Factorization theorems in Denjoy-Carleman classes associated to representations of $(\mathbb{R}^d, +)$

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Factorization theorems in modules over function algebras is an important subject with a long tradition in mathematical analysis.

A module $\mathcal{M}$ over a non-unital algebra $\mathcal{A}$ is said to have the strong factorization property if

$$\mathcal{M} = \mathcal{A} \cdot \mathcal{M} = \{a \cdot m \,|\, a \in \mathcal{A}, m \in \mathcal{M}\}.$$

It is said to have the weak factorization property if

$$\mathcal{M} = \operatorname{span}(\mathcal{A} \cdot \mathcal{M}).$$

We will present some new results about strong factorization:

1. A strong factorization theorem of Dixmier-Malliavin type for ultradifferentiable vectors of representations of $(\mathbb{R}^d, +)$.
2. We have established the strong factorization property for many families of convolution modules of ultradifferentiable functions. We will give some examples.

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Factorization in classical function algebras

- Factorization theorems on $\mathbb{T}$ go back to Salem and Zygmund.
- Rudin showed (1957-1958): $L^1(\mathbb{R}^d) = L^1(\mathbb{R}^d) * L^1(\mathbb{R}^d)$.
- Cohen (1959) extended this result to the function algebra of a locally compact abelian group G,

$$L^1(G) = L^1(G) * L^1(G).$$

- Hewitt (1964) used Cohen technique to prove a general factorization theorem for Banach modules.
- Cohen-Hewitt type factorization theorems also hold for various Fréchet modules.
- Essential hypothesis: existence of bounded approximative units on the algebra under consideration.
- Many locally convex algebras do not have bounded approximative units. Examples: many basic algebras of smooth functions occurring in analysis.

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Factorization in algebras of smooth functions

- Ehrenpreis' problem (1960):

Does $\mathcal{D}(\mathbb{R}^d)$ factorize as $\mathcal{D}(\mathbb{R}^d) = \mathcal{D}(\mathbb{R}^d) * \mathcal{D}(\mathbb{R}^d)$?

- In 1978, Rubel, Squires, and Taylor, showed that $\mathcal{D}(\mathbb{R}^d)$ has the weak factorization property, namely,

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- If $d \geq 3$, they also showed that $\mathcal{D}(\mathbb{R}^d)$ does not have the strong factorization property.
- Dixmier and Malliavin (1979): negative answer for $d = 2$.
- Yulmukhametov (1999): in contrast $\mathcal{D}(\mathbb{R}) = \mathcal{D}(\mathbb{R}) * \mathcal{D}(\mathbb{R})$ holds.
- Several authors have independently shown (Miyazaki; Petzeltová and P. Vrbová; Dixmier and Malliavin; Voigt; ...)

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Factorization on Lie groups

- Let G be a real connected Lie group.
- Dixmier and Malliavin showed (1979) that

$$\mathcal{D}(G) = \text{span}(\mathcal{D}(G) * \mathcal{D}(G))$$

and, when additionally G is nilpotent,

$$\mathcal{S}(G) = \mathcal{S}(G) * \mathcal{S}(G).$$

(hereafter: convolution = left convolution)
- Let E be a locally convex Hausdorff (sequentially complete) space and denote as $\mathrm{GL}(E)$ its group of isomorphisms.
- A group homomorphism $\pi : G \to \mathrm{GL}(E)$ such that

$$G \times E \to E, \quad (g, e) \mapsto \pi(g)e$$

is separately continuous is a representation of G on E.
- We call $e \in E$ a smooth vector if its orbit mapping

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The induced algebra representation Π

A representation of G on E induces a natural action on function convolution algebras.

- If $f \in C_c(G)$, we can define:

$$(f, e) \mapsto \Pi(f)e, \quad C_c(G) \times E \to E, \quad \text{where}$$

$$\boxed{\Pi(f)e = \int_G f(g)\pi(g)e \, \mathrm{d}\, g \in E}$$

- Note $\Pi(f_1 * f_2) = \Pi(f_1) \circ \Pi(f_2)$, where $*$ is left-convolution.
- If $\Pi(g) = L_g$ is left-translation and E is a function space,

$$(\Pi(f)e)(x) = \int_G f(g)e(g^{-1}x)dg,$$

so that $\Pi(f)e = f * e$.

