Asymptotic formulas for partitions, I.

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Tauberian Methods

by Jasson Vindas
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Abstract: These are lecture notes of the first of a series of talks that we delivered on the topic of analytical methods in combinatorial analysis. Specifically, we focus on partition problems.

The aim of the notes is to provide an introduction to Tauberian methods in partition theory. We start by making some comments about the beautiful Hardy-Ramanujan-Uspensky asymptotic formula for the number of unrestricted partitions of integers into integral parts. Then, we discuss Knopp result for weak asymptotics of partitions. The final topic concerns Ingham's approach to strong asymptotic formulas for partitions.

The introductory material presented here serves as preparation for the next lecture notes. In the last note, I will report on the solution of an open problem concerning a multiplicative partition problem arising when coding rooted trees with prime numbers.

1 Introduction

One of the simplest arithmetical questions we could ask is the number of ways of writing a positive integer n as sums of integral parts, namely, as $m+l+h+\cdots$, where $m, l, h, \ldots$ are positive integers. There are then two possible questions here, depending on whether we choose to count the order of the summands or not. If we decide that the order of the summands matters, then the question turns out to be a trivial exercise of induction: the answer is 2^{n-1} ways!

On the order hand, the problem becomes surprisingly deeper if we decide that the order does not matter for partitioning n. So, for instance the number $n=4$ has 5 partitions

$$4 = 1+1+1+1$$
$$4 = 1+1+2 \quad (= 1+2+1 = 2+1+1)$$
$$4 = 1+3 \quad (= 3+1)$$
$$4 = 2+2$$
$$4 = 4$$

In general an unrestricted partition of n (into integral parts) is a representation of the form

(1.1) $$n = m_1 \cdot 1 + m_2 \cdot 2 + \cdots + m_n \cdot n,$$

where $m_i \geq 0$. The problem of interest is then to study the number of unrestricted partitions (1.1) of n. Let us denote such a number by $p(n)$. Thus,

(1.2) $$p(n) = \#\left\{(m_1, \ldots, m_n) \in \mathbb{N}^n : n = \sum_{j=1}^{n} m_j \cdot j \right\}.$$

In 1917 [6], Hardy and Ramanujan found an asymptotic formula for $\log p(n)$. Indeed, they showed that

(1.3) $$\log p(n) \sim \pi \sqrt{\frac{2n}{3}}, \quad n \to \infty.$$

The asymptotic formula (1.3) is already quite surprising and certainly not easy to guess, but they went way further and one year [7] later they achieved the celebrated formula

(1.4) $$p(n) \sim \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{(4\sqrt{3})\,n}, \quad n \to \infty.$$

It is certainly fair to mention that (1.4) was also obtained, indepently, by Uspensky in 1920 [14].

After them, researchers tended to go into two directions. One direction was a deeper study of this and similar partition problems, the chief representative result being the exact formula of Rademacher for $p(n)$ [13]. The other approach goes in the direction of a broader treatment and considering partition problems into "non-integral parts". These notes go into the second direction. We will review classical Tauberian methods for deriving asymptotic properties of slightly more general unrestricted partition functions than (1.2).

$\boxed{2}$ Generating functions and unrestricted partitions

We shall consider in the sequel a more general partition problem than that discussed in $\boxed{1}$. Let

$$0<\lambda_1<\lambda_2<\lambda_3<\cdots$$

be an increasing sequence of real numbers tending to infinity and let N be its counting function, i.e.,

$$N(u)=\sum_{\lambda_k<u} 1 \quad (2.1)$$

For a real number $\varkappa>0$, we consider the unrestricted partition function

$$p(\varkappa), \quad (2.2)$$

that is, $p(\mu)$ is the number of representations of μ as

(2.3) $\mu = m_1\lambda_1 + m_2\lambda_2 + \cdots \qquad (m_j \geq 0)$.

Finally, we set for $u \geq 0$,

(2.4) $$P(u) = \sum_{\mu < u} p(\mu) \quad .$$

where the summation is taken over those (finitely many) $\mu < u$ such that $p(\mu) \neq 0$.

The problem of interest is then to study the relationship between asymptotic properties of N and those of either p or P.

We may translate these problems into the study of an analytic function, the generating function which intrinsically encodes the involved combinatorial information.

