{f}_0$ (in short: regularization), in accordance with the classical terminology for regularizing divergent integrals [10, 9]. Naturally, regularizations are not unique and two of them may differ by a distribution concentrated at the origin, that is, a finite sum of the form $\sum_{|m|\leq l}\delta^{(m)}\mathbf{v}_m$, where δ is the Dirac delta (scalar) distribution.

2.2. **Quasiasymptotics.** The quasiasymptotics [9, 12, 23] measure the asymptotic behavior of a distribution by asymptotic comparison with Karamata regularly varying functions. Recall a measurable real valued function, defined and positive on an interval $(0, A]$ (resp. $[A, \infty)$), is called *slowly varying* [2, 15] at the origin (resp. at infinity) if

$$\lim_{\lambda \to 0^+} \frac{L(a\lambda)}{L(\lambda)} = 1 \quad \left(\text{resp. } \lim_{\lambda \to \infty} \frac{L(a\lambda)}{L(\lambda)} = 1 \right) \text{, for each } a > 0.$$

In the next definition $\mathcal{A}(\mathbb{R}^n)$ is assumed to be a space of test functions on which the dilation is a continuous operator; we are mainly concerned with the cases $\mathcal{A} = \mathcal{D}, \mathcal{S}, \mathcal{D}^0, \mathcal{S}^0$.

Definition 2.1. *Let $\mathbf{f} \in \mathcal{A}'(\mathbb{R}^n, E)$ and let L be slowly varying at the origin (resp. at infinity). We say that:*
(i) $\mathbf{f}$ is quasiasymptotically bounded of degree $\alpha \in \mathbb{R}$ at the origin (resp. at infinity) with respect to L in $\mathcal{A}'(\mathbb{R}^n, E)$ if for each test function $\varphi \in \mathcal{A}(\mathbb{R}^n)$

$$\sup_{\lambda \leq 1} \frac{1}{\lambda^{\alpha} L(\lambda)} \left\| \langle \mathbf{f}(\lambda x), \varphi(x) \rangle \right\| < \infty \quad \left(\textit{resp. } \sup_{1 \leq \lambda} \right).$$

We write: $\mathbf{f}(\lambda x) = O(\lambda^{\alpha} L(\lambda))$ in $\mathcal{A}'(\mathbb{R}^n, E)$ as $\lambda \to 0^+$ (resp. $\lambda \to \infty$).
(ii) $\mathbf{f}$ has quasiasymptotic behavior of degree $\alpha \in \mathbb{R}$ at the origin (resp. at infinity) with respect to L in $\mathcal{A}'(\mathbb{R}^n, E)$ if there exists $\mathbf{g} \in \mathcal{A}'(\mathbb{R}^n, E)$ such that for each test function $\varphi \in \mathcal{A}(\mathbb{R}^n)$ the following limit holds with respect to the norm of E

$$\lim_{\lambda \to 0^+} \frac{1}{\lambda^{\alpha} L(\lambda)} \langle \mathbf{f}(\lambda x), \varphi(x) \rangle = \langle \mathbf{g}(x), \varphi(x) \rangle \in E \quad \left(\textit{resp. } \lim_{\lambda \to \infty} \right).$$

In such a case we write,

$$\mathbf{f}(\lambda x) = \lambda^{\alpha} L(\lambda) \mathbf{g}(x) + o(\lambda^{\alpha} L(\lambda)) \quad \textit{in } \mathcal{A}'(\mathbb{R}^n, E) \tag{1}$$

as $\lambda \to 0^+$ (resp. $\lambda \to \infty$).

We will also use the notation $\mathbf{f}(\lambda x) \sim \lambda^{\alpha} L(\lambda) \mathbf{g}(x)$ for (1). In [5] distributions having quasiasymptotic behavior at infinity are called asymptotically homogeneous generalized functions.

If $\mathcal{A} = \mathcal{S}$ or $\mathcal{D}$ in (ii) of Definition 2.1, it is easy to show [9, 12, 23] that $\mathbf{g}$ must be homogeneous with degree of homogeneity α, i.e., $\mathbf{g}(ax) = a^{\alpha} \mathbf{g}(x)$, for all $a \in \mathbb{R}_+$. We refer to the paper by Drozhzhinov and Zavialov [5] for an excellent presentation of the theory of multidimensional homogeneous distributions, we remark that their results are valid for E-valued distributions too.

