Asymptotic formulas for partitions, II.

Note Title 11/17/2011

Ingham's approach

by Jasson Vindas

Logic and Analysis Seminar

Gent, December 2, 2011

Abstract: This is the second part of the series of talks that we gave on asymptotic formulas for partitions.

The aim of this note is to discuss Ingham's approach to the problem. We start by recalling Ingham's theorem on strong asymptotics for unrestricted partitions. We then apply it to show Hardy-Ramanujan-Uspensky formula for the number of unrestricted partitions into integral parts. The main part of the text deals with a complex Tauberian theorem of Ingham for large Laplace transforms. We then show how to generate theorems for partitions from it. In the next note, we discuss a strong asymptotic formula for a multiplicative partition problem arising when coding rooted trees with prime numbers.

1 Introduction

We refer to the previous note for motivation of the material presented here.

We shall consider in the sequel the following general partition problem. Let

$$0<\lambda_1<\lambda_2<\lambda_3<\cdots$$

be an increasing sequence of real numbers tending to infinity and let N be its counting function, i.e.,

$$N(u)=\sum_{\lambda_k\leq u} 1 \qquad (1.1)$$

For a real number $\mu>0$, we consider the unrestricted partition function

$$p(\mu), \qquad (1.2)$$

that is, $p(\mu)$ is the number of representations of μ as

$$\mu=m_1\lambda_1+m_2\lambda_2+\cdots \qquad (m_j\geq 0). \qquad (1.3)$$

Finally, we set for $u\geq 0$,

$$P(u)=\sum_{\mu\leq u} p(\mu) \qquad (1.4)$$

where the summation is taken over those (finitely many) $\mu<u$ such that $p(\mu)\neq 0$.

The problem of interest is then to deduce asymptotic formulas for the functions P and P_h, where h is a positive, given by

$$(1.5) \quad P_h(u) = \frac{P(u) - P(u-h)}{h},$$

from the asymptotic properties of N.

In the special case $\lambda_k = k \in \mathbb{N}$, we obtain the classical unrestricted partition problem, where

$$(1.6)\ p(n) = \#\left\{ (m_1, \ldots, m_n) \in \mathbb{N}^n : n = \sum_{j=1}^{n} m_j \cdot j \right\}.$$

The classical Hardy-Ramanujan-Uspensky formula tells the asymptotics of (1.6), it follows the rule [3, 7, 8, 12, 14]

$$(1.7) \quad p(n) \sim \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{(4\sqrt{3})n}, \quad n \to \infty.$$

Observe also that in this case $N(x) = [x]$, but information than this is needed to establish (1.9)!

We will deduce (1.7) in $\boxed{3}$, but first, we recall Ingham's theorem for partitions.

$\boxed{2}$ Ingham's theorem for strong asymptotics.

In 1941, Ingham deduced the following result for strong asymptics of partitions from a complex Tauberian theorem (see $\boxed{4}$).

The proof of the ensuing result is postpone for $\boxed{5}$. Recall that if $h>0$, we denote

$$P_h(u)=\frac{P(u)-P(u-h)}{h}.$$ ζ is the Riemann zeta function

Theorem 1 If

$$N(u)=Au^{\alpha}+R(u), \quad (\alpha>0, A>0),$$

where

$$(2.1) \quad \int_0^u \frac{R(t)}{t}\,dt = B\log u + C + o(1), \quad u\to\infty,$$

Then, $u\to\infty$,

$$P(u)\sim\left(\frac{1-\beta}{2\pi}\right)^{\frac{1}{2}} e^{C} M^{-(B+\frac{1}{2})\beta} u^{(B+\frac{1}{2})(1-\beta)-\frac{1}{2}} e^{\frac{(Mu)^{\beta}}{\beta}},$$

where

$$\beta=\frac{\alpha}{\alpha+1}, \quad M=\left[A\alpha\Gamma(\alpha+1)\zeta(\alpha+1)\right]^{\frac{1}{\alpha}}.$$

Moreover, if $P_h(u)$ is increasing,

$$P_h(u)\sim\left(\frac{1-\beta}{2\pi}\right)^{\frac{1}{2}} e^{C} M^{-(B-\frac{1}{2})\beta} u^{(B-\frac{1}{2})(1-\beta)-\frac{1}{2}} e^{\frac{(Mu)^{\beta}}{\beta}}.$$

Remark :

P_h is certainly increasing if h is an element of $\{\lambda_k\}$.

