WAVELET EXPANSIONS AND ASYMPTOTIC BEHAVIOR OF DISTRIBUTIONS

KATERINA SANEVA AND JASSON VINDAS

Abstract. We develop a distribution wavelet expansion theory for the space of highly time-frequency localized test functions over the real line $\mathcal{S}_0(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$ and its dual space $\mathcal{S}_0'(\mathbb{R})$, namely, the quotient of the space of tempered distributions modulo polynomials. We prove that the wavelet expansions of tempered distributions converge in $\mathcal{S}_0'(\mathbb{R})$. A characterization of boundedness and convergence in $\mathcal{S}_0'(\mathbb{R})$ is obtained in terms of wavelet coefficients. Our results are then applied to study local and non-local asymptotic properties of Schwartz distributions via wavelet expansions. We provide Abelian and Tauberian type results relating the asymptotic behavior of tempered distributions with the asymptotics of wavelet coefficients.

1. Introduction

Orthogonal wavelets have shown to be a very effective tool in several areas of both pure and applied mathematics. Their usefulness often arises from their good localization properties, which in turn make them very suitable for attacking problems in time-scale analysis [2, 6, 8, 15, 17, 42]. Remarkably, wavelet expansions enjoy good pointwise convergence properties too [9, 10, 29, 41, 42].

The wavelet coefficients can be used to give intrinsic characterizations of important spaces such as L^p-spaces, Sobolev spaces, Besov spaces, Hölder-Zygmund spaces, and Triebel-Lizorkin spaces, among many others, [17]. The growth properties of wavelet coefficients play an essential role in the analysis of regularity notions for distributions: They are capable to capture the smoothness of a distribution. Convergence and other related aspects of wavelet series in various function and distribution spaces have been extensively studied, see [9, 10, 12, 17, 32, 42] (and references therein).

In this article we develop a wavelet expansion theory that is applicable not only in restricted subspaces of $\mathcal{S}'(\mathbb{R})$ but also to *arbitrary* tempered distributions. We also describe the point behavior of a distribution in terms

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of the asymptotic behavior of its wavelet coefficients. We remark that similar problems have been investigated by other authors. In [41], Walter has studied multiresolution analysis approximations of distributions in connection with the problem of pointwise convergence of wavelet series (see also [29, 30, 42]). Walter proved that multiresolution expansions converge to the point values of a distribution (in the sense of Łojasiewicz [14]), thus extending the previous results on pointwise convergence of wavelet series from [10]. These results were generalized to more general point asymptotic notions by Pilipović, Takači, and Teofanov [20]. Recent refinements of those results are provided in [21].

When dealing with a tempered distribution, there is an obvious limitation for studying its wavelet series. If the orthogonal wavelet ψ belongs to the Schwartz class of rapidly decreasing functions [26], then all the moments of ψ must vanish [6, 17, 42], and hence the wavelet coefficients of any polynomial are zero. Therefore, the association between tempered distributions and their wavelet series is not one-to-one.

In order to avoid such a difficulty, we work on the quotient of the space of tempered distributions modulo polynomials. The resulting space is $\mathcal{S}'_0(\mathbb{R})$, the dual of the space of highly time-frequency localized functions [8]. We first show the convergence of wavelet series on the space of test functions $\mathcal{S}_0(\mathbb{R})$, and we use this result to prove that the wavelet expansions of tempered distributions are actually convergent in $\mathcal{S}'_0(\mathbb{R})$. Then, we characterize the bounded sets and weakly convergent nets in $\mathcal{S}'_0(\mathbb{R})$ in terms of uniform estimates of wavelet coefficients. It should be noticed that our approach to wavelet expansions of distributions differs from that of [30, 41, 42], where it is assumed that the wavelet comes from a priori given multiresolution analysis (MRA) and multiresolution approximations are studied instead of actual wavelet series. On the other hand, we work with the actual wavelet series, and our approach is independent from an MRA.

We shall apply our distribution wavelet expansion theory to analyze asymptotic properties of distributions. The asymptotic notion to be considered in this study is the so called quasiasymptotic behavior (quasiasymptotics). This notion has found many applications in areas such as quantum field theory [43, 39, 40], PDE theory [39], and in the asymptotic study of various integral transforms [5, 19, 36, 39]. Wavelet methods in asymptotic analysis have attracted much attention, both in the classical [7, 8, 9, 18] and distributional [20, 21, 22, 23, 24, 25, 29, 38] contexts. In this paper we relate the (quasi-)asymptotic behavior of a distribution with the ones of the wavelet coefficients. In essence, the problem under consideration has an *Abel-Tauber* nature [11, 39]. We obtain general theorems in the form of necessary and sufficient condition which describe the quasiasymptotic behavior in terms of wavelet coefficients; we then derive Abelian and Tauberian results as their consequences. Our Tauberian theorems completely characterize quasiasymptotic properties of distributions.

This article is organized as follows. In Section 2 we recall some spaces of test functions and distributions along with some basic wavelet concepts. We provide in Section 3 the convergence results for wavelet expansions on the spaces $\mathcal{S}_0(\mathbb{R})$ and $\mathcal{S}_0'(\mathbb{R})$; we also study boundedness and convergence in $\mathcal{S}_0'(\mathbb{R})$ via wavelet coefficients. Section 4 presents a brief summary of definitions and basic facts about asymptotic analysis of distributions. The (quasi-)asymptotics (in $\mathcal{S}_0'(\mathbb{R})$) are characterized in Section 5. The Abelian and Tauberian theorems are given in Section 6 and Section 7, respectively.

2. Preliminaries and Notations

The set $\mathbb{H}$ denotes the upper half-plane, that is, $\mathbb{H} = \mathbb{R} \times \mathbb{R}_+$; we use the notation $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2.1. **Spaces of Functions and Distributions.**

The Schwartz spaces of test functions and distributions over the real line $\mathbb{R}$ are denoted by $\mathcal{D}(\mathbb{R})$ and $\mathcal{D}'(\mathbb{R})$, respectively; the space of rapidly decreasing smooth functions and its dual, the space of tempered distributions, are denoted by $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$, respectively. We refer to [26] for the well known properties of these spaces.

Following [8], we define the space of highly time-frequency localized functions over the real line $\mathcal{S}_0(\mathbb{R})$ as the set of those elements $\phi \in \mathcal{S}(\mathbb{R})$ for which all the moments vanish, i.e.,

$$\int_{-\infty}^{\infty} x^n \phi(x) \mathrm{d}x = 0 \ , \quad \forall n \in \mathbb{N}_0 \ .$$

It is provided with the relative topology inhered from $\mathcal{S}(\mathbb{R})$. Observe that $\mathcal{S}_0(\mathbb{R})$ is a closed subspace of $\mathcal{S}(\mathbb{R})$. It is a Fréchet space, closed under differentiation and multiplication by polynomials. Its dual space is $\mathcal{S}_0'(\mathbb{R})$. Notice that there exists a well defined continuous linear projector from $\mathcal{S}'(\mathbb{R})$ onto $\mathcal{S}_0'(\mathbb{R})$ as the transpose of the trivial inclusion from the closed subspace $\mathcal{S}_0(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})$. This map is surjective due to the Hahn-Banach theorem; however, there is no continuous right inverse for it [4]. The kernel of this projection is the space of polynomials; hence, the space $\mathcal{S}_0'(\mathbb{R})$ can be regarded as the quotient space of $\mathcal{S}'(\mathbb{R})$ by the space of polynomials. We will not introduce a notation for this map, so if $f \in \mathcal{S}'(\mathbb{R})$, we will keep calling by f the restriction of f to $\mathcal{S}_0(\mathbb{R})$.