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A representation also induces an action of the convolution algebra $\mathcal{D}(G)$ on the smooth vectors,

$$(f,e)\mapsto \Pi(f)e, \quad \mathcal{D}(G)\times E^{\infty}\to E^{\infty}, \quad \text{where}$$

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So, E^{∞} is module over $\mathcal{D}(G)$.

Theorem

If E is a Fréchet space, E^{∞} has the weak factorization property w.r.t. $\mathcal{D}(G)$, that is, $\boxed{E^{\infty}=\operatorname{span}(\Pi(\mathcal{D}(G))E^{\infty})}$.

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A representation also induces an action of the convolution algebra $\mathcal{D}(G)$ on the smooth vectors,

$$(f,e) \mapsto \Pi(f)e, \quad \mathcal{D}(G) \times E^{\infty} \to E^{\infty}, \quad \text{where}$$

$$\boxed{\Pi(f)e = \int_G f(g)\pi(g)e \, \mathrm{d}g \in E}$$

So, E^{∞} is module over $\mathcal{D}(G)$.

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If G is a compact Lie group, one always has $\boxed{E^{\infty} = (\Pi(C^{\infty}(G))E^{\infty})}$.

Strong factorization also holds in other situations, but one needs to take into account the growth of the representation.

- Let $\mathfrak{d}$ be the distance associated to a left-invariant Riemannian metric and $1 \in G$ the group identify. We write $|g| := \mathfrak{d}(1, g)$.
- If E is Banach there is n such that $\|\pi(g)\|_{L_b(E)} \leq e^{n|g|}$.
- Thus, $\boxed{\Pi(f) = \int_G f(g)\pi(g)\,\mathrm{d}\,g}$ is well defined as long as f is exponentially rapidly decreasing on G.

Theorem

If E is a Hilbert space, the representation is unitary, and G is nilpotent, then E^{∞} has the strong factorization property w.r.t. $\mathcal{S}(G)$.

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If E is a Hilbert space, the representation is unitary, and G is nilpotent, then E^{∞} *has the strong factorization property w.r.t.* $\mathcal{S}(G)$.

Analytic factorization of Lie group representations

- $e \in E$ is an analytic vector if $g \mapsto \pi(g)e$ is an analytic mapping.
- E^ω: subspace of analytic vectors.
- A representation is called an F-representation if
  - E is a Fréchet space;
  - there is a basis of continuous seminorms $(p_n)_{n\in\mathbb{N}}$ such that for each n the action $G \times (E, p_n) \to (E, p_n)$ is continuous.
- For F-representations, we get an action of the algebra of exponentially rapidly decreasing analytic functions $\mathcal{A}(G)$ on E^ω.

Theorem (Gimperlein, Krötz, and Lienau (2012))

For F-representations, E^ω has the weak factorization property w.r.t. $\mathcal{A}(G)$, that is, $\boxed{E^\omega = \operatorname{span}(\Pi(\mathcal{A}(G))E^\omega)}$.

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Analytic factorization for $(\mathbb{R}^d, +)$

The convolution algebra $\mathcal{A}(\mathbb{R}^d)$ consists of real analytic functions f admitting holomorphic extension to $\mathbb{R}^d + i]-h, h[^d$ for some $h > 0$ and satisfying

$$\sup_{|\operatorname{Im} z| \leq h} e^{n|\operatorname{Re} z|}|f(z)| < \infty, \qquad \text{for each } n \in \mathbb{N}.$$

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Curiously, $\mathcal{A}(\mathbb{R}^d) = E^\omega$ for the regular representation of $\mathbb{R}^d$ on

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Some remarks

Our results hold for more general representations than F-representations:

- projective generalized proto-Banach representations;
- inductive generalized proto-Banach representations.

Also, they apply to more general classes than that of analytic vectors:

- In fact, for Denjoy-Carlemann classes of smooth vectors;
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Projective and inductive generalized proto-Banach representations

Let $\operatorname{csn}(E)$ be collection of all continuous seminorms on E.

Definition

A representation (π, E) is said to be a projective generalized proto-Banach representation if

$$\forall p \in \operatorname{csn}(E)\, \exists q_p \in \operatorname{csn}(E)\, \exists \kappa_p > 0\, \forall x \in \mathbb{R}^d\, \forall e \in E\ :$$
$$p(\pi(x)e) \leq e^{\kappa_p |x|} q_p(e)$$

$\mathfrak{B}(E)$ stands for the collection of non-empty absolutely convex closed bounded subsets of E and for $B \in \mathfrak{B}(E)$ we denote $E_B = \operatorname{span}(B)$.