We derive in the next section Hardy-Ramanujan-Uspensky theorem from Theorem 1.

3 Hardy-Ramanujan-Uspensky formula.

Corollary 1: If $\lambda_k = k \in \mathbb{N}$, then

$$p(n) \sim \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{(4\sqrt{3})n}, \quad n \to \infty. \tag{3.1}$$

Proof: All we have to do is to estimate

$$\int_0^u \frac{R(t)}{t}\,dt,$$

where $N(u) = [u] = u + R(u)$, and then use the formula for $P_1(n) = P(n) - P(n-1) = p(n)$.

So,

$$\int_0^u \frac{[t]-t}{t}\,dt = \int_0^u \frac{[t]}{t}\,dt - u$$

$$= [u]\log u - \int_0^u \log t \, d[t] - u$$

$$= [u]\log u - \sum_{n \le u} \log n - u$$

$$= [u]\log u - \log([u]!) - u$$

But Stirling's formula [4, p.43] tells us that

$$\log N! = \left(N+\frac{1}{2}\right)\log N - N + \log\sqrt{2\pi} + O(1)$$

So we get

$$\int_0^u \frac{R(t)}{t}\,dt = [u]\log u - [u]\log[u] - \frac{1}{2}\log[u] + [u]$$

$$- \log\sqrt{2\pi} - u + O(1)$$

$$= [u]\log\left(1+\frac{u-[u]}{[u]}\right) - \frac{1}{2}\log[u] + [u] - u$$

$$- \log\sqrt{2\pi} + O(1)$$

$$= u - [u] + O(1) - \frac{1}{2}\log[u] + [u] - u - \log\sqrt{2\pi}$$

$$= \frac{1}{2}\log\left(1+\frac{u-[u]}{[u]}\right) - \frac{1}{2}\log u - \log\sqrt{2\pi} + O(1)$$

$$= -\frac{1}{2}\log u - \log\sqrt{2\pi} + O(1).$$

Thus in Theorem 1 we have that

$\alpha = 1$, $A = 1$, $B = -\frac{1}{2}$, $C = -\frac{1}{2}\log 2\pi$, $\Gamma(2)\zeta(2) = \frac{\pi^2}{6}$,

Hence $\beta = \frac{1}{2}$, $M = \frac{\pi^2}{6}$. Therefore,

$$p(n) \sim \left(\frac{1}{4\pi}\right)^{\frac{1}{2}} \left(\frac{1}{\sqrt{2\pi}}\right) \cdot \left(\frac{\pi^2}{6}\right)^{\frac{1}{2}} n^{-1} e^{2\sqrt{\frac{\pi^2}{6} n}}$$

$$= \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{(4\sqrt{3})n}, \quad n \to \infty.$$

$\boxed{4}$ Ingham's Complex Tauberian Theorem

We now discuss a powerful tool in generating theorem for partitions, namely, Ingham's Tauberian theorem from [8] (see also [10, p. 223]).

We shall consider here two functions $\varphi(s)$ and $\chi(s)$ that are analytic in some domain Ω that contains a real segment $(0,h]$. We set $\delta(s) = \operatorname{dist}(s, \partial\Omega)$; $s \in \Omega$. Moreover, we write $s = \sigma + it$. We assume:

i) $\Omega \cap \{s : |\arg s| \leq \eta\}$, for all $0 \leq \eta < \frac{\pi}{2}$.

ii) φ and χ are real and positive in $(0,h]$.

iii) $-\sigma\varphi'(\sigma) \nearrow \infty$ and $-\sigma^k \varphi'(\sigma) \searrow 0$ as $\sigma \searrow 0$ for some $k \in \mathbb{N}$.

iv) $$\frac{[\varphi''(\sigma)]^{\frac{1}{2}}}{-\varphi'(\sigma)} = o\left(\frac{\delta(\sigma)}{\sigma}\right) \quad \sigma \searrow 0$$

v) $$\varphi''(\sigma+\tau) = O(\varphi''(\sigma)) \quad \chi(\sigma+\tau) = O(\chi(\sigma))$$

uniformly for $|z|<\delta(\sigma)$, $\sigma \searrow 0$.