The space of highly localized function over $\mathbb{H}$ is denoted by $\mathcal{S}(\mathbb{H})$. It consists of those smooth functions Φ on $\mathbb{H}$ for which

$$\sup_{(b,a)\in\mathbb{H}} \left(a + \frac{1}{a}\right)^m (1+|b|)^n \left|\frac{\partial^{k+l}\Phi}{\partial a^k \partial b^l}(b,a)\right| < \infty \ ,$$

for all $m, n, k, l \in \mathbb{N}_0$. The canonical topology of this space is defined in the standard way [8].

2.2. **The Wavelet Transform of Distributions.** The *wavelet transform* of $f\in\mathcal{S}'(\mathbb{R})$ with respect to $\psi\in\mathcal{S}_0(\mathbb{R})$ is the C^{∞}-function on $\mathbb{H}$ defined by

$$\mathcal{W}_{\psi}f(b,a):=\langle f(b+ax),\bar{\psi}(x)\rangle=\Big\langle f(t),\frac{1}{a}\bar{\psi}\Big(\frac{t-b}{a}\Big)\Big\rangle,\ (b,a)\in\mathbb{H}\ . \tag{2.1}$$

Note that $\mathcal{W}_{\psi}:\mathcal{S}_0(\mathbb{R})\longmapsto\mathcal{S}(\mathbb{H})$ is a continuous linear map [8]. For an arbitrary tempered distribution $f\in\mathcal{S}'(\mathbb{R})$, one can verify that $\mathcal{W}_{\psi}f$ is a function of slow growth on $\mathbb{H}$, that is, it satisfies an estimate of the form

$$|\mathcal{W}_{\psi}f(b,a)|\leq O\left(\left(a+\frac{1}{a}\right)^{m}(1+|b|)^{n}\right)\ , \tag{2.2}$$

for some $m,n\in\mathbb{N}_0$.

Naturally, the wavelet transform (2.1) can be considered for $f\in\mathcal{S}_0'(\mathbb{R})$, since $\psi\in\mathcal{S}_0(\mathbb{R})$. A complete wavelet transform theory based on the spaces $\mathcal{S}_0(\mathbb{R})$ and $\mathcal{S}_0'(\mathbb{R})$ can be found in Hölschneider's book [8].

2.3. **Orthogonal Wavelets.** We briefly recall some concepts from the theory of orthonormal wavelet bases of $L^2(\mathbb{R})$ [2, 15, 42] (see also [6, 17]).

An *orthonormal wavelet* on $\mathbb{R}$ is a function $\psi\in L^2(\mathbb{R})$ such that the set $\{\psi_{m,n}\ :\ m,n\in\mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$, where $\psi_{m,n}(x)=2^{m/2}\psi(2^m x-n)$, $m,n\in\mathbb{Z}$. So, any $f\in L^2(\mathbb{R})$ can be written as

$$f=\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}(f,\psi_{m,n})_{L^2(\mathbb{R})}\,\psi_{m,n}\ , \tag{2.3}$$

with convergence in $L^2(\mathbb{R})$-norm. The series representation of f in (2.3) is called a wavelet series. We will denote the wavelet coefficients of f with respect to the orthonormal wavelet ψ by $c^{\psi}_{m,n}(f)$, i.e.,

$$c^{\psi}_{m,n}(f)=(f,\psi_{m,n})_{L^2(\mathbb{R})}=\int_{-\infty}^{\infty}f(x)\bar{\psi}_{m,n}(x)\mathrm{d}x,\ \ m,n\in\mathbb{Z}\ .$$

Note that the relation between the wavelet coefficients and the wavelet transform of f is given by

$$c^{\psi}_{m,n}(f)=2^{-\frac{m}{2}}\mathcal{W}_{\psi}f\left(n2^{-m},2^{-m}\right)\ . \tag{2.4}$$

Since we are interested in tempered distributions, we need the orthonormal wavelets to be elements of $\mathcal{S}(\mathbb{R})$. It is well known that every orthonormal wavelet from $\mathcal{S}(\mathbb{R})$ must belong to the space $\mathcal{S}_0(\mathbb{R})$ [6, Cor.3.7, p.75]. The existence of such wavelets was first proved by Lemarié and Meyer [13, 16]; in [16], Meyer constructed orthonormal wavelets $\psi\in\mathcal{S}(\mathbb{R})$ such that $\hat{\psi}\in\mathcal{D}(\mathbb{R})$, arising from a Littlewood-Paley MRA [17, p.25]; the corresponding multidimensional wavelets of this type were found in [13].

3. Wavelet Expansion Theory on $\mathcal{S}_0(\mathbb{R})$ and $\mathcal{S}_0'(\mathbb{R})$

In this section we provide a wavelet expansion theory for the spaces $\mathcal{S}_0(\mathbb{R})$ and $\mathcal{S}_0'(\mathbb{R})$. We show the convergence of the wavelet series on these spaces. We always assume that the orthonormal wavelet $\psi\in\mathcal{S}_0(\mathbb{R})$. Therefore, it makes sense to consider the wavelet coefficients of $f \in \mathcal{S}_0'(\mathbb{R})$, defined as usual by

$$c^{\psi}_{m,n}(f) := \left\langle f, \bar{\psi}_{m,n} \right\rangle .$$

We then use wavelet expansions to characterize boundedness and convergence in $\mathcal{S}_0'(\mathbb{R})$, provided with the strong dual topology; for this purpose, we first describe a natural isomorphism of $\mathcal{S}_0(\mathbb{R})$ with a certain space of sequences identified with the wavelet coefficients. To describe the topology on $\mathcal{S}_0(\mathbb{R})$, we use the following family of seminorms

$$\|\phi\|^{\mathcal{S}_0}_l := \sup_{x\in\mathbb{R},\, 0\leq k\leq l} (1+|x|)^l \left|\phi^{(k)}(x)\right| \ , \quad l \in \mathbb{N}_0 \ .$$

3.1. **Convergence of Wavelet Expansions on $\mathcal{S}_0(\mathbb{R})$.** We first estimate the wavelet coefficients of functions from the space $\mathcal{S}_0(\mathbb{R})$.

