A Tauberian approach to Blackwell's renewal theorem

(by Jasson Vindas, Analysis Seminar, 30-4-24, Ghent)

$\boxed{1}$ **Renewal theorem:** A typical situation we want to model is the following one: Suppose we installed a new lightbulb at $t=0$ whose lifespan is finite and positive. If the bulb stops working, we replace it, and so on.

To model this probabilistically, let $\{X_n\}_{n=1}^{\infty}$ be a sequence of non-negative independent random variables, equally distributed according to a probability law supported on $[0,\infty)$

$$F(t) = P\{X_n \leq t\}, \quad n=1,\dots$$

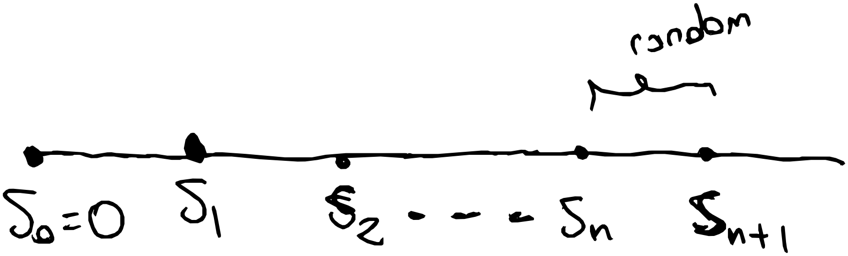
We define

$$S_n = \sum_{j=1}^{n} X_j \quad \text{called renewal } \underline{\text{epochs}} \ (S_0=0)$$

We are interested in

$$N_t = \max\{k : S_k \leq t\},$$

which in our model situation is the number of failures up to time t.



We called $N = \{N_t\}_{t \geq 0}$ the <u>renewal process</u>.

$$S_{N_t} \leq t < S_{N_t+1}$$

$A(t) = t - S_{N_t}$: age of the unit

$R(t) = S_{N_t+1} - t$ = remaining lifespan

$T(t) = A(t) + R(t)$: total lifespan.

In our problem, at time t the lightbulb in use is the $(N_t$

<u>Definition</u>: The renewal function is the expectation of each N_t:

$$U(t) = E(N_t + 1), \quad t \geq 0. ///$$

To relate U to the probability law, we notice

$$P\{N_t = n\} = P\{S_n \leq t < S_{n+1}\}$$

$$= P\{S_n \le t\} - P\{S_{n+1} \le t\}$$

$$= F^{n*}(t) - F^{(n+1)*}(t)$$

Therefore,

$$U(t) = \sum_{n=0}^{+\infty} (n+1) P\{N_t = n\}$$

$$= \sum_{n=0}^{\infty} F^{n*}(t).$$

$$\Rightarrow \quad dU = \delta + \sum_{n=1}^{\infty} dF^{n*},$$

where δ is the Dirac delta and $*$ is convolution of measures.

$\boxed{2}$ The renewal theorem. We call dF lattice if there is $\alpha > 0$ such that dF is supported on $\alpha \mathbb{N} = \{\alpha, 2\alpha, \ldots\}$. Otherwise, dF is non-lattice.

Remark. In the lattice case the maximal α is called its span.

Theorem (Renewal theorem). Let $\mu = \int_0^\infty x\, dF(x)$

① (Blackwell, 1948). If dF is non-lattice, then for each $h > 0$

$$U(x+h) - U(x) \longrightarrow \frac{h}{\mu}, \qquad x \to +\infty \qquad (3)$$

② (Kolmogorov 1938, Erdős-Feller-Pollard 1949)
If dF is lattice, the previous relation holds with $h = n\alpha$, $n \in \mathbb{N}$. ///

We give a Tauberian proof.

3 A Tauberian theorem:

Our proof is based on the following Tauberian theorems of B. Chen and myself (see more general versions at arXiv:2311.03013).

Theorem 2.

① Let S be non-decreasing with convergent Laplace-Stieltjes transform $\mathcal{L}\{dS; s\}$ on $\operatorname{Re} s > 1$. If $\operatorname{Re} \mathcal{L}\{dS; s\}$ is non-negative on $\sigma \in (-\lambda, \lambda) \times (1, 2]$ and has L^1_{loc}-boundary behavior on $1 + i(\mathbb{R} \setminus \{0\})$, then there is a s.t.

$$S(x) \sim a e^x, \quad x \to \infty.$$

② Let $F(z) = \sum_{n=0}^{\infty} c_n z^n$ be convergent in $\mathbb{D} = \{z : |z| < 1\}$. If $U(z) = \operatorname{Re} F(z)$ is non-negative, $\{c_n\}$ is real, and has L^1_{loc}-boundary behaviour on $\partial \mathbb{D} \setminus \{1\}$, then

$\lim c_n$ exists.

[4] Proof of Theorem 1.

① $S(x) = \int_{0-}^{\infty} e^x \, dU(x)$. We have

$$\mathcal{L}\{dS; s\} = \mathcal{L}\{\sum_{n=0}^{\infty} dF^{n*}; s-1\}$$

$$= \frac{1}{1-G(s)}, \quad G(s) = \mathcal{L}\{dF; s-1\}.$$

F is non-lattice $\Rightarrow G(s) \neq 1$ $\forall s \in \{s : s \neq 1 \text{ and } \operatorname{Re} s \geq 1\}$.

Also

$$\operatorname{Re} \mathcal{L}\{dS; s\} = \frac{1 - \int_0^{\infty} e^{-(1-\sigma)x} \cos(tx) \, dF(x)}{|1-G(s)|^2} \geq 0, \quad \sigma = \operatorname{Re} s.$$

Therefore, by Theorem 2 ①, there is a s.t.

$$S(x) \sim a e^x.$$

Actually $a = \lim_{\sigma \to 1} (\sigma - 1) \mathcal{L}\{dS; \sigma\} = \frac{1}{\mu}.$

Finally, $dU(t) = e^{-t}\, dS(t)$. We also write

$$r(x) = \frac{S(x)}{e^{x}} - \frac{1}{\mu} = o(1).$$ Then

$$U(x) = \frac{x+1}{\mu} + r(x) + \int_0^x r(u)\,du = \frac{x+1}{\mu} + o(1) + \int_0^x o(1)\,du$$

which ends the proof.

(2) In this case $dF(x) = \sum_{n=0}^{+\infty} p_n \delta(x - \alpha n)$

and $dU(x) = \sum_{n=0}^{\infty} q_n \delta(x - n\alpha)$ with $q_0 = 1$, $p_0 = 0$

and $\sum p_n = 1$, and finally

$$q_n = \sum_{k=1}^{n} p_k q_{n-k}.$$

The proof is completely analogous to the first part if one looks at the generating functions of $\{p_n\}$ and $\{q_n\}$. From Theorem 2 part (2) one deduces

$$\lim_{n\to\infty} q_n \text{ exists},$$

which is the statement that should be shown. ///