The generating function is then defined as (assuming the integral is convergent for $\operatorname{Re} s > 0$):

(2.5) $$F(s) = \int_0^\infty e^{-su}\, dP(u) = \sum_\mu p(\mu) e^{-\mu s} = s\int_0^\infty e^{-su} P(u)\, du.$$

We have the following "Euler product formula"

(2.6) $$F(s) = \prod_{k=1}^{\infty} \frac{1}{1-e^{-s\lambda_k}} \; .$$

[Proof of (2.6): $\prod_{k=1}^{\infty} \frac{1}{1-e^{-\lambda_k s}} = \prod_{k=1}^{\infty} \left(1+e^{-\lambda_k s}+e^{-2\lambda_k s}+\cdots\right)$

$$= \sum e^{-s(m_1\lambda_1+m_2\lambda_2+\cdots+m_k\lambda_k)} = \sum_{\mu} p(\mu) e^{-s\mu}.$$]

Now, taking logarithm in (2.6), we see that

$$f(s) = \log F(s) = -\sum_{k=1}^{\infty} \log(1-e^{-s\lambda_k})$$
$$= -\int_0^{\infty} \log(1-e^{-su})\, dN(u)$$
$$= s\int_0^{\infty} \frac{1}{e^{su}-1} N(u)\, du$$

Thus, the combinatorics of our problem looks like

(2.7) $$F(s) = e^{f(s)}$$

where,

(2.8) $$f(s) = s\int_0^{\infty} \frac{1}{e^{su}-1} N(u)\, du.$$

A typical Tauberian theorem obtains the asymptotic behavior of a function from that of an integral transformation of it. So, (2.7) and (2.8) tell us that we are in front of a Tauberian problem!

$\boxed{3}$ Weak Asymptotics for Partitions

From now on, we retain the notation introduced in $\boxed{2}$.
The title of this section refers to the problem of relating the asymptotic behaviour of N with that of $\log P$ (cf. (2.1) and (2.4)).
In 1925, Knopp [9] showed the following theorem.

We denote below by Γ the Euler gamma-function and by ζ the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

Theorem 1 Let $\alpha > 0$.

$$(3.1) \quad N(u) = \sum_{\lambda_k < u} 1 \sim A u^{\alpha}, \quad u \to \infty$$

if and only if

$$(3.2) \quad \log P(u) \sim M_\alpha A^{\frac{1}{\alpha+1}} u^{\frac{\alpha}{\alpha+1}}, \quad u \to \infty,$$

where $M_\alpha = \left(1 + \frac{1}{\alpha}\right) \left[\alpha \Gamma(\alpha+1) \zeta(\alpha+1)\right]^{\frac{1}{\alpha+1}}$ //

Knopp's proof is via the generating function.
Erdős [3] found an elementary treatment for the proof of some cases of Theorem 1.

Let us apply this result to the case $\lambda_k = k$.

Example 1 When $\lambda_k = k$, $N(u) = [u]$.

so (3.1) holds with $A=1$ and $\alpha=1$.
Since $\Gamma(2)=1$ and $\zeta(2)=\frac{\pi^2}{6}$, we obtain that

$$\log \sum_{n\leq u} p(n) \sim 2\cdot\frac{\pi}{\sqrt{6}}\sqrt{u} = \pi\sqrt{\frac{2}{3}u},$$

which is slightly weaker than (1.3), but still gives valuable information.

Theorem 1 can be deduced from the following two Tauberian theorems (due to Kohlbecker [11] in more generality, see also [1, p.230 and Sec.4.12])

We first state the connection between (3.1) and the transform (2.8). That is the content of the ensuing result:

Theorem 2 Let $\tilde{N}$ be a non-decreasing function on $[0,\infty)$ that vanishes near 0. If $\alpha>0$, then

$$\tilde{N}(u) \sim Au^{\alpha}, \quad u\longrightarrow\infty$$

iff

$$f(s) := s\int_0^{\infty}\frac{1}{e^{su}-1}\tilde{N}(u)\,du \sim A\zeta(\alpha+1)\Gamma(\alpha+1)\left(\frac{1}{s}\right)^{\alpha}, \quad s\longrightarrow 0^+.$$

Thus Theorem 2 reduces the proof of Theorem 1 to the study of the connection between (3.2) and $\log F(s)$ (cf. (2.5)).