Suppose that $\mathbf{f}_0, \mathbf{g}_0 \in \mathcal{S}^{0\prime}(\mathbb{R}^n, E)$ satisfy $\mathbf{f}_0(\lambda x) \sim \lambda^{\alpha} L(\lambda) \mathbf{g}_0(x)$ in the space $\mathcal{S}^{0\prime}(\mathbb{R}^n, E)$. Then $\mathbf{g}_0$ must be homogeneous of degree α over $\mathcal{S}^0(\mathbb{R}^n)$. Suppose now that $\alpha \notin -n - \mathbb{N}$, applying [5, Thm. 3.1, Cor. 3.2], we conclude the existence of a *unique* regularization at the origin $\mathbf{g} \in \mathcal{S}'(\mathbb{R}^n, E)$ which is homogeneous of degree α. The case $\alpha = -n - p$, $p \in \mathbb{N}$, is slightly different; from [5, Cor. 3.3], we obtain a regularization $\mathbf{g}$ which is *associate homogeneous* of order 1 and degree $-n-p$ over $\mathcal{S}(\mathbb{R}^n)$ (cf. [9, p. 74], [10], [16]); specifically, there are $\mathbf{v}_m \in E$, $|m| = p$, such that

$$\mathbf{g}(ax) = a^{-n-p}\mathbf{g}(x) + a^{-n-p}\log a \sum_{|m|=p} \delta^{(m)}(x)\mathbf{v}_m, \quad \text{for each } a > 0. \tag{2}$$

Of course, the same considerations are true for $\mathcal{D}^{0'}(\mathbb{R}^n, E)$.

2.3. **Asymptotically and Associate Asymptotically Homogeneous Functions.** The functions to be introduced here will appear naturally in Section 3. The terminology in the scalar-valued case is from [18, 19, 20, 21] (see also de Haan theory in [2]), we remark these classes of functions have already shown to be a tool of great importance in the study of asymptotic properties of one-dimensional distributions. We list some of their basic properties in Section 4 (Lemma 4.3).

Definition 2.2. *Let $\mathbf{c} : (0, A) \to E$ (resp. $(A, \infty) \to E$), $A > 0$, be a continuous E-valued function and let L be slowly varying at the origin (resp. at infinity). We say that:*
(i) $\mathbf{c}$ is asymptotically homogeneous of degree $\gamma \in \mathbb{R}$ with respect to L if

$$\mathbf{c}(a\lambda) = a^{\gamma}\mathbf{c}(\lambda) + o(L(\lambda)) \quad \textit{as } \lambda \to 0^+ \ \ (\textit{resp. } \lambda \to \infty), \quad \textit{for each } a > 0.$$

(ii) $\mathbf{c}$ is associate asymptotically homogeneous of degree 0 with respect to L if there exists $\mathbf{v} \in E$ such that

$$\mathbf{c}(a\lambda) = \mathbf{c}(\lambda) + L(\lambda)\log a\, \mathbf{v} + o(L(\lambda)) \quad \textit{as } \lambda \to 0^+ \ \ (\textit{resp. } \lambda \to \infty),$$

for each $a > 0$.
(iii) $\mathbf{c}$ is asymptotically homogeneously bounded of degree $\gamma \in \mathbb{R}$ with respect to L if

$$\mathbf{c}(a\lambda) = a^{\gamma}\mathbf{c}(\lambda) + O(L(\lambda)) \quad \textit{as } \lambda \to 0^+ \ \ (\textit{resp. } \lambda \to \infty), \quad \textit{for each } a > 0.$$

3. The Main Results: Quasiasymptotic Properties of Regularizations

The following two theorems completely describe the asymptotic properties of arbitrary regularizations when the distribution to be regularized has prescribed quasiasymptotic properties. We only state the tempered case, but the results are also valid if we replace everywhere below $\mathcal{S}$ by $\mathcal{D}$ (cf. Remark 1). In their proofs, we make use of three auxiliary lemmas which are postponed for Section 4.