Lemma 3.1. *Let $\psi \in \mathcal{S}_0(\mathbb{R})$ be an orthonormal wavelet. Then, for given $\beta, \gamma > 0$ there exists $l \in \mathbb{N}_0$ and a constant $C > 0$, which depend only on β and γ, such that*

$$\left|c^{\psi}_{m,n}(\phi)\right| \leq C \, \|\phi\|^{\mathcal{S}_0}_l \, (|n|+1)^{-\beta} \left(2^m + \frac{1}{2^m}\right)^{-\gamma} \ , \quad \forall \phi \in \mathcal{S}_0(\mathbb{R}) \ . \tag{3.1}$$

Proof. The proof follows from the relation (2.4) and the fact that $\mathcal{W}_\psi : \mathcal{S}_0(\mathbb{R}) \longmapsto \mathcal{S}(\mathbb{H})$ is a continuous linear map [8]. Therefore, for given $k, j \in \mathbb{N}_0$, $k > j$, there exists an integer l and a constant $C_{j,k} > 0$ such that

$$|c^{\psi}_{m,n}(\phi)| \leq C_{j,k} \, \|\phi\|^{\mathcal{S}_0}_l \left(1 + \frac{|n|}{2^m}\right)^{-j} \left(\frac{1}{2^m} + 2^m\right)^{-k}.$$

From the following inequalities

$$\left(1 + \frac{|n|}{2^m}\right)^{-j} \leq \begin{cases} \dfrac{1}{(1+|n|)^j}, & m < 0 \\[2ex] \dfrac{2^{mj}}{(1+|n|)^j}, & m \geq 0 \end{cases} \leq \frac{1}{(1+|n|)^j} \left(2^m + \frac{1}{2^m}\right)^j ,$$

we obtain

$$|c^{\psi}_{m,n}(\phi)| \leq C_{j,k} \, \|\phi\|^{\mathcal{S}_0}_l \, (1+|n|)^{-j} \left(\frac{1}{2^m} + 2^m\right)^{-(k-j)} .$$

Relation (3.1) follows by taking $j \geq \beta$ and $k \geq j + \gamma$. □

We now show convergence of the wavelet series in the topology of the space $\mathcal{S}_0(\mathbb{R})$.

Theorem 3.2. *Let $\phi \in \mathcal{S}_0(\mathbb{R})$ and let $\psi \in \mathcal{S}_0(\mathbb{R})$ be an orthonormal wavelet. Then ϕ can be expanded as*

$$\phi = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} c^{\psi}_{m,n}(\phi) \psi_{m,n} \ , \tag{3.2}$$

with convergence in $\mathcal{S}_0(\mathbb{R})$.

Proof. Observe that the fast decrease of the wavelet coefficients (3.1) implies that the series (3.2) converges uniformly to ϕ. If we differentiate (3.2), the derivatives are also uniformly convergent, by (3.1) once again. To show convergence in $\mathcal{S}_0(\mathbb{R})$, we need to prove that for each $l \in \mathbb{N}_0$

$$\lim_{\substack{M\to\infty \\ N\to\infty}} \left\| \phi - \sum_{|m|\le M} \sum_{|n|\le N} c^{\psi}_{m,n}(\phi)\psi_{m,n} \right\|^{\mathcal{S}_0}_{l} = 0$$

i.e.,

$$\lim_{\substack{M\to\infty \\ N\to\infty}} \sup_{x\in\mathbb{R},\, 0\le k\le l} (1+|x|)^l \Big| \sum_{|m|>M} \sum_{|n|>N} c^{\psi}_{m,n}(\phi)(\psi_{m,n}(x))^{(k)} \Big| = 0 \ .$$

We have then

$$\begin{aligned}
&\sup_{x\in\mathbb{R},\, 0\le k\le l} (1+|x|)^l \Big| \sum_{|m|>M} \sum_{|n|>N} c^{\psi}_{m,n}(\phi)(\psi_{m,n}(x))^{(k)} \Big| \\
&= \sup_{x\in\mathbb{R},\, 0\le k\le l} (1+|x|)^l \Big| \sum_{|m|>M} \sum_{|n|>N} c^{\psi}_{m,n}(\phi)(2^{\frac{m}{2}}\psi(2^m x - n))^{(k)} \Big| \\
&\le \sup_{x\in\mathbb{R},\, 0\le k\le l} (1+|x|)^l \sum_{|m|>M} \sum_{|n|>N} 2^{m(k+1/2)} |c^{\psi}_{m,n}(\phi)|\, |\psi^{(k)}(2^m x - n)| \\
&\le O(1) \sup_{x\in\mathbb{R},\, 0\le k\le l} (1+|x|)^l \sum_{|m|>M} \sum_{|n|>N} \left(2^m + \frac{1}{2^m}\right)^{k+1} \frac{|c^{\psi}_{m,n}(\phi)|}{(1+|2^m x - n|)^l} \\
&\le O(1) \sup_{x\in\mathbb{R}} (1+|x|)^l \sum_{|m|>M} \sum_{|n|>N} \frac{|c^{\psi}_{m,n}(\phi)|}{(1+|2^m x - n|)^l} \left(2^m + \frac{1}{2^m}\right)^{l+1} .
\end{aligned}$$

If we now use the elementary inequality

$$\frac{1+|x|}{1+|x-y|} \le 1+|y| \ ,$$

we obtain that

$$\begin{aligned}
\frac{1}{(1+|2^m x - n|)^l} \le \frac{(1+|n|)^l}{(1+2^m|x|)^l} &\le \begin{cases} \dfrac{2^{-ml}(1+|n|)^l}{(1/2^m + |x|)^l} \ , & m<0 \\[2ex] \dfrac{(1+|n|)^l}{(1+|x|)^l} \ , & m\ge 0 \end{cases} \\
&\le \frac{(1+|n|)^l}{(1+|x|)^l} \left(2^m + \frac{1}{2^m}\right)^l .
\end{aligned}$$

Therefore, from the last inequalities, we get

$$\sup_{x\in\mathbb{R},\,0\leq k\leq l}(1+|x|)^l\Big|\sum_{|m|>M}\sum_{|n|>N}c^{\psi}_{m,n}(\phi)(\psi_{m,n}(x))^{(k)}\Big|$$
$$\leq O(1)\sum_{|m|>M}\sum_{|n|>N}|c^{\psi}_{m,n}(\phi)|(1+|n|)^l\left(2^m+\frac{1}{2^m}\right)^{2l+1} .$$

Finally, the rapid decay of the wavelet coefficients (3.1) implies that the last term tends to 0. Indeed, it is enough to choose $\beta=l+2$ and $\gamma=2l+2$ in (3.1) to ensure that the term in the last inequality is less than $O(N^{-1}2^{-M})$. □

We obviously obtain the next corollary from Theorem 3.2.

Corollary 3.3. *Let $\psi\in\mathcal{S}_0(\mathbb{R})$ be an orthonormal wavelet. Then, the linear span of $\{\psi_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$ is dense in the space $\mathcal{S}_0(\mathbb{R})$.*

3.2. **Convergence of Wavelet Expansions on $\mathcal{S}_0'(\mathbb{R})$.** For the convergence of wavelet series expansions in the space $\mathcal{S}_0'(\mathbb{R})$, we first show the following lemma.

Lemma 3.4. *Let $\psi\in\mathcal{S}_0(\mathbb{R})$ be an orthonormal wavelet. The wavelet coefficients of $f\in\mathcal{S}_0'(\mathbb{R})$ satisfy an estimate*

$$|c^{\psi}_{m,n}(f)|\leq M(|n|+1)^{\beta}\Big(\frac{1}{2^m}+2^m\Big)^{\gamma} \tag{3.3}$$

for some $\beta,\gamma,M>0$ which depend on f.

Proof. The growth properties of $\mathcal{W}_\psi f$ on $\mathbb{H}$ imply that the wavelet coefficients satisfy an estimate of the form (2.2); the same argument as in the proof of Lemma 3.1 shows an estimate of the form (3.3) for the wavelet coefficients. □

From Theorem 3.2, Lemma 3.1, and Lemma 3.4, we easily obtain the following convergence result.