Finally, we set

$$f_0(s) = \chi(s) e^{\varphi(s)}$$

and,

$$S_0(u) = \frac{\chi(\sigma) e^{\varphi(\sigma)+u\sigma}}{\sigma\sqrt{2\pi\varphi''(\sigma)}},$$

where $\sigma = \sigma(u)$ is the unique solution of $u = -\varphi'(\sigma)$.
With this notation we have:

Theorem 2 Let S be an increasing function that vanishes for $u \le 0$. Suppose that its Laplace-Stieltjes transform converges on $\operatorname{Re} s > 0$,

$$F(s) = \int_0^\infty e^{-su} dS(s), \quad \operatorname{Re} s > 0.$$

Assume that

a) $F(\sigma) \sim f_0(\sigma)$ as $\sigma \to 0^+$ $(\sigma \in \mathbb{R})$.

b) $F(s) = O(f_0(|s|))$ as $s \to 0$
uniformly on angles $|\arg s| \le \eta < \frac{\pi}{2}$

Then,

$$S(u) \sim S_0(u)$$

Example: Suppose that $F(s) \sim e^{\varphi(s)} = e^{\frac{1}{s}}$, uniformly on angles $|\arg s| \leq \eta < \frac{\pi}{2}$. One may take Ω as the intersection of a disk with $\operatorname{Re} s > 0$.

Since $-f_0'(t) = \frac{1}{t^2} \Longrightarrow h(u) = u^{\frac{1}{2}}$. Furthermore $f_0''(t) = \frac{2}{t^3}$. So, one gets

$$S(u) \sim S_0(u) = \frac{e^{\sqrt{u} + \sqrt{u}}}{u^{-\frac{1}{2}} \sqrt{2\pi u^{\frac{3}{2}}}} = \frac{1}{\sqrt{2\pi}} \cdot u^{-\frac{1}{4}} \, e^{2\sqrt{u}}.$$

$\boxed{6}$ Proof of Theorem 1

(We start by considering the generating functions (see the first part of the notes).

First notice that

$$N(u) \leq \int_u^{2u} \frac{N(t)}{t} dt = O(u^{\alpha}) + R(2u) - R(u) = O(u^{\alpha})$$

Thus, the usual calculations with the generating functions are justified. Recall:

$$(6.1) \quad F(s) = \int_0^{\infty} e^{-su} dP(u) = \sum_n p(n) e^{-ns} = \prod_{k=1}^{\infty} \frac{1}{1 - e^{-s\lambda_k}}$$

and so

$$(6.2) \quad f(s) = \log F(s) = s \int_0^{\infty} \frac{1}{e^{su} - 1} N(u) \, du, \text{ i.e.,}$$

On the other hand,

$$(6.3)\quad F_h(s) = \int_0^\infty e^{-su}\, dP_h(u) = \frac{1-e^{-hs}}{h} F(s)$$

$$\sim s F(s).$$

Now, $N(u) = A u^\alpha + R(u)$, so

$$\log F(s) = f(s) = A \int_0^\infty \frac{s u^\alpha}{e^{su}-1}\, du + \int_0^\infty \frac{R(u)}{u} \frac{su}{e^{su}-1}\, du,$$

so if $\omega(u) = \dfrac{u}{e^u - 1}$, then $\omega(0) = 1$, and so,

$$\log F(s) = A\Gamma(\alpha+1)\zeta(\alpha+1)s^{-\alpha} - \int_0^\infty (B\log u + C) s\omega'(su)\, du$$

$$+ O(1)$$

$$= A\Gamma(\alpha+1)\zeta(\alpha+1)s^{-\alpha} + C - B\log s - B\int_0^\infty \omega'(u)\log u\, du + O(1)$$

Since, $\displaystyle\int_0^\infty \omega'(u)\log u\, du = \lim_{\varepsilon\to 0^+} \int_\varepsilon^\infty \omega'(u)\log u\, du =$

$$= \left[\frac{u}{e^u-1}\log u - \log(1-e^{-u})\right]_\varepsilon^\infty = 0,$$