Theorem 3.1. *Let L be slowly varying at the origin (resp. at infinity) and let $\mathbf{f}_0 \in \mathcal{S}^{0'}(\mathbb{R}^n, E)$ have the quasiasymptotic behavior*

$$\mathbf{f}_0(\lambda x) = \lambda^{\alpha}L(\lambda)\mathbf{g}_0(x) + o(\lambda^{\alpha}L(\lambda)) \quad \textit{in } \mathcal{S}^{0'}(\mathbb{R}^n, E) \tag{3}$$

as $\lambda \to 0^+$ (resp. $\lambda \to \infty$). Suppose that $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$ is a regularization of $\mathbf{f}_0$. Then:

(i) If $\alpha \notin -n-\mathbb{N}$ and $\mathbf{g}$ is the homogeneous regularization of $\mathbf{g}_0$, there exist $d \in \mathbb{N}$ and $\mathbf{w}_m \in E$, $|m| \leq d$, such that

$$\mathbf{f}\,(\lambda x) = \lambda^{\alpha} L(\lambda) \mathbf{g}(x) + \sum_{|m|\leq d} \frac{\delta^{(m)}(x)}{\lambda^{n+|m|}} \mathbf{w}_m + o(\lambda^{\alpha} L(\lambda)) \quad \text{in } \mathcal{S}'(\mathbb{R}^n, E). \tag{4}$$

(ii) If $\alpha = -n-p$, $p \in \mathbb{N}$, and $\mathbf{g}$ is a regularization of $\mathbf{g}_0$ satisfying (2), there exist $d \in \mathbb{N}$, $\mathbf{w}_m \in E$, $|m| \leq d$, and associate asymptotically homogeneous E-valued functions $\mathbf{c}_m$, $|m| = p$, satisfying

$$\mathbf{c}_m(a\lambda) = \mathbf{c}_m(\lambda) + L(\lambda) \log a \, \mathbf{v}_m + o(L(\lambda)), \tag{5}$$

such that

(6)

$$\mathbf{f}\,(\lambda x) = \frac{L(\lambda)}{\lambda^{n+p}} \mathbf{g}(x) + \sum_{|m|\leq d, |m|\neq p} \frac{\delta^{(m)}(x)}{\lambda^{n+|m|}} \mathbf{w}_m + \sum_{|m|=p} \frac{\delta^{(m)}(x)}{\lambda^{n+p}} \mathbf{c}_m(\lambda) + o\left(\frac{L(\lambda)}{\lambda^{n+p}}\right)$$

in the space $\mathcal{S}'(\mathbb{R}^n, E)$.

Proof. The hypothesis (3) and Lemma 4.1 imply the existence of $d \in \mathbb{N}$ such that for any $\rho \in \mathcal{S}(\mathbb{R}^n)$ satisfying $\rho^{(m)}(0) = 0$, $|m| \leq d$,

$$\langle \mathbf{f}(\lambda x), \rho(x) \rangle = \lambda^{\alpha} L(\lambda) \langle \mathbf{g}(x), \rho(x) \rangle + o\left(\lambda^{\alpha} L(\lambda)\right).$$

Let η be a fixed test function such that $\eta(x) = 1$ on a neighborhood of the origin and $\operatorname{supp} \eta \subset B(0,1)$, the ball of radius 1 centered at the origin. Given $\varphi \in \mathcal{S}(\mathbb{R}^n)$, an arbitrary test function, we set

$$T_{\varphi}(x) = \sum_{|m|\leq d} \frac{\varphi^{(m)}(0)}{m!} x^m,$$

its Taylor polynomial of order d. So, if we write $\rho(x) := \varphi(x) - \eta(x) T_{\varphi}(x)$, then we obtain

$$\begin{aligned}
\langle \mathbf{f}(\lambda x), \varphi(x) \rangle &= \lambda^{\alpha} L(\lambda) \langle \mathbf{g}(x), \rho(x) \rangle + \langle \mathbf{f}(\lambda x), \eta(x) T_{\varphi}(x) \rangle + o\left(\lambda^{\alpha} L(\lambda)\right) \\
&= \lambda^{\alpha} L(\lambda) \langle \mathbf{g}(x), \varphi(x) \rangle + \sum_{|m|\leq d} \varphi^{(m)}(0) \left\langle \mathbf{f}(\lambda x) - \lambda^{\alpha} L(\lambda) \mathbf{g}(x), \frac{x^m}{m!} \eta(x) \right\rangle \\
&\quad + o\left(\lambda^{\alpha} L(\lambda)\right).
\end{aligned}$$