Theorem 3.5. *Let $\psi\in\mathcal{S}_0(\mathbb{R})$ be an orthonormal wavelet. Then, the wavelet expansion series of $f\in\mathcal{S}_0'(\mathbb{R})$,*

$$f=\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}c^{\psi}_{m,n}(f)\psi_{m,n}\ , \tag{3.4}$$

converges in (the strong dual topology of) $\mathcal{S}_0'(\mathbb{R})$.

Proof. We will show the weak convergence of (3.4), the strong convergence follows from the Banach-Steinhaus theorem [31]. Let $\phi\in\mathcal{S}_0(\mathbb{R})$. Since $\bar{\psi}$ is also an orthonormal wavelet, we have from Theorem 3.2

$$\phi=\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}c^{\bar{\psi}}_{m,n}(\phi)\bar{\psi}_{m,n}\ ,$$

with convergence in $\mathcal{S}_0(\mathbb{R})$. Using Lemma 3.1 and Lemma 3.4, we obtain the convergence of the wavelet series with coefficients $c^{\psi}_{m,n}(f)$. Moreover,

$$\begin{aligned}\langle f,\phi\rangle &= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} c^{\bar{\psi}}_{m,n}(\phi)\left\langle f,\bar{\psi}_{m,n}\right\rangle = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} c^{\psi}_{m,n}(f)\left\langle \psi_{m,n},\phi\right\rangle \\ &= \left\langle \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} c^{\psi}_{m,n}(f)\psi_{m,n},\ \phi\right\rangle ,\end{aligned}$$

which shows (3.4).

□

Corollary 3.6. *Let $\psi\in\mathcal{S}_0(\mathbb{R})$ be an orthonormal wavelet. Then, for $f\in\mathcal{S}_0'(\mathbb{R})$ and $\phi\in\mathcal{S}_0(\mathbb{R})$*

$$\langle f,\phi\rangle = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} c^{\psi}_{m,n}(f)c^{\bar{\psi}}_{m,n}(\phi)\ .$$

3.3. **The Space of Dyadic Rapidly Decreasing Sequences.** We say that a double sequence $\{c_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$ is of dyadic rapid decrease if

(3.5)

$$\|\{c_{m,n}\}\|^{\mathcal{W}}_{l} := \sup_{(m,n)\in\mathbb{Z}^2} |c_{m,n}|\,(1+|n|)^l\left(2^m+\frac{1}{2^m}\right)^l < \infty\ ,\ \text{for all } l\in\mathbb{N}_0\ .$$

We denote the space of all sequences satisfying (3.5) by $\mathcal{W}(\mathbb{Z}^2)$. We call it the space of *dyadic rapidly decreasing* sequences. The canonical Fréchet space topology in $\mathcal{W}(\mathbb{Z}^2)$ is defined by means of the seminorms (3.5). Its dual is $\mathcal{W}'(\mathbb{Z}^2)$, the space of dyadic slowly increasing sequences. One readily verifies that the elements of the dual space $\mathcal{W}'(\mathbb{Z}^2)$ are canonically identifiable with those sequences satisfying

$$\left\|\{c'_{m,n}\}\right\|^{\mathcal{W}'}_{-l} := \sup_{(m,n)\in\mathbb{Z}^2} |c_{m,n}|\,(1+|n|)^{-l}\left(2^m+\frac{1}{2^m}\right)^{-l} < \infty\ ,\ \text{for some } l\in\mathbb{N}_0\ .$$

So, we obtain the following isomorphisms.

Proposition 3.7. *Let $\psi\in\mathcal{S}_0(\mathbb{R})$ be an orthonormal wavelet. Then:*

(i) *the linear map $c^{\psi}:\mathcal{S}_0(\mathbb{R})\longmapsto\mathcal{W}(\mathbb{Z}^2)$ which takes $\phi\mapsto\left\{c^{\psi}_{m,n}(\phi)\right\}_{(m,n)\in\mathbb{Z}^2}$ is an isomorphism of Fréchet spaces.*

(ii) *the map which takes $f\in\mathcal{S}_0'(\mathbb{R})$ to its wavelet coefficients is an isomorphism of $\mathcal{S}_0'(\mathbb{R})$ onto $\mathcal{W}'(\mathbb{Z}^2)$ for the strong dual topologies.*

Proof. (i) The continuity of the map follows directly from Lemma 3.1. The map is injective because $\{\psi_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$ is an orthonormal basis of $L^2(\mathbb{R})$. That the map is onto can be shown as in the proof of Theorem 3.2, being its inverse $\{c_{m,n}\}\mapsto\sum\sum c_{m,n}\psi_{m,n}$. Finally, one easily shows that the inverse is continuous, for instance, applying the open mapping theorem [31].

For the map from (ii), it is enough to observe that its inverse is the transpose of the isomorphism $\phi \mapsto \left\{c^{\bar{\psi}}_{m,n}(\phi)\right\}$, so the assertion follows from (i). □

Remark 3.8. We can describe $\mathcal{W}'(\mathbb{Z}^2)$ as an inductive limit of an increasing sequence of Banach spaces. For each $l \in \mathbb{N}_0$, define

$$\mathcal{W}_{-l}(\mathbb{Z}^2) := \left\{ \{c'_{m,n}\} : \|\{c'_{m,n}\}\|^{\mathcal{W}'}_{-l} < \infty \right\} ,$$

with norm $\|\cdot\|^{\mathcal{W}'}_{-l}$; then,

$$\mathcal{W}'(\mathbb{Z}^2) = \bigcup_{l\in\mathbb{N}_0} \mathcal{W}_{-l}(\mathbb{Z}^2) = \operatorname{ind} \lim_{l\in\mathbb{N}_0} \mathcal{W}_{-l}(\mathbb{Z}^2) .$$

3.4. **Characterization of Boundedness and Convergence in $\mathcal{S}'_0(\mathbb{R})$ through Wavelet Coefficients.** We now use Proposition 3.7 and Remark 3.8 to obtain a characterization of the bounded sets in $\mathcal{S}'_0(\mathbb{R})$ in terms of localization of wavelet coefficients. Note that weak boundedness is equivalent to strong boundedness, due to the Banach-Steinhaus theorem [31].

Corollary 3.9. *Let $\psi \in \mathcal{S}_0(\mathbb{R})$ be an orthonormal wavelet. A subset $\mathfrak{B} \subset \mathcal{S}'_0(\mathbb{R})$ is (strongly) weakly bounded in $\mathcal{S}'_0(\mathbb{R})$ if and only if there exist constants $C, \beta, \gamma > 0$, which depend only on $\mathfrak{B}$, such that*

$$|c^{\psi}_{m,n}(f)| \le C(|n|+1)^{\beta}\left(2^m + \frac{1}{2^m}\right)^{\gamma} , \quad \forall f \in \mathfrak{B} . \tag{3.6}$$

Proof. By assertion (ii) of Proposition 3.7, $\mathfrak{B}$ is bounded if and only if $\left\{c^{\psi}_{m,n}(f) : f \in \mathfrak{B}\right\}$ is bounded in $\mathcal{W}'(\mathbb{Z}^2)$, and since $\mathcal{W}'(\mathbb{Z}^2)$ is the inductive limit of the Banach spaces $\mathcal{W}_{-l}(\mathbb{Z}^2)$, it holds if and only if the set $\left\{c^{\psi}_{m,n}(f) : f \in \mathfrak{B}\right\}$ lies in one of the spaces $\mathcal{W}_{-l}(\mathbb{Z}^2)$ and is bounded in a $\|\,\|^{\mathcal{W}'}_{-l}$ norm, which is obviously equivalent to (3.6). □

We point out that the above result is indeed a discretized version of [8, Thrm.28.0.1], which also provides a characterization of bounded sets but by uniform localization of the wavelet transform. Therefore, Corollary 3.9 gives an improvement when orthogonal wavelets are employed.