Theorem 3 Let $\tilde{P}$ be a non-decreasing function that vanishes for $u \le 0$. Assume that its Laplace-Stieltjes transform converges for $s > 0$,

$$F(s) = \int_0^\infty e^{-su} d\tilde{P}(u) = s \int_0^\infty e^{-su} \tilde{P}(u) du .$$

If $\alpha > 0$, then

$$\log F(s) \sim B \left(\frac{1}{s}\right)^\alpha, \quad s \to 0^+,$$

if and only if

$$\log \tilde{P}(u) \sim \left(1 + \frac{1}{\alpha}\right)(B\alpha)^{\frac{1}{\alpha+1}} u^{\frac{\alpha}{\alpha+1}}.$$

We shall not prove these theorems, we rather refer to the already cited references. We have stated here only the simplest case of Theorem 1. It should be stressed that it admits a general form, due to Kohlbecker [11], involving regularly varying functions [1], in particular asymptotic comparison formulas with functions such as

$$K u^\alpha (\log u)^{a_1} (\log\log u)^{a_2} \cdots (\log_K u)^{a_K},$$

important in applications. Finally, let us mention that Kohlbecker's theorem for partitions can be conceptually better understood by employing the concept of de Bruijn conjugate slowly varying functions [2,1].

$\boxed{4}$ Strong asymptotic formulas

We now turn our attention to the problem of strong asymptotic formulas, that is, the connection between asymptotics of N and those of P. The following result was shown by Ingham in 1940 [8], it is a very strong tool in the theory of partitions. Let N and P be as before. Furthermore, for $h>0$ and u real, define

$$P_h(u)=\frac{P(u)-P(u-h)}{h}$$

Theorem 4 If

$$N(u)=Au^{\alpha}+R(u), \quad (\alpha>0, A>0),$$

where

$$\int_0^u \frac{R(t)}{t}dt=B\log u+C+o(1), \quad u\to\infty. \tag{4.1}$$

Then, $u \to \infty$,

$$P(u) \sim \left(\frac{1-\beta}{2\pi}\right)^{\frac{1}{2}} e^{C} M^{-(B+\frac{1}{2})\beta} u^{(B+\frac{1}{2})(1-\beta)-\frac{1}{2}} e^{\frac{(Mu)^{\beta}}{\beta}},$$

where

$$\beta = \frac{\alpha}{\alpha+1}, \quad M = \left[A\alpha\Gamma(\alpha+1)\zeta(\alpha+1)\right]^{\frac{1}{\alpha}}.$$

Moreover, if $P_h(u)$ is increasing,

$$P_h(u) \sim \left(\frac{1-\beta}{2\pi}\right)^{\frac{1}{2}} e^{C} M^{-(B-\frac{1}{2})\beta} u^{(B-\frac{1}{2})(1-\beta)-\frac{1}{2}} e^{\frac{(Mu)^{\beta}}{\beta}}.$$

Remark:

P_h is certainly increasing if h is an element of $\{\lambda_k\}$.

The proof of Theorem 4 depends upon a deep complex Tauberian theorem of Ingham, which we will not recall here. See Ingham's original paper [8] or Korevaar's book [10, p. 229]. We derive in the next section Hardy-Ramanujan-Uspensky theorem from Theorem 4.

$\boxed{5}$ Hardy-Ramanujan-Uspensky formula.

Corollary 1: If $\lambda_k = k \in \mathbb{N}$, then

$$(5.1) \quad p(n) \sim \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{(4\sqrt{3})n}, \quad n \to \infty$$

Proof: All we have to do is to estimate

$$\int_0^U \frac{R(t)}{t}\,dt,$$

where $N(U) = [U] = U + R(U)$, and then use the formula for $P_1(n) = P(n) - P(n-1) = p(n)$.

So,

$$\int_0^U \frac{[t]-t}{t}\,dt = \int_0^U \frac{[t]}{t}\,dt - U$$

$$= [U]\log U - \sum_{n\le U} \log n - U$$

But Stirling's formula [4, p. 43] tells us that

$$\log N! \sim \left(N + \tfrac{1}{2}\right)\log N - N + \log\sqrt{2\pi}$$

So we get

$$\int_0^U \frac{R(t)}{t}\,dt = -\frac{1}{2}\log U - \frac{1}{2}\log 2\pi$$

Thus in Theorem 4 we have that

$\alpha = 1$, $A = 1$, $B = -\frac{1}{2}$, $C = -\frac{1}{2}\log 2\pi$, $\Gamma(2)\zeta(2) = \frac{\pi^2}{6}$,

and so an easily calculation shows that (5.1) holds

$\boxed{6}$ Final remarks

The deduction of Corollary 1 was easy; however the proof of Theorem 4 remains as a difficult task. For an accesible and direct treatment, we recommend the reader also to consult Newman's book [12] (the same approach is reproduced in [4].

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