Thus, if set $\mathbf{c}_m(\lambda) = (-1)^{|m|} \langle \lambda^{-\alpha} \mathbf{f}(\lambda x) - L(\lambda) \mathbf{g}(x), x^m \eta(x)/m! \rangle$, we obtain the asymptotic formula

$$\mathbf{f}(\lambda x) = \lambda^{\alpha} L(\lambda) \mathbf{g}(x) + \sum_{|m|\leq d} \delta^{(m)}(x) \lambda^{\alpha} \mathbf{c}_m(\lambda) + o\left(\lambda^{\alpha} L(\lambda)\right), \tag{7}$$

valid now in $\mathcal{S}'(\mathbb{R}^n, E)$. We study now the asymptotic properties of the E-valued functions $\mathbf{c}_m$. For each $|m| \leq d$, we fix a test function φ_m such that $\varphi_m^{(m)}(0) = (-1)^{|m|}$ but $\varphi_m^{(j)}(0) = 0$ for $j \neq m$, $|j| \leq d$. At this point we split the proof into the cases (i) and (ii).

Case (i): $\alpha \notin -n-\mathbb{N}$.

If we evaluate $\mathbf{f}(a\lambda x)$ at $\varphi_m(x)$, $a>0$, use the homogeneity of $\mathbf{g}$, the slowly varying property of L, and apply (7), we obtain

$$\begin{aligned}
&a^{\alpha}\lambda^{\alpha}L(\lambda)\left\langle \mathbf{g}(x),\varphi_m(x)\right\rangle + a^{\alpha}\lambda^{\alpha}\mathbf{c}_m(a\lambda)+o\left(\lambda^{\alpha}L(\lambda)\right)\\
&=\left\langle \mathbf{f}(a\lambda x),\varphi_m(x)\right\rangle=\left\langle \mathbf{f}(\lambda x),\frac{1}{a^n}\varphi_m\left(\frac{x}{a}\right)\right\rangle\\
&=\lambda^{\alpha}L(\lambda)\left\langle \mathbf{g}(x),\frac{1}{a^n}\varphi_m\left(\frac{x}{a}\right)\right\rangle + a^{-n-|m|}\lambda^{\alpha}\mathbf{c}_m(\lambda)+o\left(\lambda^{\alpha}L(\lambda)\right)\\
&=a^{\alpha}\lambda^{\alpha}L(\lambda)\left\langle \mathbf{g}(x),\varphi_m\left(x\right)\right\rangle + a^{-n-|m|}\lambda^{\alpha}\mathbf{c}_m(\lambda)+o\left(\lambda^{\alpha}L(\lambda)\right).
\end{aligned}$$

Hence, for each $|m|\leq d$, the E-valued function $\mathbf{c}_m$ satisfies (i) in Definition 2.2 with $\gamma=-\alpha-n-|m|$, and so, Lemma 4.3 gives the existence of $\mathbf{w}_m\in E$ (some of them may be $\mathbf{0}$), for each $|m|\leq d$, such that

$$\mathbf{c}_m(\lambda)=\lambda^{-\alpha-n-|m|}\mathbf{w}_m+o\left(L(\lambda)\right). \tag{8}$$

Inserting (8) into (7), one gets (4). This completes the proof of the first case.

Case (ii): $\alpha=-n-p$.

Observed that (2) shows that $\mathbf{g}$ is homogeneous when acting on test functions such that $\varphi^{(j)}(0)=0$ for $|j|=p$. Thus, if $d<p$ the proceeding argument shows that $\mathbf{f}$ satisfies indeed (6) with $\mathbf{w}_m$ as before and $\mathbf{c}_m$ identically $\mathbf{0}$. We suppose now that $p\leq d$; if $|m|\neq p$, then the preceding argument applies also to show the existence of $\mathbf{w}_m$ such the $\mathbf{c}_m$ satisfy (8), which in turn implies