As a consequence of Corollary 3.9, we characterize convergent nets in $\mathcal{S}'_0(\mathbb{R})$. We state this result in the ensuing theorem.

Theorem 3.10. *Let $\psi \in \mathcal{S}_0(\mathbb{R})$ be an orthonormal wavelet. A net $\{f_\lambda\}_{\lambda\in\mathbb{R}}$ is (strongly) weakly convergent ($\lambda \to \infty$) in $\mathcal{S}'_0(\mathbb{R})$ if and only if each of the following limits exist*

$$\lim_{\lambda\to\infty} c^{\psi}_{m,n}(f_\lambda) = a_{m,n} < \infty , \tag{3.7}$$

and there exist constants $\lambda_0, C, \beta, \gamma > 0$, independent on m and n, such that

$$|c^{\psi}_{m,n}(f_\lambda)| \le C(|n|+1)^{\beta}\left(2^m + \frac{1}{2^m}\right)^{\gamma} , \quad \forall\, \lambda \ge \lambda_0 . \tag{3.8}$$

In such a case the limit functional, $\lim_{\lambda\to\infty} f_\lambda = g$*, satisfies* $c^{\psi}_{m,n}(g) = a_{m,n}$.

Proof. Assume (3.7) and (3.8). By Corollary 3.9, relation (3.8), and the Banach-Steinhaus theorem the net $\{f_\lambda\}_{\lambda\in\mathbb{R}}$ is strongly bounded in $\mathcal{S}_0'(\mathbb{R})$; on the other hand, as a consequence of (3.7), it is weakly convergent on the linear span of $\{\psi_{m,n}\}$, which turns out to be dense in $\mathcal{S}_0(\mathbb{R})$ (Corollary 3.3), hence the net is weakly convergent. The Montel property [31, p.358] of $\mathcal{S}_0(\mathbb{R})$ implies that it is a reflexive space, and therefore the net is in fact strongly convergent. Conversely, the weak convergence gives directly (3.7) while (3.8) is a consequence of Corollary 3.9. □

Remark 3.11. Let us observe that the orthogonality of $\{\psi_{m,n}\}$ has only been used in the proofs of Theorem 3.5 and Proposition 3.7. For the other results, the essential assumptions are the highly time-frequency localization of $\psi_{m,n}$ and the convergence of the wavelet series in $L^2(\mathbb{R})$, in combination with the trivial inequality

$$1 + a^{-jm} \leq \left(a^m + \frac{1}{a^m}\right)^j,$$

with $a = 2$. This yields the *open question* of whether the results of Section 3 can be generalized to the context of *wavelet frames* [2, Chap.3]. We point out a difficulty. While the building blocks for wavelet expansions with respect to orthogonal wavelets correspond to the functions $\psi_{m,n}$ themselves, for wavelet frames one should consider dual wavelet frames. As outlined in [2, p.70], the dual frame may have very poor regularity properties even if $\psi \in \mathcal{S}(\mathbb{R})$. Due to these facts, for example, the arguments given in Theorem 3.2 fail.

4. Asymptotic Behavior of Distributions

In this section we briefly explain the asymptotic notions for Schwartz distributions that we will study in the next sections. We measure the behavior of a distribution by comparison with Karamata regularly varying functions [1, 27], that is, the so called quasiasymptotic behavior of distributions [5, 19, 33, 37, 39].

Let us recall that a measurable real-valued function, defined and positive on an interval of the form $(0, A]$ (resp. $[A, \infty)$), $A > 0$, is called slowly varying at the origin (resp. at infinity) if

$$\lim_{\varepsilon\to 0^+} \frac{L(a\varepsilon)}{L(\varepsilon)} = 1 \ \left(\text{resp. } \lim_{\lambda\to\infty} \frac{L(a\lambda)}{L(\lambda)} = 1\right) \ \text{ for each } a > 0\ .$$

The standard references for slowly varying functions are [1, 27]. Observe that slowly varying functions are asymptotically invariant under rescaling at small scale (resp. large scale), and therefore, wavelet analysis is a very convenient tool for studying this class of functions [20, 21, 22, 23, 24, 25, 38].

Let L be a slowly varying function at the origin. We say that the distribution $f \in \mathcal{S}'(\mathbb{R})$ has quasiasymptotic behavior of degree $\alpha \in \mathbb{R}$ at the point $x_0 \in \mathbb{R}$ with respect to L if there exists $g \in \mathcal{S}'(\mathbb{R})$ such that for each $\phi \in \mathcal{S}(\mathbb{R})$

$$\lim_{\varepsilon \to 0^+} \left\langle \frac{f(x_0+\varepsilon x)}{\varepsilon^\alpha L(\varepsilon)},\, \phi(x) \right\rangle = \langle g(x), \phi(x) \rangle \ . \tag{4.1}$$

We also use the following convenient notation for the quasiasymptotic behavior,

$$f(x_0+\varepsilon x) \sim \varepsilon^\alpha L(\varepsilon) g(x) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{S}'(\mathbb{R}) \ ,$$

which should always be interpreted in the weak topology of $\mathcal{S}'(\mathbb{R})$, i.e., in the sense of (4.1). Sometimes, we also write

$$f(x_0+\varepsilon x) = \varepsilon^\alpha L(\varepsilon) g(x) + o(\varepsilon^\alpha L(\varepsilon)) \ \text{ as } \ \varepsilon \to 0^+ \ \text{ in } \mathcal{S}'(\mathbb{R}) \ .$$

One can prove that g cannot have an arbitrary form; indeed, it must be homogeneous with degree of homogeneity α [5, 19, 39]. We remark that all homogeneous distributions on the real line are explicitly known; indeed, they are linear combinations of either x_+^α and x_-^α, if $\alpha \notin \mathbb{Z}_-$, or $\delta^{(k-1)}(x)$ and x^{-k}, if $\alpha = -k \in \mathbb{Z}_-$, where we follow the notation from [5]. It can also be shown [37, Thrm.6.1] that if (4.1) holds just for each $\phi \in \mathcal{D}(\mathbb{R})$, then it must hold for each $\phi \in \mathcal{S}(\mathbb{R})$; therefore, the quasiasymptotic behavior at finite points is a local property. The quasiasymptotics of distributions at infinity with respect to a slowly varying function at infinity is defined in a similar manner, and the notation $f(\lambda x) \sim \lambda^\alpha L(\lambda) g(x)$ as $\lambda \to \infty$ in $\mathcal{S}'(\mathbb{R})$ will be used in this case.