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Denjoy-Carleman classes

Consider a log-convex sequence $M = (M_p)_p$ of positive numbers and

$$\omega_M(t) = \sup_{p\in\mathbb{N}} \log\left(\frac{t^p M_0}{M_p}\right), \qquad t > 0.$$

We impose the assumption:

$$0 < \liminf_{t\to\infty} \frac{\omega_M(\lambda t)}{\omega_M(t)} \leq \limsup_{t\to\infty} \frac{\omega_M(\lambda t)}{\omega_M(t)} < \infty, \quad \forall \lambda > 0.$$

Prototypical example: $M_p = (p!)^\sigma$, with $\sigma > 0$. Then, $\omega_M(t) \asymp t^{1/\sigma}$.

- A vector $e \in E$ is ultradifferentiable of class $[M]$ if its orbit mapping is (bornologically) ultradifferentiable of class $[M]$.
- $[M]$ is the common notation for both the Beurling (M) and $\{M\}$ Roumieu cases of ultradifferentiability.
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$$\omega_M(t) = \sup_{p\in\mathbb{N}} \log\left(\frac{t^p M_0}{M_p}\right), \qquad t > 0.$$

We impose the assumption:

$$0 < \liminf_{t\to\infty} \frac{\omega_M(\lambda t)}{\omega_M(t)} \leq \limsup_{t\to\infty} \frac{\omega_M(\lambda t)}{\omega_M(t)} < \infty, \quad \forall\lambda > 0.$$

Prototypical example: $M_p = (p!)^\sigma$, with $\sigma > 0$. Then, $\omega_M(t) \asymp t^{1/\sigma}$.

- A vector $e \in E$ is ultradifferentiable of class $[M]$ if its orbit mapping is (bornologically) ultradifferentiable of class $[M]$.
- $[M]$ is the common notation for both the Beurling (M) and $\{M\}$ Roumieu cases of ultradifferentiability.
- $E^{[M]}$ denotes the space of ultradifferentiable vectors of class $[M]$ of a representation.
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Factorization theorem for ultradifferentiable vectors

For $h > 0$, we define the Fréchet space

$$\mathcal{K}^{M,h}(\mathbb{R}^d) = \{\varphi \in C^\infty(\mathbb{R}^d) \mid \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{h^{|\alpha|} |\partial^\alpha \varphi(x)| e^{n|x|}}{M_{|\alpha|}} < \infty, \quad \forall n \in \mathbb{N}\}.$$

We set

$$\mathcal{K}^{(M)}(\mathbb{R}^d) = \varprojlim_{h \to \infty} \mathcal{K}^{M,h}(\mathbb{R}^d) \quad \text{and} \quad \mathcal{K}^{\{M\}}(\mathbb{R}^d) = \varinjlim_{h \to 0^+} \mathcal{K}^{M,h}(\mathbb{R}^d).$$

If $M_p = p!$, then $\mathcal{A}(\mathbb{R}^d) = \mathcal{K}^{\{M\}}(\mathbb{R}^d)$.

Theorem (Debrouwere, Prangoski, and V. (2021))

Let (π, E) be either a projective or an inductive generalized proto-Banach representation of $(\mathbb{R}^d, +)$ on a sequentially complete lcHs E. Then, $E^{[M]}$ has the strong factorization property w.r.t. $\mathcal{K}^{[M]}(\mathbb{R}^d)$

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- Our factorization theorem implies the strong factorization property for many concrete families of modules of ultradifferentiable functions.

Example:

- Let $\omega : \mathbb{R}^d \to (0, \infty)$ be a continuous weight function satisfying

$$\sup_{x \in \mathbb{R}^d} \frac{\omega(x + \cdot)}{\omega(x)} \in L^\infty_{loc}(\mathbb{R}^d).$$

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Factorization of modules of ultradifferentiable functions

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For more details, see our article:

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Related works on factorization theorems for representations:

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- H. Gimperlein, B. Krötz, C. Lienau, *Analytic factorization of Lie group representations*, J. Funct. Anal. **262** (2012), 667–681.
- H. Glöckner, *Continuity of LF-algebra representations associated to representations of Lie groups,* Kyoto J. Math. 53 (2013), 567–595.