$$\mathbf{f}(\lambda x)=\frac{L(\lambda)}{\lambda^{n+p}}\mathbf{g}(x)+\sum_{|m|\leq d,\ |m|\neq p}\frac{\delta^{(m)}(x)}{\lambda^{n+|m|}}\mathbf{w}_m+\sum_{|m|=p}\frac{\delta^{(m)}(x)}{\lambda^{n+p}}\mathbf{c}_m(\lambda)+o\left(\frac{L(\lambda)}{\lambda^{n+p}}\right) \tag{9}$$

in $\mathcal{S}'(\mathbb{R}^n,E)$. We now analyze the behavior of $\mathbf{c}_m$ when $|m|=p$, it remains to establish (5). Evaluating (9) at φ_m (defined as before) and using (2), we have

$$\begin{aligned}
&\frac{L(\lambda)}{(a\lambda)^{n+p}}\left\langle \mathbf{g}(x),\varphi_m(x)\right\rangle+(a\lambda)^{-n-p}\mathbf{c}_m(a\lambda)+o\left(\frac{L(\lambda)}{\lambda^{n+p}}\right)\\
&=\left\langle \mathbf{f}(a\lambda x),\varphi_m(x)\right\rangle=\left\langle \mathbf{f}(\lambda x),\frac{1}{a^n}\varphi_m\left(\frac{x}{a}\right)\right\rangle\\
&=\frac{L(\lambda)}{\lambda^{n+p}}\left\langle \mathbf{g}(ax),\varphi_m\left(x\right)\right\rangle+(a\lambda)^{-n-p}\mathbf{c}_m(\lambda)+o\left(\frac{L(\lambda)}{\lambda^{n+p}}\right)\\
&=\frac{L(\lambda)}{(a\lambda)^{n+p}}\left\langle \mathbf{g}(x),\varphi_m\left(x\right)\right\rangle+\frac{L(\lambda)\log a}{(a\lambda)^{n+p}}\mathbf{v}_m+(a\lambda)^{-n-p}\mathbf{c}_m(\lambda)+o\left(\frac{L(\lambda)}{\lambda^{n+p}}\right);
\end{aligned}$$

consequently, they satisfy the requirements. □

We now consider quasiasymptotic boundedness.

Theorem 3.2. *Let L be slowly varying at the origin (resp. at infinity) and let $\mathbf{f}_0 \in \mathcal{S}^{0\prime}(\mathbb{R}^n, E)$ be quasiasymptotically bounded of degree α at the origin (resp. at infinity) with respect to L in $\mathcal{S}^{0\prime}(\mathbb{R}^n, E)$. Suppose that $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$ is a regularization of $\mathbf{f}_0$. Then:*
(i) If $\alpha \notin -n - \mathbb{N}$, there exist $d \in \mathbb{N}$ and $\mathbf{w}_m \in E$, $|m| \leq d$, such that

$$\mathbf{f}\,(\lambda x) = \sum_{|m| \leq d} \frac{\delta^{(m)}(x)}{\lambda^{n+|m|}} \mathbf{w}_m + O(\lambda^{\alpha} L(\lambda)) \quad \text{in } \mathcal{S}'(\mathbb{R}^n, E).$$

(ii) If $\alpha = -n - p$, $p \in \mathbb{N}$, there exist $d \in \mathbb{N}$, $\mathbf{w}_m \in E$, $|m| \leq d$, and asymptotically homogeneously bounded E-valued functions $\mathbf{c}_m$, $|m| = p$, with respect to L such that

$$\mathbf{f}\,(\lambda x) = \sum_{|m| \leq d, |m| \neq p} \frac{\delta^{(m)}(x)}{\lambda^{n+|m|}} \mathbf{w}_m + \sum_{|m| = p} \frac{\delta^{(m)}(x)}{\lambda^{n+p}} \mathbf{c}_m(\lambda) + O\left(\frac{L(\lambda)}{\lambda^{n+p}}\right)$$

in the space $\mathcal{S}'(\mathbb{R}^n, E)$.