We may also consider quasiasymptotics in other distribution spaces. The relation $f(x_0+\varepsilon x) \sim \varepsilon^\alpha L(\varepsilon) g(x)$ as $\varepsilon \to 0^+$ in $\mathcal{S}_0'(\mathbb{R})$ means that (4.1) is satisfied just for each $\phi \in \mathcal{S}_0(\mathbb{R})$; and analogously for quasiasymptotics at infinity in $\mathcal{S}_0'(\mathbb{R})$.

We need an additional notion from quasiasymptotic analysis, that of *quasiasymptotic boundedness* [35]. It will play the role of the Tauberian hypothesis in Section 7. We say that the distribution $f \in \mathcal{S}'(\mathbb{R})$ is quasiasymptotically bounded at $x_0 \in \mathbb{R}$ of degree $\alpha \in \mathbb{R}$ with respect to the slowly varying function at the origin L if

$$f(x_0+\varepsilon x) = O(\varepsilon^\alpha L(\varepsilon)) \text{ as } \varepsilon \to 0^+ \text{ in } \mathcal{S}'(\mathbb{R}) \ ;$$

as usual, the above relation should be interpreted in the weak topology of $\mathcal{S}'(\mathbb{R})$, i.e.,

$$\langle f(x_0+\varepsilon x),\, \phi(x) \rangle = O(\varepsilon^\alpha L(\varepsilon)) \text{ as } \varepsilon \to \ 0^+ \ , \tag{4.2}$$

for each $\phi \in \mathcal{S}(\mathbb{R})$. Quasiasymptotic boundedness of degree $\alpha \in \mathbb{R}$ with respect to the slowly varying function at infinity L is defined in a similar manner. We also consider quasiasymptotic boundedness in the space $\mathcal{S}_0'(\mathbb{R})$, meaning that (4.2) is satisfied just for each $\phi \in \mathcal{S}_0(\mathbb{R})$.

5. Wavelet Characterization of Quasiasymptotics in $\mathcal{S}'_0(\mathbb{R})$

In this section we apply the results from Section 3 to the study of quasiasymptotics of distributions. Our main goal is to provide necessary and sufficient conditions for the existence of the quasiasymptotic behavior in the space $\mathcal{S}'_0(\mathbb{R})$. In Section 7, we will use the results from this section to show Tauberian type results relating the asymptotics of wavelet coefficients and the quasiasymptotic behavior in the space $\mathcal{S}'(\mathbb{R})$.

From now on, we employ the following notation. Let $\psi \in \mathcal{S}_0(\mathbb{R})$ be an orthonormal wavelet and let $f \in \mathcal{S}'(\mathbb{R})$. We denote the wavelet coefficients of $f(x_0 + \varepsilon\,\cdot)$, $x_0 \in \mathbb{R}$, by

$$c^{\psi}_{m,n}(f;\varepsilon,x_0) = \langle f(x_0+\varepsilon x), \bar{\psi}_{m,n}(x)\rangle \ ,$$

and we name them the *perturbed wavelet coefficients* of f around the point x_0. If $x_0 = 0$, then we write $c^{\psi}_{m,n}(f;\varepsilon)$; for large parameters we use λ instead of ε.

The next two theorems provide complete "Tauberian" characterizations of quasiasymptotics in $\mathcal{S}'_0(\mathbb{R})$. We first consider the case of point behavior.

Theorem 5.1. *Let $\psi \in \mathcal{S}_0(\mathbb{R})$ be an orthonormal wavelet and let L be slowly varying at the origin. For $f \in \mathcal{S}'_0(\mathbb{R})$, the following two conditions,*

(i) *the limits*

$$\lim_{\varepsilon\to 0^+} \frac{c^{\psi}_{m,n}(f;\varepsilon,x_0)}{\varepsilon^{\alpha}L(\varepsilon)} < \infty \tag{5.1}$$

exist for each $m,n \in \mathbb{Z}$, and

(ii) *there exist $\beta,\gamma,C > 0$ such that*

$$|\, c^{\psi}_{m,n}(f;\varepsilon,x_0)| \le C\varepsilon^{\alpha}L(\varepsilon)(1+|n|)^{\beta}\left(2^m + \frac{1}{2^m}\right)^{\gamma} \ , \tag{5.2}$$

for all $m,n \in \mathbb{Z}$ and $0 < \varepsilon \le 1$,

are necessary and sufficient conditions for the existence of a distribution g such that

$$f(x_0+\varepsilon x) \sim \varepsilon^{\alpha}L(\varepsilon)g(x) \quad \text{as } \varepsilon \to 0^+ \ \text{ in } \ \mathcal{S}'_0(\mathbb{R}) \ . \tag{5.3}$$

Proof. It follows immediately from Theorem 3.10 by taking the net

$$f_{\lambda} := \frac{\lambda^{\alpha} f\left(x_0 + \lambda^{-1}\,\cdot\,\right)}{L(\lambda^{-1})} \ .$$

□

A similar assertion holds for quasiasymptotics at infinity.

Theorem 5.2. *Let $\psi \in \mathcal{S}_0(\mathbb{R})$ be an orthonormal wavelet and let L be slowly varying at infinity. For $f \in \mathcal{S}'_0(\mathbb{R})$, the following two conditions,*

(i) *the limits*

$$\lim_{\lambda\to\infty}\frac{c^{\psi}_{m,n}(f;\lambda)}{\lambda^{\alpha}L(\lambda)}<\infty \tag{5.4}$$

exist for each $m,n\in\mathbb{Z}$, *and*

(ii) *there exist* $\beta,\gamma,C>0$ *such that*

$$|\,c^{\psi}_{m,n}(f;\lambda)|\leq C\lambda^{\alpha}L(\lambda)(1+|n|)^{\beta}\left(2^{m}+\frac{1}{2^{m}}\right)^{\gamma}\ , \tag{5.5}$$

for all $m,n\in\mathbb{Z}$ *and* $1\leq\lambda$,

are necessary and sufficient conditions for the existence of a distribution g *such that*

$$f(\lambda x)\sim\lambda^{\alpha}L(\lambda)g(x)\ \ \text{as } \lambda\to\infty\ \ \text{in } \mathcal{S}'_{0}(\mathbb{R})\ . \tag{5.6}$$

Proof. It follows from Theorem 3.10 by taking the net

$$f_{\lambda}:=\frac{f\,(\lambda\,\cdot\,)}{\lambda^{\alpha}L(\lambda)}\ .$$

□

It is important to point out that in Theorems 5.1 and 5.2 the wavelet coefficients of g are given by the limits (5.1) and (5.4), respectively (cf. Theorem 3.10).

We now use Corollary 3.9 to characterize quasiasymptotic boundedness in the space $\mathcal{S}'_{0}(\mathbb{R})$.