We finally obtain

$$F(s) = \chi_0(s) e^{\varphi(s)}, \quad F_h(s) \sim \chi_1(s) e^{\varphi(s)},$$

$s \to 0$, angularly, where

$$\chi_0(s) = e^{C s^{-B}}, \quad \chi_1(s) = e^{C s^{1-B}}$$

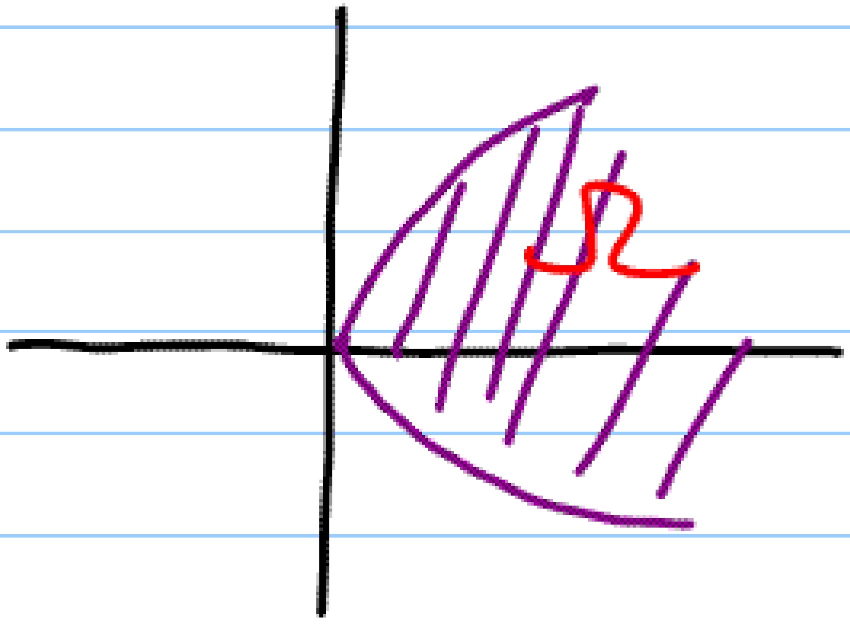
and $\varphi(s) = \frac{1}{\alpha}\left(\frac{M}{s}\right)^{\alpha}$, where

$$M = \left[A\alpha\, \Gamma(\alpha+1)\, \zeta(\alpha+1)\right]^{\frac{1}{\alpha}}.$$

We can apply Theorem 2 if we take for example

$$\Omega = \{ s = \sigma + it : \ |t| < K\sigma^{\gamma} \} \text{ with } K > 0$$

and $1 \leq \gamma < 1 + \frac{B}{2}$.



A simple calculation leads to the formulas stated in Theorem 1.

References

[1] N.H. Bingham, C.M. Goldie, J.L. Teugels, Regular variation, Cambridge University Press, 1987.

[2] N.G. de Bruijn, Pairs of slowly oscillating functions occuring in asymptotic problems concerning the Laplace transform, Nieuw Arch. Wisk (3) 7, 20-26.

[3] P. Erdős, On an elementary proof of some asymptotic formulas in the theory of partitions, Ann. of Math. 43 (1942), 437-450.
[4] R. Estrada, R.P. Kanway, A distributional approach to asymptotics. Theory and applications, second edition, Birkhäuser, 2002.
[5] G.H. Hardy, Divergent series, Clarendon Press, 1949.
[6] G.H. Hardy, S. Ramanujan, Asymptotic formulae for the distribution of integers of various types, Proc. London Math. Soc. 16 (1917), 112-132.
[7] G.H. Hardy, S. Ramanujan, Asymptotic formulae in combinatory analysis, Proc. London Math Soc. 17 (1918), 75-115.
[8] A.E. Ingham, A Tauberian theorem for partitions, Ann. of Math. 42 (1941), 1075-1090.
[9] K. Knopp, Asymptotische formeln der additiven Zahlentheorie, Schriften der Königsberger Gelehrten Gesselschaft, 2 jahr vol. 3 (1925), 45-94.
[10] J. Korevaar, Tauberian theory. A century of developments, Springer, 2004.