Proof. It is enough to set $\mathbf{g} = \mathbf{0}$ and replace o by O in the arguments given in the proof of Theorem 3.1. We leave the details of such modifications to the reader. □

Remark 1. *Theorems 3.1 and 3.2 still hold if we replace $\mathcal{S}$ by $\mathcal{D}$ everywhere in the statements. Indeed, the proofs of these assertions are identically the same as the ones for the tempered case, but now making use of Lemma 4.2 instead of Lemma 4.1.*

4. Auxiliary Lemmas

We show in this section three lemmas which were used in Section 3.

The following lemma is due to Drozhzhinov and Zavialov in the scalar-valued case [5, Lem. 2.1]; actually, a similar proof applies to the E-valued case. Denote by $\mathcal{S}_d(\mathbb{R}^n)$, $d \in \mathbb{N}$, the closed subspace of $\mathcal{S}(\mathbb{R}^n)$ consisting of functions such that all their derivatives up to order d vanish at the origin; they are provided with the relative topology inhered from $\mathcal{S}(\mathbb{R}^n)$.

Lemma 4.1. *Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$ and let L be slowly varying at the origin (resp. at infinity).*
(i) Let $\mathbf{g} \in \mathcal{S}'(\mathbb{R}^n, E)$. Suppose that the restrictions of $\mathbf{f}$ and $\mathbf{g}$ to $\mathcal{S}^0(\mathbb{R}^n)$ satisfy

$$\mathbf{f}(\lambda x) \sim \lambda^{\alpha} L(\lambda) \mathbf{g}(x) \quad \text{in } \mathcal{S}^{0\prime}(\mathbb{R}^n, E).$$

Then, there exists $d \in \mathbb{N}$ (large enough) such that the restriction of $\mathbf{f}$ to $\mathcal{S}_d(\mathbb{R}^n)$ has the same quasiasymptotic behavior in the space $\mathcal{S}'_d(\mathbb{R}^n, E)$.
(ii) Suppose that the restriction of $\mathbf{f}$ to $\mathcal{S}^0(\mathbb{R}^n)$ is quasiasymptotically bounded at the origin (resp. infinity) with respect to L in the space $\mathcal{S}^{0\prime}(\mathbb{R}^n, E)$, then there exists $d \in \mathbb{N}$ (large enough) such that the restriction of $\mathbf{f}$ to $\mathcal{S}_d(\mathbb{R}^n)$ is equally quasiasymptotically bounded with respect to L (with the same degree) in the space $\mathcal{S}'_d(\mathbb{R}^n, E)$.

Proof. For each $l \in \mathbb{N}$, define the norms

$$\|\rho\|_l = \max_{|m|\leq l} \sup_{x\in\mathbb{R}^n} \left(\frac{1}{|x|^2} + |x|^2\right)^l \left|\rho^{(m)}(x)\right|, \tag{10}$$

for $\rho \in \mathcal{S}^0(\mathbb{R}^n)$. Obviously, these norms induce on $\mathcal{S}^0(\mathbb{R}^n)$ the same topology as the one inhered from $\mathcal{S}(\mathbb{R}^n)$. For each l fixed, denote by $\mathcal{X}_l$ the completion of $\mathcal{S}^0(\mathbb{R}^n)$ with respect to the norm $\|\cdot\|_l$; then, $\mathcal{S}^0(\mathbb{R}^n) = \bigcap \mathcal{X}_l$, the intersection having also topological meaning as a projective limit. The Banach-Steinhaus theorem implies that $\mathbf{f}$ has the same quasiasymptotic behavior (resp. is equally quasiasymptotically bounded) over some $\mathcal{X}_{l_0}$. But, clearly, if d is large enough $\mathcal{S}_d(\mathbb{R}^n) \subset \mathcal{X}_{l_0}$. This shows the lemma. □

We have a similar assertion for the non-tempered case. Denote now $\mathcal{D}_d(\mathbb{R}^n) = \mathcal{D}(\mathbb{R}^n) \cap \mathcal{S}_p(\mathbb{R}^n)$, provided with the relative topology inhered from $\mathcal{D}(\mathbb{R}^n)$.