Proposition 5.3. *Let* $\psi\in\mathcal{S}_{0}(\mathbb{R})$ *be an orthonormal wavelet. Suppose that* $f\in\mathcal{S}'_{0}(\mathbb{R})$ *and* L *is slowly varying at the origin. Then,* f *is quasiasymptotically bounded of degree* $\alpha\in\mathbb{R}$ *at* $x_0\in\mathbb{R}$, *with respect to* L, *in the space* $\mathcal{S}'_{0}(\mathbb{R})$ *if and only if an estimate of the form* (5.2) *holds.*

Proof. Consider the net $f_{\lambda}:=\lambda^{\alpha}f\left(x_0+\lambda^{-1}\,\cdot\,\right)/L(\lambda^{-1})$ in Corollary 3.9. □

Proposition 5.4. *Let* $\psi\in\mathcal{S}_{0}(\mathbb{R})$ *be an orthonormal wavelet. Suppose that* $f\in\mathcal{S}'_{0}(\mathbb{R})$ *and* L *is slowly varying at infinity. Then,* f *is quasiasymptotically bounded of degree* $\alpha\in\mathbb{R}$ *at infinity, with respect to* L, *in the space* $\mathcal{S}'_{0}(\mathbb{R})$ *if and only if an estimate of the form* (5.5) *holds.*

Proof. Take the net $f_{\lambda}:=f\,(\lambda\,\cdot\,)\,/\,(\lambda^{\alpha}L(\lambda))$ in Corollary 3.9. □

6. Abelian Theorems for Wavelet Coefficients

We now consider $f\in\mathcal{S}'(\mathbb{R})$. The first Abelian result follows directly from Theorems 5.1 and 5.2 (when applied to the projection of f onto $\mathcal{S}'_{0}(\mathbb{R})$). The Tauberian counterparts will be given in the next section.

Proposition 6.1. *Let* $f\in\mathcal{S}'(\mathbb{R})$ *and let* $\psi\in\mathcal{S}_{0}(\mathbb{R})$ *be an orthonormal wavelet.*

(i) *If f has the quasiasymptotic behavior* (5.3), *then for each $m, n \in \mathbb{Z}$,*

$$\lim_{\varepsilon \to 0^+} \frac{c^{\psi}_{m,n}(f;\varepsilon, x_0)}{\varepsilon^{\alpha} L(\varepsilon)} = c^{\psi}_{m,n}(g) \ . \tag{6.1}$$

(ii) *If f has the quasiasymptotic behavior* (5.6), *then for each $m, n \in \mathbb{Z}$,*

$$\lim_{\lambda \to \infty} \frac{c^{\psi}_{m,n}(f;\lambda)}{\lambda^{\alpha} L(\lambda)} = c^{\psi}_{m,n}(g) \ . \tag{6.2}$$

We mention that other Abelian type results were obtained by one of the authors in [25]. For instance, one can show that if f has the quasiasymptotic behavior (5.3) at $x_0 = 0$, then for each fixed $n \in \mathbb{Z}$,

$$c^{\psi}_{m,n}(f) \sim \frac{C_n}{2^{m(\alpha + \frac{1}{2})}} L\left(2^{-m}\right) \quad \text{as } m \to \infty,$$

where $C_n = \langle g(x), \psi(x-n) \rangle$. A similar result holds for quasiasymptotics at infinity.

Corollary 6.2. *Let $\psi \in \mathcal{S}_0(\mathbb{R})$ be an orthonormal wavelet. Let L be slowly varying function at infinity. Suppose that $f \in \mathcal{S}'(\mathbb{R})$ has the quasiasymptotic behavior* (5.6), *then for each $n \in \mathbb{Z}$*

$$c^{\psi}_{-m,n}(f) \sim C_n 2^{m\left(\alpha + \frac{1}{2}\right)} L\left(2^m\right) \quad \textit{as} \quad m \to \infty \ ,$$

where $C_n = \langle g(x), \psi(x-n) \rangle$.

Proof. By part (ii) of Proposition 6.1, we obtain

$$c^{\psi}_{-m,n}(f) = 2^{\frac{m}{2}} c^{\psi}_{0,n}(f; 2^m) \sim c^{\psi}_{0,n}(g) 2^{m\left(\frac{1}{2} + \alpha\right)} L(2^m) = C_n 2^{m\left(\alpha + \frac{1}{2}\right)} L\left(2^m\right) \ .$$

□

7. Tauberian Theorems for Wavelet Coefficients

We now investigate the inverse (Tauberian) theorems related to Proposition 6.1. Observe that the results from Section 5 have already a Tauberian nature; however, they give a characterization of the asymptotic properties of the tempered distribution in the restricted space $\mathcal{S}'_0(\mathbb{R})$. Our main aim is now to obtain the asymptotic behavior in the space $\mathcal{S}'(\mathbb{R})$ under natural Tauberian hypotheses.

In Section 5 we obtained the asymptotic behavior of the distribution in $\mathcal{S}'_0(\mathbb{R})$ from the asymptotics of the wavelet coefficients, so our problem reduces to the following question: What information does the quasiasymptotic behavior in $\mathcal{S}'_0(\mathbb{R})$ give us about quasiasymptotics in the space $\mathcal{S}'(\mathbb{R})$? We may refer to this question as a *quasiasymptotic extension problem.* Such a problem has been recently studied in detail by the second author and collaborators [38, 34]. It is also connected with the results from [33, Sec.4].

As a result of the extension of quasiasymptotic relations, new asymptotic terms may appear in certain situations. In fact, since polynomials are invisible for the quasiasymptotics in $\mathcal{S}'_0(\mathbb{R})$, one should expect new polynomial terms to appear in the quasiasymptotic expansions in the space $\mathcal{S}'(\mathbb{R})$.

We now combine Theorems 5.1 and 5.2 with the results from [38, 34] and obtain the desired Tauberian theorems. In the next two subsections we first consider quasiasymptotics with degree $\alpha \notin \mathbb{N}_0$.

7.1. **Tauberians for Quasiasymptotics at Finite Points.** In this subsection L denotes a slowly varying function at the origin. The next two theorems describe the full asymptotic expansion of a distribution whose perturbed wavelet coefficients have the asymptotics (6.1) together with the estimate (5.2); their proofs follow at once by applying Theorem 5.1 and [38, Cor.1] (see also [34, p.354]). In Theorem 7.3 below, we actually recover the quasiasymptotic behavior, provided with the Tauberian hypothesis of quasiasymptotic boundedness at the point.

Theorem 7.1. *Let $\psi \in \mathcal{S}_0(\mathbb{R})$ be an orthonormal wavelet. Let $f \in \mathcal{S}'(\mathbb{R})$ and let $\alpha < 0$. The conditions* (5.1) *and* (5.2) *are necessary and sufficient for the existence of a homogeneous distribution g of degree α such that*

$$f(x_0 + \varepsilon x) \sim \varepsilon^{\alpha} L(\varepsilon) g(x) \ \text{ as } \ \varepsilon \to 0^+ \ \text{ in } \ \mathcal{S}'(\mathbb{R}) \ . \tag{7.1}$$

Theorem 7.2. *Let $\psi \in \mathcal{S}_0(\mathbb{R})$ be an orthonormal wavelet. Let $f \in \mathcal{S}'(\mathbb{R})$ and suppose $\alpha > 0$, $\alpha \notin \mathbb{N}$. The conditions* (5.1) *and* (5.2) *are necessary and sufficient for the existence of a polynomial p of degree less than α and a homogeneous distribution g of degree α such that*

$$f(x_0 + \varepsilon x) = p(\varepsilon x) + \varepsilon^{\alpha} L(\varepsilon) g(x) + o(\varepsilon^{\alpha} L(\varepsilon)) \ \text{ as } \ \varepsilon \to 0^+ \ \text{ in } \ \mathcal{S}'(\mathbb{R}) \ . \tag{7.2}$$

If we assume that the distribution is quasiasymptotically bounded at the point, then the polynomial terms in (7.2) do not occur.