Lemma 4.2. *Let $\mathbf{f}, \mathbf{g} \in \mathcal{D}'(\mathbb{R}^n, E)$ and let L be slowly varying at the origin (resp. at infinity).*
(i) Suppose that the restrictions of $\mathbf{f}$ and $\mathbf{g}$ to $\mathcal{D}^0(\mathbb{R}^n)$ satisfy

$$\mathbf{f}(\lambda x) \sim \lambda^\alpha L(\lambda)\mathbf{g}(x) \quad \text{in } \mathcal{D}^{0'}(\mathbb{R}^n, E).$$

Then, there exists $d \in \mathbb{N}$ (large enough) such that the restriction of $\mathbf{f}$ to $\mathcal{D}_d(\mathbb{R}^n)$ has the same quasiasymptotic behavior in the space $\mathcal{D}'_d(\mathbb{R}^n, E)$.
(ii) Suppose that the restriction of $\mathbf{f}$ to $\mathcal{D}^0(\mathbb{R}^n)$ is quasiasymptotically bounded at the origin (resp. infinity) with respect to L in the space $\mathcal{D}^{0'}(\mathbb{R}^n, E)$, then there exists $d \in \mathbb{N}$ (large enough) such that the restriction of $\mathbf{f}$ to $\mathcal{D}_d(\mathbb{R}^n)$ is equally quasiasymptotically bounded with respect to L (with the same degree) in the space $\mathcal{D}'_d(\mathbb{R}^n, E)$.

Proof. Let $\mathcal{D}(\overline{B(0,1)}) \subset \mathcal{D}(\mathbb{R}^n)$ the subspace consisting of test functions supported by the closed ball of radius 1 with center at the origin. Denote $\mathcal{D}_d(\overline{B(0,1)}) = \mathcal{D}_d(\mathbb{R}^n) \cap \mathcal{D}(\overline{B(0,1)})$, $d \in \mathbb{N}$. Since any $\varphi \in \mathcal{D}_d(\mathbb{R}^n)$ can be written as $\varphi = \varphi_d + \varphi_0$, with $\varphi_d \in \mathcal{D}_d(\overline{B(0,1)})$ and $\varphi_0 \in \mathcal{D}^0(\mathbb{R}^n)$, it is enough to show that the conclusions of the lemma are valid in one of the spaces $\mathcal{D}'_d(\overline{B(0,1)}, E)$ for the restriction of $\mathbf{f}$ to $\mathcal{D}_d(\overline{B(0,1)})$. For each l, let $\mathcal{Y}_l$ be the completion of $\mathcal{D}^0(\overline{B(0,1)})$ with respect to the norm (10); then, $\mathcal{D}^0(\overline{B(0,1)}) = \bigcap \mathcal{Y}_l$, as a projective limit. As in Lemma 4.1, we conclude the existence of l_0 such that $\mathbf{f}$ has the same quasiasymptotic behavior (resp. is equally quasiasymptotically bounded) over $\mathcal{Y}_{l_0}$. Taking d large enough $\mathcal{D}_d(\overline{B(0,1)}) \subset \mathcal{Y}_{l_0}$, which yields the result. □

One can obtain the precise asymptotic behavior of asymptotically homogeneous and homogeneously bounded functions of non-zero degree. The proof of the following lemma can be given exactly as in the scalar-valued case [18] (see also comments in [20, 21]), we choose to omit it and refer the reader to the cited papers for a proof.

Lemma 4.3. *Let L be slowly varying at the origin (resp. at infinity). (i) Assume that $\mathbf{c}$ is asymptotically homogeneous at the origin (resp. at infinity) of degree γ with respect to L.*

(i.1) *If $\gamma > 0$ (resp. $\gamma < 0$), then $\mathbf{c}(\lambda) = o(L(\lambda))$.*
(i.2) *If $\gamma < 0$ (resp. $\gamma > 0$), then $\mathbf{c}(\lambda) = \lambda^\gamma \mathbf{w} + o(L(\lambda))$, for some $\mathbf{w} \in E$.*

(ii) Assume $\mathbf{c}$ is asymptotically homogeneously bounded at the origin (resp. at infinity) of degree γ with respect to L.

(ii.1) *If $\gamma > 0$ (resp. $\gamma < 0$), then $\mathbf{c}(\lambda) = O(L(\lambda))$.*
(ii.2) *If $\gamma < 0$ (resp. $\gamma > 0$), then $\mathbf{c}(\lambda) = \lambda^\gamma \mathbf{w} + O(L(\lambda))$, for some $\mathbf{w} \in E$.*

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