Theorem 7.3. *Let $\psi \in \mathcal{S}_0(\mathbb{R})$ be an orthonormal wavelet and let $\alpha \notin \mathbb{N}_0$. Suppose that the perturbed wavelet coefficients of $f \in \mathcal{S}'(\mathbb{R})$ satisfy the condition* (5.1)*. Then, the Tauberian hypothesis*

$$f(x_0 + \varepsilon x) = O\left(\varepsilon^{\alpha} L(\varepsilon)\right) \ \text{ as } \ \varepsilon \to 0^+ \ \text{ in } \ \mathcal{D}'(\mathbb{R}) \ , \tag{7.3}$$

implies the existence of a homogeneous distribution g of degree α such that (7.1) *is satisfied.*

Proof. Quasiasymptotic boundedness in the space $\mathcal{S}'(\mathbb{R})$ obviously implies quasiasymptotic boundedness in $\mathcal{S}'_0(\mathbb{R})$, hence, by Proposition 5.3, the estimate (5.2) is satisfied. Next, if $\alpha < 0$, Theorem 7.1 gives (7.1) at once. On the other hand, if $\alpha > 0$, Theorem 7.2 implies the existence of $p(x) = \sum_{n=0}^{[\alpha]} a_n x^n$ and a homogeneous distribution g of degree α such that f satisfies (7.2). Select $\varphi \in \mathcal{S}(\mathbb{R})$ with first $[\alpha]$ moments equal 1 (such test functions always exist [3]). Testing (7.2) at φ and using (7.3), we obtain, as $\varepsilon \to 0^+$,

$$O(\varepsilon^{\alpha}L(\varepsilon)) = \langle f(x_0+\varepsilon x), \varphi(x)\rangle = \sum_{n=0}^{[\alpha]} a_n \varepsilon^n + O(\varepsilon^{\alpha}L(\varepsilon)) \ .$$

Now, select $0<\sigma<\alpha-[\alpha]$. Since $L(\varepsilon)=O(\varepsilon^{-\sigma})$ [27], we have that

$$p(\varepsilon) = O(\varepsilon^{\alpha-\sigma}) \ ,$$

and thus $p\equiv 0$. Hence (7.1) holds. □

Remark 7.4. Observe that the converse of Theorem 7.3 is also true, that is, the asymptotic relation (7.1) implies (7.3) and (5.1). Therefore, Theorem 7.3 is a complete Tauberian wavelet characterization of the local behavior of a distribution when the degree of the quasiasymptotic behavior does not belong to $\mathbb{N}_0$.

Remark 7.5. Clearly, the wavelet coefficients $c^{\psi}_{m,n}(g)$ in Theorems 7.1–7.3 are given by the limits (6.1). Moreover, g is uniquely determined by its wavelet coefficients, due to its homogeneity.

7.2. **Tauberians for Quasiasymptotics at Infinity.** Throughout this subsection, L denotes a slowly varying function at infinity. The proof of the next theorem follows directly from Theorem 5.2 and [38, Cor.2] (see also [34, p.355]).

Theorem 7.6. *Let $\psi\in\mathcal{S}_0(\mathbb{R})$ be an orthonormal wavelet. Let $f\in\mathcal{S}'(\mathbb{R})$ and let $\alpha\notin\mathbb{N}_0$. The conditions* (5.4) *and* (5.5) *are necessary and sufficient for the existence of a polynomial p, which may be chosen to be divisible by $x^{\max\{0,\,[\alpha]+1\}}$, and a homogeneous distribution g of degree α such that*

$$f(\lambda x) = p(\lambda x) + \lambda^{\alpha}L(\lambda)g(x) + o(\lambda^{\alpha}L(\lambda)) \ \text{ as } \ \lambda\to\infty \ \text{ in } \ \mathcal{S}'(\mathbb{R}) \ .$$

In the next Tauberian theorem we give an analog to Theorem 7.3. The proof is identically the same as that of Theorem 7.3, we therefore omit it.

Theorem 7.7. *Let $\psi\in\mathcal{S}_0(\mathbb{R})$ be an orthonormal wavelet and let $\alpha\notin\mathbb{N}_0$. Suppose that the perturbed wavelet coefficients of $f\in\mathcal{S}'(\mathbb{R})$ satisfy the condition* (5.4)*. Then, the Tauberian hypothesis*

$$f(\lambda x) = O\left(\lambda^{\alpha}L(\lambda)\right) \ \text{ as } \ \lambda\to\infty \ \text{ in } \ \mathcal{D}'(\mathbb{R}) \ ,$$

implies the existence of a homogeneous distribution g of degree α such that

$$f(\lambda x) \sim \lambda^{\alpha}L(\lambda)g(x) \ \text{ as } \ \lambda\to\infty \ \text{ in } \ \mathcal{S}'(\mathbb{R}) \ .$$

Remark 7.8. The same considerations of Remarks 7.4 and 7.5 apply to the case at infinity: Theorem 7.7 is a complete Tauberian wavelet characterization of the quasiasymptotic behavior at infinity when the degree does not belong to $\mathbb{N}_0$; the wavelet coefficients $c^{\psi}_{m,n}(g)$ in Theorems 7.6 and 7.7 uniquely determine g and they are given by the limits (6.2).

7.3. **Final Remark: Quasiasymptotics with Degree $\alpha \in \mathbb{N}_0$.** One can also characterize the quasiasymptotics when the degree is a non-negative integer, but in this case the description becomes more complex.

On combining once again Theorem 5.1 and [38, Cor.1] (cf. [34, p.354]), one obtains that the conditions (5.1) and (5.2) with $\alpha = k \in \mathbb{N}_0$ are necessary and sufficient for the existence of a polynomial p of degree at most $k-1$, a function $b \in C(0,\infty)$, and a distribution $g \in \mathcal{S}'(\mathbb{R})$ such that

$$f(x_0+\varepsilon x) = p(\varepsilon x)+\varepsilon^k b(\varepsilon)x^k+\varepsilon^k L(\varepsilon)g(x)+o(\varepsilon^k L(\varepsilon)) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{S}'(\mathbb{R})\ .$$

Here the distribution g is not homogeneous but associate homogeneous of order 1 and degree k [5, 28], specifically, it has the form

$$g(x) = C_- x_-^k + C_+ x_+^k + A x^k \log|x| \ , \tag{7.4}$$

for some constants. Furthermore, b satisfies

$$b(ax) = b(x) + AL(x)\log a + o(L(x)) \quad \text{as } x \to 0^+ \ , \quad \text{for each } a > 0\ . \tag{7.5}$$

Similarly, (5.4) and (5.5) with $\alpha = k \in \mathbb{N}_0$ are necessary and sufficient for

$$f(\lambda x) = p(\lambda x) + \lambda^k b(\lambda)x^k + \lambda^k L(\lambda)g(x) + o(\lambda^k L(\lambda)) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R})\ ,$$

where p is a polynomial, which may be chosen to be divisible by x^{k+1}, g has the form (7.4), and the function b satisfies (7.5) but now as $x \to \infty$.

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